Jens Høyrup

"On Parts of Parts and Ascending Continued Fractions. An Investigation of the Origins and Spread of a Peculiar System".

Filosofi og Videnskabsteori på Roskilde Universitetscenter.
3. Række: *Preprints og Reprints* 1988 Nr. 2.

(manuscript, not faithful to the original pagination)

Contents

O. Neugebauer in memoriam

I. INTRODUCTION

The following article deals with two particular ways to denote fractional numbers, one of them multiplicative (»parts of parts«) and the other multiplicative-additive (»ascending continued fractions«). They turn up in sources from several cultures and epochs, but as a standard idiom only in Arabic mathematics, where their occurrence has been amply described. In certain other contexts (Babylonia, High and Late Medieval Europe) their occasional presence has been taken note of though rarely investigated systematically. Finally, a few scattered occurrences in Ancient Greek and Egyptian sources have not been commented upon until this day.

Widespread occurrence of similar practices raises the question of interdependence versus independent development by accident or in response to analogous situations. Thus also in this case. Posing the question, however, turns out to be more easy than answering it, not least because some of the cultures to be dealt with only present us with utterly few examples of the usage, and only the combination of evidence and arguments of many kinds will allow us to construct a scenario which is at least well-founded if not definitively verified on all points.

As a by-product, the inquiry will cast new light on the origins of the Egyptian unit fraction system.

II. ISLAMIC AND POST-ISLAMIC EVIDENCE

In chapter V of Leonardo Fibonacci's *Liber abaci* (second version, 1228) a number of complex writings for fractional numbers are introduced. One of them – the others are irrelevant for the present purpose – is what later has come to be called the »ascending continued fraction« (»Aufsteigende Kettenbrüche« in German), which Leonardo exemplifies by the number

$$\frac{1\ \ 5\ \ 7}{2\ \ 6\ \ 10}$$

meaning 7 10ths plus 5 6ths of a 10th plus ½ of a 6th of a 10th[1] – in more compact writing $\{{}^{1}/_{10}\cdot[7+({}^{1}/_{6})\cdot(5+½)]\}$. In general,

$$\frac{a_3 \ a_2 \ a_1}{b_3 \ b_2 \ b_1}$$

stands for

$$\frac{a_1}{b_1} + \frac{a2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} = \frac{a_1 + \frac{a_2 + \frac{a_3}{b_3}}{b_2}}{b_1 b_2 b_3}$$

The generalization to two or four or more levels is obvious. Incidentally, the latter expression demonstrates that »ascending continued fractions« have nothing but an inverted visual image in common with genuine continued fractions.

The notations for ascending continued fractions was not invented by Leonardo but apparently in the Maghreb mathematical school, probably during the 12th century. They are discussed in ibn al-Bannā'’s 13th century *Talkhīṣ aʿmāl al-ḥisāb*[2] though without indication of the way they were to be written. Various commentaries show, however, that standardized notations were in use. In one late commentary, al-Qalaṣādī’s *Arithmetic*[3] (1448), it is furthermore required that the denominators in an ascending continued fraction stand in descending order from the right ($b_1>b_2>b_3$), as it is actually the case in Leonardo’s examples. Even though some of the examples.2(of)-2mivenof other commentators[4] do not observe this rule, which I shall denote *al-Qalaṣādī’s canon* in the following, it was probably not of al-Qalaṣādī’s own making. The purpose of the canon may have to do with the value of the first member as an approximation. The error committed by throwing away all members but the first will necessarily be less than $1/b_1$ (*a*’s are supposed to be less than corresponding *b*’s). Choosing b_1 as large as possible

1

will ensure that a_1/b_1 is a good approximation (though not necessarily the optimal approximation, cf. note 28). Thus, in Leonardo's example, dividing first by 10 ensures that the first member will at most be 0.1 off the true value (a's are always smaller than the corresponding b's). If the reverse canon ($b_1<b_2<b_3$) had been used, the result had been ${}^{1}/_{2}+{}^{1}/_{2}\cdot{}^{1}/_{2}+{}^{1}/_{2}\cdot{}^{1}/_{2}\cdot{}^{1}/_{6}$, and the error committed by taking the first member alone would have been ${}^{7}/_{24}$,0.29.

The invention of *notations* was part of the general drive of Maghreb mathematics, but verbally expressed ascending continued fractions and other composite fractional expressions belonged to the common lore of Arabic mathematics. They had been amply used and discussed in the later 10th century by Abū'l-Wafāʾ in his *Book on What Scribes, Officials and the Like Need from the Science of Arithmetic*.[5] They are also present in al-Khwārizmī's early ninth century *Algebra*[6] and as well as in the *Liber mensurationum* by one Abū Bakr, translated by Gherardo of Cremona into Latin in the 12th century and presumably written in the first place around 800 A.D.[7] Among the occurrences in al-Khwārizmī's work are the following (page references to Rosen's translation):

- P. 24, ${}^{25}/_{36}$ is transformed into »two-thirds and one-sixth of a sixth« [${}^{2}/_{3}+{}^{1}/_{6}\cdot{}^{1}/_{6}$].
- P. 45, 1 *māl* is found as »a fifth and one-fifth of a fifth« of $4\,{}^{1}/_{6}$ *māl* [${}^{1}/_{5}+{}^{1}/_{5}\cdot{}^{1}/_{5}$].
- P. 53: »three and three-fourths of twenty parts« [${}^{3}/_{20}+{}^{3}/_{4}\cdot{}^{1}/_{20}$] is transformed into »fifteen eightieths«.
- P. 54, a twelfth is expressed as »the moiety of one moiety of one-third« [${}^{1}/_{2}\cdot{}^{1}/_{2}\cdot{}^{1}/_{3}$].
- P. 72, as one of several rules for finding the circular area we find the square of the diameter minus »one seventh and half one-seventh of the same«.
- P. 88, the third of »nine dirhems and four-fifth of thing« is found to be »three dirhems, and one-fifth and one-third <of> one-fifth of thing« [$3+{}^{1}/_{5}+{}^{1}/_{3}\cdot{}^{1}/_{5}$].
- P. 99. »two-sevenths and two thirds of a seventh of the share of a son« [${}^{2}/_{7}+{}^{2}/_{3}\cdot{}^{1}/_{7}$].

The *Liber mensurationum* (which contains mostly integer numbers) presents us with the following relevant passages:

---

[5] See Youschkevitch, "Abū'l-Wafāʾ"; *idem* 1976: 25ff; or Saidan 1974. My poor Russian has not permitted me to make much use of Medovoj's fuller description (1960) of Abū'l-Wafāʾ's treatise. Nor have I been able to use Saidan's Arabic edition (1971: 64-368) of the work.

[6] Ed., transl. Rosen 1831.

[7] Ed. Busard 1968. As to the dating (built on terminological considerations), see Høyrup 1986.

- N° 19 (p. 90), *7 et dimidium septime*.
- N° 89 (p. 107), *43 et due quinte et quattuor quinte quinte*, resulting from the computation of $169-(11\,^1/_5)^2$. Similarly but in greater computational detail in N° 128 (p. 115).
- N° 113 (p. 112), the root of $^3/_{16}$ *census* is expressed as *radix octave census et medietatis octave census*.
- N° 144 (p. 118), the area of the circle is expressed as the square on the diameter minus *septimam et septime eius medietatem*. Similarly in N^{os} 146, 156 and 158 (pp. 119 and 124).

The elementary building stones of the ascending continued fractions are the »parts of parts«, the *partes de partibus* as they came to be called in the Medieval Latin tradition, i.e., expressions of the form »$^p/_q$ of $^1/_r$«. The extent to which these were natural to Arabic speakers of early Islam is demonstrated in the first treatise of the 10th century *Epistles of the Brethren of Purity*, the *Rasāʾil ikhwān al-ṣafāʾ*. In this exposition of the fundaments of arithmetic great care is taken to explain that the first of a collection of two is called a half, while the first of three is a third, that of four a fourth, and that of eleven one part of eleven; the first of twelve, however, is labeled a half of a sixth, without a single word commenting upon the reasons for or meaning of this composition. Similarly, the first of fourteen is expressed without explanation as a half of a seventh, and that of fifteen as a third of a fifth.[8]

The origin of both the parts of parts and of the ascending continued fractions has been ascribed to a variety of causes, in particular to the peculiarities of the Arabic vocabulary. Unit fractions from $^1/_2$ to $^1/_{10}$ possess a particular name of their own, while those with larger denominators require a full phrase, $^1/_n$ being expressed as »one part of *n*« or »one part of *n* parts« *unless* it can be composed from unit fractions with smaller denominators. This might indeed explain that the Arabic authors transformed the $^1/_{14}$ of Hero's (or rather pseudo-Hero's) rule for finding the circular area[9] into »half one-seventh«, and that they expressed $^1/_{25}$ as »one-fifth of a fifth«.

On the other hand, »the moiety of one moiety of one-third« is somewhat at odds with the hypothesis: Why not »one-third of a fourth«, when in the actual case the number 12 arises as $3 \cdot 4$? Or at least »one-half of a sixth«, which

[8] Transl. Brentjes 1984: 212f.

[9] The square on the diameter minus $^1/_7$ and $^1/_{14}$ of the square. *Geometrica* **24**.40, ed., transl. Heiberg 1912: 442f. Cf. *Geometrica* **17**.4, *ibid.* 332^b, 333^b.

according to Abū'l-Wafā' is to be preferred to »one-third of a fourth«,[10] and which still circumvents the difficulties created by the Arabic language while using only two factors? Al-Khwārizmī, moreover, had no particular difficulty with general fractions, at times with denominators exceeding 10, which abound even in those very calculations where the »parts of parts« turn up. The reason that the reciprocal of $^{25}/_{6}$ is expressed in the form of an ascending continued fraction on p. 45 of the *Algebra* while another ascending continued fraction is, reversely, reexpressed as $^{15}/_{80}$ on p. 53 seems simply to be that both reformulations fit the further calculations better. The conventional explanation of the use of composite expressions based *solely* on Arabic linguistic particularities is apparently insufficient, even if these particularities have evidently tainted the way the system was used.

III. CLASSICAL ANTIQUITY AND ITS LEGACY

The need for an explanation which goes beyond the peculiarities of the Arabic language is confirmed by certain older sources. One of them is the collection of arithmetical riddles in *Anthologia Graeca* XIV.[11] A study of these give the fascinating result that the types of fractional expressions used varies with the subject of the problem. Problems which refer to Greek mythology or history make use of unit or general fractions. So do all problems dealing with apples or walnuts stolen by girl friends, with the filling of jars or cisterns from several sources, with spinners', brickmakers' or gold- or silversmiths' production, with wills, and with the epochs of life – none of them make use of »parts of parts«.

»Parts of parts« and related composite expressions, on the other hand, turn up in all problems dealing with the Mediterranean extensions of the Silk Road (N^os^ 121 and 129), with the legal partition of heritages (N^os^ 128 and 143), and with the hours of the day (N^os^ 6, 139, 140, 141, and 142; N^o^ 141 is connected to astrology). A final »fifth of a fifth« is found in N^o^ 137, dealing with a catastrophic banquet probably meant to be held in Hellenistic Syria. It appears that a number of recreational problems belonging to (at least) two different contexts (providing the dress of the problems) have been brought together in the anthology, each conserving its own distinctive idiom for fractions: on one hand the traditional Greek idiom, which makes use of general and unit fractions; on the other, the

[10] Saidan 1974: 368.

[11] Ed., transl. Paton 1979. The editor of this part of the *Anthologia* was probably Metrodoros (fl. c. A.D. 500), but the single epigrams are older.

usage of the trading community and of juridical calculators (and perhaps of astrologers and makers of celestial dials), which is different.

We may list the various composite fractional expressions:[12]

- N° 6 (the hour of the day): »Twice two-third«.
- N° 121 (travelling from Cadiz to Rome): »One-eighth and the twelfth part of one-tenth«.
- N° 128 (a textually and juridically corrupt heritage): »The fifth part of seven-elevenths«.
- N° 129 (travelling from Crete to Sicily): »Twice two-fifths«.
- N° 137 (the Syrian banquet): »A fifth of the fifth part«.
- N° 139 (a dial-maker asked for the hour of the day): »Four times three-fifths«.
- N° 140 (the hour of a lunar eclipse): »Twice two-sixths and twice one-seventh«.
- N° 141 (the hour of a birth, to be used for a horoscope): »Six times two-sevenths«.
- N° 142 (The hour for spinning-women to wake up): »A fifth part of three-eighths«.
- N° 143 (The heritage after a shipwrecked traveller): »Twice two-thirds«.

We observe the character of these composite expressions is similar to but does not coincide with what we know from the Arabic texts. Firstly, of course, these do not contain integer multiples of fractions like those of N^os^ 6, 129, 139, 140, 141 and 143, and they would speak of »three fifths of an eighth«, not of »a fifth part of three-eighths«. Secondly, the Arabic sources mostly follow the canon made explicit by al-Qalasādī, while for instance N° 121 of the *Anthologia* does not – and $^1/_{12}$ they split further, *viz* into $^1/_2$ of $^1/_6$, into $^1/_3$ of $^1/_4$ or even, as we have seen, into $^1/_2$ of $^1/_2$ of $^1/_3$.

Most likely, the integer multiples of the *Anthologia* are to be explained from the recreational character of the arithmetical riddles: by being unusual, the multiples make the riddles more funny or more obscure at first sight – it is hardly imaginable that »two-thirds« would be expressed as »twice two-sixths« for any everyday purpose. The demands of versification may have played a supplemen-

[12] I follow Paton's translation, even though a somewhat more literal translation of some fractional expressions *could* be made. Paton's concessions to English rhythm are immaterial for the present purpose.

tary rôle – but since problems with a traditional »Greek« subject make no use of the stratagem hardly more than a supplementary rôle.

The deviation from »al-Qalaṣādī's canon«, however, gives no impression of grotesquerie and can therefore not be an effect of the recreational purpose of the epigrams. It is thus probable that it reflects the daily usage of the practitioners trading in »parts of parts«, which will not have respected the later Arabic canon and customs in full.

Another, Latin source of interest for our purpose is the Carolingian collection *Propositiones ad acuendos iuvenes* conventionally ascribed to Alcuin and dating from c. A.D. 800.[13] Chronologically, it is roughly contemporary with al-Khwārizmī and probably with the *Liber mensurationum*. The material, however, appears to be inherited from late Antiquity, and the Carolingian scholar (be it Alcuin or somebody else connected to the Carolingian educational effort) has only acted as an editor.

A brief exposition of the global character of the collection will serve the double purpose of locating its composite fractional expressions with respect to their background and of introducing some notions concerning the function of recreational problems from which the further discussion will benefit. In general, the collection is highly eclectic, bringing together material and methods from a variety of traditions, combining at times mutually incompatible approximations within the same problem solution.[14] Of particular interest in the present context is the very diverse network of connections behind the arithmetical problems. N° 13, dealing with 30 successive doublings of *1*, points back to a very similar problem from Old Babylonian Mari[15] and eastward to the Arabo-Indian chess-board problem and even to China. N^{os} 5, 32-34, 38-39 and 47 all belong to the type of »A hundred fowls« known from earlier Chinese and contemporary or earlier Indian sources[16] and presented by Abū Kāmil as a type of question »circulating among high-ranking and lowly people, among scholars and among the uneducated, at which they rejoice, and which they find new and beautiful;

[13] Ed. Folkerts 1978.

[14] See Høyrup 1987: 291 n. 38 (»an/42.9« in line 9 from bottom should read »and«).

[15] Published in Soubeyran 1984: 30. The connection and similarities between the Carolingian doublings and those from other epochs and places (except China) are discussed in detail in Høyrup 1986: 477-479. On China, see Thompson 1975: V, 542 (Z 21.1), or Høyrup 1987: 288f.

[16] See the survey in Tropfke/Vogel 1980: 613-616.

one asks the other, and he is then given an approximate and only assumed answer, they know neither principle nor rule in the matter«.[17] Other problems too point to the »oral technical literature«, the treasure of recreational problems shared and carried by the community of traders and merchants interacting along the Silk Road, the combined caravan and sea route reaching from China to Spain.[18]

Connections to the *Anthologia graeca* and thus to the Greco-Roman orbit are also present. Most significant is probably N^{o} 35, which is a puzzle on heritages – one of the types, we remember, which made use of multiples of parts. It can be traced back to Roman jurisprudential digests, even though the editor of the *Propositiones* has got the solution wrong[19]

A final type represented by N^{os} 2, 3, 4, 40 and 45 seems to by-pass what we know from the *Anthologia graeca* and point directly to Egyptian traditions (even though matters may in reality be more complex, cf. below, p. **???**). Admittedly, when expressed in algebraic symbolism the problems in question are of a type identical with the one dominating the *Anthologia graeca*, both being represented by first degree equations. The equations of the *Anthologia*, however, are variations on the pattern

$$x\cdot(1-{}^{1}/_{p}-{}^{1}/_{q}-{}^{1}/_{r})=R$$

(*p*, *q*, and *r* being integers), while N^{os} 2, 3, 4, 40 and 45 of the *Propositiones* build on the scheme

$$x\cdot(n+\alpha+\beta)=T$$

(*n* being an integer larger than 1 and α and β being unit fractions or »parts of parts«). Both types possess analogues in the Ancient Egyptian Rhind Mathematical Papyrus[20]. The former type corresponds approximately to N^{os} 24-27 and 31-34; these are problems which consider an unspecified quantity or »heap« (*ʿḥʿ*), and which only differ from those of the *Anthologia* by adding the unit fractions

---

[17] My English translation from Suter 1910: 100.

[18] The classification of recreational mathematics as a parallel to folk-tales and riddles, and thus as a special *genre* of oral literature, is discussed in Høyrup 1987: 288f and 1990: 74f.

The influence of eastern trading routes on the stock from which the *Propositiones* were drawn is also made clear by problems N^{os} 39 and 52, dealing, respectively, with the purchase of animals (including camels) *in oriente* and with transport on camel back.

[19] Folkerts 1978: 33.

[20] Ed., transl. Chace et al 1929.

instead of subtracting them. The first-degree problems of the *Propositiones* just spoken of, on the other hand, belong to the same type as Rhind Mathematical Papyrus N^{os} 35-38, problems dealing with the *ḥeḳat*-measure.[21]

The reason for this lengthy presentation of the *Propositiones* and of a particular group of first-degree problems is that four of the five problems in this group employ »parts of parts«:

– N^{o} 2: *medietas medietatis, et rursus de medietate medietas* (meaning $^{1}/_{2}\cdot{}^{1}/_{2}+{}^{1}/_{2}\cdot{}^{1}/_{2}\cdot{}^{1}/_{2}$)

– N^{o} 3: *ter et medietas tertii* ($^{1}/_{3}+{}^{1}/_{2}\cdot{}^{1}/_{3}$)

– N^{o} 4: *medietas medietatis* ($^{1}/_{2}\cdot{}^{1}/_{2}$).

– N^{o} 40: *medietatem de medietate et de hac medietate aliam medietatem* ($^{1}/_{2}\cdot{}^{1}/_{2}+{}^{1}/_{2}\cdot{}^{1}/_{2}\cdot{}^{1}/_{2}$)

Composite fractions thus seem to go naturally with this problem type. On the other hand, they occur nowhere else, neither in the problems which point to the »Silk Road corpus«, nor those which remind of one or the other group from the *Anthologia graeca*, nor in the inheritance problem. One observes that al-Khwārizmī's predilection for taking successive halves instead of a simple fourth is equally present here, and is even extended to the use of $^{1}/_{2}$ of $^{1}/_{3}$ instead of $^{1}/_{6}$. This is all the more remarkable since the simple terms *quadrans* and *sextans* were at hand,[22] and the composite *quarta pars* and *sexta pars* are actually used in other parts of the text (e.g., N^{os} 8 and 47). It will also be noticed that three of the four cases are rudimentary ascending continued fractions.

IV. BABYLONIA

Some scattered instances of »parts of parts« and of simple ascending continued fractions can thus be dug out from sources belonging to or pointing back to classical Antiquity though not to the core of Greek mathematical culture.[23] Antecedents for the fuller use of ascending continued fractions, on

[21] Another group from the *Propositiones*, consisting of N^{os} 36, 44 and 48, deviates from both models but comes closest to the *ḥeḳat*-type.

[22] In the sense that the use of these subdivisions of the *as* as names for abstract fractions is described explicitly in the preface to the fifth-century *Calculus* of Victorius of Aquitania (ed. Friedlein 1871: 58f).

[23] This peripheral status of the Greek »parts of parts« is borne out by Ananias of Shirak's 7th century arithmetical collection (ed., transl. Kokian 1919), a work strongly dependent on contemporary Byzantine teaching. »Parts of parts« are

the other hand, must be looked for further back in time – much further, indeed.

They can be found in the Babylonian tablet MLC 1731, which was analyzed by Abraham Sachs,[24] and which dates from the Old Babylonian period (c. 2000 to c. 1600 B.C.; the mathematical texts belong to the second half of the period). It presents us with the following examples of composite fractions:[25]

- Nº 1: »One-sixth of one-fourth of [the unit] a barleycorn«.
- Nº 3: »One-fourth of a barleycorn and one-fourth of a fourth of a barleycorn«.
- Nº 4: »One-third of a barleycorn and one-eighth of a third of 20«.[26]
- Nº 5: »Two-thirds of 20 and one-eighth of two-thirds«.
- Nº 6: »A barleycorn and one-sixth of a fourth of 20«.
- Nº 7: »A barleycorn, two-thirds of 20 and one-eighth of two-thirds of 20«.
- Nº 9: »17 bar<leycorns>, one-third of 20, and one-fourth of a third of a barleycorn«.

All these composite expressions result from the conversion of numbers belonging to the »abstract« sexagesimal system into metrological units. Sachs has convincingly pointing out that the notation in question is used because no unit below the barleycorn existed[27] – fractions could not be expressed in terms of a smaller unit, as done in other conversions to metrological notation. Still, the tablet shows that the parlance of »parts of parts« *was* at hand, and even that there was an outspoken tendency to make use of ascending continued fractions

---

as absent from this work as from the »Greek« problems of the *Anthologia graeca*.

[24] Sachs 1946. Besides the fractional expressions of that tablet, the article presents and discusses similar usages in other Babylonian tablets.

[25] In my translation of Babylonian texts, I follow the following conventions:
- »The *n*'th part« renders the expression »igi-*n*-gál«.
- Fractions and numbers written with numerals ($^2/_3$, $^1/_2$, etc.; 1, 2, etc.) renders special cuneiform signs for these fractions and numbers.
- Fractions and numbers written as words render corresponding expressions in syllabic writing.

[26] In all metrological systems, the barleycorn is 0;0,0,20 times the fundamental unit. »20« is thus a shorter way to write »a barleycorn«.

[27] Except in the system of weights, where ½ barleycorn existed as a separate unit – cf. Sachs 1946: 208f and note 16. Most likely, however, the text is concerned with area units (among other things because the numbers to be converted are obtained as products of two factors, both of which vary from problem to problem).

rather than of sums of unit fractions with denominators below 10.[28] We observe that two-thirds is the only general fraction to turn up, while everything else consists of unit fractions and their combinations,[29] and that »al-Qalaṣādī's canon« is inverted – be it accidentally or by principle.

This tablet presents us with the most systematic Old Babylonian use of composite fractions. It is not quite isolated, however, and scattered occurrences can be found here and there in other Old Babylonian tablets.

One instance was pointed out by Sachs: YBC 7164 N° 7 (line 18), where the time required for a piece of work is found to be »$^2/_3$ of a day, and the 5th part of $^2/_3$ of a day«.[30]

In another text from the Yale collection, »parts of parts« (though no ascending continued fractions) occur in all five times: YBC 4652 N^{os} 19-22,[31] problems of riddle-character dealing with the unknown weight of a stone. Here, »the 3d part of the 7th part«, »the 3d part of the 13th part«, »the 3d part of the 8th part« (twice) and »$^2/_3$ of the 6th part« turn up. We observe that the ordering of factors agrees with »al-Qalaṣādī's canon«, and that even a »13th part« is present. (Babylonian, in contrast to Arabic, had a name for this fraction).

In the series text YBC 4714, N° 28, line 10 (and probably also in the damaged text of N° 27), »a half of the 3d part« turns up in the statement.[32] This is evidently meant as a step toward greater complexity from the previous problems having »the *n*'th part« (n=7, 4, and 5) in the same place.

---

[28] In N° 4, the result could have been given as »$^1/_4+^1/_8$« (or as »$^1/_4+^1/_2\cdot{}^1/_4$«). In N^{os} 5 and 7, »$^1/_2+^1/_4$« (or »$^1/_2+^1/_2\cdot{}^1/_2$«), and in N° 9, »$^1/_4+^1/_6$« would have been possible.

The actual choices of the texts secure that the first member alone approximates the true value as closely as possible. They demonstrate that »al-Qalaṣādī's canon«, even though ensuring that the first member of the expansion is a *fair* approximation, of course does not guarantee it to be *optimal*.

[29] Naturally enough, this reminded Sachs of the Egyptian unit fraction system (as also borrowed by the Greeks): Even there, $^2/_3$ is treated on a par with the submultiples $^1/_2$, $^1/_3$, $^1/_4$, etc. He did not make much of the fact that »$^1/_3$ of $^1/_5$« would be no number to an Egyptian scribe but a problem with the solution »$^1/_{15}$«. Nor was he apparently aware that much closer parallels to his notation could be found in the Arabic orbit.

[30] MCT, 82. Discussed in Sachs 1946: 212.

[31] MCT, 101.

[32] MKT I, 490.

A text of special interest is the Susa tablet TMS V.[33] All the way through the tablet, sequences of numbers are used as abbreviations for complex numerical expressions involving parts of parts. Recurrent from section to section (albeit with some variation), 13 times in total, is the following series (the right column gives the interpretation)

a: »2«	2
b: »3«	3
c: »4«	4 (cf. the different meaning in **g**)
d: »$^2/_3$«	$^2/_3$
e: »$^1/_2$«	$^1/_2$
f: »$^1/_3$«	$^1/_3$
g: »4«	$^1/_4$
h: »$^1/_3$ 4«	$^1/_3$ of $^1/_4$
i: »7«	$^1/_7$
j: »2 7«	2 times $^1/_7$
k: »7 7«	$^1/_7$ of $^1/_7$
l: »2 7 7«	2 times $^1/_7$ of $^1/_7$
m: »11«	$^1/_{11}$
n: »2 11«	2 times $^1/_{11}$
o: »11 11«	$^1/_{11}$ of $^1/_{11}$
p: »2 11 11«	2 times $^1/_{11}$ of $^1/_{11}$
q: »11 7«	$^1/_{11}$ of $^1/_7$
r: »2 11 7«	2 times $^1/_{11}$ of $^1/_7$
s: »$^2/_3$ $^1/_2$ $^1/_3$ 11 7«	$^2/_3$ of $^1/_2$ of $^1/_3$ of $^1/_{11}$ of $^1/_7$
t: »2 $^2/_3$ $^1/_2$ $^1/_3$ 11 7«	2 times $^2/_3$ of $^1/_2$ of $^1/_3$ of $^1/_{11}$ of $^1/_7$

In section 10 we also find

A: »1 $^2/_3$«	1 plus $^2/_3$
B: »1 $^1/_2$«	1 plus $^1/_2$

[33] TMS, 35-49. The tablet has probably been prepared toward the end of the Old Babylonian period.

C: »1 $^1/_3$«	1 plus $^1/_3$
D: »1 4«	1 plus $^1/_4$
E: »1 $^1/_3$ 4«	1 plus $^1/_3$ of $^1/_4$
F: »1 7«	1 plus $^1/_7$
G: »1 2 7«	1 plus 2 times $^1/_7$
H: »1 7 7«	1 plus $^1/_7$ of $^1/_7$
I: »1 2 7 7«	1 plus 2 times $^1/_7$ of $^1/_7$
J: »2 $^1/_2$«	2 plus $^1/_2$
K: »3 $^1/_3$«	3 plus $^1/_3$
L: »4 4«	4 plus $^1/_4$ (not $^1/_4$ of $^1/_4$)
M: »7 igi-7«	7 plus $^1/_7$
N: »7 2 igi-7«	7 plus 2 times $^1/_7$

In all cases, the expressions multiply the side of a square (literally: count the number of times the side is to be taken).

In order to make *his text* as unambiguous as possible, the scribe has followed a fairly strict format, most clearly to be seen in ***t*** and ***N***: starting from the right, he lists (with increasing *denominator*) those fractions which in full writing would be written igi-*n*-gál, and which he abbreviates as the integer numeral *n;* next come, in increasing *magnitude*, fractions possessing their own ideogram ($^1/_3$, $^1/_2$ and $^2/_3$). This entire section of the sequence is to be understood as »parts of parts«. Then follows an (optional) integer numerator (>1), and finally an (equally optional) integer addend. As long as the numerator is kept at 2 and the addend at 1, the system is unambiguous. If we violate these restrictions (as in **c** and **L**), however, it stops being so. Inside the text, the systematic progress eliminates the ambiguities; if used as a general notation, on the other hand, the system would lead to total confusion – a fact which is obviously recognized by the scribe, since he introduces *ad hoc* the sign igi in **M** and **N**.

These observations entail the conclusion that we are confronted with a specific, context-dependent shorthand, not with a standardized notation for general fractions, as claimed by Evert Bruins.[34] Behind the shorthand, moreover, sticks not just general fractions but the *system of »parts of parts«*; the summation required by the ascending continued fractions, on the other hand, is not visible

[34] TMS, 36.

through the notation.

In the end of the above-mentioned article, Sachs[35] reviews a number of Seleucid notarial documents making use of composite expressions often involving »parts of parts« (all examples apart from N° 15 deal with the sale of temple prebends corresponding to parts of the day):

(1) »A fifth of a day and a third from a 15th of a day«.

(2) »A sixth, an 18th, and a 60th«.

(3) »A 30th, and a third from a 60th«.

(4) »A half from three quarters«.

(5) »A fifth from two thirds«.

(6) »Two thirds of a day and an 18th of a day«.

(7) »A sixth and a ninth of a day«.

(8) »A 20th from one day, of which a sixth from a 60th of a day is lacking«.

(9) »A 16th and a 30th of a day«, added to »a 16th of a day«, giving »an eighth and a 30th of a day«.

(10) »An eighth from a seventh«.

(11) »A half from an eighteenth«.

(12) »A third from a twelfth«.

(13) »An 18th from a seventh«.

(14) »A twelfth from a seventh«.

(15) »A half from a twelfth« (as a share of real estate).

Sachs rightly observes that the system seems less strict than the old one. In cases where the number is expressed as a sum, no particular effort is made to assure that the first member is an optimal approximation, nor to follow the strict pattern of an ascending continued fraction.[36] From the present perspective, it may be of interest that all »parts of parts« *except those involving the irregular* $^1/_7$ respect

[35] 1946: 213f. In the present case I take over Sachs's translation, except that I translate *ina* as »from« instead of »in«.

[36] Thus, N° 2 could have been rearranged as $^1/_5+^1/_{45}+^1/_{60}$ or, alternatively, as $^1/_5$, $^1/_6$ $_6\cdot$ $/_5$ N° 7 either as $^1/_4+^1/_9\cdot{}^1/_4$ or as $2\cdot{}^1/_9+^1/_2\cdot{}^1/_9$. N^{os} 3 and 6 only need reformulation and no rearrangement in order to agree with the pattern of ascending continued fractions.

»al-Qalaṣādī's canon«.[37] The Arabic avoidance of denominators larger than 10, of course, is not observed.

V. EGYPT

Its building stones being unit fractions with small denominators, the »parts of parts« scheme has often been connected to the Egyptian unit fraction system. In its mature form, as we know it from Middle Kingdom through Demotic sources, however, the Egyptian system had no predilection for those small denominators which it is the purpose of the »parts of parts« scheme to achieve. The Egyptians, furthermore, were not interested in such splittings where the first member can serve as a good first approximation, whereas a fair first approximation is a key point in the extension of the »parts of parts« into ascending continued fractions (as we met it already in the Old Babylonian tablet, cf. note 28). Attempts to explain the schemes of »parts of parts« and ascending continued fractions by reference to the Egyptian unit fractions system thus appear to be misguided.

»Parts of parts« as discussed above are not common in Egypt. In fact, I only know of three places where the usage is employed *to indicate a number*[38] (cf. below on other applications). The first of these is Rhind Mathematical Papyrus (RMP), Problem 37, one of the *ḥeḳat*-problems which were mentioned above in connection with the *Propositiones ad acuendos iuvenes*: »Go down I [i.e., a jug of unknown capacity] times 3 into the *ḥeḳat*-measure, $^1/_3$ of me is added to me, $^1/_3$ of $^1/_3$ of me is added to me, $^1/_9$ of me is added to me; return I, filled am I. Then

[37] »May be of interest« but need not, at least as far as the history of *mathematical* ideas and notations is concerned. Indeed, in an article discussing some of the same examples and a number of others Denise Cocquerillat (1965) points out that the expressions are chosen in a way which will make the merchandise look as impressing as possible to a mathematically naive customer. The governing principle may thus have been sales psychology rather than any general idiomatic preference.

[38] True enough, as pointed out by a referee, these numbers are no pure numbers: they represent the value of one quantity measured by another – a ḥeḳat-measure gauged by a jug, the toll on a herd of cattle as part of the original herd, the number $2^2/_3$ measured by the number 20. But this is precisely what numbers are mostly used for in daily practice, in Ancient Egypt as elsewhere, and also the way numbers most often occur within calculations in Egyptian mathematical texts.

what says it?«.[39] The second is Problem 67 of the same papyrus, »Now a herdsman came to the cattle-numbering, bringing with him 70 heads of cattle. The accountant of cattle said to the herdsman, Small indeed is the cattle-amount that thou hast brought. Where is then thy great amount of cattle? The herdsman said to him, What I have brought to thee is: $^2/_3$ of $^1/_3$ of the cattle which thou hast committed to me ...«.[40] The third example of »parts of parts« used to indicate a number, finally, belongs in the Moscow Mathematical Papyrus (MMP), Problem 20, where $2^2/_3$ is told to be $\frac{1}{5}$ *of* $\frac{2}{3}$ of 20.[41]

The latter example is put into perspective in RMP, »Problem« 61B, which explain the method to find $^2/_3$ of any unit fraction with odd denominator, and uses $\frac{2}{3}$ *of* $\frac{1}{5}$ as a paradigm.[42] The $\frac{1}{5}$ *of* $\frac{2}{3}$ which appears as a regular number in the MMP is thus (reversion of factors apart, which was trivial to the Egyptians) not recognized as such in the RMP, N° 61B: a composite expression like $\frac{1}{5}$ *of* $\frac{2}{3}$ was be considered *a problem* and no number *per se* (a problem whose answer is $\frac{1}{10}+\frac{1}{30}$). The same observation can be made on RMP, »Problem« 61, which is in fact a tabulation of a series of solutions to such problems.[43]

A final use of what appears a first like composite fractional expressions α of β turns up in the description of reversed metrological computations and

---

[39] Chace et al 1929, Plate 59. The grammatical construction used is $\frac{1}{3}$ *n* $\frac{1}{3}$, the indirect genitive, which is also used in expressions like $\frac{1}{10}$ *of this 10* (RMP 28), $\frac{1}{2}+\frac{1}{4}$ *of cubit* (RMP 58), $\frac{1}{3}+\frac{1}{5}$ *of this 30* (MMP, 3), etc. (here as in all transcriptions of Egyptian unit fraction sums I modernize the writing; the original text merely juxtaposes the denominators with the superscript dot meaning *ro*, »part«.). This construction should be distinguished from the reverse construction *z n 5*, »persons until [a total of] 5« discussed by Graefe (1979).

We observe that the sequence $\frac{1}{3}$ *and* $\frac{1}{3}$ *of* $\frac{1}{3}$ suggests the idea of ascending continued fractions (as do the successive *medietates* in the related *Propositiones*-problems).

[40] *Ibid.*, Plate 67. I have straightened somewhat the opaque language of the extremely literal translation.

[41] Ed., transl. Struve 1930: 95.

[42] Chace et al 1929, Plate 83.

[43] $^2/_3$ of $^2/_3$, $^1/_3$ of $^2/_3$, $^2/_3$ of $^1/_3$, $^2/_3$ of $^1/_6$, $^2/_3$ of $^1/_2$, etc. (*loc. cit.*). Peet (1923: 103f) makes a point out of a terminological distinction inside the table, which uses the construction α *of* β in cases where α is $^2/_3$ or can be obtained from $^2/_3$ by halving or successive halvings, but a construction β, its α (β α . f) in other cases. Since some of the formulations have been corrected by the scribe it seems indeed that the distinction is determined by a specific canon (which, as we observe, is broken by the $\frac{1}{5}$ *of* $\frac{2}{3}$ of MMP 20).

conversions (RMP 44, 45, 46 and 49). As an example we may take RMP 45,[44] which connects the two. A granary is known to contain 1500 *khar* and is supposed to have a square base of 10 cubits by 10 cubits (1 *khar* is $^{2}/_{3}$ of a cube cubit), and the height is looked for. The calculation then proceeds in the following steps:

1	*1500;*
$^{1}/_{10}$	*150;*
$^{1}/_{10}$ of $^{1}/_{10}$ of it	*15;*
$^{2}/_{3}$ of $^{1}/_{10}$ of $^{1}/_{10}$ of it:	*10.*

The key to the calculation is provided by Problem 44, which supplies the corresponding direct computation of the content of a cubic container of 10 cubit by 10 cubit by 10 cubit: the volume is first computed as $10 \cdot 10 \cdot 10$ [cube cubits] and then transformed into $1000+½\cdot 500=1500$ *khar*. A solution of the reverse Problem 45 by geometric reasoning would have to go through these steps in reverted order, transforming first the volume of 1500 *khar* into 1000 cubic cubits, and then dividing by the area of the base or, alternatively, by length and width separately. The text, as we see, proceeds differently, reversing the multiplications of Problem 44 one by one without changing their order. The reversal is thus taking place at the level of computational steps, where the order of divisions does not matter, and not on that of analytical reasoning. The composite expression »$^{2}/_{3}$ of $^{1}/_{10}$ of $^{1}/_{10}$« is not meant as another way to express the number $^{1}/_{150}$ but rather as a way to recapitulate the sequence of computational steps (in other words: To display the algorithm to be used).[45] Its single constituents ($^{2}/_{3}$, $^{1}/_{10}$ and $^{1}/_{10}$) are numbers but the composition is neither an authentic number nor a numerical expression to be transformed into a number (a »problem« in the sense which makes »$^{2}/_{3}$ of $^{1}/_{2}$« a problem and »$^{1}/_{10}+^{1}/_{30}$« the answer in RMP 61B).[46]

Though exceptional, the few occurrences of composite fractional expressions *used as legitimate numbers* are sufficient proof that the schemes of »parts of parts«

[44] Chace et al 1929, Plate 67.

[45] What looks like »parts of parts« and ascending continued fractions in the Indian *śulva sútras*, e.g. in the passage customarily interpreted as an approximation $^{1}2$¸$1+^{1}/_{3}+^{1}/_{34}-^{1}/_{3\cdot 4\cdot 34}$, has the same character, i.e., it is a prescription of a (geometric) procedure and no arithmetical number in itself (see *Baudháyana śulva sútra*, ed. Thibaut 1875: II,21). Genuine »parts of parts« are absent from Indian mathematics (as confirmed to me by Guy Mazars in a private communication).

[46] The non-numerical function of the composite expressions is confirmed by the non-observance of the canon deduced by Peet from RMP 61 in RMP 44, 45, 46 and 49, which all speak of *$^{1}/_{10}$ of $^{1}/_{10}$* (44-46 also have *$^{2}/_{3}$ of $^{1}/_{10}$ of $^{1}/_{10}$*).

and ascending continued fractions are indeed connected to Egypt though not to be explained with reference to the preferred unit fraction notation of the Egyptian scribes. The Egyptians *were* able to understand »parts of parts« not only as problems or as sequential prescriptions but also as numbers in their own right. When would they do so?

It is difficult to deduce a rule from only three isolated instances. At least two of the present cases, however, are not isolated but embedded in a specific context, on which I shall make some observations in order to answer the question.

Firstly, the *ḥekat*-problems are formulated as riddles. When searching the Rhind Papyrus for other riddles I only found one – *viz* the cattle problem in N° 67 (this is actually how I first discovered my second instance). Stylistically, these five problems are intruders into a problem collection which is otherwise written in didactically neutral style.

Secondly, we note that the »$^2/_3$ of $^1/_3$« of the cattle-problem is put into the mouth of the herdsman and not into that of the accountant-scribe (similarly, the »$^1/_3$ of $^1/_3$« is put into the mouth of a jug).

Thirdly, the similarity was already noted between the *ḥekat*-problems and those problems of the *Propositiones* which make use of »parts of parts«. The *ḥekat*-problems are thus connected to the whole fund of recreational mathematics.

All this matches a comprehension of recreational mathematics as a »pure« outgrowth of practitioner's mathematics.[47] »Parts of parts« appear to have belonged to non-technical, »folk« parlance, i.e., to the very substrate from which the riddles of recreational problems were drawn. Scribal mathematics, on the other hand, made use of the highly sophisticated scheme of unit fractions; this was a technical language, and the tool which the scribe would use to solve the recreational riddles even when these were formulated in a different idiom.[48]

A parallel to the Old Babylonian situation is obvious. Even here, the ascending continued fractions appeared when the result of calculations in the »technical system« of sexagesimals had to be transformed into »practical« units, while the »parts of parts« turned up in the statement of the riddles on stones of unknown weight, and when supplementary complication had to be added to purely mathematical problems.

»Parts of parts« *could* have arisen as a non-technical simplification and

---

[47] See Høyrup 1990: 66-71.

[48] The *$^1/_5$ of $^2/_3$* of MMP 20, it is true, turns up inside the calculation. It looks like a slip, like the reformulation of a description of computational steps (which in the present case would rather give *$^2/_3$ of $^1/_5$*) inspired by non-scholarly but familiar idiom.

consecutive extension of the unit fraction system, inspired by the sequential prescriptions of reversed computational schemes. Alternatively, it could be the basis from which the unit fraction system had developed. It is as yet not possible to decide the question with full certainty. Strong chronological arguments can be given, however, for the priority of the folk parlance and the secondary character of the unit fraction system. In order to see that we will have to determine the epoch during which the latter system was developed – a question which has never been seriously approached before.

The unit fraction system is used in fully developed form in the RMP. The original from which this papyrus has been copied is dated to the Middle Kingdom, i.e. to the early 2nd millennium. Other papyri computing by means of the unit fraction system, some of them genuine accounts and not materials for teaching or tables for reference, belong to the same period. By this time, general unit fractions had thus become a standard tool for scribal calculators.[49]

Older sources, however, are almost devoid of unit fractions. Old Kingdom scribes made use of metrological sub-units and of those fractions which are *not* written in the standardized way (i.e., $^1/_n$ written as the numeral *n* below the sign *ro*), *viz* $^2/_3$, $^1/_2$, and $^1/_3$.[50] Only the Fifth Dynasty Abū Sir Papyri (24th century B.C.) present us with the unit fractions $^1/_4$, $^1/_5$ and $^1/_6$.[51] At the same time, however, they present us with striking evidence that the later *system* was not developed. The sign for $^1/_5$, indeed, appears in the connection »$^1/_5$ $^1/_5$«, meaning $^2/_5$. Later, $2\cdot(^1/_5)$ (or, as it is expressed in the RMP, »2 called out of 5«) would be no number but a problem, the solution of which was $^1/_3+^1/_{15}$ – about one-third of the text of the Rhind Mathematical Papyrus is in fact occupied by the solution of $^2/_n$, *n* going from 3 to 101.[52] There are thus good reasons to believe that a notation for simple

[49] The scribal corrections in RMP 61 would suggest, however, that the canon deduced by Peet may only have emerged after the writing on the original, but before the copy was made.

[50] My main basis for this description of Old Kingdom sub-unity arithmetic is the material presented in Sethe 1916.

[51] I am indebted to Professor Wolfgang Helck for referring me to the publications on the Abū Sir Papyri. The fractional signs in question are found in Posener-Kriéger & de Cenival 1968: Plates 23-25, cf. translation in Posener-Kriéger 1976 and the discussion in Silberman 1975.

[52] Silberman (1975: 249) suggests that the writing be explained as a product of scribal ignorance. In view of the central position occupied in Egyptian arithmetic by doubling and ensuing conversion of fractions this is about as plausible as finding a modern accountant ignorant of the place value system.

aliquot parts was gradually being extended toward the end of the Old Kingdom, but was not yet developed into its mature form. True, Reineke[53] thinks that it will have been needed in the complex administration of the Old Kingdom, and thus dates the development to the first three dynasties. As far as I can see, however, real practical tasks are better solved by means of metrological sub-units (which are standardized and can thus be marked out on measuring instruments). The advantage of the unit fraction system is *theoretical*; it will only become manifest in the context of a school system.

This conclusion is supported by analysis of the pyramid problems of the RMP (N^os^ 56, 57, 58, 59A, 59B, 60). Those of them which appear to deal with »real«, traditional pyramids, i.e., which have a slope close to that of Old Kingdom pyramids (N^os^ 56-59B) measure the slope in adequate metrological units (*viz* palms [of horizontal retreat per cubit's ascent].[54] The result of N° 60, which deals with some other, unidentified structure, is given as a dimensionless, abstract number. At the same time, the dimensions of the first five, »real« pyramids are given without the unit, as it would be adequate for master-builders who knew what they were speaking about; N° 60 states the data as numbers of cubits, as suitable for a teacher instructing students who do not yet know the concrete practices and entities spoken about. It is thus likely that the author of the papyrus took over the first 5 problems with their metrological units from an older source but created or edited the final, abstract problem himself.[55]

The time when teaching changed from apprenticeship to organized school teaching is fairly well-established.[56] Schools were unknown in the Old Kingdom (if we do not count the education of sons of high officials together with the royal princes), which instead relied upon an apprentice-system. Only after the collapse of the Old Kingdom do we find the first reference to a school (and the absence of a God for the school shows that schools only arose when the Pantheon had reached its definitive structure). By the time of the early Middle Kingdom, on the other hand, scribal education is school education. There is thus a perfect coordination between the changing educational patterns, the move from metrological toward pure number, and the development of the full unit fraction system as far as it is reflected in the sources.

---

[53] 1978: 73f.

[54] See the comparison of real and »Rhind« slopes in Reineke 1978: 75 n. 28.

[55] This is also plausible from »a serious [conceptual] confusion [which] has taken place« in the text of N° 60, and which is pointed out and discussed by Peet (1923: 101f).

[56] See Brunner 1957: 11-15, and Wilson in Kraeling & Adams 1960: 103.

It is therefore fairly certain that the systematic use of unit fractions was a quite recent development when the original of the Rhind Papyrus was written – and implausible, as a consequence, that a non-technical usage built on »parts of parts« should already have been derived from it. On the other hand, the traces of an incipient use of the unit fraction notation in the Abū Sir Papyri fits a development starting from a set of elementary aliquot parts in popular use but extending and systematizing this idiom in agreement with the requirements of school teaching.

VI. A SCENARIO

The single occurrences of »parts of parts« and ascending continued fractions are easily established. When it comes to questions of precedence and to possible connections, however, conclusions will have to be built on indirect evidence and on plausibility. Instead of proposing candidly a theory and claiming it to be necessary truth I shall therefore propose a *scenario* and, in cases where this is needed, try to evaluate the merits of alternative interpretations. Instead of treating the matter in chronological order I shall begin with the most obvious, leaving the more intricate matters to the end.

Most obvious of all are the connections within Western Asia. The Old Babylonian »parts of parts« and ascending continued fractions are so close to the usage later testified in Arabic sources that the existence of unbroken habits in the Babylonian-Aramaic-Arabic-speaking region is beyond reasonable doubt. The minor differences between canons and materializations of shared principles can easily be explained as effects of the peculiarities of the single languages and from the use of different computational tools or techniques.

In the early Islamic period, the composite fractions belonged with the »finger-reckoning« tradition and thus with the non-scholarly discourse of merchants and other practical reckoners.[57] One may assume this to have been the case already in earlier times – not least because most of the Old Babylonian occurrences suggest so. The intense interaction of merchants along the Silk Road, which was able to carry a shared culture of recreational problems, will also have been able to spread a Semitic merchants' usage to traders and calculators of neighbouring civilizations. The early rôle of the Phoenicians and the persistent participation

[57] After the mid-eleventh century, the originally separate »finger-reckoning« and »Hindu« reckoning« traditions merged (cf. Høyrup 1987: 309-11). Al-Qalaṣādī, like Leonardo Fibonacci, would hence combine the two.

of Syrian and other Near Eastern merchants in Mediterranean trade, in particular, will have been an excellent channel for the spread of the system to the West (as it was probably the channel through which a shared system of finger-reckoning spread from the Near East to the whole Mediterranean region and as far as Bede's Northumbria[58]). The striking coincidence that problems from the *Anthologia graeca* concerning parts of the day refer to the very usage which also turns up in Seleucid calculations dealing with that subject, as well as the references to astrology and to dial-makers in the *Anthologia*, suggests that not only traders but »Chaldean« astrologers and instrument-makers were involved in the spread of the usage from the Near Eastern to the Greek orbit.

To the Greek orbit, but not general spread *within* the orbit of Greek culture. The reason that we can speak of striking coincidences is, in fact, that no such spread took place. »Parts of parts« and derived expressions are restricted to those very domains where their original practitioners employed them, using probably an idiom borrowed together with other professional instruments from the Near East. Other domains were not affected.

The above argument presupposes that diffusion took place, and that a channel for that diffusion has to be found. Caution requires, however, that this presupposition be itself examined critically. After all, »parts of parts« seem to be an idea close at hand. Everybody who understands the fractions will also understand their composition, we should think. Ascending continued fractions, furthermore, is a generalization of the metrological principle of descending sub-units; any culture possessing a linearly ordered and multi-layered metrology should be able to invent them.

So it seems. But the actual evidence contradicts the apparent truisms. Greek Antiquity, though having demonstrably the schemes before its eyes, did not grasp at a notation which was so near at hand. It accepted the notation in a few select places – precisely the ones to where it can be assumed to have been brought. But the Greeks did not like it. For everyday use, they stuck to the Egyptian system; for mathematical purposes, they developed something like general fractions; and in astronomy, they adopted the Babylonian sexagesimal fractions.

The same holds for Latin Europe. The *Propositiones* became quite popular and influenced European recreational mathematics for centuries. But a 14th century problem coming very close to those dealing with *medietas et medietas medietatis* transforms this number into »$^1/_2$ and $^1/_4$«.[59] The usage »at hand« did not spread – on the contrary, it was resorbed.

[58] References in Høyrup 1987: 291.

[59] Ms. Columbia X 511 A13, ed. Vogel 1977: 109.

The ascending continued fractions had a similar fate. As told above, they were taken over from Arabic arithmetic as an obligatory subject in Italian arithmetic from Leonardo onwards without acquiring ever any importance. Outside Italy, only Jordanus de Nemore tried to naturalize them as part of theoretical mathematics. He did so in his treatises on »algorism«, computation with Hindu numerals. For this purpose he invented a special concept »dissimilar fractions«. To explain what the concept was about he connected it precisely to systems of metrological sub-units.[60] Not even his closest followers, however, appear to have found anything attractive in the idea, and no echo whatsoever can be discovered. Ascending continued fractions, no more than »parts of parts«, came naturally to the minds of Medieval European reckoners and mathematicians.

If a concept cannot spread inside a given culture but remains restricted to a very particular use (ultimately to be resorbed) it is not likely to have been invented by this culture – at least not if there is no specific need for it in the context where it establishes itself. On this premise the »parts of parts« occurring in the *Anthologia graeca* and the *Propositiones* can safely be assumed to be there as the result of a borrowing.

In the case of the *Anthologia*, as we have seen, the only conceivable source is Western Asia; as far as the *Propositiones* are concerned, the question of the direct channel is less easily decided. As we have observed, composite fractions are absent even from the problems inspired by the Eastern trade. Only one specific type of riddle employs them – a type which ultimately points toward Ancient Egypt and not to the trading network. During the Achaemenid and Hellenistic eras, however, Egyptian and Western Asiatic methods and traditions had largely been mixed up. Even if the composite fractions of the *Propositiones* can ultimately be traced to Egypt, the way from Aachen to Egypt may therefore have passed through anywhere between Kabul and Seville.

Tracing the composite fractions of the *Anthologia* to the Semitic-speaking world of Western Asia and those of the *Propositiones* to Egyptian sources brings us back to the most intricate question: How did these (or, more precisely: the Babylonian and Egyptian usages) relate to each other?

We have found the traces of an Old Kingdom Egyptian and as well as an Old Babylonian »folk« usage of elementary aliquot parts (including $^{2}/_{3}$). We have seen, moreover, that these were combined in both cultures into »parts of parts«; that they were expanded at least in Babylonia into a system of ascending continued fractions, and that they presumably provided Middle Kingdom Egypt

[60] See the preface to *Demonstratio de minutiis*, ed. Eneström 1913. Cf. Høyrup 1988: 337f.

with the foundation on which the full unit fraction system was built.

In principle, the Babylonian and Egyptian composite fractions may have developed in complete independence; two arguments, however, contradict this assumption. For one thing, »parts of parts« seem not to come naturally to an »average« culture, if we trust the Greek, Latin and Italian evidence. The Ancient Mesopotamian compositions appear, moreover, to be strictly bound to the *Babylonian* language. Third millennium Sumerian texts employ elementary unit fractions freely; but they never combine them as »parts of parts«; these, and the ascending continued fractions, only appear when mathematical traditions carried by the Babylonian language took possession of the scribal school in the Old Babylonian epoch. *Shared origins* or at least *shared roots* are thus more credible than full independence.

Shared origins are by no means excluded. Both the Semitic (including the Babylonian) and the Ancient Egyptian languages belong to the Hamito-Semitic language family. Furthermore, a socio-cultural need for simple fractions can reasonably be ascribed to the (presumably pastoral) carriers of the language before the Semitic and the Egyptian branch broke away from each other.[61] Already at this early epoch, the habit of combining them as »parts of parts« *may* also have existed, even though the (scarce) comparative evidence suggest no need for such arithmetical subtleties in a non-monetary economy. Alternatively, diffusion of the habit via trade routes from one culture to the other at a later moment can be imagined: during the fourth and as well as the third millennium B.C., connections existed, in all probability via Syrian territory.[62]

Yet whether such commercial links were able to influence the development of arithmetical idioms is an open question. They may have involved a whole chain of intermediaries. An argument in favour of diffusion through trading contacts one way or the other (or from an intermediary) could be the common »institution« of recreational mathematics, which is not likely to have existed when the Semitic and Egyptian branches of the family separated (probably no later than the fifth millennium); but since Babylonian and Egyptian scribes have only the institution but no members (i.e., problem-types) in common, independent development of the recreational *genre* as a response to the similar social environments of professional reckoners – i.e., *shared* (sociological) *roots* of the *genre* is an alternative explanation at least as near at hand as *shared origins*

[61] See the table of shared vocabulary in Diakonoff 1965: 42-49, and other shared vocables mentioned elsewhere in the book. Common property is, *inter alia*, the term *ḥsb*, »to count«, »to reckon«, »to calculate«.

[62] See Moorey 1987 on the 4th millennium, and Klengel 1979: 61-72 on the third.

through common descent or through diffusion.

Similarly, *shared roots* (though linguistic or computational and not sociological) may be the better explanation that composite fractions are found in both Egypt and Babylonia. As one will remember, the objection against fully independent development of systems of composite fractions was founded on the observation that the creation of a scheme of »parts of parts« is *not* near at hand, in spite of what might look like reasonable *a priori* expectancies. Strictly speaking, however, this observation was only made *on a Greek, Latin, or Italian* linguistic background and on the background of the computational techniques and tools in common use in classical Antiquity and Medieval Europe. But developments in Egypt and Babylonia will not have been fully independent: they will have taken place on structurally similar linguistic backgrounds, and maybe on the background of shared techniques and tools. A common heritage of Babylonians and Egyptians could be a set of elementary fractions and a pattern of linguistic or computational habits being naturally open to specific developments – in particular the development of a scheme of »parts of parts«.[63] This would be *parallel developments from shared roots*.

Summing up we may conclude with a high degree of certainty that later occurrences of »parts of parts« and ascending continued fractions outside the Egypto-Semitic area are due to borrowings from developed usages (in some cases distorting or rudimentary borrowings). We may also assume that the parallel Semitic and Egyptian idioms can be ascribed to a shared heritage. But we cannot know whether the shared heritage was *an actual way to speak* about fractional entities or *only a potential scheme* inherent in language structures or computational practices. Personally, I confess to be inclined toward belief in the potential

[63] In his book (1965) on the Hamito-Semitic language family, Diakonoff mentions many instances where different languages of the family have developed similar features independently; thus as complex a phenomenon as the *pluralis fractus* (p. 68). We might speak of »structural causation«, the effect of shared linguistic structures determining that specific developments are near at hand and compatible with general linguistic habits.

»Structural causation«, however, need not be linguistic. Non-linguistic instruments for accounting and computation (be they mental or material) may in the same way open the way for specific inventions and block others which are not compatible with existing habits, tools or conceptualizations.

Knowledge of the way fractions are spoken about in other Hamito-Semitic languages might seem to offer a way to distinguish linguistic from non-linguistic causation. However, native and ethnically conscious Berber speakers studying mathematics whom I interviewed in Algeria confessed to speak about fractions in Arabic and to be ignorant of any Berber idiom for fractions.

scheme.

BIBLIOGRAPHY AND ABBREVIATIONS

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Brentjes, Sonja, 1984. "Die erste Risâla der Rasâʾil Ih[redacted]wân as-Ṣafâʾ über elementare Zahlentheorie: Ihr mathematischer Gehalt und ihre Beziehungen zu spätantiken arithmetischen Schriften". *Janus* **71**, 181-274.

Brunner, Hellmut, 1957. *Altägyptische Erziehung.* Wiesbaden: Otto Harrassowitz.

Busard, H. L. L., 1968. "L'algèbre au moyen âge: Le »Liber mensurationum« d'Abû Bekr". *Journal des Savants*, Avril-Juin 1968, 65-125.

Chace, Arnold Buffum, Ludlow Bull & Henry Parker Manning, 1929. *The Rhind Mathematical Papyrus.* II. Photographs, Transcription, Transliteration, Literal Translation. *Bibliography of Egyptian and Babylonian Mathematics* (Supplement), by R. C. Archibald. *The Mathematical Leather Roll in the British Museum*, by S. R. K. Glanville. Oberlin, Ohio: Mathematical Association of America.

Cocquerillat, Denise, 1965. "Les calculs pratiques sur les fractions à l'époque seleucide". *Biblioteca Orientalis* **22**, 239-242.

Diakonoff, Igor. M., 1965. *Semito-Hamitic Languages.* Moscow: Nauka Publishing House.

Djebbar, A., 1981. *Enseignement et recherche mathématiques dans le Maghreb des XIII^{e}-XIV^{e} siècles (étude partielle).* (Publications mathématiques d'Orsay, 81-02). Orsay: Université de Paris-Sud.

Eneström, Georg, 1913. "Das Bruchrechnen des Nemorarius". *Bibliotheca Mathematica*, 3. Folge **14** (1913-14), 41-54.

Folkerts, Menso, 1978. "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*. Überlieferung, Inhalt, Kritische Edition". *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band, 6. Abhandlung.

Friedlein, Gottfried, 1871. "Der Calculus des Victorius". *Zeitschrift für Mathematik und Physik* **16**, 42-79, 253-254, Tabelle I-IX.

Graefe, Erhart, 1979. "Ein unerkannter Gebrauch des indirekten Genitivs in Zahlkonstruktionen", pp. 174-184 *in* Manfred Görg & Edgar Pusch (eds), *Festschrift Elmar Edel.* (Ägypten und Altes Testament. Studien zur Geschichte, Kultur und Religion Ägyptens und des Alten Testament). Bamberg.

Heiberg, J. L. (ed., trans.), 1912. Heronis *Definitiones* cum variis collectionibus. Heronis quae feruntur *Geometrica.* (Heronis Alexandrini Opera quae supersunt omnia, IV). Leipzig: Teubner.

Høyrup, Jens, 1986. “Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra”. *Erdem* **2**:5 (Ankara), 445-484.

Høyrup, Jens, 1987. “The Formation of »Islamic Mathematics«. Sources and Conditions”. *Science in Context* **1**, 281-329.

Høyrup, Jens, 1988. “Jordanus de Nemore, 13th Century Mathematical Innovator: An Essay on Intellectual Context, Achievement, and Failure”. *Archive For History of Exact Sciences* **38**, 307-363.

Høyrup, Jens, 1990. “Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon”. *History of Science* **28**, 63-86.

Klengel, Horst, 1979. *Handel und Händler im alten Orient*. Leipzig: Koehler und Amelang.

Kokian, P. Sahak (ed., trans.), 1919. “Des Anania von Schirak arithmetische Aufgaben”. *Zeitschrift für die deutsch-österreichischen Gymnasien* **69** (1919-20), 112-117.

Kraeling, Carl, & Robert McC. Adams (eds), 1960. *City Invincible*. A Symposium on Urbanization and Cultural Development in the Ancient Near East Held at the Oriental Institute of the University of Chicago, December 4-7, 1958. Chicago: University of Chicago Press.

MCT: O. Neugebauer / A. Sachs, *Mathematical Cuneiform Texts*. (American Oriental Series, vol. 29). New Haven, Connecticut: American Oriental Society, 1945.

Medovoj, M. I., 1960. “Ob arifmetičestkom traktate Abu-l-Vafy”. *Istoriko-Matematičeskie Issledovanija* **13**, 253-324.

MKT: O. Neugebauer, *Mathematische Keilschrift-Texte*. I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937.

Moorey, P. R. S., 1987. “On Tracking Cultural Transfers in Prehistory: The Case of Egypt and Lower Mesopotamia in the Fourth Millennium BC”. Pp. 36-46 *in* M. Rowlands, M. Larsen & K. Kristiansen (eds), *Centre and Periphery in the Ancient World*. (New Directions in Archaeology). Cambridge: Cambridge University Press.

Paton, W. R. (ed., trans.), 1979. *The Greek Anthology*. Volume V. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann. 1st ed. 1918.

Peet, T. Eric, 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058*. Introduction, Transcription, Translation and Commentary. London: University Press of Liverpool.

Posener-Kriéger, Paule, & Jean Louis de Cenival, 1968. *Hieratic Papyri in the British Museum*. Fifth Series, *The Abu Sir Papyri*. Edited, together with Complementary Texts in Other Collections. London: The Trustees of the British Museum.

Posener-Kriéger, Paule, 1976. *Les Archives du temple funéraire de Néferirkare-kakaï (les papyrus d’Abousir)*. Traduction et commentaire. 2 vols. Paris: Institut Français d’Archéologie Orientale du Caire.

Reineke, Walther Friedrich, 1978. “Gedanken zum vermutlichen Alter der mathematischen Kenntnisse im alten Ägypten”. *Zeitschrift für ägyptische Sprache und Altertumskunde* **105**, 67-76.

Rosen, Frederic (ed., trans.), 1831. The *Algebra* of Muhammad ben Musa, Edited and Translated. London: The Oriental Translation Fund.

Sachs, Abraham J., 1946. “Notes on Fractional Expressions in Old Babylonian Mathematical Texts”. *Journal of Near Eastern Studies* **5**, 203-214.

Saidan, Ahmad S., 1971. *ʿIlm al-ḥisāb al-ʿarabī*. Amman. Not seen.

Saidan, Ahmad S., 1974. “The Arithmetic of Abū’l-Wafāʾ”. *Isis* **65**, 367-375.

Sethe, Kurt, 1916. *Von Zahlen und Zahlworten bei den Alten Ägyptern, und was für andere Völker und Sprachen daraus zu lernen ist*. (Schriften der Wissenschaftlichen Gesellschaft in Straßburg, 25. Heft). Straßburg: Karl J.Trübner.

Silberman, David P., 1975. “Fractions in the Abu Sir Papyri”. *Journal of Egyptian Archaeology* **61**, 248-249.

Soubeyran, Denis, 1984. “Textes mathématiques de Mari”. *Revue d’Assyriologie* **78**, 19-48.

Souissi, Mohamed (ed., trans.), 1969. Ibn al-Bannāʾ, *Talkhīṣ aʿmāl al-ḥisāb*. Texte établi, annoté et traduit. Tunis: L’Université de Tunis.

Souissi, Mohamed (ed., trans.), 1988. Qalasādī, *Kašf al-asrār ʿan ʿilm ḥurūf al-ġubār*. Carthage: Maison Arabe du Livre.

Struve, W. W., 1930. *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau*. Herausgegeben und Kommentiert. (Quellen und Studien zur Geschichte der Mathematik. Abteilung A: Quellen, 1. Band). Berlin: Julius Springer.

Suter, Heinrich, 1910. “Das Buch der Seltenheiten der Rechenkunst von Abū Kāmil al-Miṣrī”. *Bibliotheca Mathematica*, 3. Folge **11** (1910-1911), 100-120.

Thibaut, G. (ed., trans.), 1875. “The Śulvasútra of Baudháyana”. I: *The Pandit* **9** (1874-75), 292-298. II: *The Pandit* **10** (1875-76), 17-22, 44-50, 72-74, 139-146, 166-170, 186-194, 209-218. III: *The Pandit*, New Series **1** (1876-77), 316-324, 556-578, 626-642, 692-706, 761-770.

Thompson, Stith, 1975. *Motif-Index of Folk-Literature. A Classification of Narrative Elements in Folktales, Ballads, Myths, Fables, Mediaeval Romances, Exempla, Fabliaux, Jest Books and Local Legends*. I-VI. Rev. and enl. edition. London: Indiana University Press.

TMS: E. M. Bruins & M. Rutten, *Textes mathématiques de Suse*. (Mémoires de la Mission Archéologique en Iran, XXXIV). Paris: Paul Geuthner, 1961.

Tropfke, J. / Vogel, Kurt, et al, 1980. *Geschichte der Elementarmathematik*. 4. Auflage. Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.

Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13)*. (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München:

Deutsches Museum.

Vogel, Kurt, 1982. “Zur Geschichte der Stammbrüche und der aufsteigenden Kettenbrüche”. *Sudhoffs Archiv* **66**, 1-19.

Youschkevitch, Adolf P., “Abū’l-Wafāʾ”. *Dictionary of Scientific Biography* I, 39-43. New York: Scribner, 1970.

Youschkevitch, Adolf P., 1976. *Les mathématiques arabes (VIII^e-XV^e siècles)*. (Collection d’Histoire des Sciences, 2). Paris: J. Vrin.

Addendum

An error in Hero’s *Metrika* I.xxxi [ed. Schöne 1903: 76 l.13] suggests thinking in terms of composite fractions: $^{2}/_{7}$ (β ζ’)is replaced by $^{1}/_{14}$ (ιδ’)