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A

"Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought".

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JENS HØYRUP

Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought I

Til Sara og Janne

Abstract

Through a broad structural analysis and a close reading of Old Babylonian mathematical "procedure texts" dealing mainly with problems of the second degree it is shown that Old Babylonian "algebra" was neither a "rhetorical algebra" dealing with numbers and arithmetical relations between numbers nor built on a set of fixed algorithmic procedures. Instead, the texts must be read as "naive" prescriptions for geometric analysis–naive in the sense that the results are seen by immediate intuition to be correct, but the question of correctness never raised–dealing with measured or measurable but unknown line segments, and making use of a set of operations and techniques different in structure from that of arithmetical algebra.

The investigation involves a thorough discussion and re-interpretation of the technical terminology of Old Babylonian mathematics, elucidates many terms and procedures which have up to now been enigmatic, and makes many features stand out which had not been noticed before.

The second-last chapter discusses the metamathematical problem, whether and to which extent we are then entitled to speak of an Old Babylonian *algebra*; it also takes up the over-all implications of the investigation for the understanding of Old Babylonian patterns of thought. It is argued that these are not *mythopoeic* in the sense of H. and H. A. Frankfort, nor *savage* or *cold* in a Lévi-Straussian sense, nor however as abstract and modern as current interpretations of the mathematical texts would have them to be.

The last chapter investigates briefly the further development of Babylonian "algebra" through the Seleucid era, demonstrating a clear arithmetization of the patterns of mathematical thought, the possible role of Babylonian geometrical analysis as inspiration for early Greek geometry, and the legacy of Babylonian "algebraic" thought to Medieval Islamic algebra.

Introduction

The following contains an account of a broad investigation of the terminology, methods, and patterns of thought of Old Babylonian so-called *algebra*. I have been engaged in this investigation for some years, and circulated a preliminary and fairly unreadable account in 1984, of which the item (Høyrup 1985) in the bibliography of the present article is a slightly corrected reprint. I have also presented the progress of the project in the four Workshops on Concept Development in Babylonian Mathematics held at the Seminar für Vorderasiatische Altertumskunde der Freien Universität Berlin in 1983, 1984, 1985 and 1988, and included summaries of some of my results–without the detailed arguments–in various contexts where they were relevant.

This article is then meant to cover my results coherently and to give the details of the argument, without renouncing completely on readability. Admittedly, the article contains many discussions of philological details which will hardly be understandable to historians of mathematics without special assyriological training, but which were necessary if philological specialists should be able to evaluate my results; I hope the non-specialist will not be too disturbed by these stumbling–stones. On the other hand many points which are trivial to the assyriologist are included in order to make it clear to the non–specialist why current interpretations and translations are only reliable up to a certain point, and why the complex discussions of terminological structure and philological details are at all necessary. I apologize to whoever will find them boring and superfluous.

It is a most pleasant duty to express my gratitude to all those who have assisted me over the years,–especially Dr. Bendt Alster, Dr. Aage Westenholz and Dr. Mogens Trolle Larsen of Copenhagen University, and to Professor, Dr. Hans Nissen, Professor, Dr. Johannes Renger, Dr. Robert Englund, and Dr. Kilian Butz of Freie Universität Berlin, together with all participants in the Berlin Workshops, not least the indefatigable Professor Jöran Friberg of Göteborg University, Professor Marvin Powell of Northern Illinois University and Professor, Dr. Wolfgang Lefèvre. Special thanks are due to Professor, Dr. von Soden for giving always in the briefest possible delay kind but yet precise criticism of every preliminary and unreadable paper I sent him, and for adding always his gentle advice and encouragement.

Everybody who followed the Berlin Workshop will know that Dr. Peter Damerow of the Max-Planck-Institut für Bildungsforschung, Berlin, deserves the greatest gratitude of all, to which I can add my personal experience as made over the last six years.

The intelligent reader will easily guess who remains responsible for all errors.

I dedicate the work to my daughters Sara and Janne, for reasons which have nothing to do with mathematics, Babylonia or Assyriology, but much with our common history over the years.

April 21, 1989

Abbreviations

Detailed bibliographic information will be found in the bibliography.

ABZ	Assyrisch-babylonische Zeichenliste (=Borger 1978)
AHw	Akkadisches Handwörterbuch (=von Soden 1965)
BiOr	Bibliotheca Orientalis
CAD	Chicago Assyrian Dictionary
GAG	Grundriss der akkadischen Grammatik (=von Soden 1952)
GEL	A Greek-English Lexicon (=Liddell – Scott 1968)

HAHw	Hebräisches und Aramäisches Handwörterbuch (=Gesenius 1915)
JCS	Journal of Cuneiform Studies
JNES	Journal of Near Eastern Studies
MCT	Mathematical Cuneiform Texts (=Neugebauer — Sachs 1945)
MEA	Manuel d'épigraphie akkadienne (=Labat 1963)
MKT	Mathematische Keilschrift-Texte, I–III (=Neugebauer 1935)
RA	Revue d'Assyriologie et d'Archéologie Orientale
ŠL	Šumerisches Lexikon, I–III (=Deimel 1925)
SLa	The Sumerian Language (=Thomsen 1984)
TMB	Textes mathématiques Babyloniens (=Thureau-Dangin 1938)
TMS	Textes mathématiques de Suse (=Bruins — Rutten 1961)
WO	Die Welt des Orients
ZA	Zeitschrift für Assyriologie und vorderasiatische Archäologie
ZDMG	Zeitschrift der Deutschen Morgenländischen Gesellschaft

Bibliography

Allotte de la Fuÿe 1915: F. M. Allotte de la Fuÿe, Mesures agraires et formules d'arpentage à l'époque présargonique, in: RA 12 [1915], 117–146.

Baqir 1950: T. Baqir. An Important Mathematical Problem Text from Tell Harmal, in: Sumer 6 [1950], 39–54.

Baqir 1950a: T. Baqir, Another Important Mathematical Text from Tell Harmal, in: Sumer 6 [1950], 130–148.

Baqir 1951: T. Baqir, Some More Mathematical Texts from Tell Harmal, in: Sumer 7 [1951], 28–45.

Baqir 1962: T. Baqir, Tell Dhiba'i: New Mathematical Texts, in: Sumer 18 [1962], 11–14, pl. 1–3.

Bauer 1967: J. Bauer, Altsumerische Wirtschaftstexte aus Lagasch. Inauguraldissertation, Philosophische Fakultät der Julius-Maximilian-Universität, Würzburg 1967.

Beck 1978: B. E. F. Beck, The Metaphor as a Mediator Between Semantic and Analogic Modes of Thought, in: Current Anthropology 19 [1978], 83–97, 20 [1979], 189–191 (incl. discussion).

Borger 1978: R. Borger, Assyrisch-babylonische Zeichenliste, Kevelaer — Neukirchen-Vluyn 1978 (Alter Orient und Altes Testament, 33).

Brentjes — Müller 1982: S. Brentjes — M. Müller, Eine neue Interpretation der ersten Aufgabe des altbabylonischen Textes AO 6770, in: NTM. Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin 19, H. 2 [1982], 21–26.

Bruins 1953: E. M. Bruins, Revisions of the Mathematical Texts from Tell Harmal, in: Sumer 9 [1953], 241–253.

Bruins — Rutten 1961: E. M. Bruins — M. Rutten, Textes mathématiques de Suse, Paris 1961 (Mémoires de la Mission Archéologique en Iran, XXXIV).

Bruins 1966: E. M. Bruins, Fermat Problems in Babylonian Mathematics, in: Janus 53 [1966], 194–211.

Bruins 1971: E. M. Bruins, Computation in the Old Babylonian Period, in: Janus 58 [1971], 222–267.

Bruins 1983: E. M. Bruins, On Mathematical Terminology, in: Janus 70 [1983], 97–108.

Bubnov 1899: N. Bubnov (ed.), Gerberti postea Silvestri II papae Opera Mathematica (972–1003), Berlin 1899.

Busard 1968: H. L. L. Busard, L'algèbre au moyen âge: Le *Liber mensurationum* d'Abû Bekr, in: Journal des Savants, Avril–Juin 1968, 65–125.

Chicago Assyrian Dictionary (CAD): The Assyrian Dictionary of the Oriental Institute of the University of Chicago, Chicago — Glückstadt 1956ff.

Deimel 1923: A. Deimel, Die Inschriften von Fara, II. Schultexte aus Fara, in Umschrift herausgegeben und bearbeitet, Leipzig 1923 (Wissenschaftliche Veröffentlichungen der Deutschen Orient-Gesellschaft, 43).

Deimel 1925: A. Deimel, Šumerisches Lexikon, I–III, Rome 1925–1937 (Scripti Pontificii Instituti Biblici).

Dijksterhuis 1956: E. J. Dijksterhuis, Archimedes, Copenhagen 1956 (Acta Historica Scientiarum Naturalium et Medicinalium, vol. 12).

Diophanti Alexandrini Opera omnia edidit et latine interpretatus est Paulus Tannery, I–II, Leipzig 1893–1895. Cf. Ver Eecke 1926, Sesiano 1982, and Rashed 1984.

Edzard 1969: D. O. Edzard, Eine altsumerische Rechentafel (OIP 14,70), in: Röllig 1969: 101–104.

Euclidis *Data* . . . edidit H. Menge (Euclidis Opera omnia VI), Leipzig 1896.

Euclidis *Elementa* . . . edidit I. L. Heiberg (Euclidis Opera omnia I–V), Leipzig 1883–1888.

Falkenstein 1959: A. Falkenstein, Das Sumerische, Leiden 1959 (Handbuch der Orientalistik, 1. Abt., 2. Bd., 1. u. 2. Abschnitt, Lfg. 1).

Förtsch, 1916: W. Förtsch, Altbabylonische Wirtschaftstexte aus der Zeit Lugalanda's und Urukagina's, Leipzig 1916 (Vorderasiatische Schriftdenkmäler der Königlichen Museen zu Berlin, Heft XIV, 1. Hälfte).

Foster 1982: B. R. Foster, Archives and Record-keeping in Sargonic Mesopotamia, in: ZA 72 [1982], 1–27.

Frankfort – Frankfort – Wilson – Jacobsen – Irwin 1946: H. Frankfort – H. A. Frankfort – J. A. Wilson – Th. Jacobsen – W. A. Irwin, The Intellectual Adventure of Ancient Man. An Essay on Speculative Thought in the Ancient Near East, Chicago 1946.

Friberg (forthcoming), Mathematik (to appear in: Reallexikon der Assyriologie).

Gadd 1922: C. J. Gadd, Forms and Colours, in: RA 19 [1922], 149–159.

Gandz 1939: S. Gandz, Studies in Babylonian Mathematics, II. Conflicting Interpretations of Babylonian Mathematics, in: Isis 31 [1939], 405–425.

Gandz 1948: S. Gandz, Studies in Babylonian Mathematics, I. Indeterminate Analysis in Babylonian Mathematics, in: Osiris 8 [1948], 12–40.

Gesenius 1915: Gesenius, W., Hebräisches und Aramäisches Handwörterbuch über das Alte Testament. Bearbeitet von F. Buhl, 16. Aufl., Leipzig 1915.

Goetze 1946: A. Goetze, Number Idioms in Old Babylonian, in: JNES 5 [1946], 185–202.

Goetze 1951: A. Goetze, A Mathematical Compendium from Tell Harmal, in: Sumer 7 [1951], 126–155.

Gundlach – von Soden 1963: K.-B. Gundlach – W. von Soden, Einige altbabylonische Texte zur Lösung „quadratischer Gleichungen", in: Abhandlungen aus dem mathematischen Seminar der Universität Hamburg 26 [1963], 248–263.

Heronis Alexandrini Opera quae supersunt omnia, Vol. IV. Heronis *Definitiones*, Heronis quae feruntur *Geometrica*, edidit J. L. Heiberg, Leipzig 1912.

Hofmann 1970: J. E. Hofmann (ed.), François Viète, Opera mathematica recognita à Francisci von Schooten, Hildesheim – New York 1970.

Høyrup 1980: J. Høyrup, Influences of Institutionalized Mathematics Teaching on the Development and Organization of Mathematical Thought in the Pre-Modern Period, in: Materialien und Studien. Institut für Didaktik der Mathematik der Universität Bielefeld 20 [1980], 7–137.

Høyrup 1982: J. Høyrup, Investigations of an Early Sumerian Division Problem, c. 2500 B.C., in: Historia Mathematica 9 [1982], 19–36.

Høyrup 1983: J. Høyrup, Review of Unguru – Rowe 1981, in: Zentralblatt für Mathematik 504 [1983], 11f.

Høyrup 1984: J. Høyrup, Review of Brentjes – Müller 1982, in: Zentralblatt für Mathematik 517 [1984], 7.

Høyrup 1985: J. Høyrup, Babylonian Algebra from the View-Point of Geometrical Heuristics. An Investigation of Terminology, Methods, and Patterns of Thought, Roskilde ²1985.

Høyrup 1985a: J. Høyrup, Varieties of Mathematical Discourse in Pre-Modern Socio-Cultural Contexts: Mesopotamia, Greece, and the Latin Middle Ages, in: Science and Society 49 [1985], 4–41.

Høyrup 1986: J. Høyrup, Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: On the Origins of Islamic Algebra, in: Erdem 2 (Ankara 1986) 445–484.

Høyrup 1988: J. Høyrup, *Dynamis* and *mitḫartum*. On Analogous Concepts in Greek and Old Babylonian Mathematics, Roskilde 1988 (Filosofi og Videnskabsteori på Roskilde Universitetscenter, 3. Række: Preprints og Reprints, 1988, nr. 1.
Høyrup 1989: J. Høyrup, Zur Frühgeschichte algebraischer Denkweisen. Ein Beitrag zur Geschichte der Algebra, in: Mathematische Semesterberichte 36 [1989], 1–46.
Høyrup, 1989a: J. Høyrup, Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon, Roskilde 1989 (Filosofi og Videnskabsteori på Roskilde Universitetscenter, 3. Række: Preprints og Reprints, 1989, nr. 1.
Hunger 1968: H. Hunger, Babylonische und assyrische Kolophone, Kevelaer – Neukirchen-Vluyn 1968 (Alter Orient und Altes Testament, 2).
Jacobsen 1976: Th. Jacobsen, The Treasures of Darkness. A History of Mesopotamian Religion, New Haven – London 1976.
Krasnova 1966: S. A. Krasnova (ed.), Abu-l-Vafa al Buzdžani, Kniga o tom, čto neobchodimo remeslenniku iz geometričeskich postroenij, in: A. T. Grigor'jan – A. P. Juškevič (eds.), Fiziko-matematičeskie nauki v stranach vostoka. Sbornik statej i publikacij. Vypusk I (IV), Moscow 1966, 42–140.
Labat 1963: R. Labat, Manuel d'épigraphie akkadienne (signes, syllabaire, idéogrammes), Paris [4]1963.
Lambert 1953: M. Lambert, Textes commerciaux de Lagash, in: RA 47 [1953], 57–69, 105–120.
Larsen 1983: M. T. Larsen, The Mesopotamian Lukewarm Mind. Reflections on Science, Divination and Literacy, in: F. Rochberg-Halton (ed.), Language, Literature, and History: Philological and Historical Studies Presented to Erica Reiner, New Haven 1983 (American Oriental Series, Vol. 67), 203–225.
Levey 1966: M. Levey, The ALGEBRA of Abū Kāmil, Kitāb fī al-jabr wa'l-muqābala, in a Commentary by Mordecai Finzi. Hebrew Text, Translation, and Commentary, Madison – Milwaukee – London 1966.
Lévi-Strauss 1972: C. Lévi-Strauss, The Savage Mind, London 1972 (French 1st ed. 1962).
Liddell – Scott 1968: H. G. Liddell – R. Scott, A Greek-English Lexicon. Ninth Edition (1940), with a Supplement, Oxford 1968.
Limet 1973: H. Limet, Étude de document de la période d'Agadé appartenant à l'Université de Liège, Paris 1973 (Bibliothèque de la Faculté de Philosophie et Lettres de l'Université de Liège, fasc. CCVI).
Luckey 1941: P. Luckey, Ṯābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen, in: Berichte der Mathematisch-Physikalischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig 93 [1941], 93–114.
Mahoney 1971: M. S. Mahoney, Babylonian Algebra: Form vs. Content, in: Studies in History and Philosophy of Science 1 [1971/1972], 369–380 (Essay review of Neugebauer 1934, on occasion of the reprint edition 1969).
Marcus 1980: S. Marcus, The Paradoxical Structure of the Mathematical Language, in: Revue Roumaine de Linguistique 25 [1980], 359–366.
Nesselmann 1842: G. H. F. Nesselmann, Versuch einer kritischen Geschichte der Algebra, 1. Theil. Die Algebra der Griechen, Berlin 1842.
Neugebauer 1932: O. Neugebauer, Zur Transkription mathematischer und astronomischer Keilschrifttexte, in: AfO 8 [1932/1933], 221–223.
Neugebauer 1932a: O. Neugebauer, Studien zur Geschichte der antiken Algebra I, in: Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung B: Studien 2 [1932/1933], 1–27.
Neugebauer 1934: O. Neugebauer, Vorlesungen über Geschichte der antiken mathematischen Wissenschaften, I. Vorgriechische Mathematik, Berlin 1934 (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 43).
Neugebauer 1935: O. Neugebauer, Mathematische Keilschrift-Texte, I–III, Berlin 1935–1937 (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung A: Quellen, 3: 1–3).
Neugebauer – Sachs 1945: O. Neugebauer – A. Sachs, Mathematical Cuneiform Texts, New Haven 1945 (American Oriental Series, 29).

Pauly-Wissowa: Paulys Realencyclopädie der Classischen Altertumswissenschaften. Neue Bearbeitung begonnen von Georg Wissowa, Stuttgart 1893ff.

Peet 1923: T. E. Peet, The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary, London 1923.

Powell 1972: M. A. Powell, Sumerian Area Measures and the Alleged Decimal Substratum, in: ZA 62 [1972/1973], 165–221.

Powell 1976: M. A. Powell, The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics, in: Historia Mathematica 3 [1976], 417–439.

Procli Diadochi in Primum Euclidis Elementorum Librum Commentarii ex recognitione Godofredi Friedlein, Leipzig 1873.

Rashed 1984: R. Rashed (ed.), Diophante, Les Arithmétiques, Tôme III (Livre IV), Tôme IV (Livres V, VI, VII). Texte établi et traduit, Paris 1984.

al-Rawi – Roaf 1984: F. N. H. al-Rawi – M. Roaf, Ten Old Babylonian Mathematical Problems from Tell Haddad, Hamrin, in: Sumer 43 [1984, published 1987], 175–218.

Renger 1976: J. Renger, Hammurapis Stele „König der Gerechtigkeit". Zur Frage von Recht und Gesetz in der altbabylonischen Zeit, in: WO 8 [1975/1976], 228–235.

Röllig 1969: W. Röllig (ed.), *Lišān mitḫurti*. Festschrift Wolfram Freiherr von Soden zum 19. VI. 1968 gewidmet von Schülern und Mitarbeitern, Kevelaer – Neukirchen-Vluyn 1969 (Alter Orient und Altes Testament, 1).

Rosen 1831: F. Rosen (ed.), The Algebra of Muhammed ben Musa, Edited and Translated, London 1831.

Rudhardt 1978: J. Rudhardt, Trois problèmes de géométrie, conservés par un papyrus genevois, in: Museum Helveticum 35 [1978], 233–240.

Sachs 1952: A. Sachs, Babylonian Mathematical Texts II–III, in: JCS 6 [1952], 151–156.

Saggs 1960: H. W. F. Saggs, A Babylonian Geometrical Text, in: RA 54 [1960], 131–146.

Sayılı, 1958: A. Sayılı, Sâbit ibn Kurra'nin Pitagor Teoremini Tamimi, in: Belleten 22 [1958], 527–549.

Sayılı 1960: A. Sayılı, Thâbit ibn Qurra's Generalization of the Pythagorean Theorem, in: Isis 51 [1960], 35–37.

Sayılı 1962: A. Sayılı, Abdülhamid ibn Türk'ün katașık deklemlerde mantıkî zaruretler adlı yazısı ve zamanın cebri (Logical Necessities in Mixed Equations by ʿAbd al Ḥamîd ibn Turk and the Algebra of his time), Ankara 1962 (Türk Tarih Kurumu Yayınlarından, VII. seri, no. 41).

Scheil 1915: V. Scheil, Les tables igi x gal-bi, etc., in: RA 12 [1915], 195–198.

Sesiano 1982: J. Sesiano, Books IV to VII of Diophantus' Arithmetica in the Arabic Translation Attributed to Qusṭa ibn Lūqā, New York – Heidelberg – Berlin 1982 (Sources in the History of Mathematics and Physical Sciences, 3).

Sesiano 1986: J. Sesiano, Sur un papyrus mathématique grec conservé à la Bibliothèque de Genève, in: Museum Helveticum 43 [1986], 74–79.

Steinkeller 1979: P. Steinkeller, Alleged GUR.DA = ugula-géš-da and the Reading of the Sumerian Numeral 60, in: ZA 69 [1979], 176–187.

Suter 1922: H. Suter, Beiträge zur Geschichte der Mathematik bei den Griechen und Arabern, Erlangen 1922 (Abhandlungen zur Geschichte der Naturwissenschaften und der Medizin, Heft 4).

Szabó 1969: Á. Szabó, Anfänge der griechischen Mathematik, München – Wien – Budapest 1969.

Tanret 1982: M. Tanret, Les tablettes "scolaires" découvertes à Tell ed-Dēr, in: Akkadica 27 [1982], 46–49.

Thomsen 1984: M.-L. Thomsen, The Sumerian Language. An Introduction to its History and Grammatical Structure, Copenhagen 1984 (Mesopotamia, 10).

Thureau-Dangin 1897: F. Thureau-Dangin, Un cadastre chaldéen, in: RA 4 [1897], 13–27.

Thureau-Dangin, 1934: F. Thureau-Dangin, Notes assyriologiques. lxxvii – Nombres ordinaux et fractions en Accadien. lxxviii – Carré et racine carré, in: RA 31 [1934], 49–52.

Thureau-Dangin 1936: F. Thureau-Dangin, Review of Neugebauer 1935 (MKT I–II), in: RA 33 [1936], 55–61.
Thureau-Dangin 1936a: F. Thureau-Dangin, L'équation du deuxième degré dans la mathématique babylonienne d'après une tablette inédite du British Museum, in: RA 33 [1936], 27–48.
Thureau-Dangin 1938: F. Thureau-Dangin, Textes mathématiques babyloniens, Leiden 1938 (Ex Oriente Lux, Deel 1).
Thureau-Dangin 1940: F. Thureau-Dangin, L'origine de l'algèbre, in: Académie des Belles-Lettres, Comptes rendus 1940, 292–319.
Thureau-Dangin 1940a: F. Thureau-Dangin, Notes sur la mathématique babylonienne, in: RA 37 [1940], 1–10.
Tropfke – Vogel 1980: J. Tropfke, Geschichte der Elementarmathematik, Bd. 1: Arithmetik und Algebra. Vollständig neu bearbeitet von K. Vogel, K. Reich, H. Gericke, Berlin – New York [4]1980.
Unguru – Rowe 1981: S. Unguru – D. E. Rowe, Does the Quadratic Equation Have Greek Roots? A Study of „Geometric Algebra", „Application of Areas", and Related Problems, in: Libertas Mathematica 1 [1981], 1–49, 2 [1982], 1–62.
Vajman 1961: A. A. Vajman, Šumero-vavilonskaja matematika. III–I tysjačeletija do n. è., Moscow 1961.
van der Waerden 1961: B. L. van der Waerden, Science Awakening, Groningen [2]1961.
Ver Eecke 1926: P. Ver Eecke (ed.), Diophante d'Alexandrie, Les six livres arithmétiques et Le livre des nombres polygones. Œuvres traduites pour la première fois du Grec en Français avec une introduction et des notes, Bruges 1926.
Vitruvius, On Architecture, Edited and Translated by F. Granger, I–II, London – Cambridge, MA 1970 (Loeb Classical Library, 251, 280).
Vogel 1933: K. Vogel, Zur Berechnung der quadratischen Gleichungen bei den Babyloniern, in: Unterrichtsblätter für Mathematik und Physik 39 [1933], 76–81.
Vogel 1936: K. Vogel, Bemerkungen zu den quadratischen Gleichungen der babylonischen Mathematik, in: Osiris 1 [1936], 703–717.
Vogel 1959: K. Vogel, Vorgriechische Mathematik, II. Die Mathematik der Babylonier, Hannover – Paderborn 1959 (Mathematische Studienhefte, 2).
Vogel 1960: K. Vogel, Der „falsche Ansatz" in der babylonischen Mathematik, in: Mathematisch-Physikalische Semesterberichte 7 [1960], 89–95.
Vogel 1968: K. Vogel, Chiu chang suan shu. Neun Bücher arithmetischer Technik. Ein chinesisches Rechenbuch für den praktischen Gebrauch aus der frühen Hanzeit (202 v. Chr. bis 9 n. Chr.), Braunschweig 1968 (Ostwalds Klassiker der exakten Wissenschaften, N. F. Bd. 4).
von Soden 1939: W. von Soden, Review of Thureau-Dangin 1938 (TMB), in: ZDMG 93 [1939], 143–152.
von Soden 1952: W. von Soden, Grundriss der akkadischen Grammatik, Rome 1952 (Analecta Orientalia, 33).
von Soden 1952a: W. von Soden, Zu den mathematischen Aufgabentexten vom Tell Harmal, in: Sumer 8 [1952], 49–56.
von Soden 1964: W. von Soden, Review of Bruins – Rutten 1961 (TMS), in: BiOr 21 [1965], 44–50.
von Soden 1965: W. von Soden, Akkadisches Handwörterbuch, Wiesbaden 1965–1981.
Witmer 1983: T. R. Witmer (ed.), François Viète, The Analytic Art. Nine Studies in Algebra, Geometry and Trigonometry from the Opus Restituae Mathematicae Analyseos, seu Algebrâ Novâ, translated, Kent, Ohio 1983.
Woepcke 1855: F. Woepcke, Analyse et extraits d'un recueil de constructions géometriques par Aboûl Wafâ, in: Journal Asiatique, 5ième série 5 [1855], 218–256, 309–359.

I. The starting point: Numbers or lines—in method and in conceptualization

For almost 60 years it has been known that the Babylonians of the Old Babylonian period[1] (and later) knew and solved equations of the second degree[2]—like this[3]

Obv. II, 1.	Length and width added is 14 and 48 the surface.	$x+y=14$ $x\cdot y=48$
2.	The magnitudes are not known. 14 times 14 (is) 3′16°.[4] 48 times 4 (is) 3′12°.	$14\cdot 14=196$ $48\cdot 4=192$
3.	3′12° from 3′16° you substract, and 4 remain. What times what	$196-192=4$
4.	shall I take in order to (get) 4? 2 times 2 (is) 4. 2 from 14 you subtract, and 12 remain.	$\zeta^2=4\rightarrow\zeta=2$ $14-2=12$
5.	12 times 30′ (is) 6. 6 is the width. To 2 you shall add 6, 8 is it. 8 is the length.	$12\cdot {}^1/_2=6=y$ $2+6=8=x$

1 The Old Babylonian period spans the time from c. 2000 B.C. to 1600 B.C. (middle chronology). The mathematical texts dealt with in this paper belong (with the exception of the Seleucid text presented first) to the time from c. 1800 B.C. to c. 1600 B.C.

2 Anachronisms are lurking everywhere when one speaks of Babylonian mathematics in modern terms. The Babylonians did not classify their problems according to degree. They have related classifications, but the delimitations deviate somewhat from ours, and they have another basis. "Equations", on the other hand, is a fully adequate description even of the Old Babylonian pattern of thought, if only we remember that what is equated is not pure number but the entity and its measuring number: Combinations of unknown quantities equal given numbers or, in certain cases, other combinations of unknown quantities.

3 BM 34568 No 9 (BM 34568 refers to the museum signature, No. 9 to the number of the problem inside the tablet as numbered in the edition of the text). The text was published, transliterated, translated and discussed by O. Neugebauer in MKT III 15ff. The numbers in the margin refer to the position of the text on the tablet: Obserse/reverse, column No, line No. The text is Seleucid, i.e. from around the 3rd century B.C. The translation is a literal retranslation of O. Neugebauer's German translation as given in MKT III. So, it renders the way in which Babylonian algebra is known to broader circles of historians of mathematics.—All translations given below will be my own direct translations from the original language.

4 For the transcription of the sexagesimal place value numbers found in the text I follow F. Thureau-Dangin's system, which in my opinion is better suited than O. Neugebauer's for the purpose of the present investigation: 3° is the same as 3, 3′ the same as $3\cdot 60^{-1}$, 3″ means $3\cdot 60^{-2}$, etc. 3′ means $3\cdot 60^{1}$, 3″ equals $3\cdot 60^{2}$, etc. The notation is an extension of our current degree-minute-second-notation, which anyhow descends directly from the Babylonian place value system.—I use the notation as a compromise between two requirements: For the convenience of the reader, the translations must indicate absolute place; this is not done in the original cuneiform, but so few errors are made during additive operations that the Babylonians must have possessed some means to keep track of orders of magnitude. On the other hand, the zeroes necessary in the conventional transcription introduced by O. Neugebauer (1932) (3,0;5 instead of F. Thureau-Dangin's 3\5′ and the Babylonian 3 5) are best avoided in an investigation of Babylonian patterns of thought, where such zeroes had no existence. Admittedly, the situation is quite different in an investigation of mathematical techniques, especially the techniques of mathematical astronomy, with special regard to which O. Neugebauer introduced his notation.

This short text will serve to locate the central question of the present paper. Apart from the statements of the problem and of the result, the text contains nothing but the description of a series of numerical computations – it can be characterized as an exemplification of an algorithm. Even problems 18 and 19 of the same tablet (MKT III,16f.), which describe a procedure abstractly, do so on the purely algorithmic level: "Take length, width and diagonal times length, width and diagonal. Take the surface times 2. Subtract the product from the ⟨square on length, width and⟩ diagonal. Take the remainder times one half...". There are no explanations of the way the solution is found, no justification of the steps which are made and, so it seems, no indication whatever of the pattern of thought behind the method.

Now it is an old observation that traditional algebraic problems can be solved by basically different (though often homomorphic) methods. So, if we look at a problem of the type $x+y=a, x \cdot y=b$, we would of course solve it by manipulation of symbols. Most Latin and Arabic algebras of the Middle Ages, from al-Khwārizmī onwards, would formulate it that "I have divided 10 into two parts, and multiplying one of these by the other, the result was 21"[5]; in order to obtain the solution, they would call one of the numbers "a thing" and the other "10 minus a thing", and by verbal argument ("rhetorical algebra") they would transform it into the standard problem "10 things are equal to 10 dirhems and a square", the solution of which was known from a standard algorithm. Diophantos would speak more abstractly of "finding two numbers so that their sum and product make given numbers"[6]; he would exemplify the method in a concrete case, "their sum makes 20 units, while their product makes 96 units", and he would proceed until the complete solution by purely rhetorical methods, formulated however by means of a set of standardized abbreviations ("syncopated algebra"[7]).

In the so-called "geometric algebra" of the Greeks, geometrical problems of the same structure are solved.[8] So, in Euclid's *Data*, prop. 85 it is demonstrated by stringent geometrical construction that "if two lines contain a given surface in a given angle, and their sum is also given, then they must both be given".[9]

Quite different geometry is used by al-Khwārizmī to justify the standard algorithms by means of which he solves the basic mixed second-degree equations. To avoid any confusion with the much-discussed "geometrical algebra" I will propose the term "naive geometry".[10] Since this concept will be fundamental for the following, I shall present it more fully.

[5] Al-Khwārizmī, *Algebra*, tr. Rosen 1831: 41.

[6] *Arithmetica* I, xxvii.

[7] The term is due to Nesselmann (1842: 302ff.), who also introduced the more current "rhetorical algebra".

[8] Irrespective of the question whether "geometric algebra" was or was intended to be an "algebra".

[9] Cf. also *Elements* II 5. An analogue of the corresponding algebraic problem in one unknown is found in *Data*, prop. 58, and in *Elements* VI 28.

[10] In a preliminary discussion paper (Høyrup 1985) I spoke of "geometrical heuristics". I have also pondered "visual" or "intuitive geometry". After much reflection, however, I have come to prefer "naive geometry" as relatively unloaded with psychological and philosophical connotations.

In order to justify his solution to the equation "square and roots equal to number", al-Khwārizmī explains the case "a Square and ten Roots are equal to thirty-nine Dirhems".[11] The number 39 is represented by a composite figure: A square of side equal to the unknown "Root" and two rectangles of length 5 ($=10 \div 5$) and width equal to the Root, positioned as shown in Fig. 1 (full-drawn line). The gnomonic figure is completed by addition of a square equal to $5^2=25$ (dotted line), the whole being then a square of area $39+25=64$. Its side being $\sqrt{64}=8$, the unknown Root will be $8-5=3$.

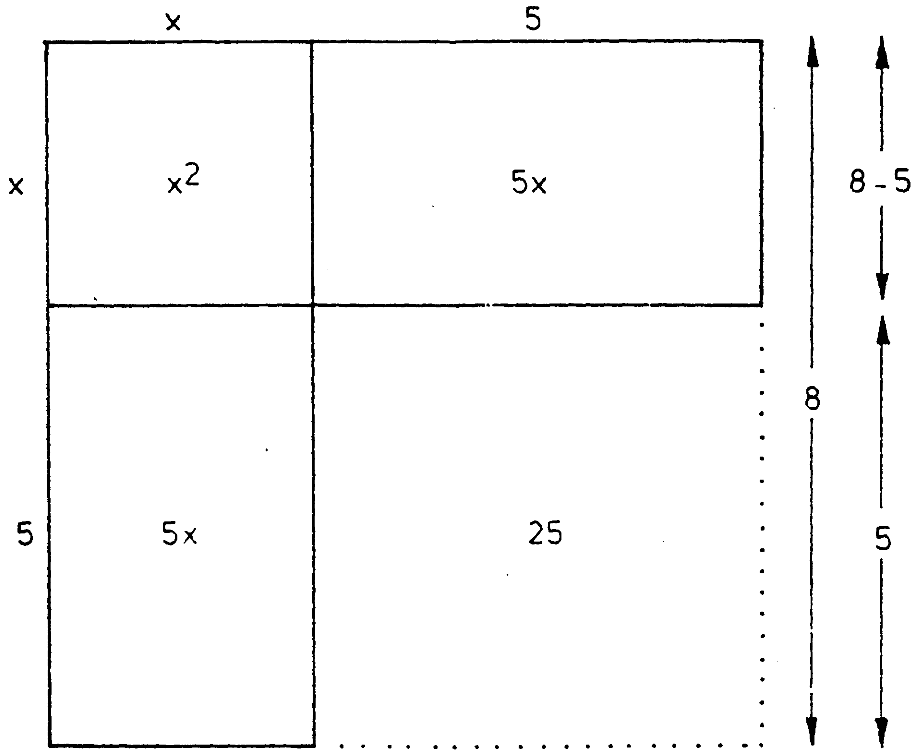


Fig. 1

We may feel comfortably sure that the argument behind our Babylonian algorithm was not of the Euclidean brand–Babylonian geometric texts show no trace at all of Euclidean argumentation. We can also safely exclude the hypothesis that the Babylonians made use of symbolic algebra.[12] Finally, we can

[11] See his *Algebra*, tr. Rosen 1831: 13–16.

[12] The immediate argument for this is that symbolic algebra requires a level of abstraction which appears to be totally alien to Babylonian thought. If this seems too much of an argument *ex silentio*, it can be added that symbolic algebra is *grosso modo* akin in structure to arithmetico-thetorical algebra. So, even if we upkeep the possibility of symbolic algebra as a silent hypothesis, the arguments which will be given later against an arithmetico-rhetorical interpretation will also exclude symbolic translations of the latter.–On the same account, an "abacus" representation of Babylonian algebra with counters representing the coefficients of the products and powers of the unknowns can be discarded. In itself, the "abacus interpretation" might have a certain plausibility, since material calculi had been used for common reckoning and/or computation in earlier epochs in Mesopotamia. Nothing, however, but the writing material, pebbles instead of ink, distinguishes such a representation from the syncopated algebra of Diophantos or the further development and schematization of the same principle found in Medieval Indian algebra. Arguments against an arithmetico-rhetorical interpretation of Babylonian algebra will hence also be arguments against an arithmetical "abacus algebra".–I shall return below to the possibility of a geometric "abacus algebra" related to the Greek "figurate numbers".

also be confident that some kind of argument lays behind the text. Random play with numbers might of course lead to the discovery of a correct algorithm for a single type of equation, and such an algorithm could then be transmitted mechanically. Still, the equation-types of Babylonian mathematics are so numerous, and the methods used to solve them so freely varied that random discovery cannot explain them. Some mental (and perhaps also physical) representation must have been at hand which could give a meaning to the many intermediate numbers of our algorithm (196, 4, 192, 4, 2, 12, $^1/$) and to the operations to which they are submitted.

We cannot, however, read out of the text whether this representation was of rhetorico-arithmetical character or should be described as naive geometry. Truly, the "length", "width" and "surface" might seem to suggest the latter possibility. But even Diophantos used a geometrical vocabulary ("square", "application") which was only meant to suggest the arithmetical relations involved. Similarly, the Arabic and Latin algebras of the Middle Ages would speak indifferently of a second power as "square" or "property" and of a first power as "thing" or "root", intending nothing but suggestive words which might fill the adequate places in the sentences. So, no conclusion is possible on that level.

The procedure leaves us in no better situation. It is easy to devise a rhetorical method which yields the numbers of the text as intermediate results, viz. a verbal translation of this:

$x+y=14; \qquad xy=48.$
$(x+y)^2=196; \qquad 4xy=192$
$(x-y)^2=(x+y)^2-4xy=196-192=4$
$x-y=\sqrt{4}=2$ (the length is normally supposed to exceed the width; hence, no double solution will arise)
$2y=(x+y)-(x-y)=14-2=12$
$y=\frac{1}{2}\cdot 12=6$
$x=(x-y)+y=2+6=8$

It is, however, just as easy to devise a geometrical figure on which the correctness of the solution and of the single steps can be argued naively (see Fig. 2). Here, a geometrical counterpart of every single number occurring in the calculation can be found. So, the algorithm leaves us in a dead end: It fits equally well to a rhetorical argument by arithmetical relations and to an argument by naive geometry.

Concerning another aspect of the question arithmetic/naive geometry we are no better off than in the case of the method, namely regarding the conceptualization of the problem itself: Was it seen as a problem of unknown numbers, represented perhaps by the dimensions of a geometric figure, or shall it be taken at its words, as a problem really concerned with unknown dimensions of such a figure?

That this latter question must be separated from that of the character of the method can be seen from comparison with other algebraic traditions. It is clear that Modern mathematics thinks of a set of equations like $x+y=14$; $x\cdot y=48$ as concerned with numbers, and that we understand the operations used to solve it as purely arithmetical operations. So, the basis of Modern algebra is

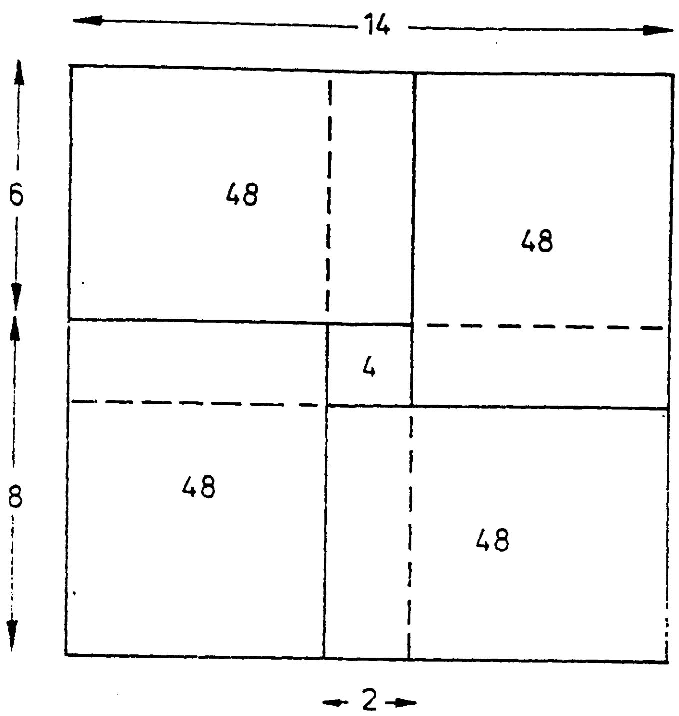


Fig. 2

arithmetical in conceptualization as well as method.[13] It is equally clear that we meet with lots of concrete problems, e.g. concerned with spatial extensions, which we translate into algebra and then solve by algebraic methods. In such cases, our conceptualization is concrete, e.g. geometrical, but our method is arithmetical–concrete entities are represented by abstract numbers.

On the whole, the same description would fit the Medieval algebraic tradition, with one important exception: The al-Khwārizmīan justification of the solution to the mixed second-degree equations (cf. above). There, the conceptualization of the problems is as arithmetical as everywhere else in al-Khwārizmī's algebra, but the method is naive geometry, where lines and surfaces represent the abstract numbers. Basic conceptualization and method need not coincide.

To anybody reading Babylonian "algebraic" sources it will be obvious that the conceptualizations of the problems are as varied as those of Modern algebra. Some are quite concrete geometrical problems: Partitions of triangular or quadrangular fields, calculations of the volumes of siege ramps, etc.; some are formulated as pure number problems, concerned e.g. with a pair of numbers belonging together in a table of reciprocals. The main body of texts, finally, deal with "lengths", "widths" and "surfaces" which cannot *a priori* be interpreted at face value, nor however as arithmetical dummies. Anyhow, there can be no reasonable doubt that these latter problems represent the basic conceptualization of Babylonian algebra, and that their "lengths" etc. are the entities which represent real lengths as well as numbers when such magnitudes occur in other problems.

[13] By "Modern" I mean "post-Renaissance", in the case of algebra specifically "post-Vieta". I disregard what mathematicians would call "modern" (abstract, "post-Noether") algebra as irrelevant to the present discussion: It is, at least in classical senses of these words, neither arithmetical nor geometric, be it in basic conceptualization or in method, although it is, primarily, an abstract extrapolation from arithmetical conceptualization and method.

There are, then, two main aspects of the problem investigated below: Firstly, whether the method used in Old Babylonian algebra was arithmetical (rhetorical or related) or naive-geometrical. Secondly, whether its basic conceptualization was arithmetical or geometrical.[14] Around these basic questions a web of other related and derived discussions will be spun, in order to give an allround picture of the discipline.

II. The obstacles

Neither the terminology nor the procedure of the problem translated above, would permit us to decide this question, or just to approach it. In this respect it is similar to a great many other Babylonian texts. For half a century, in has therefore been the prevailing opinion among historians of mathematics that at least the surviving and published texts will not permit us to solve the dilemma arithmetic/geometry. At the same time, most historians have implicitly or explicitly tended to favour the fully arithmetical hypothesis[15]—with the partial exception of K. Vogel, A. A. Vajman and B. L. van der Waerden.[16]

Until the Summer 1982, I shared these common opinions and prejudices, as I would now call them. At that time, however, I was inspired, by an interpretation of a puzzling text[17] and by a critical question from P. Damerow for my reasons, to look for traces of geometrical thought in other texts. Since then my knowledge of the language has improved so much that I have come to regard my original textual inspiration as totally wrong.[18] But like another Columbus I had the good luck to hit land on a course which I had chosen for bad reasons. A close reading of the texts, and the use of methods closer to those of contemporary human sciences (linguistics and structural semantics as well as literary analysis) than to those traditionally used in the history of Ancient mathematics, revealed that the arithmetical hypothesis cannot be upheld. As it is always more

[14] It should be emphasized that the investigation deals only with the algebraic texts. There is no reason to doubt the purely numerical character of many of the table texts; but the numerical character of texts like Plimpton 322 (MCT 38) does not permit us to conclude that algebraic problems, too, were understood and solved arithmetically. Similarly, it cannot be doubted that a number of texts deal with real geometric problems,—but even there generalizations are not automatically justified.

[15] Among the most explicit, Thureau-Dangin (1940: 302) states that the problems dealing with geometrical figures do so because "a plane figure will easily give rise to a second-degree equation", but that the problems are still "purely numerical", just like the indeterminate equations of Diophantos' *Arithmetica* VI, for which right triangles function merely as a pretext.

[16] So, van der Waerden (1961: 71f.) suggests hypothetically that certain basic algebraic identities may have been proved geometrically ($\{a-b\}\,\{a+b\}=a^2-b^2$, etc.). The conjecture is accepted by Vajman (1961: 168f.). At the same time, however, B. L. van der Waerden distinguishes the method of proof from the conceptualization, stating that the "thought processes of the Babylonians were chiefly algebraic (i.e. arithmetico-algebraic—J. H.). It is true that they illustrated unknown numbers by means of lines and areas, but they always remained numbers".

[17] IM 52301, the inscription on the edge as interpreted by Bruins (1953: 242f., 252).

[18] Cf. the revised transliteration and the new discussion of IM 52301 in Gundlach – von Soden 1963: 253, 259f.

difficult to verify than to falsify, I cannot claim that the investigation has proved a specific geometrical interpretation to be correct. Still, the geometrical reading gets very strong support, and I think it can be taken for sure that the Old Babylonian algebra must at least have been structurally isomorphic to a representation by naive geometry, while the arithmetical representation is only a homomorphism.

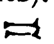
It will be clear from the following that my results could not have been found without methodological innovations. So, we should not wonder that the evidence against arithmetical thought has gone largely unnoticed for 50 years, and that the interpretation which O. Neugebauer characterized as a "first approximation" in 1932[19] has stood unchallenged since then.

This may sound cryptic to readers who are not familiar with the cuneiform script and texts, and may require an explanation. The Babylonian texts were written in a Semitic language (Akkadian) which has been dead as a literary language for two millennia (and as a spoken language even longer), with strong, at times all-dominating admixtures of loanwords from another language (Sumerian), which was probably already dead around c. 1800 B.C. except as a literary language used by the restricted circle of scribes, and of which no relative is known. Even the interpretation of the Akkadian language is far from completed, and the situation for Sumerian is still worse.[20] To add to the confusion, the script used consists of signs which may stand for one or, normally, several phonetic values, not necessarily close to one another, and for one or often several semantic ("ideographic") values, i.e. values as word signs ("logograms")[21] for Sumerian words and semantically related Akkadian words. The connection between the different values is rooted in semantic affinity, in phonetic affinity in either of the two languages, or simply in the conflation of originally separate signs.[22] To all this may come trite problems of legibility, due to careless writing or to bad preservation of the tablets.

[19] Neugebauer 1932a: 6.

[20] So, no real Sumerian dictionary exists to this day.

[21] The prevailing tendency has been to leave the conception of ideograms and to claim that the cuneiform signs when not used phonetically would stand for, and be read as, specific Akkadian words. The difference between an ideogram and a logogram is as the difference between "+" and "viz.": The first sign will of course always be read by words, depending on the situation as "plus", "added to", "and", or something similar; only in the specific additive meaning, however, can it replace the spoken word "and"; it is no logogram, it corresponds to an operational concept which is not identical with any verbal description. "Viz.", on the other hand, is a real logogram for "namely".—No doubt, the logographic interpretation describes the normal non-phonetic use of cuneiform signs adequately. At least in mathematical texts, however, certain signs must be understood as ideograms, not as logograms, as I shall exemplify below (cf. notes 57f. and note d to TMS XVI A; cf. also SLa 25f., on similar phenomena in non-mathematical contexts).

[22] The sign  may be taken as an example. The conventional sign name is KAŠ, the name given to it in ancient sign lists. It may stand for Sumerian kaš, "beer" (Sumerian words are usually transliterated in spaced types), and for the possessive suffix -bi; the latter reading is used in Sumerian as an approximate syllabic writing for the compound b+e > bé, "says it" (or rather "it is said"). These three uses have given rise, respectively, to logographic use in Akkadian texts for the corresponding words *šikarum*, *-šu/-ša* and *qabûm*, together with the derived *šū/šūatu*, "this", a function in which Su-

Happily, the system was also ambiguous for the Babylonian scribes themselves, and they developed certain aids for avoiding the ambiguities (phonetic complements to logograms; semantic determinatives). Furthermore, inside texts belonging to a specific type and period, the range of possible values of a given sign is strongly restricted. The restrictions, however, have to be discovered; hence, extensive knowledge of a whole text-type is required before the single text can be safely transliterated into syllabic Latin writing.

On this background, the immensity of the task solved in the 1930es by O. Neugebauer and F. Thureau-Dangin will be seen: To decipher the phrasing of the mathematical texts, and to discover the mathematical meaning of the terms. First when this is done in a way which can be relied upon can the question of conceptualization be raised in earnest.

Raised . . . but hardly solved by direct methods. Just because the language of the single text-type is specific, we must regard the terminology as technical or semi-technical. We know from modern languages that the semantic contents of a technical term are not necessarily unravelled by etymological studies. The etymology of "perpendicular" would lead us to the pending plumb-line and thus to the vertical direction. *A posteriori* we can understand the way from here to the right angle–but we cannot predict *a priori* that "vertical" will change into "right angle", nor can we even be sure that a modern geometer thinks of verticality when he uses the standard-phrase and raises a perpendicular.[23]

The situation is not very different in Akkadian, or in Semitic languages in general. An example from the Hebrew on which I shall draw below will show this. *'bq* has, as a verb, the meaning "to fly away". Hence we have nominal derivations "(light) dust" and "pollen" (HAHw, 7[a]); from "light dust" probably the tablet covered with light dust or sand, the "dust abacus", and from here apparently the "abacus" in general.[24] Who would imagine that the heavy table on which stone calculi are moved was, etymologically, "something flying away"?

Truly, the character of Semitic languages is such that the basic semantic implications of the root from which a word derives are rarely or never lost quite of sight–they are conserved at least as connotations. Such conservations are forced upon the users of the language by its very structure.[25] But a requirement

merian bi can also be used. In the Old Babylonian period it will also be found with the phonetic values *bi*, *bé*, *pí*, and *pé* (accents and subscript numbers are used to distinguish different writings of the same syllable). In later periods, it can also be used phonetically as *gaš*, *kaš* and *kás*.–To this comes the role in a number of composite sign groups used logographically: different specified sorts of beer; innkeeper; song(?); etc. Finally, the sign may represent twice the surface unit èše, written . (After MEA and ABZ No 214, and a commentary from B. Alster).

[23] To know whether he thinks concretely through the standard-term we would have to investigate whether he avoids using it when constructing the orthogonal to a non-horizontal line; i.e., we would have to investigate the structure of his total terminology and its use in various situations.

[24] See Pauly-Wissowa I(i), 5. HAHw quotes the Semitic root in Hebrew, Arabic and Aramaic. It appears to be absent in Akkadian.

[25] The Semitic languages combine–with special clarity and richness in the system of verbs and their derivations–fixed, mainly consonantal roots carrying the semantic basis, with a huge variety of prefixes, infixes (among which the vowels, which are sub-

that there should be a semantic umbilical chord between the general and the technical meaning of a term can at most be used as a control with hindsight, when the technical meaning has already been interpreted tentatively. It can tell nothing in advance.

In principle, technical terms should therefore be interpreted from technical texts. Here, more than anywhere else, the Wittgensteinian dictum should be remembered: "Don't ask for the meaning–ask for the use". Then, however, we are led into a vicious circle: Our sole access to the use of the technical terms is the body of texts, which only tell us about the use if we understand their terms. As long as two conflicting interpretations of the terminology both permit coherent understanding of use and meaning, neither can be rejected. And indeed, if we believe in an arithmetical interpretation of Babylonian algebra, we are led to an arithmetical interpretation of the unknown terms denoting its operations, and thus to a confirmation of our initial beliefs; initial belief in a geometrical interpretation is, however, equally selfconfirming.

Let us take an example, the phrase

10 *itti* 10 *šutākil-ma*: 1′40°.[26]

itti can be translated "together with", and the enclitic particle *-ma* by "and then" or "and thus", or it can simply be represented (as I shall do in the following) by ":". So, the phrase can be partially translated as

10 *šutākil* together with 10: 1′40°,

and so we know that *šutākil* represents an operation which from 10 and 10 creates 1′40° (=100), either an arithmetical multiplication of pure numbers, or a geometrical operation creating a rectangle with sides 10 and 10 and a corresponding surface of 100. The form can also be recognized as the imperative of a reciprocative causative stem derived from *akālum*, "to eat", or from *kullum* "to hold" (in which case the transcription ought to be *šutakīl*).[27] Hence we have the interpretation

"Make 10 and 10 eat/hold each other: 100,"

or, if we do not see what "eating" or "holding" has to do with the matter, and if

mitted to change) and suffixes determining not only grammatical category but also many semantic displacements which in Indo-European languages are not subject to morphological regularity. The actual functioning of such a system requires that its speakers apprehend subconsciously all the derivations of a root as belonging to one scheme, in the way an English four-year old child apprehends "whistled" as a temporal displacement of the semantic basis "whistle" according to a general scheme, as revealed by her construction of forms like "goed" instead of "went".

26 VAT 8390 rev. 21 (MKT I 337).

27 The former interpretation is suggested by the use of the Sumerian kú, "to eat", as a logogram for the term (cf. below section IV.2). For this reason it is normally accepted today, cf. von Soden 1964: 50, and AHw, *kullu(m)* and *akālu(m)*.–The latter interpretation was proposed by F. Thureau-Dangin (e.g. TMB 219), who explained the logographic use of kú as a pun-like transfer, inspired by coincident Št-forms for *kullum* and *akālum* (cf. TMB 232f.). Such transfers are in fact not uncommon in cuneiform writing (cf. above, note 22), and hence a derivation from "holding" cannot be outruled. –As it will appear below, a relation to another term *(takīltum)* appears to rule out the derivation from "eating", while a connection to "holding" makes perfect sense (cf. below, section IV.3). On the other hand, A. Westenholz expects that *kullum* would give rise to the form *šutkīl* and not to *šutakīl*–which I cannot make agree, however, with a number of derivations from *ḫiaqum*. Most safely, the question is left open.

we want to keep the question explicitly open, we may represent the semantic basis through a dummy XX:

"Make 10 and 10 XX each other: 100."

In both ways, we get something like idiomatic English as translation of the phrase. Still, concerning the question arithmetical versus geometrical interpretation we are no more wise.

Truly, most standard terms of Babylonian algebra look less opaque than "mutual eating/holding". "To append" x to y, "to pile up" x and y; "to tear out" or "to cut off" x from y or to see "how much y goes beyond x"; "to break x to two"; all of these can, as descriptions of additive and substractive procedures and of halving, respectively, be interpreted concretely, and all seem to suggest an imagination oriented toward something manifest, e.g. the procedures of naive geometry, rather than an abstract arithmetical understanding. But so do the Latin etymologies of "addition" and "subtraction"; like these, several of the Akkadian terms were established as standard expressions, and some may have been fixed translations of age-old terms. There may have been as little concrete substance left in them as there remains of lead in a right angle.

On the level of single terms and their applications the texts are thus not fit to elucidate the conceptual aspects of Babylonian algebra and mathematics.

III. The structural and discursive levels

Originally, I started my search for traces of naive-geometrical thought precisely at the level of single-term applications and literal meanings, and I was soon able to draw the negative conclusions just presented. At the same time, however, the close reading of the texts had led me to some real clues. One of these was the structure of the total mathematical terminology used in the Babylonian algebraic texts.[28] The other has to do with what could be called the "discursive aspect" of the texts (as opposed to technical and terminological aspects): The way things are spoken of and explained, the organization of explanations and directives, and metaphorical and other non-technical use of seemingly technical terms.[29]

[28] A simple instance of such structural analysis was suggested in note 23 as a means to investigate whether a modern user of geometrical terminology associates the "raising" of a perpendicular with the literal meaning of this term.

[29] This paradoxical phrase should perhaps be clarified. An important characteristic of a technical term is fixed semantic contents and relative absence of connotations and analogic meanings. Technical terms when applied as such are not open-ended. Even in modern mathematics, however, technical terms are also used metaphorically and in other ways departing from their technical semantics. This happens during theoretical innovation, when the technical terminology has to adapt to new conceptual structures. It also occurs in informal discussion and didactical explanation when truth is not to be stated but to be discovered or conveyed. These are processes which always require compromise with pre-existent understanding, and therefore such non-technical displacements of meaning reveal something about this understanding. (Cf. for certain aspects of this discussion Beck 1978 and Marcus 1980).

The Babylonian mathematical texts abound in examples of such derived meanings and applications of terms to an extent which suggests that we are not confronted with

The clues implied by the discursive aspect of the texts can only be demonstrated on specific examples, and I shall postpone their presentation. Part of the evidence provided by the structural analysis can, on the other hand, be explained in abstract form. Instead of retelling my Odyssey through the texts completely and from the beginning,[30] I shall therefore present some basic results abstractly before going on to a selection of texts in order to penetrate further. Exemplifications and supplementary arguments will be given on the basis of these texts.

In current English, the expressions "*a* times *b*" and "*a* multiplied by *b*" describe the same process–they are synonyms. Which one to choose in a given situation is a matter of style–as will be demonstrated by the fact that person A may choose the one in a situation where person B would choose the other, or that the choice depends on audience (school children versus mathematicians) or medium (oral or written, popular or scholarly). We have two different expressions at our disposal, but we have only one mathematical concept.

The Babylonians had many multiplicative expressions: *šutākulum* (whence *šutākil*); *našûm*; íl; nim; *eṣēpum*; tab; a-rá; UL.UL; UR.UR. The matter has, to my knowledge, never been discussed explicitly, but it has been taken for granted and selfevident that all[31] described the same concept.[32]

As long as an arithmetical conceptualization was itself taken for granted, and taken for granted to such an extent that the mere possibility of alternative conceptualizations was not recognized, this automatic conflation of all multiplicative concepts was unavoidable: In an arithmetical conceptualization there is only one operation to be described, there can be only one concept.[33]

Still, selfevident as it has appeared to be, the conflation is not true to Babylonian mathematical thought. The terms are not synonyms, the choice among them is restricted by other criteria than those of style, taste and dialect.

Truly, some sets of terms are synonyms. íl is the Sumerian equivalent of *našûm*, "to raise", and it is used logographically in exactly the same functions (which makes it debatable whether we are entitled to speak of a different term–*našûm* and íl are rather full and shorthand writings of the same Akkadian term). nim, Sumerian equivalent of *elûm*, "to be high" and used even for its factivitive stem "to elevate", is used instead in a few texts (here, then, another term for

a real technical terminology after all, that few terms possess a basic, really fixed technical meaning. Instead, most terms should probably be regarded as open-ended expressions which in certain standardized situations are used in a standardized way. This will be amply exemplified below.

30 This is, *grosso modo*, the way I go through the subject in my preliminary presentation (Høyrup 1985) of the problem and of my results. The outcome is rather opaque.

31 With the partial exception of *eṣēpum* and its logographic equivalent tab, the original meaning of which is "to duplicate", and which in phrases "duplicate x to n" means "multiply x by (the positive integer) n" if interpreted arithmetically.

32 It should, however, be emphasized that both O. Neugebauer and F. Thureau-Dangin show great intuitive sensitivity to the shades of the vocabulary in MKT and TMB. I remember no single restitution of a broken text in either of the two collections which does not fit the results of my structural investigation.

33 Disregarding the possibility to distinguish between multiplications involving only integers, multiplications where one factor at least is an integer, and multiplications of wider classes of numbers. In fact, all Babylonian terms except *eṣēpum* (and tab) can be applied for the "multiplication" of any number by any other number.

the same concept is in play–the equivalence is semantic but no longer logographic). Similarly, *šutākulum* (with the logographic writing kú) is replaced by UL.UL in certain texts and by UR.UR in others. But while the choice of a term inside a group is free, the choice of the group from which a term shall be taken is subject to clear rules–rules which in a geometrical interpretation of the procedures are easily stated.

IV. Basic vocabulary and translational principles

Most other classes of arithmetical operations are also subdivided in Old Babylonian mathematical thought, if we are to judge from the Old Babylonian vocabulary.[34] As a preparation for the presentation of the texts, I shall summarize in schematic form the basic vocabulary and its subdivisions, indicating in rough outline the use of each subclass. I shall also give the "standard translations" of the terms which I am going to use in my translations of texts in the following chapters, together with the translations of the terms given in AHw.[35]

IV. 1. Additive operations

Two different "additions" are distinguished. The first is described by the term *waṣābum* (AHw "hinzufügen"), and it is used when something is added to an entity the identity of which is conserved through the process (the nominal derivative *ṣibtum* designates *inter alia* the interest, which does not change the identity of the capital to which it is added). The Sumerian daḫ is used as a logogram. In order to avoid associations to the modern abstract concept of addition, I use the standard translation "to append" for both terms.

The other addition is designated by *kamārum* (AHw "schichten, häufen"). It is used when several entities are accumulated into one "heap" (cf. the etymology of "accumulation" from "cumulus"), which is identical with neither of them. gar-gar and UL.GAR are both used ideographically in the same function[36], apparently as pure logograms. For standard translations of all three terms I use "to accumulate".

While no separate name for the sum of an "identity-conserving" addition is found (for good reasons, of course), the "accumulation" can be designated by various derivations of *kamārum*: *kimrātum*, a feminine plural[37] (whence my

[34] The vocabulary of the later (Seleucid) mathematical texts is very different, and can indeed be taken as an indication that the mathematical conceptualizations had changed through and through during the centuries which separate the two periods. Cf. below, section X.2.

[35] In order to emphasize the purely Old Babylonian character of the summary I write all Akkadian verbs and nouns with "mimation", i.e. with the final *-m* which was lost in later centuries.

[36] Literally, the Sumerian gar-gar means something like "to lay down (gar) repeatedly"; possibly, the UL of UL.GAR is due to a sound shift from UR = ur, *inter alia* "to collect" (ŠL II No 575.9), which would lead to an interpretation of UL.GAR as a composite verb "to lay down collectedly" (maybe an artificial "pseudo-Sumerogram").

[37] Cf. section VIII.2, the notes to AO 8862, for reasons why the single sum has to be understood as a plural.

standard-translation "things accumulated"), *nakmartum* (standard translation "accumulated") and *kumurrûm* ("accumulation"). gar-gar and UL.GAR can both serve logographically in the same functions.

IV. 2. Subtractive operations

Subtractions too may and may not conserve identity. The "non-conserving" subtraction compares two different entities, by means of the expression *mala x eli y itter*, "as much as *x* over *y* goes beyond" (from *watārum*, "übergroß, überschüssig sein/werden", with the logograms SI and dirig). The most common term for the "identity-conserving" subtraction is *nasāḫum*, "ausreißen", with logographic equivalent zi. I shall use the standard translation "tear out". Another term with the same function (but apparently a slightly different shade) is *ḫarāṣum*, "abschneiden" (etc.), st. transl. "cut off". In specific situations, a variety of other terms may occur.

IV. 3. Multiplicative operations

The standard expression of the multiplication tables is "*x* a-rá *y*" where *x* and *y* are pure numbers. It is also found in a few of the problem texts (normally in double constructions, cf. below). The semantic base is rá, "to go" (cf. Danish *gange*, "times", from *gå*, "to go", and the analogous Swedish terms). After having used initially the modernizing standard translation "*x* times *y*" for "*x* a-rá *y*" I have opted for "*x* steps of *y*", mainly because even Seleucid texts remember this sense of the term, as revealed by their use of a genitive for the second factor (cf. below, section X.2, BM 34568 No 9; cf. also note 38).

The term *eṣēpum* (AHw "verdoppeln") and its equivalent tab "to duplicate", i.e. "to take once more", whence even the extension "to repeat several times", was already mentioned. It is used for multiplications of any concrete entity by a positive and not too large integer, and apparently meant as a concrete repetition of that entity. When used to "make multiple", it occurs in phrases like "*X ana n eṣēpum*", "to repeat *x* until *n*", or "*x* a-rá *n* tab", "to repeat *x n* steps" (the deviating use of a-rá will be noticed[38]). In all cases, I use the standard translation "to repeat".

The third group is made up of *našûm* ("(hoch)heben, tragen"), its Sumerian equivalent íl (the normal logogram for *našûm*), and the Sumerian nim, apparently also used logographically in certain texts. As mentioned above, the latter term means originally "be high", equivalent of Akkadian *elûm*. In mathematical contexts it is in all probability used as a pseudo-Sumerogram for the (factitive) D-stem *ullûm* of this word.[39] These terms are used for the normal calcula-

[38] A similar use of Akkadian *alākum*, "to go", as a substitute for *eṣēpum* is found in several Susa texts (among which TMS IX, translated below in section VIII.3). In one of them (viz. TMS VII) the "step" which is gone repeatedly appears to be designated a-rá.

[39] Originally, Thureau-Dangin suggested the conjecture that nim might be used for the factitive or causative Š-stem *šūlûm* (TMB 239). However, the headline of the Susa list

tion of concrete quantities by multiplication: When multiplying by the tabulated constant (igi-gub) factors; when multiplying by a reciprocal as a substitute for division (cf. below); in all situations involving a factor of proportionality; and when the areas of trapeziums, triangles and trapezoids are found.[40] As standard translations I use "to raise" for *našûm* and íl (the alternative "to carry" cannot be brought into semantic harmony with nim). For nim I use "to lift".

The connection between "raising" and multiplication is not obvious to the modern mind. Several clues exist in the texts, however, which connect the usage to Babylonian technical practice.

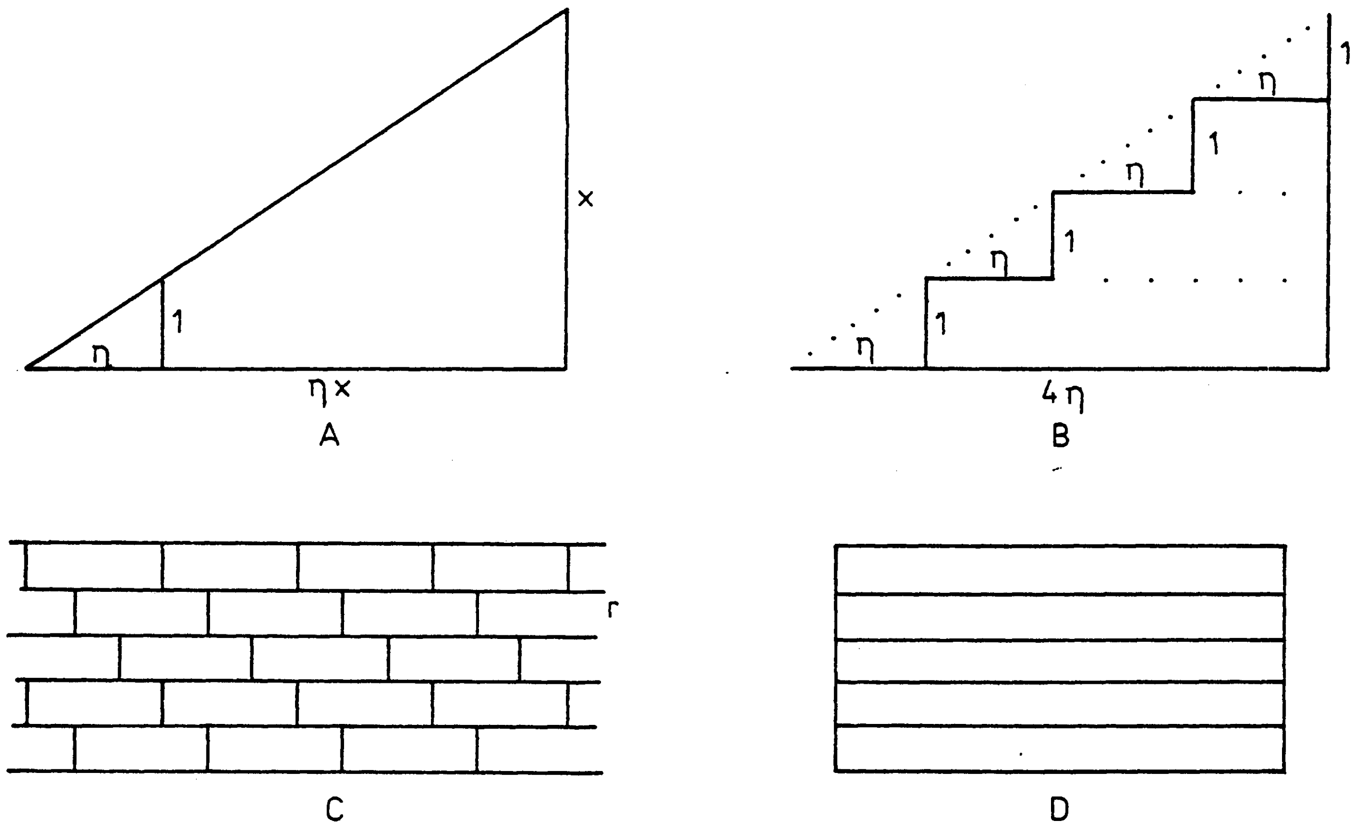


Fig. 3

One clue derives from the way volumes are calculated. If the base is quadratic, rectangular or circular, it is normally "spanned" by length and width (or found as $^1/_{12}$ of the area spanned by the circular circumference with itself). The multiplication with the vertical dimension, however, is a "raising" or "lifting". In itself, this already speaks to the imagination—raising *is* vertical movement. Furthermore, *ullûm* (and hence nim, cf. above) is precisely the term used when a wall is elevated *n* brick layers (AHW 208 b$^{11-14}$).

Another clue is provided by the use of the expression "íl^{tum} of 1 cubit (height)" (íl with phonetic complement *-tum*, indicating a derivation from *našûm* with ending *-tum*, e.g. *našitum*, a substantivized participle meaning "that which raises") as a measure for the

of constant (igi-gub-) factors claims to contain "igi-gub, that of making anything high" (TMS III 1), using the infinitive *ullûm* of the constantly factitive D-stem. Since the Š-stem is furthermore used (in AO 17264, MKT I 126f., and in Haddad 104,III,7, al-Rawi—Roaf 1984) in the sense of making a square-root "come up" as a result, nim ∼*ullûm* is probably to replace F. Thureau-Dangin's conjecture.

40 As we shall see below, the area of a rectangle is presumably also found by "raising", although the operation is normally not made explicit.

inverse gradient of a slope, i. e., the length one has to progress horizontally in order to attain an elevation of 1 cubit.[41]

Fig. 3A shows the situation, demonstrating the role of the íl^{tum} (η) both as a factor of proportionality and as "that which raises the slope 1 cubit". Fig. 3B shows the same in a less sophisticated manner (for which reason it is used occasionally in modern elementary teaching), closer to the Babylonian term than the Greek-type Figure 3A.

Comparison of Fig. 3B and Fig. 3C shows that the "raising of a slope" and the "raising of a wall" can easily be imagined as the same process. Figure 3D, finally, dem onstrates how the conception of a rectangular area as consisting of unit strips which is testified by the terminology (cf. below, section VII.2) can make one assimilate even area calculation to the same scheme.

Sargonic and earlier mathematical texts contain many area computations but never any term for multiplication. Brickwork and slope calculations seem to have arisen later—the oldest mathematical brick text known is from Ur III.[41a] We may imagine that explicit multiplicatory terminology was introduced together with these "new multiplications", and that it was then also used metaphorically for other similar calculations, be it area computations or arguments of proportionality. In this connection one should remember that not only the use of igi-gub-factors but also the computation of a/b by means of a table of reciprocals (cf. section IV.6) builds on proportionality.

The last group of multiplicatory operations is made up by *šutākulum*, "to make eat/hold each other", and its various semantic cognates: ì-kú-kú and ì-kú (its logograms), UL.UL and UR.UR. Some further cognates turn up below under the heading "squaring". In the algebra-texts, these terms are only used when an entity which may be considered a "length" is multiplied by another which is a "width", or by itself. That is, in a geometric interpretation of the texts it is used when a rectangle or a square is considered, in fact, as we shall see below, when it is produced. To a modern mind it might be tempting to interpret this as an indication that the term is used for the calculation of an area, since this involves the multiplication of two quantities of dimension length. The falseness of such an interpretation is, however, obvious from the way the areas of triangles, trapeziums and trapezoids are found: As soon as calculated average lengths are multiplied, the term used is *našûm*, íl or nim.

The interpretation of *šutākulum* understood as "mutual eating" is less than self-evident. Truly, an idea which was advanced by S. Gandz[42] in order to explain the use of *ukullûm*, "ration of food", as a term for the inverse gradient of a slope, could be extended as a last resort: In Hebrew, a field covered by vines is said to be "eaten" by the vines.[43] Similarly, a "mutual eating" inherent in *šutākulum* could be read as "mutual covering". To "make length and width cover each other" should then mean "to make them define/confine" a surface—viz. a rectangular surface, since it is fully described by length and width. The case where "length and length" are made eat/hold/cover each other,[44] on the other hand, turns out to describe the construction of an irregular quadrangle.

[41] BM 85196 rev. II 11 (MKT II 46).

[41a] N.C. 304, see Vajman 1961: 246f., cf. for the dating Friberg (forthcoming) § 4.5.

[42] Gandz 1939: 417f.

[43] The same idea of covering a piece of land is indeed seen in the Old Babylonian measurement of a slope by the "*ukullûm* eaten in 1 cubit", i.e. covered per cubit height (VAT 6598 rev. I 18, in MKT I 279, cf. TMB 129).

[44] YBC 4675 obv. 1 (MCT 44) has the expression *šumma* a. šà uš uš i-kú", "when a length and a length eat/hold a surface", referring to a surface stretched by two (dif-

It would, however, seem much more obvious to conceptualize the situation as a length and a width (or a length and another length) "holding" together the rectangle (or trapezoid) in question. In either case the geometrical contents of the metaphor is the same, the two lines confining together a surface. As standard translation I shall use the phrase "make *A* and *B* span" (which should be neutral with regard to the two possible derivations though slanted towards "holding"). Two texts (VAT 8390 and AO 8862, cf. below) make explicit that surface construction is meant, telling that "length and width I have made span: A surface I have built".

The ideogram ì-kú-kú seems to derive simply from the reciprocity of the Št-stem (the form ì-kú being a mere abbreviation: it is mainly used in the utterly compact "series texts"). UR.UR and UL.UL have the same repetitive structure; their semantics is probably best explained in connection with the concepts for squaring, to which we shall turn next.

As it will be seen below, the term *takīltum* (read as *šakīltum* in MKT I), which turns up in specific connections during the solution of second-degree-equations, must be related to *šutākulum*; I shall use the term untranslated. Detailed discussions of its meaning and use must await its occurrence in the texts. At present it should only be observed that according to all available evidence it cannot derive from *akālum*, which forms no D-stem. Its close connection to *šutākulum* implies that the derivation same must hold for the latter term (in which case, by the way, the correct transcription will be *šutakūl(l)um*), cf. note 27).

IV. 4. Squaring and square-root

The two fundamental verbs belonging to this area are si₈, "to be equal", and *maḫārum*, "gegenübertreten (as an adversary, as an equivalent)" etc. From the mid-third millennium onwards, si₈ is used to denote a square as (a quadrangular figure with) equal sides. At approximately the same early epoch, it is also seen to denote the equality of the lengths alone or the widths alone in quadrangles.[45] In the Old Babylonian texts, it is found with a prefix as íb-si.[46], literally a verbal form, probably meaning "it makes equal". It is used when square-roots are extracted, at times inside constructions where it stands clearly as a verb, at times seemingly as a noun identifying the square-root itself. In YBC 6504 (MKT III 22f.) and in the "series texts" it is used for (geometrical or arithmetical) squaring (cf. note 63), and in one text[47] it denotes an indubitable geometric square.

ferent) lengths, i.e. to an irregular quadrangular surface. Later in the same text (rev. 15) the term *šutākulum* itself stands as a complete parallel to the use (in rev. 6) of *epēšum*, "to make", "to produce" (viz. a quadrangular surface). In neither case is any multiplication to be found.

45 On the denotation of squares, see Deimel 1923 No 82 (cf. MKT I 91, and Powell 1976: 430) and Edzard 1969. On the equality of lengths alone or widths alone, see Allotte de la Fuÿe 1915: 137ff.

46 Occasionally ba-si₈. This term is, however, more common in connection with cube roots.

47 BM 15285, passim (MKT I 137f.). The geometrical character of the squares is certain

To a modernizing mathematical interpretation this looks like primitive confusion: The Babylonians use the same term for a square (number) and its square root. Such a reading is, however, anachronistic, due to a pattern of thought which would have looked confused to a Babylonian: We conflate the geometrical figure characterized by equal and mutually orthogonal sides with one of its attributes, viz. the area which can be ascribed to it (the square "is" 25 m^2, while it "has" a side of 5 m). The Babylonians conflate the figure with another attribute, viz. with its side (the square figure "is" 10 nindan, while it "has" an area of 1 iku = 100 nindan^2). Following a proposal by J. Friberg[48], I shall use the standard translation "equilateral" in cases where the term is used as a noun. This should avoid the wrong connotations following from the use of words bound up with our own conceptual distinctions and conflations. When the term is used as a verb, I shall use "to make equilateral"–the reasons for this will be given below on the basis of the texts.

maḫārum itself is mostly used in mathematical texts in the sense of "correspond to/confront (as equal)" ("confront" will be my standard translation). The derivation *mitḫartum* (a nominal derivation, "thing characterized by correspondence/counterposition") is used to denote a square, i.e., as we shall see in the following chapter, a geometrical square–once again identified with its side and possessing an area.[49] I shall use the standard translation "confrontation", in agreement with a conception of the square as a "situation" determined by confronting equals. The verbal Št-stem *šutamḫurum* ("to make correspond to/make confront each other") is used for the process of squaring with only one number or length as the object. I shall use the standard translation "make confront itself", viz. so that a square is formed.

A final important derivative is *meḫrum* (for which gaba(-ri) appears to be used logographically), "that which corresponds to/confronts its equal". Its function is best explained in connection with occurrences in the texts, so I shall postpone it. As standard translation I use "counterpart".

A number of other terms and signs belong to the same semantic field. LAGAB (written KIL in MKT and TMS) is used in one text[50] to indicate equality between shares in a field partition; in the "Tell Ḥarmal compendium"[51] and in one of the Susa texts[52] it denotes the usual square figure ("being" a length and "possessing" an area. Basing myself on the Tell Ḥarmal compendium I shall treat it as a logogram for *mitḫartum*, giving it the same standard translation.[53] NIGIN (written KIL.KIL in MKT) is used in one

both because they are spoken of as positioned and because they are drawn on the tablet. Shifts between the two terms show that íb-si$_8$ is intended here as a logogram for the Akkadian word *mitḫartum* (cf. immediately below). In the "algebraic" problem text Str. 363 (MKT I 244), where the scribe has done his best to find (and, one may suspect, to construct) Sumerian logograms to express his Akkadian thought, the same equivalence íb-si$_8$ ~ *mitḫartum* is used.

[48] Private communication.

[49] See e.g. BM 13901 passim (several problems are translated below).

[50] AO 17 264 obv. 2f. (MKT I 126).

[51] Goetze 1951.

[52] Texte V, TMS 35ff. All three occurrences are late Old Babylonian, AO 17 264 possibly even early Kassite.

[53] The sign is indeed a "confrontation" of equal lines: . It is thus probable that its ideographic equivalence with *mitḫartum*, rather than being connected to its use as a

Susa text[54] exactly as LAGAB, for the square figure. In the larger part of the Susa corpus it could be replaced by *šutamḫurum*, as also in some genuine Babylonian texts.[55] Finally it is found in a couple of Susa texts with two factors[56], corresponding to the use of *šutākulum*. This practical equivalence with several semantically related yet glossarially distinct terms makes it impossible to consider it a real logogram for any of its equivalences; hence, NIGIN is an example of a non-logographic ideogram.[57] Since the sign can replace *lawûm*, "umgeben", *saḫārum*, "sich wenden", "herumgehen" and its derivative *siḫirtum*, „Umkreis", I shall propose the standard translation "make surround", viz. surround a square or rectangular figure, and square or rectangular "surrounding", depending on the word class required by context.

UR.UR is found in certain texts in constructions similar to those with *šutākulum*.[58] UR itself is found in another late Old Babylonian or early Kassite text[59] in the sense of "squaring", and in general non-mathematical language it can be used logographically (with various complements) for *ištēniš*, "like one", "together" (<*ištēnum* "one"), for *mitḫāriš* "correspondingly" (i.e. "equally" or "simultanously", <*maḫārum*, cf. above),

logogramm for *lawûm*, "to surround" (in which case its Sumerian reading is nigín), is to be considered directly iconic.—In any case, the use of the sign in AO 17264 (cf. note 50) must be considered secondary, derived from the habitual association of the quadratic figure with equality. In this connection it is perhaps worthwhile remembering that the sign for si$_8$ was also originally (and still in Old Babylonian inscriptions on stone) a square standing on a corner (and , respectively). Even this sign would thus have directly iconic connotations.—It should be observed that the evidence for logographic equivalence from the Tell Ḥarmal compendium is evidence for the way it was read aloud but not necessarily for complete identity (nowadays, "+" may be read aloud as "and", but the context will show that addition is meant). Precisely this text, indeed, contains syllabic writings of terms which in other texts are invariably written with Sumerograms (*šiddum* for uš, *nārum* for íd).

[54] Texte VI, TMS 49ff.

[55] BM 85194 (MKT I 143ff.) and BM 85196 (MKT II 43ff.).

[56] Texte IX 5 and 12, and Texte XXI 4 (TMS 63 and 108). The edition transcribes as *šutamḫurum* and translates as *šutākulum*!

[57] Cf. above, note 21. The ideographic role of the sign in connection with squaring and "rectangularization" should of course be distinguished from its logographic role inside other semantic fields.

The sign is , a repeated LAGAB. As in the logogram ì-kú-kú, the repetition looks like an intentional graphic repetition of the reciprocity of the Št-stems *šutākulum* and *šutamḫurum* or perhaps a representation of the use of two lines to stretch the square or rectangle. Cf. also note 58 on UL.UL and UR.UR.

[58] YBC 4662 and 4663 passim (MCT 69, 71f.). In YBC 4662, the term occurs in the construction *x* a-rá *x* UR.UR.a; however, in several other constructions (appending, i.e. an additive operation; raising) the tablet writes a-rá instead of *ana*, due perhaps to a dictation or writing error; so, I guess that the original intention was *x ana x* .... In YBC 4663, the term when used for squaring gives the factor only once (3°15′ UR. UR.ta), but for once *šutākulum* is used in the same way in that tablet (rev. 20). On the other hand, while the tablet has uš sag UR.UR.ta (ta ~ *ina*, "from"/"by means of"), it writes uš *u* sag *šutākil* (*u* ~ "and"); UR.UR can therefore not be a pure logogram for *šutākulum*, instead the whole phrase is written as an ideographic syncope.

A. Goetze (MCT 148) counts the two tablets among the early Southern ones. Both, however, state results with the word *tammar*, "you see", as do the texts belonging to his group VI and other Northern texts (cf. below, note 84).

As in the case of ì-kú-kú as a logogram for *šutākulum*, the repetitive structure of UR.UR is probably to be read as a (pseudo-) Sumerian rendition of the reciprocity of the Št-form *šutamḫurum* —or, rather, as a way to render in Sumerian grammar a geomet. rical idea which is rendered in Akkadian by the Št-stem, and rendered badly so, as the verb has only one object.

[59] AO 17264 obv., 13f. (MKT I 126, cf. TMB 74).

and for *nakrum* "enemy", probably derived from the association of this concept with *maḫārum* (cf. above).[60] Because of the ideographic but probably not logographic equivalence with *maḫārum* I propose the standard translation "oppose".

UL.UL is found in 7 tablets[61], in all of which it is used for squarings, in a way which could make it a logogram for *šutamḫurum*. But in one of them[62] it is also used in the same role as *šutākulum*, and in another[63] it is also used as a substitute for íb-si$_8$ in a situation where this term could be translated "as a square" or "squared", and where it is kept apart from *šutākulum* and its relatives. So, we have to do with yet another ideogram to which no well-defined logographic value can be ascribed.

Once more, the term appears to point to the idea of confrontation of equal forces. Originally the sign represents a lowered bull's head, corresponding to the reading ru$_5$ (used logographically for *nakāpum*, "to butt"). UL.UL should then be read ru$_5$-ru$_5$, viz. as a logogram for *itkupum* 'to butt each other", "to join battle"[64], and figuratively thus "to confront". Since this latter term is already used, I shall propose a distinct but semantically analogous standard translation, "to make encounter".

IV. 5. Halving

As it is later seen in Medieval elementary arithmetic, halving is a separate operation in Old Babylonian mathematics, or, rather, it occurs as a specific operation in certain specific connections. Chief among these are the bisection of a side or of a sum of opposing sides when areas of triangles or quadrangles are calculated, and the halving of the "coefficient of the first-degree term" in the treatment of second-degree equations. The term used is the verb *ḫepûm*, "zerbrechen", in connections like "break into two" or "half of x break" (where I have used the standard translation "break"). Certain texts use the Sumerogram gaz.

The half resulting from a "breaking" operation is designated *bāmtum* (occasionally abbreviated or Sumerianized BA.A), a term which I shall translate "moiety". It is distinguished from the normal half, *mišlum* (~šu-ri-a), which designates the number $^1/_2=30'$ as well as that half of an entity which is obtained through multiplication by 30'.[65]

60 All three values appear to belong originally to UR$_5$, but all are also testified for UR—cf. the terms in question in AHw, and MEA, No 401 (UR$_5$) and No 575 (UR).
It may be worth noticing that the original sign for UR$_5$ still used on stone in the Old Babylonian period was a square standing on a corner: ◈.

61 Str. 363 passim (MKT I 244f.); Str. 368 rev. 5, 8 (MKT I 311); VAT 7532 obv. 19 (MKT I 295); VAT 7535 rev. 17 (MKT I 305); VAT 7620 passim (MKT I 315); YBC 6504 passim (MKT III 22f.).

62 Str. 363 rev. 15f.: . . . 20 *u* 1 UL.UL-*ma* 20 / 40 *u* 5 UL.UL-*ma* 3'20° . . . . Furthermore, in obv. 9 of the same tablet a relative clause refers back to UL.UL by a syllabic *šutākulum*.

63 YBC 6504. In the first two problems of the tablet, íb-si$_8$ is used in the statement, while *šutākulum* is used for squarings in the prescription of the procedure, and íb-si$_8$ turns up when towards the end a square-root is taken. In the third and fourth problems, UL.UL is used both in statement and procedure for squarings, while íb-si$_8$ is still used for the square-root.

64 See CAD *nakāpu*. I am grateful to A. Westenholz for pointing out this meaning of UL.UL to me, whose implications I had overlooked.

65 One place where the distinction between "halving" and "division by 2" (i.e. multi-

According to parallels from other Semitic languages, *bāmtum* was originally a designation for a rib-side or for the slope of a mountain ridge. Probably because such a side or slope can be apprehended as one of two opposing sides or slopes, the term is used in a variety of situations where an entity splits naturally or customarily into two parts, or where e.g. a building is composed of two wings. In mathematical texts, it is used similarly for the semi-sum of opposing sides in a trapezium or the semi-diameter of a circle —all being halves of entities falling naturally or by customary procedure into two "wings".
Below, we shall also see it in an important role in the treatment of second-degree equations (section V.2. on BM 13901 No 1, and passim).

IV. 6. Division

As it is well known, Babylonian mathematics possessed no genuine operation of division. Division was a problem, no procedure. If the divisor b of a problem a/b was regular, i.e. if it could be written in the form $2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma}$, in which case its reciprocal would be written as a finite sexagesimal fraction, and if it was not too big, $1/b$ would be found in agreement with the standard table of reciprocals[66], and a/b would be found by "raising" $1/b$ to a. If b was irregular [67], or if it was complicated to be recognized as regular, a mathematical problem text would simply formulate the division as a problem, "what shall I pose to b which gives me a?", and next state the solution–since normal mathematical problems were constructed backwards from known solutions, the ratio would always be expressible and mostly known.

Two concepts are important in connection with the method of reciprocals: That of the reciprocal itself, and that of the process through which it is found. The reciprocal of n is spoken of as igi n gál-bi, at times abridged to igi n gál or simply igi n. The literal meaning of the expression is unclear, but it is

plication by 2^{-1}) is especially obvious is Str. 367 rev. 3f. (MKT I 260). A clear distinction between *bāmtum* and *mišlum* is found in the tablets AO 8862 (below, section XIII.2) and BM 13901 (MKT III 1–5). A single tablet (YBC 6504, MKT III 22f.) uses šu-ri-a where others have *bāmtum*.

66 The standard table of reciprocals lists the reciprocals of the regular numbers from 1 to 1′21° (=81) (cf. MKT I 9ff.). It can be legitimately discussed whether our term "table of reciprocals" is anachronistic. Indeed, one table, which appears to antedate 1850 B. C. (MKT I 10 No 4), seems to express the idea that not $1/n$ but $60/n$ is tabulated (Scheil 1915: 196). As argued by Steinkeller (1979: 187), another table with phonetically written numbers suggests the same idea (in MKT I 26f.). On the other hand, such conceptualizations of early tables have no necessary implications for the understanding which Old Babylonian calculators had of the tables used in their own times, and two observations combined suggest that they did in fact apprehend their own tables as tabulations of the numbers $1/n$. Firstly, they used the tables for divisions, i.e. for multiplications with these numbers. Secondly, there is textual evidence that they possessed a specific concept for the number $1/n$, as distinct from a general "n'th part" of something (cf. below, note 69).

67 A few tables containing approximate reciprocals of certain irregular numbers exist: YBC 10529 lists reciprocals of regular as well as irregular numbers between 56 and 1′20° (MCT 16). M 10, John F. Lewis Collection, Free Libr. Philadelphia gives reciprocals of 7, 11, 13, 14 and 17 (Sachs 1952, 152). Apparently, however, such approximations are not used in the Old Babylonian mathematical texts, and since the irregular divisors of these texts always divide the dividends, such use would indeed lead to errors which could not go unnoticed.

testified as early as c. 2400 B.C. in the sense of "the n'th".[68] Some Old Babylonian mathematical texts use it both in this general sense as "the n'th of some quantity", and in the special sense of "$1/n$" regarded as a number, but in a way which distinguishes the two.[69] There is therefore no doubt that the Old Babylonian calculators had a specific concept for the number $1/n$, which I shall designate by the standard quasi-translation "igi of n". The general sense I shall render simply by "the n'th part".

To "find" a reciprocal is spoken of by the verb *paṭārum* "(ab)lösen, auslösen" with the logographic sumerogram du_8. In F. Thureau-Dangin's opinion[70], this term should be understood in analogy with the modern metaphor "to solve a problem". However, in two texts the term is also used subtractively[71], in a

68 VAT 4768 and VAT 4675, published by Förtsch (1916 N^os 65 and 175), transliterated and translated by Bauer (1967: 508–511). The texts belong to the fourth year of Lugalanda, and speak of 1/4 šekel silver and 1/6 šekel silver, by the phrase igi *n* gál. Similar contemporary evidence (also from Lagaš) is found in Lambert 1953:60, 105, 106, 108, 110 (1/3, 1/4 and 1/6 šekel of silver or lead) and Allotte de la Fuÿe 1915: 132 (1/4 sar of land).–All these tablets antedate the first known occurrences of sexagesimal reciprocals by some 350 years, and they antedate by c. 200 years a school text which suggests that the ideas behind the sexagesimal system were on their way but not yet mature nor formulated around 2200 B.C. (Limet 1973 No 36; cf. commentaries in Powell 1976:426f. and Høyrup 1982:28). We can therefore confidently infer that the general sense of a reciprocal is a secondary derivation. This undermines the only plausible yet grammatically somewhat enigmatic explanation of the term given to date, one offered by Bruins (e.g. 1971:240): Literally, the phrase igi 6 gál-bi 10-àm could mean "in the front of 6 is: 10 is it", i.e. "in front of 6 is found what (in the table of reciprocals)? 10". This explanation would interchange basic and derived meaning, and unless unexpected evidence turns up which moves the tables of reciprocals back into the mid-third millennium, it cannot be upheld. –Truly, Bruins (1983: 105, and earlier) points to two Old Babylonian texts which write the Akkadian term *pāni*, "in front of", in order to designate the reciprocal. (So does also Haddad 104, see al-Rawi–Roaf 1984, section 0.4.3). Certain Old Babylonian scribes hence appear to have held the same hypothesis as Bruins concerning the origin of the expression. But Old Babylonian scribes may as easily have constructed a scholarly pseudo-etymology as they can have guessed correctly a conceptual development which had taken place some 800 years before their own time. In any case, current logographic use of igi for *pānum* may easily have led them astray to an erroneous "folk etymology".

69 Str. 367 (MKT I 259f.) speaks in obv. 3 of "the third part" of a length in a complete phrase igi 3 gál, while the reciprocals of 4, 1, 3, 2, 3′ 20″ and 1`12° are spoken of (passim) simply as igi *n*. The same distinction is made in VAT 7532 and VAT 7535 (MKT I 294f. and 303ff.); here, even the n'th part of the number 1 is spoken of in the complete phrase when this number 1 is taken to represent an unknown length, and the part hence understood as a fraction of something, not as a reciprocal (a number). In BM 85210 rev. I 0–12 (MKT I 221f.), the "n'th part of m" is also spoken of by the complete expression and the reciprocals simply by igi n; but furthermore, while the finding of the latter is spoken of by the usual term du_8 (~*paṭārum*, "to detach", cf. below), the process producing the former is designated by zi (~*nasāḫum*, "to tear out"). BM 85194 (rev. I 28, rev. III 2f., and passim; MKT I 143ff.) speaks of both "part" and "reciprocal" by means of the abbreviated expression, but distinguishes by means of the differentiation between zi and du_8.

70 Thureau-Dangin 1936: 56.

71 In Str. 367 (MKT I 259f.) a triangle of area 21′36° is "detached" from a trapezium of area 36′, leaving a rectangle of area 14′25°. The other subtractive occurrence is Str. 362 obv. 15 (MKT I 240).

way which is only explained by the literal sense "detach". To "find the reciproca of *n*" is thus to be understood as "to detach the *n*'th part (from 1)"[72], a phrase that shall be my standard translation.

The division by an irregular number calls for few terminological commentaries. The term "pose" (my standard translation for *šakānum* ~ gar, see below) is no term for multiplication; at times, the multiplication to be performed is implicitly understood in the expression, but more often it is stated explicitly.[73] In the latter cases, the term used belongs invariably to the "raising"-class (*našûm*, íl, nim).

The same was the case when a dividend was multiplied by the reciprocal of a divisor, even when one side of a rectangle is found from the area and the other side.[74] Apart from the (purely arithmetic) distinction between regular and irregular divisors, division is one thing, and it is the inverse of raising. Nothing corresponding to the distinction between four different "multiplications" is found. This could be interpreted as evidence that the Babylonians understood their division as a common, purely arithmetic inversion of all four multiplications, the isomorphism between which they have of course recognized. Still, since such an understanding would rather lead to use of the purely arithmetical term a-rá, it seems to be a better explanation that the real multiplicative operation was "raising", while the other three classes were in reality something else which could not be reversed (as we shall see below, there are good reasons to apprehend "repetition" as real repetition of the concrete entity, and "spanning" as a constructive procedure; neither of these procedures is of course reversible).

IV. 7. Variables, derived variables, and units

Besides the above-mentioned terms for arithmetical operations, a number of basic concepts and appurtenant terms can profitably be presented in advance and briefly discussed. A first group contains the standard names for unknown quantities ("variables"), the way to label new variables, and the units.

By speaking of standard names for unknown quantities I want to emphasize once more that the Babylonians formulated algebraic problems dealing with

[72] Cf. also the subtractive conceptualization of the process "to find the *n*'th part of *m*" in BM 85210 and BM 85194 (see note 69).—Further evidence against F. Thureau-Dangin's assumption comes from the way the finding of a square-root is spoken of: You are requested to "make the equilateral come up" *(šūlûm < elûm)*; you "take" it *(laqûm)*; or the question is asked, "what the equilateral" (*minûm* íb-si$_8$). Had *paṭārum* meant simply "to solve" an arithmetical problem, nothing would have prevented the Babylonians from using it also for the solution of the problem $x \cdot x = A$.

[73] VAT 8389 obv. II 6—9 (below, section VII.1); VAT 8391 rev. I 28—30 (below, section VII.2); VAT 8512 rev. 1—5 (MKT I 341); VAT 8520 obv. 24f., rev. 25f. (MKT I 346f.); Str. 363 passim (MKT I 244f.).

[74] Str. 367 rev. 11 (MKT I 260); VAT 8512 obv. 10—12 (MKT I 341). A possible exception is AO 6770, N° 1, lines 5—7. Still, since no really satisfactory interpretation of this text has been given, it can hardly serve as evidence for anything. Improved transliteration and bibliography of earlier treatments of this text will be found in Brentjes—Müller 1982 (cf. Høyrup 1984 for reasons why even this latest interpretation is problematic).

many types of quantities: Numbers, prices, weights of stones, etc. One set of such unknown quantities, however, belongs with the "basic conceptualization" of Old Babylonian algebra, as unknown abstract numbers represented by letters belong with our own basic conceptualization (cf. chapter I).

These basic variables are of course the length and the width. They form a fixed pair. "Length" translates uš (very rarely written phonetically with the Akkadian term *šiddum*, "Seite, Rand; Vorhang"). "Width" translates sag, literally "head, front" (the rare corresponding Akkadian term is *pūtum*).[75] Both terms appear in surveying texts from Early Dynastic Lagaš[76]; surveying is thus the distant point of origin of the Old Babylonian second-degree algebra which should not necessarily be confused with its Old Babylonian conceptualization.

Problems in only one variable are basically formulated as concerned with a square identified with its side: *mitḫartum*, LAGAB, or NIGIN (see above, section III.4, "squaring and square-root"). In two Susa texts, the side of the square is occasionally spoken of explicitly as uš, "length", of the "square figure".[77]

In problems in one as well as two variables, the "second-degree-term" is spoken of by the same expression, a-šà, "field". Like "length" and "width", it is almost invariably written by the sumerogram, but in a number of places it occurs with a phonetic complement indicating a purely logographic use for the Akkadian *eqlum*.[78], [79] I shall use the standard translation "surface" as I want

[75] Strictly speaking, the Akkadian terms are not just rare. Excepting the Tell Ḥarmal compendium (which has uš ~ *šiddum*, but on which see note 53) they are never used as names for the standard variables but only in a couple of texts dealing with real rectangles: Db_2–146, obv. 3 (in Baqir 1962: Pl. 3; *šiddum* alone) and IM 53965, passim (in Baqir 1951; both terms). On the use of *pātum* (plural of *pūtum*) to designate the sides of a real square in BM 13901, No 23, cf. below, section V.4. Three final occurrences deal with carrying distances for bricks and the width of a canal.

[76] See the texts from c. 2400 B.C. published and discussed by Allotte de la Fuÿe (1915). A difference between the Early Dynastic surveying texts and the Old Babylonian standard algebra problems should be noted: While the latter tell us that they deal with a rectangle simply by speaking of uš and sag without any epithet, implying thereby that there is only one length and one width, the former will normally present all four sides of a quadrangle, and if a pair of opposing sides are equal they will with one exception which seems most hastily written tell explicitly that this is uš si_8, "lengths being equal", or sag si_8. "widths being equal".

[77] Even though the length is spoken of explicitly, the same lines of the text will identify the "confrontation" (LAGAB) itself with a number, viz. with the same number as the "length". Here as everywhere, square figure and side are conceptually conflated. So TMS V, obv. II.1: "The CONFRONTATION *and* 1/11 of *my* length accumulated: 1", i.e. confrontation = length = 55′.

[78] On the other hand, the terms uš and sag are on the same and other sorts of evidence not real logograms for *šiddum* and *pūtum* (cf. above).

[79] Like uš and sag, a-šà is used already in Early Dynastic texts (cf. note 72). It seems plausible that this rooting in an old tradition should be linked causally with the all-dominating Sumerographic writing (in fact, full phonetic writing of *eqlum* is as absent as phonetic Akkadian writing of uš and sag). In contrast, the unknown "confrontation" in problems of one unknown is not written by the traditional Sumerogram si_8 (cf. note 45). This appears to indicate that theoretical algebraic problems among which the problems of one unknown are important did not arise until the Old Babylonian age,

to avoid the connotations associated with the word "area": A number which describes or measures a surface. Such distinction between entity and measuring number is apparently not true to Babylonian thought.

A number of texts use terms like "length", "width" or "surface" for a succession of different numbers (in cases where we would use successively x and $\bar{x}$, etc.). In such cases the two different "lengths" can be distinguished by an epithet appended to one of them: lul corresponding to Akkadian *sarrum*, which is used in TMS XI and XXIV; standard translation "false") or *kīnum* (~gi-na; standard translation "true"). The use of these terms is best elucidated in connection with their occurrence in specific texts.

Another term with a related function is kúr, a Sumerogram used logographically for *nakārum*, "anders, fremd, feindlich sein, werden", and for its various derivatives. It turns up in certain series texts when a "second" or "modified" width occurs besides the width first considered. I shall propose the standard translation "alternate".

In contrast to Modern algebra, the seemingly pure numbers reveal themselves in certain texts as numbers counting a multiple of the basic unit of length, the nindan[80] (1 nindan equals c. 6 m). In problems concerned with volumes, however, the vertical dimension is measured in "cubits" (*ammatum* ~kuš = $^1/_{12}$ nindan), even when the problem is nothing but "disguised algebra". Areas are measured correspondingly in the unit sar = nindan^2, volumes in (volume-) sar=nindan^2 · kuš[81], i.e., a surface of 1 sar covered to the height of 1 kuš.

IV. 8. Recording

A large number of terms are used when given quantities and intermediate and final results are announced and taken note of. Some of them are mutually distinct, some are used inside the mathematical texts as "practical synonyms" (although they are not synonymous in their general use).

Most important is *šakānum*, "hinstellen, (ein)setzen, anlegen; versehen mit", and its Sumerogram gar. It may well have a precise technical meaning in the mathematical texts, but since this sense can only be approached by indirect means, I shall use a semantically rather neutral standard translation, "to pose".

The term is often used after the statement of a problem, when the given numbers are "posed" before calculations begin–they appear to be taken note of in some manner as a preparation for operations. Similarly, intermediate results are occasionally "posed" (but then mostly "posed to" or "posed by" a length etc.,

or at least that they arose among Akkadian speakers—in which connection it may be of interest that a specific Akkadian record-keeping system, distinct from the contemporary Sumerian system, was in use during the Sargonic era (see Foster 1982: 22—25). A similar conclusion could be drawn from the greater part of the basic algebraic vocabulary, which is written alternatingly in phonetic and ideographic writing, but where the latter writing is reconstructed and not traditional Sumerian.

80 Written GAR in MKT and NINDA in TMB. Cf. Powell 1972: 198f. on the transliteration nindan.

81 More complete information on the Old Babylonian metrological system will be found in TMB (pp. xiii—xvii) and MCT (pp. 4—6).

cf. below). In one case, even final results are recorded by "posing".[82] Finally, the term is invariably used in divisions by an irregular divisor, cf. above, section III.6.

The recording of intermediate results can also be spoken of by the verb *lapātum*, "eingreifen in, anfassen, schreiben" (rarely, it can also be used for the recording of a given number).[83] I shall use the standard translation "to inscribe".

The verb *nadûm*, "werfen, hin- niederlegen", is used in two apparently different functions, one of which might look as a "practical synonym" for *šakānum* and *lapātum*. In some texts, when the "equilateral" (i.e. square-root), of a number has been found, it is "laid down" in two copies, to one of which is added, and from the other of which is subtracted.[84] Two texts use "posing" in the same function, and four employ *lapātum* in a related way.[85] On the other hand, however, *nadûm* is never used in the other functions of these terms.

The other use of *nadûm* is in the tablet BM 15285[86], where the drawing of indubitably geometrical squares, circles and triangles is referred to by the term.

Even outside the domain of mathematical texts, similar uses of the term are known: "Bauten usw. anlegen"; "(Fang)netz auslegen"; "(auf Tafel usw.) eintragen, einzeichnen"; "Grundriß aufzeichnen".[87] I shall use the standard translation "to lay down", which shall therefore be read as "to lay down (in writing or drawing)". Since the former use is restricted to the laying down of entities which in the geometrical interpretation of the texts are the sides of squares, it is my guess that the real meaning in all mathematical texts is simply "to draw".

A specific phrase for recording an (invariably intermediate) result is *rēška likīl* "may your head retain (it)" (from *rēšum*, "Kopf, Haupt; Anfang, ...", and *kullum* "(fest)halten"). Apparently, the term is reserved for the storing of intermediate results of linear transformations (cf. below, section VII.2.).

The appearance of a result can be announced in various ways. It can be said that a number "comes up for you" (standard translation of *illiakkum*, from *elûm* "auf-, emporsteigen", Stative "hoch sein"), or that a calculation "gives" a certain result (my standard translation of *nadānum* "geben"; and of the Sumerogram sum). Finally, the result can be announced by the term *tammar* "you see" (from *amārum* "sehen"). The choice appears to depend exclusively on the geo-

[82] YBC 6504, passim (MKT III 22f.). In the same text, intermediate results too are "posed".

[83] IM 52301 obv. 19f. (below, section X.1); the text is rather late and contains several other deviations from normal usage); IM 54478 obv. 7 (Baqir 1951: 30). In the newly discovered text from Tell Haddad (Haddad 104 IV 9, 17, 29; in al-Rawi–Roaf 1984) the form *lupput* (D-stem, stative) is used of numbers which "stand written down" in a table of constant factors.

[84] VAT 8520 obv. 21, rev. 20 (MKT I 346f.); YBC 6967 obv. 11. Cf. below sections V. 1 and VIII. 4. A slightly different phrasing is found in IM 52301 rev. 5 and 10 (cf. note 79) and in Db_2 –146, 4 and 13 (Baqir 1962: Pl. 3), and another possibly in TMS XVII 12.

[85] "Posing" stands precisely as *nadûm* in TMS XIII, 10 (cf. correction to the line in von Soden 1964:49) and in IM 53965 rev. 7 (Baqir 1951:39). In AO 8862 II 21f. (MKT I 110), BM 13901 obv. II 8 (MKT III 2), YBC 4662 obv. 21 and 33 (MCT 71), and in YBC 4663 rev. 23 (MCT 69), finally, the "equilateral" is "inscribed until twice".

[86] Most recent edition with addition of a large fragment in Saggs 1960.

[87] AHw, article *nadû(m)* III, §§ 20, 22, 24.

graphical and chronological origin of the text (and in certain texts perhaps on personal taste).[88] The mathematical functions of all three coincide.

Very often, a result appears simply as a number, announced by no special word or at most by the enclitic particle *-ma* appended to the foregoing phrase. A single text uses the Sumerogram for "posing", gar (cf. above, note 82).

IV. 9. Structuration

The terms discussed till here were all concerned with the "arithmetical" level of the texts, that of single calculations. Another group of terms belongs to the meta-level which makes the texts "algebraic", and which structures the texts.

All those texts which describe a problem together with its solution start by stating the problem, after which the procedure is described. The former is written in the first person (viz. the teacher), past tense (only the excess of length over width will invariably be stated in the present tense). The procedure is formulated in the second person (the student), present tense, or the imperative, by a person (the instructor) who refers to the teacher in the third person. The statement has no special name, but the procedure is designated *epēšum* with Sumerographic equivalent kì. The term is the infinitive of a verb („machen, tun; bauen") used as a noun; when the description is finished, the derived term *nēpešum* is used. For *epēšum* I shall use the standard translation "the making", for *nēpešum* "the having-been-made".

Inside the description of the procedure, the statement of the problem may be quoted in justification of certain steps being made. This is done by the phrase "he has said", using the verb *qabûm*, sagen, „befehlen", which functions simply as a quotation mark.

Three terms are traditionally interpreted as indications that we pass from

[88] sum and *nadānum* are found in the texts to which A. Goetze ascribes for linguistic reasons an early, southern origin (groups I–IV, see MCT 146–151). *tammar* is found in his group VI ("northern modernizations of southern (Larsa) originals"), in the Susa texts of TMS and in a number of the late (and northern) Tell Ḥarmal texts (in Baqir 1950a and 1951); the early Tell Ḥarmal text IM 55357 (Baqir 1950:41–43) uses igi-dù, a logogram for *tammar*, mistaken by homophony for igi-du$_8$, which is used in the same function in YBC 4669 (rev. I 5–7; MKT III 27) and YBC 4673 (rev. III passim; MKT III 31); these too are probably northern, cf. MKT I 387f. and 123f. *illiakkum* and related derivations from *elûm* are found in Goetze's group V ("northern characteristics", maybe somewhat older than the group VI texts); in the remaining late Tell Ḥarmal texts (Baqir 1951); and finally in the early northern texts Db$_2$–146 (Baqir 1962: Pl. 3) and Haddad 104 (al-Rawi – Roaf 1984). – Only very few exceptions to these clear-cut rules are found. The group I text YBC 7997 (MCT 98) aligns *nadānum* and *elûm*, the former being used for final results alone; another group I text (YBC 4675, with the parallel fragment YBC 9852–MCT 44f.) uses *elûm* exclusively. *tammar* is used alongside with *nadānum* in YBC 4662, which A. Goetze locates in his group II (Larsa?), and it is used alone in MLC 1950 (MCT 48), which shares a specific Sumerian standard phrase with a number of texts belonging to group III but is otherwise unlocated. Finally, *tammar* and *elûm* are found together in one late Tell Ḥarmal text (IM 54559; Baqir 1951:41), while igi alone is found in VAT 672 (MKT I 267), a fragment with other stylistic peculiarities and containing too little Akkadian to allow for linguistic analysis.

one section of the statement or the procedure to the next: *saḫārum* ("sich wenden, herumgehen" etc.), *târum* ("sich umwenden, umkehren, zurückkehren; (wieder) werden zu"), and nigín(-na) (the sign LAGAB), which on other text-types is testified as a logogram for both. However, as first pointed out to me by A. Westenholz in connection with the use of *saḫārum* in AO 8862 (see section VIII.2), the denotation of this term and mostly also of *târum* appear to be much more precise and concrete, viz. real movement around a field. This also fits some of the occurrences of nigín but not all of them; it seems that the same technicalization which has led to logographic writing and, apparently, to conflation of the two terms as synonyms covered by the same logogram, has also reduced it to a textual delimiter. In order to make these distinctions visible I shall use the standard translations "go around" for *saḫārum* and "turn back" for both *târum* and nigín.

The hypothetical-deductive structure of the complex problem + procedure may be expressed by terms like *šumma* ("wenn, falls", standard translation "if"–also the recurrent term of the hypothetico-deductive omen texts), *inūma* ("als, wenn usw."; standard translation "as") and *aššum* ("wegen, weil usw."; standard translation "since"). Most often, it is left implicit–the statement appears as a fact, and after a phrase "You, by your making" comes an equally descriptive (occasionally jussive) procedure-part.

The equality necessary to establish an equation is normally implied the particle *-ma* followed by a numerical value (the "right-hand side" of the equation) (cf. above, chapter II). As stated there, I shall render *-ma* by the sign ":". If two unknown quantities are equated, the term *kīma* ("wie; als, wenn, daß", standard translation "as much as") can be found.

A term for equality which may function as sort of bracket is *mala* ("entsprechend (wie), gemäß;" standard translation "so much as"), used in the expression "so much as x over y goes beyond", meaning $(x-y)$.

The numerical value of a quantity can be asked for in two ways, either by the question "*x mīnum*" (*mīnum*, "was"; standard translation "what"; Sumerographic equivalent en-nam) or by a question like "*kī maṣi x*" (*kī*, „wie, als, daß"; *maṣûm*, "entsprechen, genügen, ausreichen"; standard translation of the combined expression "corresponding to what"). In a few texts, the student is asked to "make the equilateral (square-root) of x come up" (*x basā-šu šūli*).

IV. 10. The "conformal translation"

Obviously, the shades and distinctions just described in IV. 1 to IV. 9 cannot be rendered in a translation, in particular not in a translation into a non-Semitic language. One cannot achieve at the same time a one-to-one correspondence for single terms and an acceptable English sentence, not to speak of the rendition of grammatical categories. It is thus for good reasons that O. Neugebauer restricted the role of the translation to that of a general guide, "selbstverständlich genau genug, um den Inhalt korrekt erfassen zu können, nicht aber, um die Feinheiten der Terminologie und Grammatik daran ablesen zu können".[89]

[89] MKT I, viii. MKT III 5 continues "Wer terminologiegeschichtliche Studien an Hand einer Übersetzung machen will, dem ist doch nicht zu helfen".

Therefore, an investigation of Babylonian mathematics which tries to go beyond mathematical contents and penetrate patterns of thought and conceptualizations must necessarily rely on texts in the original language. On the other hand, the presentation of the results at least to the non-assyriologist must by the same necessity approach the question through a modern language.

Since the results of my investigation can only be documented and partly only explained with reference to original texts, translations are necessary. Since, on the other hand, the translations cannot be allowed to loose those shades and distinctions which cannot be translated into idiomatic English, I have chosen a compromise somewhere between a code and a real translation: All words except a few key terms are rendered by English words; a given expression is in principle always rendered by the same English expression, and different expressions are rendered differently with the only exception that well-established logographic equivalence is rendered by coinciding translation but distinct typography, while possibly mere ideographic equivalence is rendered by translational differentiation. Terms of different word class derived from the same root are rendered (when the result is not too awkward) by derivations from the same root.[90] These translations are the "standard translations" presented above. Furthermore, syntactical structure and grammatical forms are rendered as far as possible by corresponding structure and grammatical forms; the simple style of the mathematical texts makes this feasible. Expressed in mathematicians' *argot*, this sort of pseudo-translation could be called a "conformal translation".

Each line of the translation is followed by a transliteration of the original text. Here, as in current usage, phonetic Akkadian is written in italics. Sumerian words and Sumerograms (i.e., Sumerian words used logographically or ideographically for Akkadian speech) are given in spaced writing; and signs which can neither be interpreted one way or the other either because they should not be, or because our knowledge is insufficient are written in small capitals. In order to follow the principle of conformity as far as possible, and in order to facilitate the comparison of translation and transliteration, the same typographical distinctions are used in the translation. So, *kamārum* is translated "to *accumulate*"; g a r - g a r will be found as "to a c c u m u l a t e" (or another adequate form—often Sumerograms etc. are found with no phonetic or grammatical complements indicating which grammatical form to choose); and UL.GAR is rendered "to ACCUMULATE". Ideograms written with an Akkadian phonetic complement are translated in mixed writing. So, a - š à^{lam} is translated as "s u r *f a c e*". The result violates all ideals of typographic beauty, but it should make it relatively easy for the reader who wants to do so to acquire quickly a rudimentary feeling of the original formulation.

According to analogous considerations, each number is rendered in the translation the way it stands in the original text: Standard sexagesimal numbers are written in the extended degree-minute-second-notation described in note 4. In the transliteration, the same numbers are given more faithfully, with no indication of absolute place. Number words, including words for ordinal numbers and fractions, are rendered by words. Special signs for fractions are written as

[90] So, *epēšum* and the logogram k ì when used as verbs are rendered "to make", the infinitives used as nouns by "the making", and *nēpešum* by "the having-been-made".

modern fractional symbols, $^1/_2$, $^1/_3$ etc. Ordinals and fractions written on the tablet as a number followed by a phonetic or grammatical complement are written 1st, 2nd, etc.

Of course, considerations of intelligibility put some constraints on the principle of conformity. Prepositions cannot always be rendered in the same way, nor can a number of particles which structure the Akkadian sentences (relative pronouns etc.). Certain details of the syntactical structure (e.g. the postpositive adjective) have to be given up. Furthermore, definite and indefinite articles and other English grammatical elements have to be inserted into the translation. Such insertions stand as normal writing, without spacing, emphasis and capitals.[91] In the case of ideograms without complements even markings of grammatical person etc. are written that way. Other, genuine explanatory insertions are given as normal writing in parenthesis.

In the transliterations, all restitutions of damaged passages are of course indicated by square brackets. In order not to make the typographical appearance of the translations too disorganized, I have omitted there all indications of such restitutions, when they are taken over from the original publications of the texts, and when I find them firmly established. Since the restitutions of MKT, TMB and MCT were made with great care, mainly from parallel passages of the same tablets, this holds for most restitutions. Restitutions for which I am responsible myself and restitutions which I consider more or less uncertain are indicated clearly even in the translations.

The English terms used as standard translations of Akkadian terms are normally chosen in a way which respects the use of the latter in non-mathematical texts, and which at the same time shows the possible metaphorical use of the term in a mathematical context. A possible alternative would have been a translation by modern technical terms (e.g. "plus" for *kamārum* "added to" for *waṣābum* "multiply$_1$", "multiply$_2$", . . ., "multiply$_n$" for the variety of multiplicative operations and terms). The point of my choice is not that the Akkadian terms were necessarily used as metaphors and not technically. It is that the technical function of a Babylonian term must be learnt from its own context, not by imposition from the outside of inadequate, modernizing categorizations. Indeed, one need not work for very long with a term like "to append" before one forgets most of the concrete connotations and apprehends its single occurrences technically.

The basic vocabulary for arithmetical operations, for the announcement and recording of given numbers and results and for the structuration of the texts was presented above together with the standard translations of the single terms. For the sake of clearness, it is listed again in short form in Table 1, where the ordering corresponds to the above discussion. Table 2 lists all terms for which a standard translation is used in the translations of sections V–X, ordered alphabetically according to the standard translations. Table 3 contains the same material but ordered alphabetically according to the transliterated original language.

[91] So, in a genitive construction like íb-si$_8$ 15′, the preposition "of" is given in normal writing, "the e q u i l a t e r a l of 15′". *mišil* uš will be translated "*half of* the l e n g t h", because the construct state *mišil* indicates a genitive construction, although no genitive marker is joined to uš.

Table 1. Basic vocabulary

Akkadian	Sumerian etc.	Standard translation	use
1. additive operations			
waṣābum	daḫ	to append	"identity-conserving addition"
kamārum	gar-gar /UL.GAR	to accumulate	"identity-cancelling addition"
kimrātum		things accumulated	sum by *kamārum* etc.
nakmartum		accumulated	"
kumurrûm	gar-gar /UL.GAR	accumulatiou	"
2. subtractive operations			
eli . . . watārum	ugu . . . dirig/SI	over . . . go beyond	"subtraction" by comparison
nasāḫum	zi	to tear out	"subtraction" by removal
ḫarāṣum		to cut off	"
3. multiplicative operations			
	a-rá	steps of	number times number
eṣēpum	tab	to repeat	multiplication by positive integer (concrete repetition)
našûm	íl	to raise	calculation by multiplication
	nim	to lift	"
šutākulum	ì-kú(-kú)	to make span	"multiplication" of a "length" by a "width" ("rectangularization")
takīltum		*takīltum*	cf. below, sections V. 1–2
4. squaring and square-root			
	íb-si$_8$	equilateral/ to make equilateral	square-root; geometrical square identified with the length of the side
(maḫārum)		to confront	equality of value, shares (etc.)
šutamḫurum		to make confront itself	formation of a square
mitḫartum	LAGAB(?)	confrontation	square identified with the side
meḫrum	gaba(-ri)	counterpart	"second side of a square"
	NIGIN	to make surround/surrounding	like *šutamḫurum*, *mitḫartum* and (rarely) *sutākulum*
	UR.UR	to oppose	like *šutamḫurum* (and *šutākulum*)
	UL.UL	to make encounter	like *šutamḫurum*, íb-si$_8$ (and *šutākulum*)

Akkadian	Sumerian etc.	standard translation	use
5. halving			
ḫepûm	gaz	to break	bisection
bāmtum	ba/BA.A	moiety	"natural half"; result of bisection
6. division			
(igûm)	igi *n* (gál (-bi))	igi of *n* / *n*'th part	The fraction 1/*n* considered as a number / 1/*n* of something
paṭārum	du$_8$	to detach	To find the reciprocal (to take out 1/*n* from 1)
7. variables, derived variables, units			
(šiddum)	uš	length	one of the two basic "variables"
(pūtum)	sag(-ki)	width	the other basic "variable"
mitḫartum	LAGAB	confrontation	the "variable" in second-degree problems of 1 unknown
	NIGIN	surrounding	"
(eqlum)	a-šà	surface	product, square, and any quantity which in a geometric interpretation *is* a surface
	lul	false	(optional) epithet to a length, width etc. different from the one first considered
kīnum	gi-na	true	(optional) epithet which designates a return to the original use of a term[a]
(nukkurum)	kúr	alternate	a second "variable" within a category already in use
	nindan	nindan	unit of horizontal length, c. 6 m
ammatum	kùš	cubit	1/12 nindan, unit of height and depth, c. 50 cm
	sar	sar	nindan^2 / nindan$^2\cdot$kuš
8. recording etc.			
šakānum	gar	to pose	presumably material notation and/or drawing, cf. above
lapātum		to inscribe	
nadûm		to lay down	
rēška likīl		may your head retain	memorization of intermediate results in linear transformations
illi(-akkum)		comes up (for you)	announcements of a result
nadānum	sum	to give	
tammar	igi-du$_8$/dù	you see	

Akkadian	Sumerian etc.	standard translation	use
9. structuration			
epēšum	kì	to make/making	designates the procedure to be used to solve a problem
nēpešum		having-been-made	designates the procedure when performed
qabûm		to say	quotation mark
saḫarum		to go around	apparently the pacing off of a field, by which its dimensions are found
târum	nigin(-na)	to go back	designates the passage to another part of the procedure – concretly or abstractly
šumma		if	marks a deductive structure
inūma		as	,,
aššum		since	,,
kīma		as much as	equality
-ma		:/that	after verbs: consecution, consequence (result, equality); after nouns: emphasis
mala	a-na	so much as	a rhetorical "bracket"; equality
mīnum	en-nam	what	asks for a value
kī maṣi		corresponding to what	,,

[a] In one geometrical text (YBC 8633, in MCT 53), the term "true length" designates that side of a triangle which is closest to being perpendicular to the "width".

Table 2. The standard translations ordered alphabetically

The table is intended to be comprehensive with regard to the texts translated below. Only pronouns and pronominal suffixes are left out intentionally.

The table includes a number of terms which were not represented in the below translations, but which would be useful for other texts belonging to the genre. For this open-ended enterprise, no completeness was of course aimed at.

It should be emphasized once more that this is a table of standard translations, i.e. a key to the translated texts. It is not meant to be a dictionary, and no listing of meanings.

:	*-ma* (after a verb)
accumulate, to	*kamārum*/gar-gar/UL.GAR
accumulated, the	*nakmartum*
accumulation	*kumurrum*/gar-gar/UL.GAR
add	(Seleucid: *ṭepû*/tab)
Akkadian	*akkadûm*
alternate	kúr
and	*u*
append, to	*waṣābum*/daḫ
appended, the	*wuṣubbûm*
as	*inūma*
as much as	*kīma*/gim (nam)
ask, to	*šâlum*
bandûm	*bandûm*
break, to	*ḫepûm*/gaz
break off, to	*ḫaṣābum*
bring, to	*wabālum*
build, to	*banûm*
bur	*burûm*/bùr$^{\text{(gán)}}$
by	*ina*/-ta
change	*takkirtum*
collect (taxes, rent) to	*makāsum*
come up, to	*elûm*
confront, to	*maḫārum*
confrontation	*mitḫartum*/LAGAB/íb-si$_8$ in series texts
contribution	*manātum*
corresponding to what	*kī maṣi*
counterpart	*meḫrum*/gaba(-ri)
cubic equilateral	ba-si$_8$/-si
cubit	*ammatum*/kuš
cut away, to	*kašātum*

cut down, to	*nakāsum*/kud
cut off, to	*ḫarāṣum*
detach, to	*patārum*/du$_8$
diminish, to	*maṭûm*
each	ta-àm
equilateral	íb-si$_8$/-si//*basûm*// ba-si$_8$/-si
false	*sarrum*/lul
first . . . second	*ištēn* . . . *šanûm*
(1st . . . 2nd)	1(kam) . . . 2(kam)
follow, to	*redûm* (as "to make follow", *ruddûm*)
four	*erbûm*
fourth (part)	*rabītum*
from	*ina*/-ta
front	*pūtum*
gin	*šiqlum*/gín
give, to	*nadānum*/sum
go, to	*alākum*/rá
go around, to	*saḫārum*
go away, to	*tebûm*
go beyond, to	*watārum*/dirig/SI
grain	*še'um*/še
great, (to be (come))	*rabûm*/gal
gur	gur
half	*mišlum*/šu-ri-a
hand	*qātum*
having-been-made	*nēpešum*
head	*rēsum*/sag
head retain, may your	*rēska likil*
here	*annikī'am*
if	*šumma*
igûm	*igûm*/igi
igibûm	*igibûm*/igi-bi
inscribe, to	*lapātum*
inscription	*nalpattum*
inside	*libbum*
integrity	*šulmum*
itself	*ramānišu*
know, to	*edûm*/zu
lay down, to	*nadûm*
leave, to	*ezēbum*/tag$_4$
left-over	*šittatum*
length	(*šiddum*)/uš
lift, to	nim
lower	*šaplûm*/ki(-ta)
make, to/making	*epēšum*/kì
make confront itself, to	*šutamḫurum*
make cubic, equilateral	(-e) ba-si$_8$/-si
make encounter	UL.UL
make equilateral	(-e) íb-si$_8$/-si
make follow (additively), to	*ruddûm* (D-stem of *redûm*)
make span, to	*šutākulum*/ì-kú(-kú)
make surround, to	NIGIN
meadow	*tawirtum*/garim
mina	*manûm*
moiety	*bāmtum*/BA.A
name	*šūmum* (Seleucid MU)
nindan	nindan (=GAR)
no(t) (negating a word or part of proposition)	*la*/nu
not (negating a proposition)	*ul*(*a*)/nu
now	*inanna*
one	*ištēnum*
one . . . the other	*ištēn* . . . *ištēn*
oppose, to	UR.UR
out from	*ištu*
over	*eli*/u-gù
over-going	*elēnu*
part, *n*'th	igi *n* (gál(-bi))
pose, to	*šakānum*/gar
raise, to	*našûm*/íl
reed	*qanûm*/gi
remain, to	(Seleucid: *riāḫu*)
remainder	*šapiltum*/íb-tag$_4$
repeat	*eṣēpum*/tab
retain, to	*kullum*
sar	sar
say, to	*qabûm*/dug$_4$
saying	dug$_4$/TUK
second/2nd	*šanûm*/2(kam)
see, to	*amārum*, cf. "you see"
seventh	*sebītum*
sila	*qa*/sìla
since	*aššum*
so	*kīam*
so much as	*mala*/a-na
span	see "make span"
steps of	a-rá
surface	*eqlum*/a-šà
surrounding	NIGIN
take	*laqûm*
takīltum	*takīltum*
tear out, to	*nasāḫum*/zi
that	*-ma* (after a noun)
that of	*ša* ("emphatic genitive")
things accumulated	*kimrātum*
third (part)	*šalšum*
three	*šalašum*
threescore	*šuššum*
to (prep.)	*ana*/-ra
together with	*itti*
trapezium	sag-ki-gu$_4$
true	*kīnum*/gi-na
turn back, to	*târum*/nigín(-na)
twice	*šinišu*
two	*šina*

two-third	*šinipâtum*	wāṣûm	*wāṣûm*
until	*adi*	what	*mīnum*/en-nam
upper	*elûm* /an(-ta/-na)	which	*ša*
various (things)	ḫi-a/ḫa	width	(*pūtum*)/sag(-ki)
wāṣītum	*wāṣītum*	you see	*tammar*/igi-dù/-du$_8$

Table 3. Sumerian and Akkadian terms with equivalences and standard translations

Cf. introductory remarks to Table 2. Only logographic equivalences testified in mathematical texts are listed.

In the translations of the texts, each term is written in the same typography as the (transliteration of the) term it translates.

adi	until
A-ENGUR (~*tawirtum*?)	meadow (?)
akkadûm	Akkadian
alākum (~rá)	to go
amārum	to see
ammatum (~kùš)	cubit
an(-ta/-na) (~*elûm*)	upper
ana (~-ra)	to
a-na (~*mala*)	so much as
annikī'am	here
a-rá	steps of
a-šà (~*eqlum*)	surface
aššum	since
BA(.A) (~*bāmtum*)	moiety
ba-si$_8$/si (~*basûm*)	(cubic) equilateral
bāmtum (~BA.A)	moiety
bandûm	bandûm
banûm	to build
basûm (~ba-si)	equilateral
bùr$^{\text{gán}}$ (~*būrum*)	bur
būrum (~bùr$^{\text{gán}}$)	bur
daḫ (~*waṣābum*)	to append
(ugu ...) dirig (~*eli ... watārum*)	(over ...) go beyond
du$_8$ (~*paṭārum*)	to detach
dug$_4$ (~*qabûm*, TUK)	to say/saying
edûm (~zu)	to know
elēnu	over-going
eli (~u-gù)	over
elûm (~an)	upper
elûm	to come up (as a result)
en-nam (~*mīnum*)	what
eperum (saḫar)	earth
epēšum (~kì)	to make/making
eqlum (~a-šà)	surface
erbûm	four
eṣēpum (~tab)	to "repeat"
ezēbum (~tag$_4$)	to leave
gaba(ri) (~*meḫrum*)	counterpart
gal (~*rabûm*)	great
GAM (~*šuplum*)	depth
GAM (a-rá)	(Seleucid) steps of
gar (~*šakānum*)	to pose
gar-gar (~*kamārum*)	to accumulate/accumulation
garim (~*tawirtum*)	meadow
gaz (~*ḫepûm*)	to break
gi (~*qanûm*)	reed
gim (nam) (~*kīma*)	as much as
gín (~*šiqlum*)	gin
gi-na (~*kīnum*)	true
giš	giš (= 1` nindan)
gur	gur
ḫa = ḫi-a, q. v.	
ḫarāṣum	to cut off
ḫaṣābum	to break off
ḫepûm (~gaz)	to break
ḫi-a	various (things)
íb-si$_8$	(make) equilateral (in statements of series texts: ~*mitḫartum*)
íb-tag$_4$ (~*šapiltum*)	remainder
igi (~*igûm*)	igûm
igi *n* (gal-bi)	igi of *n*/*n*'th part
igi-bi (~*igibûm*)	igibûm
igibûm (~igi-bi)	igibûm
igi-dù/-du$_8$ (~*tammar*)	you see
igûm (~igi)	igûm
íl (~*našûm*)	to raise
ina (~ -ta)	from/by
inanna	now
inūma	as
ištēnum	one
ištēn ... ištēn	one ... the other

ištēn . . . šanûm	the first . . . the second
ištu	out from
itti	together with
-kam	(ordinal suffix)
kamārum (~gar-gar, ~UL.GAR)	to accumulate
kašātum	to cut away
ki (~*itti*)	together with
KI (~*qaqqarum*)	ground
ki(t a) (~*šaplûm*)	lower
kì (~*epēšum*)	to make/making
kī maṣi	corresponding to what
kīam	so
kīma (~gim(nam))	as much as
kimrātum	things accumulated
kīnum (~gi-na)	true
kú-kú (kú in series texts)	to make span
kud (~*nakāsum*)	to cut down
kullum	retain
kumurrûm (~gar-gar; ~UL.GAR)	accumulation
kúr (~*nukkurum*?)	alternate
kùš (~*ammatum*)	cubit
la (~nu)	not
LAGAB (~*mitḫartum*)	confrontation
lapātum	to inscribe
laqûm	to take
libbum	inside
lul	false
-ma (after a verb)	:
-ma (after a noun)	that
maḫārum	to confront
makāsum	to collect (taxes, rent)
mala (~a-na)	so much as
manātum	contribution
manûm	mina
maṭûm (~lal)	to diminish
meḫrum (~gaba(-ri))	counterpart
mīnum (~en-nam)	what
mišlum (~šu-ri-a)	half
mitḫartum (~LAGAB: in series texts ~ib-si$_8$)	confrontation
nadānum (~sum)	to give
nadûm	to lay down
nakāsum (~kud)	to cut down
nakmartum	accumulated
nalpattum	inscription
nasāḫum (~zi)	to tear out
našûm (~il)	to raise

nēpešum	having-been-made
NIGIN	to make surround/surrounding
nigin, nigín	see šu-nigin, šu-nigín
nigín(-na) (~*târum*)	to turn back
nim (~*ullûm*?)	to lift
nindan	nindan
nu (~*la*, ~*ul*(*a*))	not
paṭārum (~du$_8$)	to detach
pūtum (cf. sag)	front
qa (~sìla)	sila
qabûm (~dug$_4$)	to say
qanûm (~gi)	reed
qaqqarum (~KI)	ground
qātum	hand
-ra (~*ana*)	to
rá (~*alākum*)	to go
rabītum	fourth (part)
rabûm (~gal)	great
ramānišu	itself
redûm, see *ruddûm*	
rēška likīl	may your head retain
rēšum (~sag in certain contexts)	head
riāḫu (Seleucid)	remain
ruddûm	to make follow (additively)
sag (~*rēšum*)	head
sag (-ki)	width
sag-ki-gu$_4$ (~*absammikum*?)	trapezium
saḫārum	to go around
sar	sar
sarrum (~lul)	false
sebītum	seventh (part)
SI (~dirig, ~*watārum*)	go beyond
sìla (~*qa*)	sila
sum (~*nadānum*)	to give
ṣeḫērum (~tur)	to be (come) small(er)
ṣeḫrum (~tur)	small
ša	which/that of
šakānum (~gar)	to pose
šalašum	three
šalšum	third (part)
šâlum	to ask
šanûm	second
šapiltum (~íb-tag$_4$)	remainder
šaplûm (~ki-ta)	lower
še (~*še'um*)	grain
še'um (~še)	grain
šiddum (cf. uš)	length
šina	two
šinedâtum	two-third

šinišu	twice
šiqlum (~gín)	gin
šittatum	left-over
šū	that
šulmum	integrity
šumma	if
šūmum (Seleucid ~MU)	name
šu-nigin, šu-nigín	total
šuplum (~GAM)	depth
šu-ri-a (~*mišlum*)	half
šuššum	sixty
šutākulum (~ì-kú (-kú))	to make span
šutamḫurum	to make confront itself
-ta (~*ina*)	from/by
ta-àm	each
tab (Seleucid ~*ṭepû*)	to add
tab (~*eṣēpum*)	to "repeat"
tag$_4$ (~*ezēbum*), cf. íb-tag$_4$	to leave
takīltum	takīltum

takkirtum (cf. kúr)	change
tammar (~igi-dù/du$_8$)	you see
târum (~nigín (-na))	to turn back
tawirtum (~garim)	meadow
tebûm	to go away
TUK, see dug$_4$	
tur (~*ṣeḫrum*)	small
ṭepû (Seleucid ~tab)	to add
u	and
u-gù (~*eli*)	over
ul(a) (~nu)	not
UL.GAR (~*kamārum*)	to accumulate/accumulation
UL.UL	to make encounter
UR.UR	to oppose
uš	length
wabālum	to bring
waṣābum (~daḫ)	to append
wāṣītum/wāṣûm	wāṣītum/wāṣûm
watārum (~dirig)	go beyond
wuṣubbûm	the appended
zi (~*nasāḫum*)	to tear out
zu (~*edûm*)	to know

Altorientalische Forschungen	17	1990	2	262–354

Jens Høyrup

Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought II*

Til Sara og Janne

V. The discourse: Basic second-degree procedures

As stated in section III, the discursive level of Old Babylonian algebra can only be discussed on the basis of actual instances of this discourse. In the present and the following chapters, I shall therefore present a number of texts, translated according to the principle of "conformity" in order to map the original discourse as precisely as possible if the material is not to be presented in the original language. Direct linguistic and philological commentaries are given as notes immediately below the translation of the single texts.

I do not aim at complete coverage of Old Babylonian mathematics. Most practical applications fall outside the scope of the article, and so do the table texts. The application of the specific methods of Old Babylonian algebra to genuine geometric problems are left aside for later treatment, as are most of the "complex" algebraic applications of the basic techniques.[92] Finally, with a single exception only procedure texts are taken into account: Texts which give nothing but the statement of a problem or a series of such statements give little information as long as our understanding of concepts and terminology remains at the present level.

On the other hand, in relation to the class of simple "length-width"-procedure texts the coverage can be regarded as fairly representative. Truly, each text taken into account brings some new information; still, what is left out appears to me to belong to the category of details and shades, which may await subsequent investigation. The basic features of Old Babylonian elementary "length-width-algebra" can, I hope (and think), be presented adequately on the basis of the present selection of texts.

V.1. YBC 6967 (MCT, 129)

The problem deals with a pair of numbers belonging together in the table of reciprocals, the *igûm* and the *igibûm*. The Sumerian forms igi and igi-bi mean „the igi" and "its igi"; they are used most of the way through the text, but a

* For the first part see p. 27–69 of this volume.

[92] I discuss the problems of two-dimensional geometrical conceptualizations and methods and a number of complex algebra problems in my preliminary (1985: 41–63, 105.1 to 105.42).

syllabic *i-gu-um* in rev. 5 indicates that the terms are to be read as Akkadianized loanwords though mostly written logographically.[93] Their product (the "surface" of obv. 9) is supposed to be 1' (=60), or at least an odd power of 60, not 1°. In conformal translation and transliteration, the text runs as follows (to facilitate mathematical understanding, the left margin gives a totally anachronistic commentary in symbolic algebra – *igibûm*$=x$, *igûm*$=y$):

Obverse

$[x \cdot y=60,]\ x-y=7$	1. The igibûm *over* the igûm 7 *goes beyond* [igi-b]i *eli* igi 7 *i-ter*
$x?\ y?$	2. igûm *and* igibûm *what?* [igi] *ù* igi-bi *mi-nu-um*
	3. *You*, 7 *which* the igibûm *a[t-t]a* 7 *ša* igi-bi
	4. over the igûm *goes beyond* ugu igi *i-te-ru*
$\frac{x-y}{2}=3^1/_2$	5. *to two break:* 3° 30′. *a-na ši-na ḫi-pí-ma* 3,30
$\left(\frac{x-y}{2}\right)^2=12^1/_4$	6. 3° 30′ *together with* 3° 30′ 3,30 *it-ti* 3,30
	7. *make span:* 12° 15′ *šu-ta-ki-il-ma* 12,15
$\left(\frac{x-y}{2}\right)^2+x\cdot y=\left(\frac{x+y}{2}\right)^2=72^1/_4$	8. *To* 12° 15′ *which comes up for you* *a-na* 12, 15 *ša i-li⟨-a⟩-kum*
	9. 1' the sur*face append:* 1' 12° 15′. [1 *a*-šà-*l*]*a-am ṣí-ib-ma* 1, 12, 15
$\frac{x+y}{2}=\sqrt{72^1/_4}=8^1/_2$	10. The equilateral of 1' 12° 15′ *what?* 8° 30′. [íb-si$_8$ 1], 12, 15 *mi-nu-um* 8, 30
	11. 8° 30′ *and* 8° 30′ *its counterpart lay down:* [8, 30 *ù*] 8, 30 *me-ḫe-er-šu i-di-ma*

Reverse

	1. 3° 30′ the *takīltum* 3, 30 *ta-ki-il-tam*
$\frac{x+y}{2}-\frac{x-y}{2}=8^1/_2-3^1/_2$	2. *from* the *one tear out* *i-na iš-te-en ù-su-uḫ*

[93] Another text dealing with *igûm* and *igibûm* is VAT 8520 (MKT I 346f.). There, the names of the two unknowns are written syllabically throughout the tablet, while "part" and "reciprocal" are referred to by the usual ideogram igi. This leaves little doubt that the two ideas were, and thus have to be, kept apart, if not in spoken language then at least as concepts.

$\frac{x+y}{2}+\frac{x-y}{2}=8^1/_2+3^1/_2$	3. *to* the *other append* *a-na iš-te-en ṣi-ib*
$8^1/_2+3^1/_2=12$ $8^1/_2-3^1/_2=5$	4. The *first* is 12, the *second* is 5. *iš-te-en* 12 *ša-nu-um* 5
$x=12$, $y=5$	5. 12 is the igibûm, 5 is the *igûm*. 12 igi-bi 5 *i-gu-um*

If "going beyond" is interpreted as arithmetical difference, "breaking" as arithmetical halving, "making span" as arithmetical multiplication, "surface" as arithmetical product, "equilateral" as arithmetical square root, and *takīltum* as a factor (in agreement with the interpretation "that which is made span"), most of this text could agree with an arithmetical interpretation of Old Babylonian algebra. A few points remain, however, which always have been seen as peculiar. Why is the "counterpart" of the square-root introduced? And why are these two copies of the number 8° 30′ kept so strictly apart as a "first" and a "second" in rev. 2–4?

If a naive-geometric interpretation of the procedure is made, these two questions are immediately solved (cf. Fig. 4): Since the product of *igûm* (y) and *igibûm* (x) is spoken of as a surface, they are to be regarded as width and length of a rectangle. That amount by which the length "goes beyond" the width is bisected together with the adjacent part of the rectangle. and the outer half is moved to a position where it "spans" a rectangle (actually a square) together with the inner half. The area of the resulting gnomon is still 1`. When it is appended to the square spanned by the two halves (of area $(3° \, 30')^2 = 12° \, 15'$), we get a greater square of area 1` 12° 15′. The side producing this square, or, rather, as we shall see below, the side produced by the area when the latter is understood as a square figure and thus identified with its side, is $\sqrt{1`\ 12° \, 15'} = 8° \, 30'$. It is "laid down" (possibly "drawn", cf. section IV.8) together with its "counterpart" (heavy lines). When "that which was made span" the small square (the *takīltum*) is "torn out" from the vertical heavy line (its secondary position) we get the width (the *igûm*). When it is appended to the horizontal heavy line (its original position) we get the length (the *igibûm*).

It will be noticed that not a single word of the description is superfluous or enigmatic when this interpretation is applied. It can also be noticed that an alternative formulation, the "first" and "second" 3° 30′ appended to and born out from the same 8° 30′ (e.g. the horizontal heavy line) would be less meaningful, producing two lines equal to but not identical with length and width. As it actually stands, the text tells us first to tear out the quantity 3° 30′ in one place and next to append this same quantity, now at our disposal, in another place.[93a]

This sense-making use of "first" and "second" holds throughout the many texts where they are used. That can scarcely be a random phenomenon. So, an

[93a] This (invariable) precedence of the tearing process was observed by Vajman (1961: 100), who also pointed out the implication that the same concrete quantity must be involved in tearing and appending.

In one text translated below the addition comes first, viz. IM 53201 N° 2 rev. 12f. (section X.1). But precisely in this case the objects of the two operations are the two different "moieties" of an excess. In truth an exception which confirms the rule.

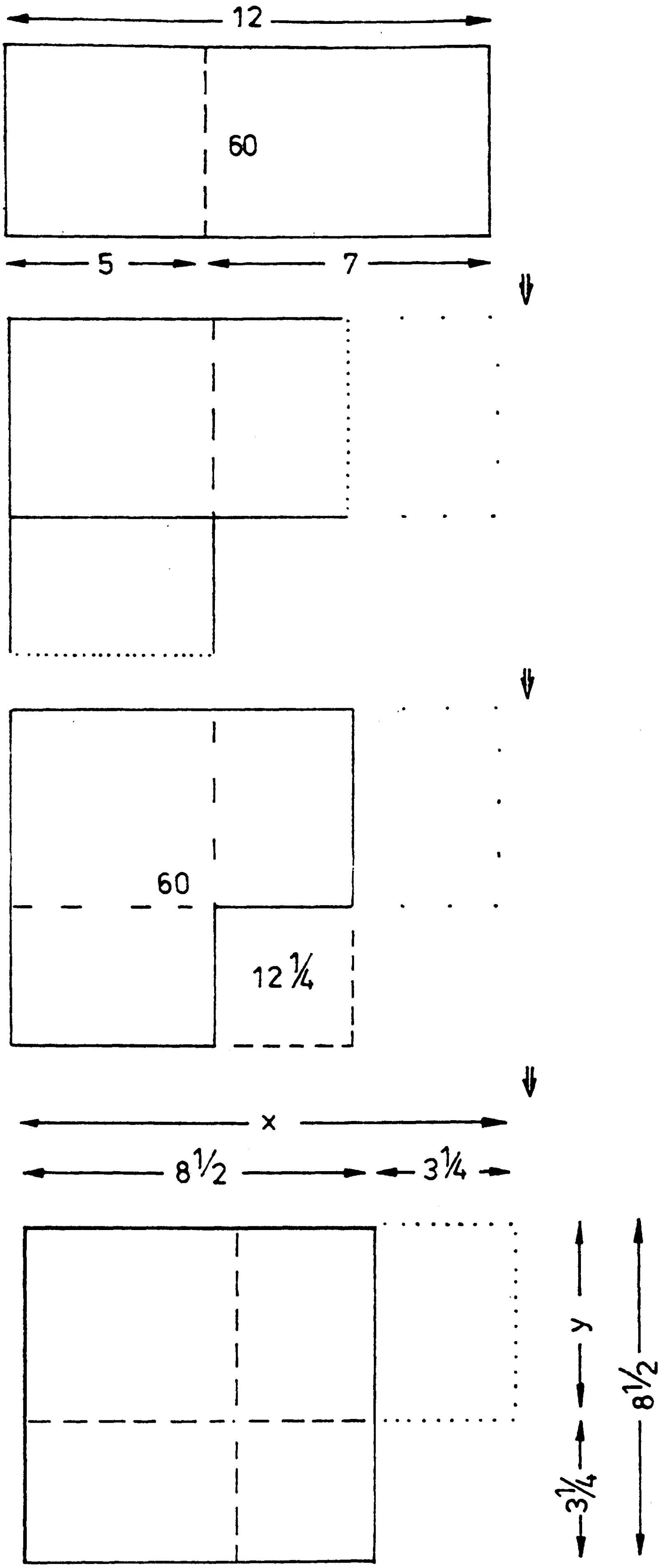


Figure 4. The geometrical interpretation of YBC 6967.

interpretation of the duplication 8° 30′ as nothing but a preparation for two different arithmetical calculations can hardly hold good–in that case, we could expect instances of "first 3° 30′ appended to first 8° 30′, second 3° 30′ born out from second 8° 30′", and other variations of the same sort. In fact, they are never found.

In other respects too, our text is representative of a whole group of procedure texts. As already observed above (section IV.8), the term "to lay down" is always reserved to that process which corresponds to the "drawing of the heavy lines"; if only a number was taken note of for use in an arithmetical calculation, how are we to explain that e.g. the numbers submitted to the operations "appending" and "tearing out" are never "laid down"? Similarly, it is a general feature that 3° 30′ is appended to 8° 30′–that quantity which is moved is appended to that which stays in place. The difference is not one of relative magnitude–as we see in obv. 8f., a greater quantity may well be appended to a smaller quantity; neither is it just a question of fixed habits – when gnomon and square are joined (a situation where both entities are already in place), either can be appended[94]; only where the geometrical interpretation requires that one addend remains in place and one is moved is it apparently impossible to exchange the roles of the two addends. Finally, the concept of a "counterpart" is reserved to roles similar to that which it plays in obv. 11 of the present text; in the case of bisections ("breakings") preparing a purely linear operation it is not used.[95]

As we see, all three features are easily explained inside a geometric interpretation. It is, on the other hand, very difficult to find reasons explaining them if an arithmetical interpretation is taken for granted; and it is extremely improbable that the random selection of surviving sources has created a fixed pattern which did not exist originally–our material is not that small.

It will be observed that the text appears to describe a constructive procedure, not argumentation on a ready-made figure like Fig. 2. It will also be seen that the procedure coincides *grosso modo* with that described by al-Khwārizmī (cf. Fig. 1, AoF 17 [1990], 36).

V.2. BM 13901, N° 1 (MKT III,1: cf. TMB, 1)

BM 13901 contains a series of problems dealing with one or more squares. The first of these is a precise analogon to the one quoted in Chapter I from al-Khwārizmī. It runs as follows:

Obverse I

$x^2+x={}^3/_4=45'$

1. The sur*face and my confrontation I have accumulated:* 45′. 1 the *wāṣītum*[a]
a-šà^{l}[^{am}] *ù mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45-e 1 *wa-ṣi-tam*

94 In various problems from BM 13901 (below), the supplementary square is appended to the gnomon; in VAT 8520, as in the present text, the gnomon is appended.

95 The sole exception from this general rule is IM 52301 (obv. 12, rev. 10). This is only one of several reasons to regard this late tablet as a symptom of changing conceptualizations (cf. below, note 113, section X.1, and note 176).

$^1/_2 \cdot 1 = 30'$, $(30')^2 = 15'$	2. *you pose*. The *moiety*[b] *of* 1 *you break*, 30′ *and* 30′ *you make span*, *ta-ša-ka-an ba-ma-at* 1 *te-ḫi-pi* 30 *ù* 30 *tu-uš-ta-kal*
$x^2 + 2 \cdot 30' \cdot x + (30')^2 = 15' + 45' = 1$; $x + 30' = \sqrt{1} = 1$	3. 15′ *to* 45′ *you append:* 1 makes 1 equilateral[c]. 30′ *which you have made span* 15 *a-na* 45 *tu-ṣa-ab-ma* 1-[e] 1 íb-si$_8$ 30 *ša tu-uš-ta-ki-lu*
$x = 1 - 30' = 30'$	4. *in* the *inside of*[d] 1 *you tear out:* 30′ the *confrontation*. *lìb-ba* 1 *ta-na-sà-aḫ-ma* 30 *mi-it-ḫar-tum*

[a] *wāṣītum* is a nominal derivation from *waṣûm*, "herausgehen, fortgehen ... herauswachsen ... hervortreten, herausragen". The term itself means something going out, including something projecting from a building. Since the mathematical application of the term has never been explained before, I have left it untranslated.

[b] The use of a term for a "wing", a "natural" instead of a mere arithmetical half is noteworthy.

[c] "1 makes 1 equilateral" translates "1-e 1 íb-si$_8$". The use of the "agentive suffix" -e (which occurs commonly in this connexion) appears to indicate not only that the verbal character of the term íb-si$_8$ is still present to the Old Babylonian calculator, but also that the first "1" is considered the agent of a transitive verb, while the second "1" must be seen as the object. Cf. Thureau-Dangin 1936a: 31 note 3, which also quotes an instance of the phrase *mi-nam* íb-si$_8$ where a square-root is asked for; here, too, the square-root must be the object of an act since it is asked for in the accusative. (So also the Susa and most Tell Ḫarmal texts).

A number of other texts, however, ask for the square-root by the phrase íb-si$_8$ x *mi-nu-um* (e.g. YBC 6967, obv. 10) or íb-si$_8$ x en-nam (e.g. VAT 8390, passim, and VAT 8520, obv. 20, rev. 19). *mi-nu-um* is an indubitable nominative; in the latter texts, the other occurrences of en-nam are indubitable nominatives, while corresponding accusatives are written phonetically as *mi-nam*. In such cases (and when the term is used in the generalized sense of "solution" to an equation), íb-si$_8$ must apparently be read as a noun, and I shall translate "the equilateral of x how much".

In a few late OB and in one early northern text, the alternative term ba-si$_8$, originally a verb too, has been adopted into Akkadian as a loanword *basûm*, which is regarded completely as a boun – cf. IM 52301, No 2, note d (below, section X.1).

[d] Thureau-Dangin (1936a: 31 note 4) explains the form *lìb-ba* (ŠÀ.BA) as *libba*, the construct state of a locativic accusative. Another possible interpretation reads ŠÀ = sag$_4$ ∼ *libbum*, BA = ba < bi-a, compound possessive + locativic suffix (cf. SLa § 182).

We observe that the "confrontation" is in fact identical with the side of the square, while the area of that figure is spoken of by a separate concept, "the surface".

When this usage is accepted, the procedure is *grosso modo* mapped by the arithmetico-symbolic interpretation in the left margin. However, it remains fully unclear why the number 1 should be spoken of as something "projecting" or "going away". Another puzzle is the choice of the term *bāmtum*, "moiety", when the normal term *mišlum*, "half", is used everywhere in the tablet when one entity is the half of another entity.

If we try a geometric interpretation, the intention of both terms can be made clear (see Fig. 5).

As in al-Khwārizmī, a geometric summation of a square and a number a of sides requires that the number a is understood as having the dimension of a length. This is shown in the first step of the figure, where the "confrontation" is represented by the area of a rectangle of length 1 and width x. The figure makes it immediately obvious that the number 1 is something which projects. The only question which is left open is whether it projects from the square or from the width x[96] (as we shall see below, the latter possibility must be preferred).

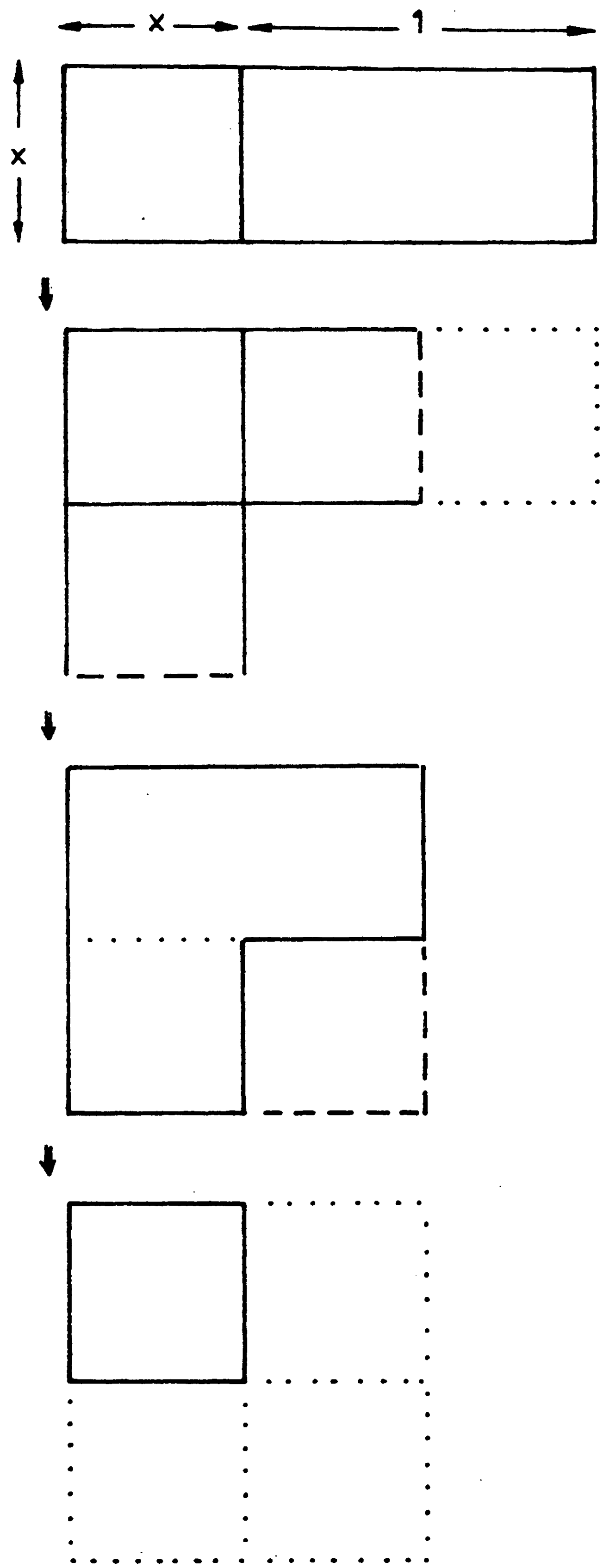
From here, the procedure is exactly parallel to that of YBC 6967 and Figure 4. Comparing the two texts we can even see why the need for the term *wāsītum* arises: while the problem of two unknowns could speak of that by which "x goes beyond y", the corresponding geometrical quantity 1 ("that by which $x+1$ goes beyond x") has no obvious designation in the problem of one unknown – if not, precisely, *wāṣītum*. This is then posed and next "broken" (i.e. bisected), and the outer half is moved so that a square is spanned. This square is appended to the gnomon resulting from the preceding manipulations of the figure, in order to produce another square. The side of this great square is found (literally: the result 1 of the appension produces 1 as "equilateral"). Finally, the quantity which spanned the complementary square[97] is removed ("torn out"), and the unknown side of the original square (the original "confrontation") is left.

Concerning the "moiety", the situation in the figure is evidently related to the origin of the term. By the very nature of the problem, the appended rectangle consists of two "wings", of which one is to be broken off and moved.

According to both F. Thureau-Dangin and O. Neugebauer, the tablet belongs

[96] In its own way, this confirms O. Neugebauer's old intuition. F. Thureau-Dangin suggested very tentatively (1936a: 31 n.1) that *wāṣītum* might designate absolute unity as distinct from 1′, 1‵ etc.). Against this, O. Neugebauer (MKT III 11) raised the objection that only absolute unities belonging with problems of one unknown were designated *wāṣītum*. Instead, he suggested that the term might designate a certain class of coefficients of value *1*. Irrespective of the precise interpretation, indeed, the "projection" is a coefficient *1* of dimension (length), multiplication by which transforms a linear quantity into a quantity of dimension [length^2].

[97] In YBC 6967, this quantity was spoken of by the noun *takīltum*, here however by the relative clause "which you have made span", *ša tuštakkilū*. This parallel (which is repeated copiously) confirms the close relation between "making span" (*šutākulum*) and *takīltum*.



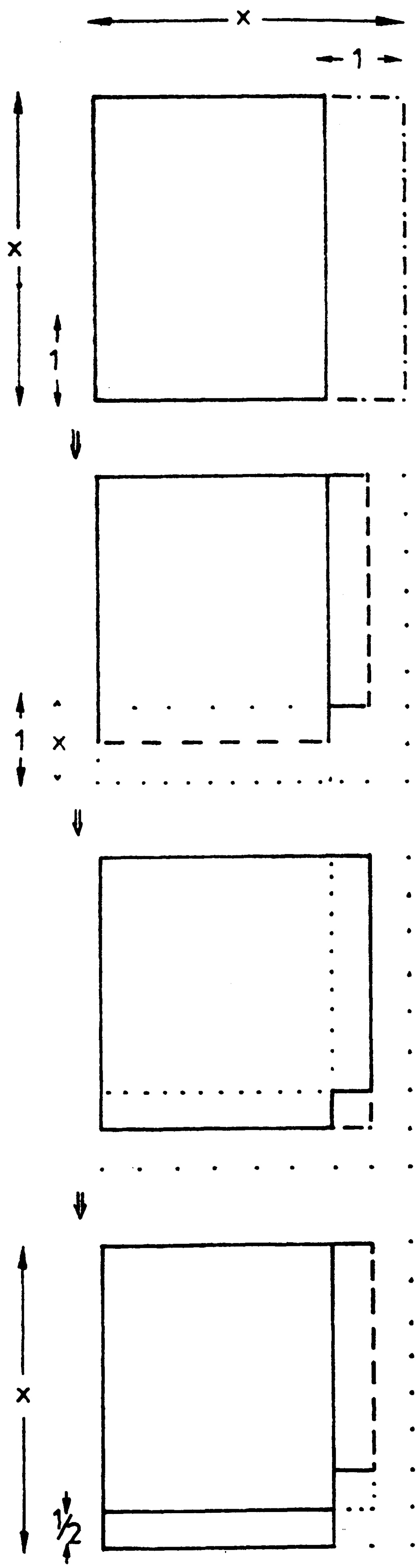


Figure 5. The geometrical interpretation of BM 13901 N° 1.

Figure 6. The geometrical interpretation of BM 13901 N° 2 (distorted proportions).

together with AO 8862 to the oldest stratum of Old Babylonian mathematics.[98] A. Goetze's linguistic analysis ascribes to both a southern origin, probably Larsa.[99]

V.3. BM 13901, N° 2 (MKT III, 1; cf. TMB, 1)

The second problem of the tablet subtracts a side instead of adding it. The text runs as follows:

	Obverse I
$x^2-x=$ 14ˋ 30°	5. *My confrontation inside of* the surface *I have torn out:* 14ˋ 30°. 1 the *wāṣītum* *mi-it-ḫar-ti lìb-bi* a-šà [*a*]*s-sú-uḫ-ma* 14, 30 1 *wa-ṣi-tam*
$^1/_2 \cdot 1 = 30'$ $(30')^2 = 15'$	6. *you pose.* The *moiety of* 1 *you break,* 30′ *and* 30′ *you make span;* *ta-ša-ka-an ba-ma-at* 1 *te-ḫi-pi* 30 *ù* 30 *tu-uš-ta-kal*
$x^2 - 2 \cdot 30' \cdot x + (30')^2 =$ 14ˋ 30° 15′ $x - 30' = \sqrt{14ˋ\ 30°\ 15'} =$ 29° 30′	7. 15′ *to* 14ˋ 30° *you append:* 14ˋ 30° 15′ makes 29° 30′ equilateral. 15′ *a-*⌜*na* 14, 30 *tu-ṣa-*]*ab-ma* 14, 30, 15-e 29, 30 íb-si$_8$
$x = 29° 30' + 30' = 30°$	8. 30′ *which you have made span to* 29ˋ 30° *you append;* 30 the *confrontation.* 30 *ša tu-uš-ta-ki-lu a-na* 29, 30 *tu-ṣa-ab-ma* 30 *mi-it-ḫar-tum*

Once again, the text is *grosso modo* mapped by the arithmetico-symbolic interpretation. Only the problem of the "1 which projects" is left open, together with the question why only the "coefficient" of the first-degree term is "posed", and the choice of the term "moiety".

If the imagery inherent in the terminology ("appending", "tearing out", "breaking", "making span") is taken at face value, we are led to a geometric procedure which solves even these problems (see Fig. 6). From the square, a rectangle of length x and width *1* is removed. The area of the remaining rectangle is 14ˋ 30°. Since the length of this rectangle exceeds the width by *1*, a strip of this width is bisected, and its outer wing is moved so as to transform the known area into a gnomon. The small square spanned by the two halves of the strip is appended, and so we get a square of known area. Its side is found, and the half-strip which was moved in order to span the small square is appended again. This gives us the original length of the rectangle, and thus the side x of the square.

The geometrical procedure is of course the same as that of Fig. 4 and Fig. 5: The area of a rectangle is given, together with the difference between its length and its width. The excess of length over width is bisected, and the rectangle is transformed into a gnomon, for which the area and the side of the lacking square are known. The area of the lacking square is then found and added to the gnomon,

[98] Thureau-Dangin 1936a: 27; MKT III 10. The criteria are language and writing.
[99] In MCT 148, 151.

transforming it into a square of known area. The side of this square is calculated, and the original length (Fig. 6), width (Fig. 5) or both (Fig. 4) can finally be found. Indeed, the only difference between the cases (as seen from the geometrical interpretation) concerns the entity asked for.

It is still not to be seen whether the *wāṣītum* should be understood as that width *1* which must project from the length in order to transform it into an area which can be torn out, or perhaps as the excess of rectangular length over rectangular width. In any case, it has a definite role to play in the procedure (and as stated above, the former possibility will turn out to be correct). In the geometric interpretation the question thus disappears why only the coefficient *1* to the linear term is posed—the *wāṣītum* is no numerical coefficient.

Once again, then, the aritmetic-algebraic interpretation allows us to understand the main mathematical progress of the calculation but not the details of the formulation; the approach through naive geometry, on the other hand, allows us to understand both the mathematical progress and the discursive organization of the texts.

V.4. BM 13901, N° 23 (MKT III, 4f.; cf. TMB, 17f.)

The three previous problems presented the standard way to solve the basic mixed second-degree equations. The present one exemplifies that the Babylonians would sometimes leave the standard methods.

The problem adds the four sides of a square to the surface – not 4 times the side, but explicitly *the four sides*:

Reverse II

$x^2+4\cdot x=41'\,40''$	11. *In* a sur*face, the four fronts and* the sur*face*[a] *I have accumulated:* 41′ 40″ a-šà^{lam} *p*[*a*]-*a*[*-at er-bé-et-tam* ù a-š] à^{lam} *ak-mur-ma* 41, 40
	12. 4, *the four fronts, you inscribe.* The igi of 4 is 15′ . 4 *pa-a-at er*[*-bé-e*]*t-tam t*[*a-la-p*]*a-at* igi 4 gál-bi 15
$^1/_4x^2+x=10'\,25''$	13. 15′ *to* 41′ 40″ *you raise:* 10′ 25″ *you inscribe.* 15 *a-na* 41, 40 [*ta-n*]*a-ši-ma* 10, 25 *ta-la-pa-at*
$(^1/_2x+1)^2=10'\,25''+1$ $=1^\circ\,10'\,25''$	14. 1 the *wāṣītum you append:* 1° 10′ 25″ makes 1° 5′ equilateral
$^1/_2x+1=\sqrt{1^\circ 10' 25''}=1^\circ 5'$	1 *wa-ṣi-tam tu-ṣa-ab-ma* 1, 10, 25-e 1, 5 íb-si$_8$
$^1/_2x=1^\circ\,5'-1=5'$	15. 1 the *wāṣītum which you have appended* *you tear out:* 5′ *to two* 1 *wa-ṣi-tam ša tu-iṣ-bu ta-na-sà-aḫ-ma* 5 *a-na ši-na*
$x=2\cdot 5'=10'$	16. *you repeat:* 10′ nindan *confronts itself*[b] *te-ṣi-ip-ma* 10 nindan *im-ta-ḫa-ar*

19*

[a] This passage is very unusual, indeed without parallel in mathematical texts, and thus of special interest. First there is the initial statement that we are dealing with a surface. In itself, the use of an accusative form here is not impossible; most plausibly it is to be interpreted as an locativic accusative (cf. GAG § 146). However, in other cases where the subject of a problem is stated this is always done by Sumerograms without any complement (uš sag e.g. in AO 8862, túl-sag in BM 85200+VAT 6599). In the present case, it seems to be important to stress either the use of an accusative form or the specific Akkadian pronunciation—even though the whole tablet is dominated by syllabic writing, complements are attached to a-šà only when they are needed to impede misunderstanding. The use of pure Sumerograms in parallel texts indicate that there was no general need to display an accusative case explicitly; most probably, then, the complements are meant to indicate the use of an Akkadian archaism.

The "fronts" translate *pāt*, plural (construct state) of *pūtum*. This word is often considered an equivalent of sag, my standard translation of which is "width". Only extremely few texts, however, use the Akkadian word instead of the Sumerogram, and none of them belong to the category of standard "length and width"-problems (see above, note 75). Even occurrences of the Sumerogram with an Akkadian phonetic complement are strictly absent. The use of the term *pūtum* in our text must thus intend something explicitly different from the technical concept "width"—perhaps another archaism. Hence my use of the literal translation "front".

The numeral "four" is in status rectus and postponed. This literary stylistic figure appears to belong to situations where the number is an invariable epithet, i.e. where n items belong invariably together ("the seven mountains", cf. GAG § 139i), whence "*the four*" instead of "the *four*".

[b] The term is *imtaḫḫar* (or possibly *imtaḫar*, the preterite form), Gt-stem of *maḫārum*, "to confront".

This time, the arithmetico-algebraic interpretations lead into real trouble. Indeed, if a "square" is only a second power, there is no reason to speak of *the four* fronts (or widths); neither is there any reason to leave the normal concept of the "confrontation" for that of "front", nor to specify in this case alone that we are dealing with a "surface".

Of course, an arithmetical interpretation can map the mathematical procedure. But it offers no explanation why normal terminology and procedure are given up in this specific case; in fact, the deviation is so astonishing that O. Neugebauer suspected it to have arisen by a combination of mistakes which happen to make sense.[100] Finally, the place of the problem on the tablet (among the complicated variations and not among the simple cases of one variable) is an enigma; so is also the "repetition to two" in a place where an arithmetic interpretation would expect a "raising" (cf. the problem discussed immediately below.)

The geometric interpretation, especially as it is made clear by the term *wāṣitum*, solves many of these problems (cf. Fig. 7). First of all it is clear that a geometric

[100] MKT III 14.

square possesses four sides, which can be regarded as "fronts". Moreover, if we take the text at its words and add four rectangles of length *1* and width x instead of one rectangle of length *4* and width x, or two of dimensions *2* times x, as we would normally expect), we get a geometrical configuration which differs from the normal square-plus-sides dealt with in the beginning of the tablet–and thus a reason that the problem is listed among the complicated variations.

The occurrences of the *wāṣītum* confirm that the cross-form configuration is indeed thought of: If we follow the text, we can imagine the multiplication by $^1/_4$ in lines 12f. as a quartering, as shown in the second step on the figure. At first, this is of course only a possibility. In line 14, however, the *wāṣītum* is appended, i.e., not any number *1* but a square 1^2 identified with the *wāṣītum*; such a square is shown in the third step[101], where it completes the quartered cross as a square. No other configuration than the cross would allow so literal a reading of the text, – and since the occurrence of the *wāṣītum* in line 14 does not refer to any earlier occurrence, it must refer to the entity itself, not to anything obtained from or equal to the "projection".

In the next step of line 14, the side of the completed square is found, and in line 15 the same *wāṣītum* is torn out. This rules out F. Thureau-Dangin's conjecture, viz. that the term may simply fix the order of magnitude to 1° (one need not fix the order of magnitude of a number which is identical with a number previously used), and it confirms that the square which was appended in line 14 is identified with its side: if a squaring of *1* had been left out by error in line 14, the invariable epithet would have been "which you have made span" instead of "which you have appended" (cf. problems N° 1 and 2 from the tablet as quoted above).

The tearing-out of the *wāṣītum* leaves half the side of the square (in the right position). It is "repeated to two", and indeed repeated quite concretely[102], in agreement with the situation of the figure, giving us one of the fronts. It is, however, not spoken of as a "front", nor designated by the normal term "confrontation" (*mitḫartum*). Instead, it is stated that 10′ is that which "confronts itself" – presumably because no "confrontation" was spoken of explicitly in the statement of the problem; instead four "fronts" have been supposed to "confront each other as equals".

Curiously enough, al-Khwārizmī uses the same figure as an alternative argument for the solution of the problem "square and roots equal to number" (cf. above, section 1). Here, instead of distributing the rectangle $10 \cdot x$ as shown

[101] We notice that the current identification of a square with its side can explain that the *wāṣītum* itself is appended, and not a "1" spanned by the *wāṣītum* together with itself. At the same time we observe that the entity which is "appended" must be the concrete geometric piece of surface, not a number measuring its magnitude: Such a number would, even to the Babylonians, have to be found via one of the "multiplicatory" processes "making span" or "raising", as are all "surfaces". Due to the configuration, however, there is no need to "make the *wāṣītum* span", i.e. to make it form a rectangle (in fact a square): The square is already there, spanned by the corner of the cross–there is no need to prescribe its construction.

[102] The specification "to two" shows that the original sense of *eṣēpum* (to duplicate, i.e. to repeat once) has been absorbed into the generalization "to repeat N times". Genuine duplication has been left behind.

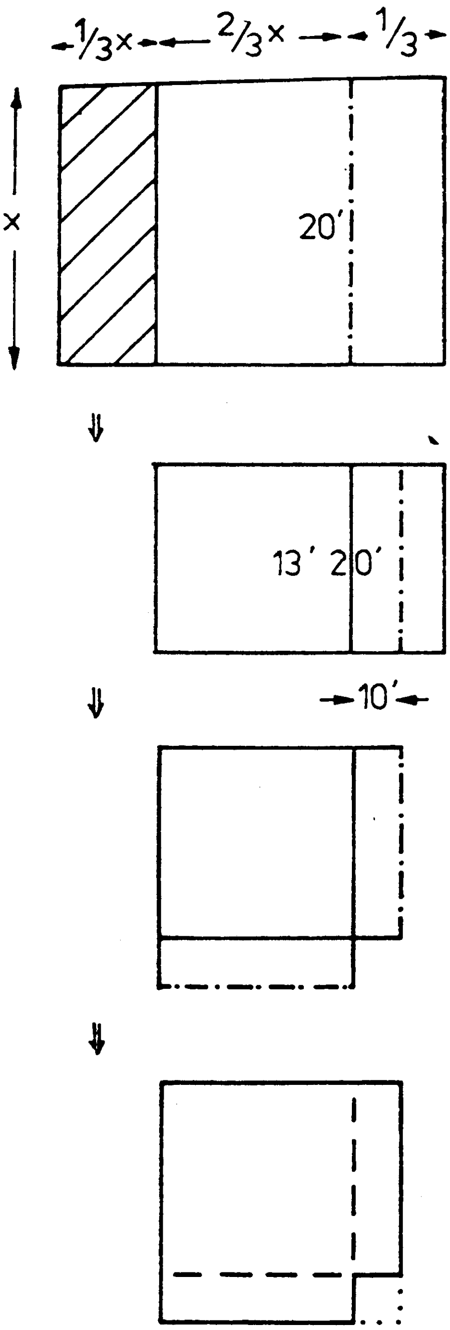


Fig. 8. The geometrical interpretation of BM 13901 N° 3.

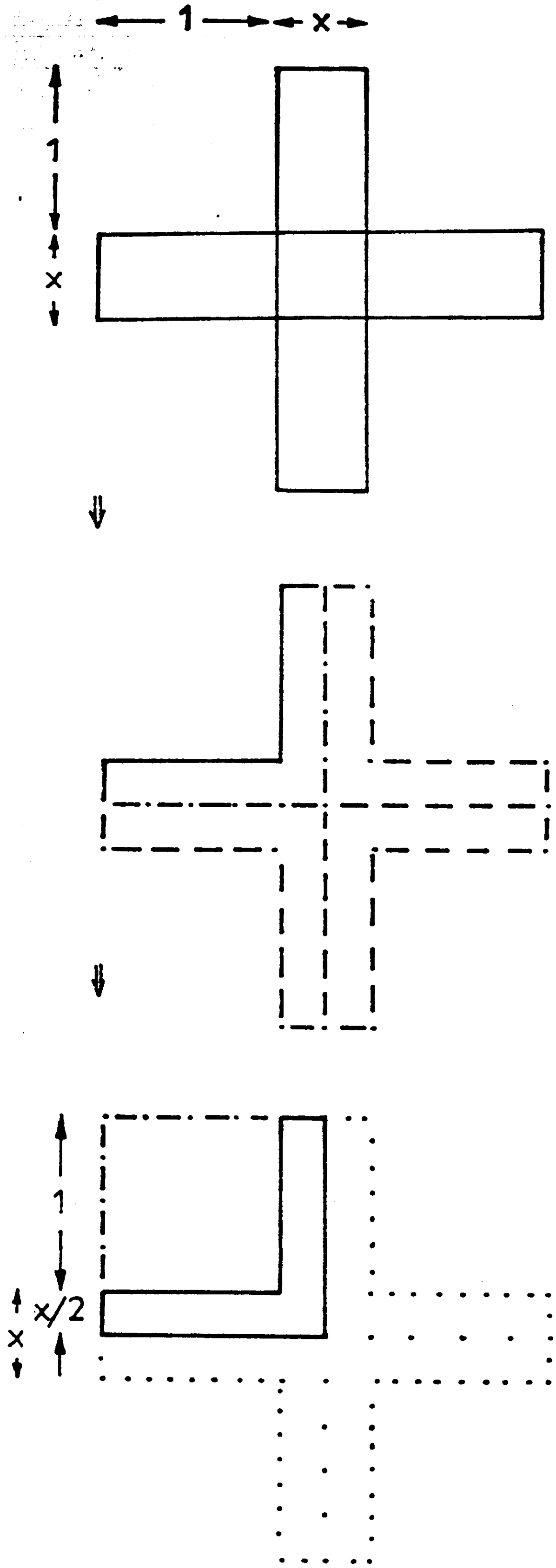


Figure 7. The geometrical interpretation of BM 13901 N° 23.

in Fig. 1, he distributes it as four rectangles $2^1/_2 \cdot x$ along the four edges of the square.[103]

[103] See Rosen 1831: 13–15. It cannot be decided on the basis of the al-Khwārizmī-text and the present Old Babylonian text alone whether the recurrence of two Old Babylonian methods in al-Khwārizmī's *Algebra* is due to coincidence or to continuous tra-

The text brings us somewhat closer to the precise meaning of the *wāṣitum*. It cannot be the excess of rectangular length over rectangular width. Possibly, it could be the length of any of the four projections from the central square; that would, however, agree poorly with the use in problem 2 of the tablet (see above). So, we are led towards the interpretation of the *wāṣitum* as that projecting width *1* which transforms a length into a rectangle of equal area.

One problem in the text is not elucidated by the naive-geometric interpretation in itself, viz. the initial "in a surface" or "concerning a surface".[103a] If nothing but the area of an ordinary square is meant, this indication is superfluous, and not to be expected after 22 problems which all deal with square areas without mentioning them explicitly beforehand. Together with other evidence which will be presented in section X.4, however, the apparent archaisms of the language may offer an explanation: *eqlum* is not only a (semi-)technical term for a mathematical area but also the everyday term for a field. All evidence combined suggests that the problem is a surveyors' recreational problem, maybe from a tradition which was older than—perhaps even a source for—Old Babylonian scribal school "algebra". The initial *eqlam* can be understood as an indication that we are dealing with a field-surveying problem (albeit an artificial one), and the apparent archaism perhaps as a reference to age and tradition or perhaps to oral or dialect usage (locativic and similar accusatives are more common and long-lived in Assyrian than in Babylonian).

V.5. BM 13901, N° 3 (MKT III, 1; cf. TMB, 1f.)

The above problems can all be classified as "normalized mixed second-degree equations". The present problem shows the habitual Old Babylonian way to deal with a non-normalized equation. The text runs as follows:

Obverse I

$(1-{}^1/_3)x^2+{}^1/_3x=20'$

9. The *third of* the surface *I have torn out:* the *third of* the *confrontation to* the *inside*
ša-lu-uš-ti a-šà *as-sú⟨-uḫ-ma⟩ ša-lu-uš-ti mi-it-ḫar-tim a-na lìb-bi*
10. *of* the sur*face I have appended:* 20′. 1 the *wāṣitum you pose*
a -šà*[lim]* *ú-ṣi-ib-ma* 20-e 1 *wa-ṣi-tam ta-ša-ka-an*

$1-{}^1/_3=1^\circ-20'=40'$

11. The *third of* 1 the *wāṣitum*, 20′ *you tear out:* 40′ *to*

dition. As I shall show in section X.4, however, another algebraic text roughly contemporary with al-Khwārizmī's shows continuity with the Old Babylonian tradition even down to the choice of grammatical forms, while displaying the same interest as the present problem in the four sides of squares and rectangles. This leaves little doubt that al-Khwārizmī too was inspired by the same old tradition.

103a Initially I believed so, reading the text as "The surface of the four fronts and the surface I have accumulated . . .", interpreting the "surface of the four fronts" as the total surface of the "arms" of the cross. I am grateful to A. Westenholz for pointing out the grammatical objections to this reading.

$(40'x)^2+40'\cdot 20'x$ $=13'\,20''$	20′ *you raise;* *ša-lu-uš-ti* 1 *wa-ṣi[-tim* 20 *ta-na-sà-aḫ-ma]* 40 *a-na* 20 *ta-na-ši*
$(40'x)^2+2\cdot 10'\cdot(40'x)$ $+(10')^2=$ $13'\,20''+1'\,40''=15'$	12. 13′ 20″ *you inscribe.* The *moiety of* 20′, the *third which you have torn out*[a] 13, 20 *ta-la-pa-at* [*ba-ma-at* 20 *ša-l*]*u-uš-tim ša ta-sú-ḫu*
$(40'x+10')^2=15'$	13. *you break:* 10′ *and* 10′ *you make span,* 1′ 40″ *to* 13′ 20″ *you append* *te-ḫi-pi* 10 [*ù* 10 *tu-uš-ta-kal* 1, 40] *a-na* 13, 20 *tu-ṣa-ab*
$40'x+10'=\sqrt{15'}=30'$ $40'x=30'-10'=20'$	14. 15′ makes 30′ equilateral. 10′ *which you have made span in* the *inside of* 30′ *you tear out:* 20′. 15-e 30 [ib-si$_8$ 10 *ša tu-uš-ta-ki-lu lìb-ba* 30] *ta-na-sà-aḫ-ma* 20
$(40)^{-1}=1^\circ\,30'$ $x=1^\circ\,30'\cdot 20'=30'$	15. The igi of 40′, 1° 30′ *to* 20′ *you raise:* 30′ the *confrontation.* igi 40 gál-b[i 1, 30 *a-na* 20 *ta-na-ši-ma* 30] *mi-it-ḫar-tum*

[a] Both for mathematical reasons and because of the many parallel passages of the tablet, this "have torn out" must be a writing error for "you have appended", *tu-iṣ-bu*.

The problem is of the type $\alpha x^2+\beta x=\gamma$. In Medieval (Arabic and Latin) algebra, such an equation would be normalized as $x^2+(\beta/\alpha)x=(\gamma/\alpha)$. The method here is different, a fact which has often been regarded as astonishing, although the same procedure is used by Diophantos and Hero[104]: Instead of x, αx is taken as the quantity looked for, and the equation is transformed into $(\alpha x)^2+\beta\cdot(\alpha x)=\alpha\gamma$. In the end, x is found from αx through multiplication by the reciprocal of α.

The application of the arithmetical interpretation raises a problem: The multiplications by α and α^{-1} are expressed by means of the term "to raise", while that of $(\beta/2)$ by $(\beta/2)$ (of 10′ by 10′) is expressed by "making span". Another problem is presented through the way the equation is transformed: As most of us would immediately feel, and as it is confirmed by the Medieval algebras, in a rhetorico-arithmetic representation it is easier to keep track of a reduction to normalized form that of the actual "change of variable". Finally, of course, the *wāṣitum* remains a stranger to any arithmetical interpretation, as does the distinction of a "moiety" from a "half".

As usual, we shall try to apply a representation by naive geometry—see Fig. 8. If we look at lines 12–14 of the text, it is clear that they follow the normal "square-plus-sides"-procedure (cf. section V.1 and Fig. 5). So, we must interpret the text geometrically in such a way that this situations comes about.

[104] Diophantos, *Arithmetica* VI, vi. Hero, *Geometrica* 21, 9f. The Diophantine and Heronian parallels have been pointed out by K. Vogel (1936: 714; 1959: 49).

Line 9f. states the problem. In line 10, furthermore, the *wāṣītum* is "posed", and since no "projection" from the square is 1, we can now be sure that the term designates that projection from a line which creates the rectangle of equal area, as suggested above. An area of one third of the side is then a rectangle of width "the third of *1* the *wāṣītum*", i.e. 20′, and length x. This corresponds to line 11 where, however, an ellipsis turns up, as the third of the *wāṣītum* is identified with that third (of the surface) which is to be "torn out"; that such a confusion is really there is confirmed in line 12. So, the "coefficient to x^2" (α) is found to be $1°-20'=40'$.

In the last part of line 11, this factor is applied to the total non-shaded area ($^2/_3x^2+{}^1/_3x=20'$). This can be apprehended geometrically as the first transformation of the figure, where the scale factor 40′ ($={}^2/_3$) is applied in the vertical direction. This operation transforms the rectangle $x \cdot {}^2/_3x$ into a square $^2/_3x \cdot {}^2/_3x$. In the same process, the appended rectangle $^1/_3 \cdot x$ is transformed into a rectangle $^1/_3 \cdot {}^2/_3x$. That is, we have obtained the required situation "square-plus-sides", and the number of "sides" is unchanged. The rest of the procedure is by now well-known: The appended rectangle is bisected and its outer wing moved so as to "span" a square of area 1′ 40″. This area is appended to the gnomon, the area of which is $40' \cdot 20' = 13'\ 20''$. The area of the resulting square is 15′, and its side therefore 30′. From this, the side 10′ of the square which was "spanned" is "torn out", leaving 20′ as the side of the square $(\alpha x)^2$. Hence, αx is 20′ and x itself is found through division by the scale factor 40′, i.e. through multiplication by its inverse 1° 30′, to be 30′.

This solves all the problems raised by the arithmetical interpretation. First of all, it is clear that the multiplication by a scaling factor or its inverse is different from the geometrical process "to span a square". If the conceptualization and method of Old Babylonian algebra are geometric, a terminological distinction between the two is next to obligatory.

Next, the geometrical interpretation leads us to prefer the "Diophantine" to the "Medieval" reduction: If the non-shaded part were to be transformed into a "square-plus-sides" through Medieval reduction, the change of scale would have to be in the horizontal direction. This would affect the width of the appended rectangle, which goes into the further calculations; on the other hand, the "Diophantine" transformations affects only its length which is anyhow irrelevant.[105]

[105] This simplification of the geometrical prodecure is not in general accompanied by calculatory simplification: The multiplication $\beta \cdot \alpha^{-1}$ is dispensed with, it is true; but the final inverse scaling would be dispensed with in the "Medieval" reduction. Only cases where α is an irregular which does not divide β and γ would be harder—indeed impossible—to deal with "Medievally".

Of course, such arguments of conceptual simplicity should be used with care. We cannot conclude in that way that Diophantos made use of geometric representations. By his syncopated rhetorics he could keep track of problems much more complicated than the present one. But the Babylonian texts were made neither by nor for mathematicians of Diophantine stature; they were school texts, made for scribe students, comparable in giftedness and interests to the students of Medieval merchant ("abacus") schools, we may guess. If the latter were unable to use the Diophantine method in a rhetorical representation, there is no reason to believe that Babylonian students were any better off.

Finally, of course, the *wāṣītum* is no stranger but a must for a geometrical interpretation (with or without a name), and the "moiety" is a natural half, a "wing".

On the other hand, the geometrical interpretation raises two new questions. The first of these concerns the semantic range of the term "raising": Is it restricted to multiplications which can be regarded as changes of scale, or is it wider? This cannot be answered from the present text, but as discussed above (section IV.3) the range is indeed much wider. (Cf. also below, section V.8.)

The second question concerns the figure: Did the Babylonians draw or imagine a series of different diagrams, as they are shown in Fig. 8? Or were they able to conceptualize the same representation first as a rectangle with sides x and $^2/_3x$, and next as a square with both sides equal to $^2/_3x$? It is equally impossible to answer this second question on the basis of the present text (or to give a definitive answer on the basis of any text I know). Yet, as I shall argue in chapter VI, indirect evidence suggests that the Babylonians were fully able to conceptualize a drawn rectangle as a diagram for a square.

The geometrical technique which appears to be used in the first examples and in al-Khwārizmī's justification can be described as a "cut-and-paste"-procedure. The same technique is used in the present example for those operations which are described by the terms "to tear out", "to append", "to break" and "to make span". The "raisings" of line 11 and 15, however, belong with another technique, of which special notice should be taken: A technique of proportionality, which in relation to the geometric representation can be described as a uni-directional "change of scale"; I shall use the term "scaling" for the technique.[106]

V.6. BM 13901, N° 10 (MKT III, 2f.; cf. TMB, 4)

The above examples were all concerned with mixed second-degree equations. We shall now turn to homogeneous problems—first to BM 13901 N° 10.

	Obverse II
$x^2+y^2=21°\ 15'$	11. The surfaces *of my two confrontations I have accumulated:* 21° 15′. a-šà *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* 21, 15
$y=(1-{}^1/_7)x={}^6/_7x$	12. *confrontation to confrontation*, the *seventh*[a] *it has diminished.* *mi-it-ḫar-tum a-na mi-it-ḫar-tim si-bi-a-tim im-ṭi*
$x=7z \quad y=6z$	13. 7 *and* 6 *you inscribe.* 7 *and* 7 *you make span*, 49. 7 *ù* 6 *ta-la-pa-at* 7 *ù* 7 *tu-uš-ta-kal* 49
$x^2=49z^2$ $y^2=36z^2$	14. 6 *and* 6 *you make span*, 36 *and* 49 *you accumulate:* 6 *ù* 6 *tu-uš-ta-kal* 36 *ù* 49 *ta-ka-mar-ma*
$x^2+y^2=(49+36)z^2=$ $1` 25°z^2=21°\ 15'$	15. 1` 25°. The igi of 1` 25° *is not detached. What to* 1` 25° 1, 25 igi 1, 25 *ú-la ip-pa-ṭa-ar mi-nam a-na* 1, 25

[106] The method is closely related to the method of a "single false position", which was also used by the Babylonians as a purely arithmetical technique (cf. K. Vogel 1960).

$15' \cdot 1`\,25^\circ = 21^\circ\,15'$	16. *shall I pose which* 21° 15′ *gives me?* 15′ makes 30′ equilateral.
$z^2 = 15'$, $z = \sqrt{15'} = 30'$	*lu-uš-ku-un ša* 21, 15 *i-na-di-nam* 15-e 30 íb-si$_8$
$x = 7 \cdot 30' = 3^\circ\,30'$	17. 30′ *to* 7 *you raise:* 3° 30′ the *first confrontation.* 30 *a-na* 7 *ta-na-ši-ma* 3, 30 *mi-it-ḫar-tum iš-ti-a-at*
$y = 6 \cdot 30' = 3$	18. 30′ *to* 6 *you raise:* 3 the *second confrontation.* 30 *a-na* 6 *ta-na-ši-ma* 3 *mi-it-ḫar-tum ša-ni-tum.*

[a] The form is a plural, *sebiātim*, cf. Thureau-Dangin 1934: 49, and Goetze 1946: 200.

A geometrical interpretation of the procedure is shown in Fig. 9. The first step, that of finding the set of proportionate numbers, looks like a purely arithmetical "single false position": A number from which one seventh is easily taken away is 7, and the removal of the seventh leaves 6.[107] These numbers are "inscribed", an expression which was also used in N° 23 and N° 3, where the areas

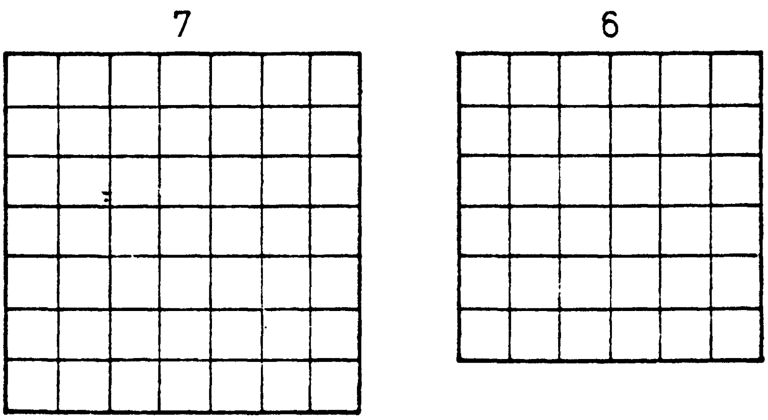


Figure 9. The geometrical interpretation of BM 13901 N° 10.

found by quartering and scaling were "inscribed". In agreement with Babylonian habits as expressed on tablets with drawings[108], we may image inscriptions along the edges of squares, as shown on the figure. The process can be so interpreted that a unit is imagined in which the lengths of the squares are 7 and 6, respectively. Such a conceptualization could follow as an extrapolation from common experience with metrological conversions. The respective areas are found (by "making span") in the square of this unit, as 49 and 36; the total area when measured so will then be $49 + 36 = 1`\,25^\circ$. In the basic area unit it is known to be 21° 15′. So, the square of the imagined unit (the area of the small squares) is $21^\circ\,15'/1`\,25^\circ = 15'$; hence its side will be $\sqrt{15'} = 30'$, and those of the two original squares $7 \cdot 30' = 3^\circ\,30'$ and $6 \cdot 30' = 3$.

Fundamentally, this conceptualization subdivides the given squares of the problem directly. An alternative interpretation could be that two auxiliary squares are imagined, of "real" sides 7 and 6. Their areas are found and added; the ratio between this and the original total area is calculated, etc.

It is impossible to decide from the text which interpretation to prefer. From

[107] The same pattern of thought is made explicit e.g. in VAT 7532 rev. 6f.
[108] Str. 367 (MKT I 259f.) may be quoted as an example.

the view-point of mathematics, they are of course equivalent.[109] My intuitive feeling is that the former is the more plausible, as it is conceptually simpler—it is easier to draw the subdivisions of an existing square, to point to it and speak about it, than to make non-mathematicians understand an abstract ratio and the reason why its square-root should be taken. As we shall see in the following examples, there is also direct evidence that the Babylonians used subdivisions and alternative "units" rather than ratios.

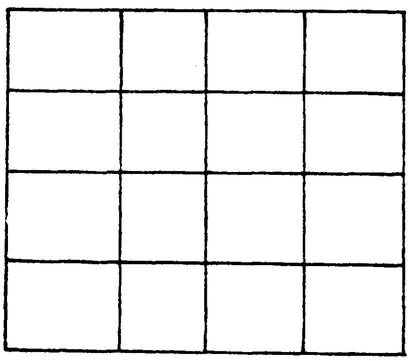
In any case, the text presents us with a third technique besides the cut-and-paste procedures and the scaling: The calculation of total "coefficients"—here the "number of small squares". Below, we shall meet in section VII.3, TMS XVI, the expression "as much as there is of" entity x, as an explicit formulation of this concept.—We notice that the number is found by "accumulation", not by "appending". The same holds for the calculation of the true total area in line 11. In both cases, indeed, none of the addends possesses an "identity" which is conserved through the process. It seems plausible, too, that "accumulation" is a more genuinely arithmetical process than "appending", adding measuring numbers, while "appending" affects only concrete though measured entities.

In order to point to a practice with which the Babylonians were utterly familiar, and which is structurally analogous to the accumulation of a coefficient, I shall speak of the "accounting technique".

V.7. BM 15285, N° 10 (MKT I, 138; MKT II, Plate 4)

BM 15285 is (part of) a large tablet where the areas of various subdivisions of a square of side 1 are asked for. The present problem is clearly related to a particular aspect of the argument of the previous problem, and it can serve to elucidate the questions left open there.

The text is accompanied by a figure, which I show in the left margin (traced after the photo in MKT II).



1. 1 the length, *a confrontation*
   [1 uš *mi-i*]*t-ḫa-ar-tum*
2. In its inside, 16 *of* a *confrontation*[a]
   sag$_4$-ba 16 *mi-it-ḫa-ar-tim*
3. *I have laid down.* Its surface what?
   ad-di a-šà-bi en-nam

[a] The form is a genitive singular.

The figure shows us precisely the subdivision of a square into smaller squares which was suggested as the first interpretation of the procedure of the previous problem. So, this interpretation is at least corroborated.

[109] Expressed in terms of the arithmetico-symbolic representation aligned with the translation, the former interpretation makes the variable z the side of the small square, one seventh of the side of the first original square. According to the latter interpretation, z is the ratio between the sides of the original and the auxiliary squares.

Another interesting point is the use of the singular genitive in line 2. True enough, H. W. F. Saggs suggests that we have to do with a simple writing error, but that appears to be excluded by the singular -bi in line 3.[110] The small squares appear to be regarded as repetitions of an identical entity—a unit of accounting. Even in this respect, the present and the previous text are related.

V.8. VAT 8390, N° 1 (MKT I, 335f.; cf. TMB, 112f.)

A final homogeneous second-degree problem is VAT 8390, N° 1[111]:

Obverse I

$xy = 10'$	1. Length *and* width *I have made span:* 10' the surface [uš ù sag] *uš-ta-ki-il-ma* 10 a-šà
$x^2 = 9 \cdot (x-y)^2$	2. The length *to itself I have made span:* [uš *a*]*-na ra-ma-ni-šu uš-ta-ki-il-ma*
	3. A surface *I have built* [a-šà] *ab-ni*
	4. *So much as* the length over the width *goes beyond* [*ma*]*-la* uš u-gù sag *i-te-ru*
	5. *I have made span, to* 9 *I have repeated:* *uš-ta-ki-il a-na* 9 *e-ṣi-im-ma*
	6. *As much as that* surface *which* the length *by itself* *ki-ma* a-šà-*ma ša* uš *i-na ra-ma-ni-šu*
	7. *has been made span*[a]. *uš-t*[*a*]*-ki-lu*
	8. The length *and* the width what? uš ù sag en-nam
	9. 10' the surface pose 10' a-šà gar-ra
	10. *and* 9 (to) *which he*[b] *has repeated* pose: ù 9 *ša i-ṣi-pu* gar-ra-*ma*
$\sqrt{9} = 3$ $[x = 3 \cdot (x-y)]$	11. The equilateral of 9 (to) *which he has repeated* what? 3. íb-si$_8$ 9 *ša i-ṣi-pu* en-nam
$x = 3z$	12. 3 *to* the length pose. 3 *a-na* uš gar-ra

110 See Saggs 1960: 139. According to SLa § 101, the use of -bi as a plural possessive suffix is apparently restricted to collective nouns ("people" and the like), and the same holds for the use of the singular status rectus after numbers above 10 (GAG § 139h). Strictly speaking, then, we have to do with either 16 copies of the same square or 16 practically identical squares.

111 N° 2 of the same tablet is a strict parallel—translated into symbolic algebra, the condition $x^2 = 9 \cdot (x-y)^2$ is replaced by $y^2 = 4 \cdot (x-y)^2$. The parallelism makes all restitutions of damaged passages certain.

I follow the improved readings given by Thureau-Dangin (1936: 58, repeated in TMB).

$\tilde{y}=[x=]\ 3z$	13. 3 *to* the width pose. 3 *a-n*[*a* s]ag gar-ra
$[(x-y)={}^1/_3x=1\cdot z]$	14. *Since "so much as* the length over the width *goes beyond* *aš-sum ma-*[*la* uš] u-gù sag *i-te-ru*
	15. *I have made span", he has said* *uš-ta-k*[*i-il*] *iq-bu-ù*
$y=\tilde{y}-(x-y)=3z-z$	16. 1 *from* 3 *which to* the width *you have posed* 1 *i-na* ˹3 *ša-a-n*]*a* sag *ta-aš-ku-nu*
	17. *tear out:* 2 *you leave.* *ú-*[*sú-uḫ-m*]*a* 2 *te-zi-ib*
$y=2z$	18. 2 *which you have left to* the width pose. 2 *ša t*[*e-z*]*i-bu a-na* sag gar-ra
$xy=3z\cdot 2z=6z^2$	19. 3 *which to* the length *you have posed* 3 *ša a-na* uš *ta-aš-ku-nu*
	20. *to* 2 *which to* the width *you have posed* raise, 6 *a-na* 2 *ša* ⟨*a-na*⟩ sag *ta-aš-ku-nu* íl 6
$6^{-1}=10'$	21. The igi of 6 *detach:* 10′. igi 6 *pu-ṭur-ma* 10
$z^2=10'\cdot 10`=1`\,40^\circ$	22. 10′ *to* 10` the surface raise, 1` 40°. 10 *a-na* 10 *a*-šà íl 1, 40
$z=\sqrt{1`\,40^\circ}=10$	23. The equilateral of 1` 40° what? 10. íb-si$_8$ 1, 40 en-nam 10

Obverse II

$x=3z=3\cdot 10=30$	1. 10 *to* 3 *which to* the length *you have posed* 10 *a-na* 3 *š*[*a a-na* uš *ta-aš-ku-nu*]
	2. raise, 30 the length. íl 30 u[š]
$y=2z=2\cdot 10=20$	3. 10 *to* 2 *which to* the width *you have posed* 10 *a-na* 2 *ša a-na* sag *ta-aš-*[*ku-nu*]
	4. raise, 20 the width íl 20 sag
Proof:	5. *If* 30 the length, 20 the width *šum-ma* 30 uš 20 sag
	6. the surface what? a-šà en-nam
$xy=30\cdot 20=10`$	7. 30 the length *to* 20 the width raise, 10` the surface. 30 uš *a-na* 20 sag íl 10 a-šà
$x^2=30\cdot 30=15`$	8. 30 the length *together with* 30 *make span:* 15` 30 uš *it-ti* 30 *šu-ta-ki-il-ma*
$x-y=30-20=10$	9. 30 the length over 20 the width *what goes beyond?* 10 *it goes beyond.* 30 uš u-gù 20 sag *mi-nam i-tir* 10 *i-tir*

$(x-y)^2 = 10 \cdot 10 = 1' \, 40°$	10. 10 *together with* 10 *make span:* 1' 40°. 10 *it-ti* [10 *šu*]-*ta-ki-il-ma* 1, 40
$9 \cdot (y-x)^2 = 9 \cdot 1' \, 40°$ $= 15'$	11. 1' 40° *to* 9 *repeat:* 15' the surface. 1, 40 *a-na* 9 *e-ṣi-im-ma* 15 a-šà
$x^2 = 9 \cdot (x-y)^2$	12. 15' the surface is *as much as* 15′ the surface *which* the length 15 a-šà *ki-ma* 15 a-šà *ša* uš
	13. *by itself has been made span.* *i-na ra-ma-ni-šu uš-ta-ki-lu*

[a] Taken by itself, the phrase "*ša* uš *ina ramanišu uštākilu*" could perhaps also be interpreted as "which I made the length span by itself". The preposition *ina* occurs, however, in connection with *šutākulum* in all four occurrences of the relative clause in question and nowhere else in the tablet (nor anywhere else, as far as I can find out). Elsewhere in the tablet *šutākulum* stands with *u, ana* and *itti*. The propability that this distribution should have come about randomly is extremely small ($2.3 \cdot 10^{-4}$ in a reasonable stochastic model). Furthermore, the occurrences in obv. II, 12f. and rev. 23f. stand in passages where the context requires the second person singular (because imperatives are pointed at) if the subject of the clause is not uš. Hence, the form cannot be the usual Št (II) (causative, reflexive), but must be Št (I) (passive of causative), of which this preterite form coincides with that of Št (II).
[b] The choice of "he" instead of "9" as the subject of the doubling is enforced by related passages in VAT 8520, obv. 7, 9, 11, rev. 8, 10.

As usually, the main lines of the procedure can be mapped by the arithmetical representation. On a number of points, however, it is inadequate: Why is a width equal to the length of 3 introduced in I, 13 (if this is at all the meaning of the expression "pose to"?)? Which principles govern the use of the three multiplicatory terms ("making span"; "raising"; and "repeating to n"? Why are so many different entities spoken of as "surfaces"? Normally, such words stand as epithets which serve to identify a number; this is also the case in I, 22, where "10' the surface" is kept apart from "10′ [the igi of 6]". But this function can only be hindered when x^2 and $9 \cdot (x-y)^2$ are also labeled "surface" (I, 2f.; II, 11f.).[112] So, in some sense or other, all these entities must be "surfaces".

Further: Why are the "surfaces" "built", while other complex expressions are not?[113] And why are "posing" (e.g. "posing 10' the surface", in I, 9) und "posing

112 In AO 8862 N° 1 (translated below, section VIII.2), even the inhomogeneous expression $xy + x - y$ is a "surface"; so, the meaning of the term cannot be that of "product". Linear expressions, on the other hand, are never called surfaces; so, a generalized sense of "function" or "combined expression" is equally excluded. The sense "polynomium of the second degree" would of course be adequate, but much too abstract to be expected in a Babylonian context.

113 Indeed, with one exception, only "surfaces" are "built" in Old Babylonian algebraic texts (VAT 8390 and AO 8862 in MKT I; YBC 4608 in MCT; TMS XVII). The exception concerns IM 52301, the deviations of which from normal usage were already mentioned above (note 95) (cf. also below, section X.1.).

to" (e.g. "posing 3 to the length", in I, 12) carefully distinguished all the way through the tablet? All these finer points of the formulation make no sense in the arithmetical interpretation. Several appear to call for a geometric reading, and indeed, a geometric representation answers all the questions, while at the same time giving us supplementary insight in the relation between "raising" and "making span".

The geometric representation which appears to be described in the text is shown in Fig. 10, the relation of which to the 16 squares of BM 15285 N° 10 is

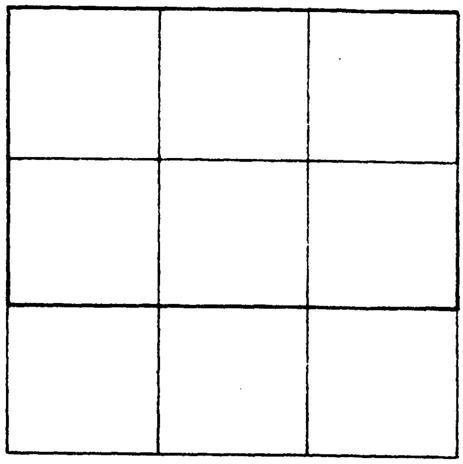


Figure 10. The geometrical interpretation of VAT 8390 N° 1.

obvious. The "repetition to 9" of the square on the excess of length over width is clearly seen to be a concrete repetition, no multiplicatory calculation. A width related to the number 3, and another width similarly related to 2, are clearly seen on the figure. And of course, all the "surfaces" are indeed surfaces in the most literal sense.

We observe that the numbers which are "posed" in I.9–10 are "real values"– the real surface of the rectangle, and the number of repetitions of the small square. The numbers which are "posed to" length and width (in I.12, 13 and 18), on the other hand, are not real values of the lengths and widths in question. It might seem as if "false values" (in the sense of a "false position") were "posed to" the entity for which they are assumed; still, according to normal Babylonian usage, later references (like that of I.19) could then be expected to quote the assumed numbers as values ("3 the length which you have posed", or perhaps "3 the false length which you have posed"). So, we are led towards the interpretation that "posing x to A" means "writing the number x along the entity A", as it was suggested in Fig. 9 (cf. note 108). Once again, the interpretation of the procedure of BM 13901 N° 10 as a subdivision rather than a comparison with an auxiliary figure is supported.

In one respect, the geometric interpretation changes the expectations which might be derived from the previous examples. When length and width, length together with length or excess together with excess give rise to rectangles or squares in I.1–5 they are "made span". So also in the proof, in II.8, 19, when the length and the excess are squared. But in I.20, the number of small squares is calculated by "raising 3 to 2", and in II.7, "30 the length" is "raised to 20 the width". What is the difference? Are the terms synonymous in spite of all contrary evidence?

The clue has to do with the term "to build", and with the way triangular and trapezoidal areas are found. Only when a length and a width (or two other lines) have been "made span", is a surface said to have been "built". Conversely, when the area of a triangle, a trapezium or a trapezoid is calculated, the term used is invariably "raising". So, firstly, the terms cannot be synonymous. And, secondly, one of them must belong with the process of building and the other with calculation. In other word, the process "to make span" is to be understood literally, as a process of *construction*, and to "build" means "to construct" (in agreement with the Latin etymology of the latter word). "Raising", on the other hand, means "calculating by multiplication".

This agrees well with the use of the terms in our text. In the beginning, the rectangle, the square on the length and the square on the excess are all constructed anew–none of them existed before. The number following the construction measures the area of the surface constructed–so, the calculation of this area is implied by the construction process[114], but it remains something different. In I.20, when the numbers 2 and 3 are multiplied and the number of small squares in the rectangle thus calculated, the rectangle is already there; hence, 3 is "raised to" 2, they are not "made span". Cf. also BM 13901, No 23: the *wāṣītum*-corner is already there, there is no need to construct it, nor is the *wāṣītum* "made span" (see above, note 101).

In the proof, the rectangle is still supposed to be there. In II.7, the length is "raised to" the width. The squares on length and excess, on the other hand, are "spanned". Since the same pattern repeats itself accurately in the second problem, this can hardly be an accident. So, the squares are not there to the same extent as the rectangle–either because only the rectangle is drawn, while the other figures are only imagined (3 and 2 being "posed" successively to the same width?), –or because everything is imagined, but the rectangle is more familiar as the basic figure and therefore still present to the inner eye. In any case it is made plausible that no complete figure like that of Fig. 10 was really drawn. Part of the procedure, if not all of it, was performed as mental geometry.

VI. The question of drawings

At this point it seems natural to ask whether the Babylonians left any traces of drawings like those of Fig. 4 to 10. The answer is, if we confine ourselves to algebraic texts like those to which these figures belonged[115], a clear no.

This might seem to present a problem to the geometrical hypothesis. Truly much "geometric" manipulation can have been performed mentally (and part of it must have been so performed, it appears from the above). But skill in mental geometry can only be acquired through familiarity with materialized geometry. So, a geometric interpretation of Babylonian algebra implies as its basis a physically palpable representation of this geometry.

[114] In AO 8862 (see section VIII.2), the calculation is at times made explicit as a separate process after the construction.

[115] The case of BM 15285 № 10 (see above, section V.7) is different. The whole tablet deals with areas of indubitably geometrical figures; no scaling and no cut-and-paste procedures appear to be involved.

On the other hand, drawings are also absent from the tablets in other cases where we can be sure that the argument presupposes a geometric figure. True enough, some real geometric problems are accompanied by a drawing. Still, this drawing is only an illustration of the statement of the problem, not of the procedure. Even in cases where auxiliary lines or appended figures are supposed by the argument they are left out from the drawing.[116] Furthermore, when the verbal statement of a geometric problem appears to be sufficiently clear, the sketch of the geometric situation is often dispensed with.

Even in cases where we can be sure that drawings have been made, they are thus absent from the tablets. This raises the question, where else they can have been made? Which medium can be imagined where drawings would leave no archaeological traces?

Several possibilities are open. The Greek drawings made in the sand are, at least from the anecdotes concerning the death of Archimedes, part of general lore.[117] For Mesopotamia, too, the use of the sand of the school courtyard has been proposed, namely as the medium for models of cuneiform signs in the basic scribal education.[118] Still, another possibility suggested by the Greeks is perhaps more interesting: The dust abacus, or its cognate, the wax tablet. As explained above (Chapter II), the Greek term ἄβαξ, "abacus", is in all probability derived from the semitic root *'bq*, "to fly away", "light dust". On that background it seems plausible that the Greeks have first met the abacus in the form of a dust-board, and that they have done so in the Western Semitic area.[119] As cultural connections between Syria and Mesopotamia were numerous—even much of the metrological system was shared and eventually taken over by the Greeks—use of the same device in Mesopotamia is at least a strong possibility. As to the wax tablet, it was certainly used in Mesopotamia in later times.

Whatever the medium of drawings corresponding to the solution of definitely geometric problems may have been, it left no traces, at least no traces which have been discovered until now. So, we need not worry much because no drawings corresponding to the solution of algebraic problems have been excavated.

116 So in VAT 8512 (MKT I 341, cf. Gandz 1948: 36f. or Vogel 1959: 72), an auxiliary rectangle is attached to the triangle spoken of in the enunciation. In this text, by the way, even the verbal explanation which states the problem is left without the support of a sketch of the situation. Indeed, the problem as stated is clear and unambiguous and requires no sketch. The far less clear exposition of the procedure (less clear at least to modern interpreters) has not given rise to any explanatory drawing.

117 Admittedly, the association of Archimedes with drawings in the sand are probably due to an ancient misunderstanding (see Dijksterhuis 1956: 30—32). Still, this very misunderstanding shows that geometrical drawings were at times made in the sand. The same is clear from an anecdote told by Vitruvius (*De architectura* VI, i, the story of the shipwrecked philosopher Aristippus finding geometric figures in the sand of the Rhodian shore).

118 In the Old Babylonian school excavated in Tell ed-Dēr, the exercise tablets of the higher teaching levels contain the instructor's model and the student's attempt to imitate in parallel. The tablets belonging to the elementary level (stylus exercises, "Silbenalphabet A", "Syllabar a") contain no instructor's model, and Tanret (1982: 49) proposes that the models have instead been drawn "dans le sable de la cour".

119 Even though Proclos is not very reliable as a source for the early period in Greek mathematics, his statement could be mentioned that arithmetic was first developed by the Phoenicians (*In Primum Euclidis . . . Commentarii* 65^{3-5}).

On the other hand, drawings have been excavated which show us something about the probable character of the geometric support for algebraic as well as geometric problem solution,–to wit the field plans. The autography of one of these, as well as a redrawing in correct proportions[120], would show us how.

The first feature of the plan to be observed is perhaps the subdivision into right triangles, right trapeziums and rectangles. Subdivisions are of course not easy to do without when a natural area has to be measured, but the plan shows

– that right triangles and trapeziums were looked for, not any triangle and trapezium. In the latter case, a height would have to be measured; right figures, on the other hand, are fully described by length and width (in the case of right trapeziums two widths, "upper" and "lower").

– that the right angles of the partial figures were clearly marked on the figure, while no care was taken to render other angles correctly.[121]

– and that the Babylonians were perfectly aware of the possibility to use auxiliary lines which were calculated, not measured. The calculation also shows awareness of the imprecision arising during measurement, since the dimensions of the partial figures are calculated in two different ways and the average found—whence the two writing directions for the partial areas.

Another striking feature is the total lack of care for a faithful rendering of proportions. A line is, so it seems, described by the number written unto it, if it is a line of importance for the determination of "lengths" and "widths" of the partial figures. One and the same line on the figure can even have two different numbers written unto it–this is the case of the line delimiting the two triangles to the uttermost left: the numbers alone tell us that two different lines in the terrain are meant.

This lack of care for correct proportions has some curious effects. At bottom of the plan, the hypotenuse of a right triangle continues directly as the skew side of a trapezium. In reality, the two lines are at an angle somewhat below 120°. To state things a bit sharply, the Babylonians did not make a drawing of the terrain in their field plans: They made a s t r u c t u r a l d i a g r a m, showing relevant lines, stating their lengths by inscribed numbers, and indicating their mutual relation with respect to the intended area calculation by visually right angles between lengths and widths.

Similar structural diagrams are also often made as a support for the verbal statement of geometrical problem texts. A glaring example of the difference

[120] A plan of the fields belonging to the district Šulgi-sipa-kalama, from the tablet MIO 1107, published, redrawn and discussed by Thureau-Dangin (1897).

[121] So, the repeated claims of S. Gandz (e.g. 1939: 415ff.), F. Thureau-Dangin (e.g. TMB xvii) and E. M. Bruins (e. g. TMS 4) that the Babylonians possessed no concept corresponding to our concept of quantifiable angles is not contradicted by the field plan. In all probability, the claims are correct for the Old Babylonian period. So, a theoretical concept of the right angle must also be considered absent. But clearly, a practical concept of the right angle, as the correct angle relevant for area measurement, must have existed according to the field plan and according to much other evidence, including architectural structures and the expression "the four winds", i.e. four cardinal points. Somewhat pointedly, a Babylonian "right" angle can be claimed to be the opposite of a "wrong" angle.

between the real figure and the diagram interpreting the structure of the problem is YBC 4675.[122]

Naturally, this does not mean that the Babylonians were unable to make real geometrical drawings when they wanted to, or that they did not recognize a geometrical square. This is shown by the rich variety of geometrical forms drawn on the table BM 15285, of which one example was discussed in section V. 7. Still, the use of structural diagrams instead of drawings in the field plans and in the geometrical problem text suggests that the geometrical drawings or imaginations which possibly supported the solution of algebraic problems may very well have been of the diagram type. The first step in the reduction of Fig. 6, the redrawing in reduced vertical scale, need not have been performed in drawing. At the evidence of field plans etc. we may surmise that the Babylonians can have been able to imagine the left section of the unshaded part of the figure first as a rectangle and next as square, while the right section would in both steps be considered an appended "one third of the side". At the same time, they will have known that the changed conception of the whole figure would correspond to a reduced area: No longer $20'$ but $40' \cdot 20' = 13'\ 20''$.

Before leaving the problem of "drawings" we should take note of the fact that geometrical configurations can be represented materially by other means than through lines traced on a soft or colour-receiving surface. Some details of the Babylonian formulations could be read as hinting at a representation through small sticks or pieces of reed. I think especially of the identification of rectangular figures and their side and of the bisection through "breaking". It is also possible to make a pebble-representation of geometric figures in Greek style and to perform naive-geometric "algebraic" argumentation on such figures–and there exists indeed some vague evidence that early Greek calculators did so, inspiring thereby the development of the theory of figurate numbers.[123]

So, even though lines traced in sand, dust or wax appear to be the most plausible candidates for a representation of naive-geometric algebra it should be remembered that they are not the only possible candidates.

VII. The first degree

All texts discussed up to this point were of the "second degree", if we translate them into modern formalism, and such problems are the main concern of the whole investigation. To a large extent, however, Babylonian mathematics dealt with real-life problems, which in the Babylonian context were of the first degree; furthermore, the more complex second-degree-problems involve transformations and equations of the first degree. Both in order to locate the use of naive-geometric

[122] MCT 44f. and Plate 26.

[123] I deal with this question in my 1988: 24ff. It should be observed, firstly, that the multi-digit numbers occurring in many Old Babylonian algebra problems make them unsuited for precise representation through pebble patterns; and secondly, that the Babylonian procedure descriptions do not fit the most natural progress of a solution by pebbles.

methods correctly in relation to the complete structure of Babylonian mathematics and in order to grasp the methods of the complex second-degree-problems it is therefore of importance to get some idea of the techniques and ways of thought of Babylonian first-degree mathematics.

The present chapter presents two groups of texts suited for that purpose. Firstly I present two procedure-texts stemming from a larger group of problems all built on the same concrete data; they are sufficiently complex to admit of some insight into the patterns of thought employed. Secondly come two texts (stemming from a single tablet) reporting a didactical explanation of the transformations of a first-degree-"equation".

On the basis of the insights gained from these texts it will be possible to proceed to further second-degree-problems involving supplementary first-degree-transformations, which will give us a more complete picture of the relations between first- and second-degree-techniques.

VII.1. VAT 8389 N° 1 (MKT I, 317f.; improvements from Thureau-Dangin 1936: 58)

The problem deals with a domain composed of two partial fields of areas S_i and S_{ii}. The first field yields a rent in kind amounting to $r_i = 4$ gur of grain per bur, while the second yields $r_{ii} = 3$ gur per bur.[124] In the present problem, the total area is told to be $S_i + S_{ii} = 30'$ (sar), while the difference between the total rents yielded by the two fields is given as $R_i - R_{ii} = 8'\ 20^\circ$ (sila). (1 bur $= 30'$ sar, 1 gur $= 5'$ sila).

Obverse I

$r_i = 4$ gur/bur	1. *From* 1 bur 4 gur of grain *I have collected.* *i-na* bùr$^{\text{gán}}$ 4 še-gur *am-ku-us*
$r_{ii} = 3$ gur/bur	2. *From* 1 *second* bur 3 gur of grain *I have collected.* *i-na* bùr$^{\text{gán}}$ *ša-ni*[*-im*] 3 še-gur *am-ku-us*
$R_i - R_{ii} = 8'\ 20^\circ$ (sila)	3. The *grain* over the *grain* $8^\circ\ 20'$ *goes beyond.* *še-um* u-gù *še-im* 8, 20 *i-tir*
$S_i + S_{ii} = 30'$ (sar)	4. *My* meadows[a] I have accumulated: 30'. garim-*ia* gar-gar-*ma* 30
	5. *My* meadows what? garim-*ú-a* en-nam
The value of the practical unit bur is "posed" repeatedly in the	6. 30' the *bur* pose. 20' the *grain which he has collected* pose. 30 *bu-ra-am* gar-ra 20 *še-am ša im-ku-sú* gar-ra

[124] The verb translated "to collect" in my translation is *makāsum*, "Ertragungsteil, -abgabe einheben". MKT reads *malāsum*, "ausrupfen"; I follow Thureau-Dangin's correction (1936: 58), which shows the perspective to be not that of the peasant or the overseer-scribe but that of the landlord or his accountant. Neither this nor F. Thureau-Dangin's other corrections interferes with the mathematical structure of the text (cf. also MKT III 58).

"mathematical" unit sar, while the specific rents are posed directly, without the intermediate calculation, as $r_{\text{i}}[=4\cdot 5']=20'$ (sila/bur) $r_{\text{ii}}[=3\cdot 5']=15'$ (sila/bur)	7. 30' the *second bur* pose. 30 *bu-ra-am ša-ni-am* gar-ra 8. 15' the *grain which he has collected* pose. 1[5] *š[e-a]m š[a] im-ku-sú* gar-ra
Similarly, $R_{\text{i}}-R_{\text{ii}}$ and $S_{\text{i}}+S_{\text{ii}}$ are "posed"	9. 8' 20° *which* the *grain* over the *grain goes beyond* pose 8, 20 *š[a] še-um* u-gù *še-im i-te-ru* gar-ra 10. *and* 30' the *accumulation of* the surfaces of the meadows[b] pose *ù* 30 *ku-mur-ri* a-šà garim-meš gar-ra-*ma*
The total surface $S_{\text{i}}+S_{\text{ii}}=30'$ (sar) is bisected into two partial fields of 15' and 15', and the respective rents are calculated under the assumption that the original specific rents hold good for these two fields:	11. 30' the *accumulation of* the surfaces of the meadows 30 *ku-mur-ri* a-šà garim-meš 12. *to two break:* 15'. *a-na ši-na ḫi-pí-ma* 15 13. 15' *and* 15' *until twice* pose: 15 *ù* 15 *a-di ši-ni-šu* gar-ra-*ma*
First the specific rents are recalculated in units of sila/sar (expressed as "false grain"). Next the hypothetical total rents R'_{i} and R'_{ii} are found through multiplication with the hypothetical areas of 15' (sar):	14. The igi *of* 30', the *bur, detach:* 2″. igi 30 *bu-ri-im pu-ṭur-ma* 2 15. 2″ *to* 20', the grain *which he has collected* 2 *a-na* 20 še *ša im-ku-sú* 16. raise, 40′, the false *grain*; *to* 15' *which until twice* íl 40 *še-um* l[ul] *a-na* 15 *š[a] a-d[i] ši-ni-šu* 16a. *you have posed* *ta-aš-ku-nu*
$R'_{\text{i}}=10'$ (sila)	17. raise, 10' *may your head retain.* íl 10 *re-eš-ka [l]i-ki-il* 18. The igi *of* 30', the *second bur, detach:* 2″. igi 30 *bu-ri-im ša-ni-im pu-ṭur-ma* 2 19. 2″ *to* 15', the *grain which he has collected* 2 *a-na* 15 *še-im ša im-ku-sú* 20. raise, 30′, the false *grain*; *to* 15' *which until twice* íl 30 *še-um* lul *a-na* 15 *ša a-di ši-ni-šu*
$R'_{\text{ii}}=7'\ 30°$ (sila)	20a. *you have posed* raise, 7' 30°. *ta-aš-ku-nu* íl 7, 30
The difference between the hypothetical total rents is found: $R'_{\text{i}}-R'_{\text{ii}}=10'-7'\ 30° =2'\ 30°$	21. 10' *which your head retains* 10 *ša re-eš-ka ú-ka-lu* 22. over 7' 30° *what goes beyond?* 2' 30° *it goes beyond.* u-gù 7, 30 *mi-nam i-tir* 2, 30 *i-tir*

This difference falls 8' 20° − 2' 30° = 5' 50° short of the real difference

23. 2' 30° *which it goes beyond from* 8' 20°
 2, 30 *ša i-te-ru i-na* 8, 20
24. *which* the *grain* over the *grain goes beyond*
 ša še-um u-gù *še-im i-te-ru*

Obverse II

1. *tear out:* 5' 50° *you leave.*
 ú-sú-uḫ-ma 5, 50 *te-zi-ib*
2. 5' 50° *which you have left*
 5, 50 *ša te-zi-bu*
3. *may your head retain*
 re-eš-ka li-ki-il

The increase of the difference between the total rents is found for a transfer of 1 sar from the second to the first field: R'_{i} increases by 40′, R'_{ii} decreases by 30′, and hence the difference increases by 40′ + 30′ = 1° 10′ (sila). The required total transfer is found through a division by 1° 10′ to be 5' (sar), which is then added to the first hypothetical partial field and subtracted from the second in order to yield the real "meadows": $S_{i} = 15' + 5' = 20'$ (sar) $S_{ii} = 15' - 5' = 10'$ (sar)

4. 40′, the *ch[ange,] and* 30′, [the *change*][d]
 40 *ta-ki-i[r-tam ù*[1] 30 [*ta-ki-ir*]*-tam*
5. accumulate*:* 1° 10′. The *igûm*[e] *I know not.*
 gar-gar-*ma* 1, 10 *i-gi-a[m ú-ul i-de]*
6. *What to* 1° 10′ *shall I pose*
 mi-nam a-na 1, 10 *lu-uš-ku[-un]*
7. *which* 5' 50° *which your head retains gives me?*
 ša 5, 50 *ša re-eš-ka ú-ka-lu i-na-di-nam*
8. 5' pose. 5' *to* 1° 10′ raise,
 5 gar-ra 5 *a-na* 1, 10 íl
9. 5' 50° *will it give you*
 5, 50 *it-ta-di-[k]um*
10. 5' *which you have posed from* 15' *which until twice*
 5 *ša t[a]-aš-ku-nu i-na* 15 *ša a-d[i] ši-ni-šu*
11. *you have posed, from one tear out*
 ta-aš-ku-nu i-na i[š]-te-en ú-sú-uḫ
12. *to* the *other append.*
 a-na iš-te-en ṣ[í]-im-ma
13. The *first* is 20', the *second* is 10'.
 iš-te-en 20 *ša-nu-um* 10
14. 20' is the surface of the *first* meadow, 10' is the surface of the *second* meadow.
 20 *a*-šà garim *iš-te-at* 10 a-šà garim *ša-ni-tim*

Proof:
The total rents R_{i} and R_{ii} are found for the values $S_{i} = 20'$ sar, $S_{ii} = 10'$ sar (by renewed calculation of the "false grains")

15. *If* 20' is the surface *of* the *first* meadow,
 šum-ma 20 a-šà garim *iš-te-at*
16. 10' the surface *of* the *second* meadow, *their grains* what?
 10 a-šà garim *ša-ni-tim še-ú-ši-na* en-nam
17. The igi *of* 30', the *bur, detach:* 2″.
 igi 30 *bu-ri-im pu-ṭur-ma* 2
18. 2″ *to* 20', the *grain which he has collected*
 2 *a-na* 20 *še-im ša im-ku-s[ú]*

19. raise, 40′; *to* 20', the surface *of* the *first* meadow
íl 40 *a-na* 20 a-šà garim *i*[*š-te-at*]
20. raise, 13' 20° the *grain, that of* 20', the surface of the meadow.
íl 13, 20 *še-um ša* [20 a-šà garim]
21. The igi of 30', the *second bur, detach:* 2″.
igi 30 *bu-ri-im ša-ni-*[*im pu-ṭur-m*]*a* 2
22. 2″ *to* 15', the *grain which he has collected,* raise, 30′.
2 *a-na* 15 *še-*[*im ša im-ku-sú* í]l 30
23. 30′ *to* 10', the surface *of* the *second* meadow,
30 *a-na* 10 a[-šà garim *ša-ni-tim*]
24. raise, 5' the *grain, that of* 10', the surface *of* the *second* meadow.
íl ⌊5⌋ *še-*[*u*]*m* [*ša* 10 a-šà garim *ša-ni-tim*]

Finally, the difference between the rents of the two meadows is found to be 8' 20° as required.

25. 13' 20° [(the grain of the first meadow)][f]
13, 20 [*še-um* (*ša*/a-šà) garim *iš-te-at*]
26. over 5 the *grain* [(of the second meadow)]
u-gù [5] *še*[*-im* (*ša*/a-šà) garim *ša-ni-tim*]
27. *what goes beyond?* 8' 20° *it goes beyond.*
mi-nam i-tir [8, 20 *i-tir*]

[a] "Meadow" translates garim (∼*tawirtum*), "(Feld-)Flur, Umland, Umgebung". This name for a specific sort of field is possibly used because the normal name for a field (*eqlum*) is reserved in mathematical contexts for the meaning "surface" (cf. the last paragraph of section V.4). The same word is used for partial fields in VAT 8512 (see von Soden 1939: 148), in a context where parallel texts would make us expect A-ENGUR. This led Thureau-Dangin (1940a: 4f.) to the conjecture that the latter sign might in mathematical texts be a logogram for *tawirtum,* and not as usually (with the reading íd) for *nārum,* "Fluß, Wasserlauf, Kanal"; according to the Tell Ḥarmal compendium, however, the sign group was read *nārum,* "river" etc., even when a partial field was meant (IM 52916, rev. 15f., in Goetze 1951: 139).

[b] The plural of the "fields" is indicated by the suffix -meš, which in the living Sumerian language had been reserved to a plurality of persons (cf. Falkenstein 1959: 37). Obviously, the Sumerograms of the text are abbreviations for Akkadian words, and not evidence of an unbroken Sumerian mathematical tradition. Cf. also SLa, 63, § 76.

[c] "Grain" is in the nominative form, *še'um.* So, for once we are allowed by this happy apposition to interprete the common construction where a single number stands both as the result of one operation and as the object of the next: In the present case at least, the number is made explicit as a result, and is then implicitly understood in the next phrase.

This observation makes sense of a peculiar usage of the tablet BM 13901, viz. the use of the Sumerian agentive suffix -e as a separation sign between numbers. Indeed, O. Neugebauer made this explicit in his translation (e.g. in N° 1, Obv. I.1, translating the passage *ak-mur-ma* 45-e 1 *wa-ṣi-tam* as "habe ich addiert und

0;45 ist es. 1, den Koeffizienten". Since the suffix is only used when a separation of a result from a succeeding number is required, I chose to regard the main function of the sign as a separation indicator, and absorbed it into the interpunctuation of the translation. There is, however, little doubt that a secondary agentive connotation is also implied by the sign.

[d] "Change" translates *takkirtum,* my conjectural restitution of the damaged words of the line. Both O. Neugebauer and F. Thureau-Dangin suggest *ta-ki-il-tam,* because this word was known to them as a mathematical term, which seemed to make some sense, since they interpreted *šutākulum* simply as multiplication and *takīltum* hence as a "factor". The profounder understanding of the terms makes this reading meaningless and hence problematic. The only other word listed in AHw which seems to fit the remaining signs of the line is *takkirtum,* "Änderung". It is absent from other mathematical texts, but it turns out to make excellent sense in connection with a mathematical argument for which parallels are even more absent from our text material.

The term derives, indeed, from the D-stem of *nakārum,* viz. *nukkurum,* "(ver)-ändern", "bessern", "weitergeben", "anderswohin bringen"., etc. Now, in certain series texts the epithet kúr was applied to a "second" or "modified" width (cf. section IV.7). The Sumerogram is in general use for *nakārum* and its derivatives, but in the mathematical texts it appears to stand for the verbal adjective *nukkurum* of the D-stem. It is thus no wonder if the corresponding *nomen actionis* should belong to the mathematical idiom. Still, the restitution is conjectural. Truly, A. Westenholz finds it to fit the photograph at least as well as the old reading; but another trained eye, viz. that of W. von Soden, rejects it as impossible (personal communications).

[e] The text appears to distinguish the igi, i.e. the reciprocal of a number (an abstract mathematical concept), from the table value *igûm,* a very manifest entity. The latter term, in fact, turns up when the absence of the value from the table of reciprocals is stated. So does even the following text. Cf. YBC 6967, above, section V. 1, which deals precisely with table values.

[f] The double bracket [(. . .)] is used for a restitution of a passage where no parallel passages indicate the precise words of the original.

The mathematical commentary aligned with the translation shows that all steps of the procedure can be interpreted very concretely.[125] In principle, the text can of course also be followed by an abstract symbolic calculation, in the way its correctness is proved in MKT. But the text contains many steps which are superfluous if we suppose the real procedure to have been abstractly algebraic or arithmetical, so for instance the recalculation of the specific rents per sar in each case separately. The very complexity of the procedure points in the same direction: Why should the system

$$S_{\text{i}}+S_{\text{ii}}=30' \qquad \frac{5'}{30'}\,(4S_{\text{i}}-3S_{\text{ii}})=8'\ 20^{\circ}$$

125 A concrete interpretation of the procedure was, as far as I know, first proposed by van der Waerden 1961: 67.

be solved via calculation of the quantity $S_{\mathrm{i}}-\frac{S_{\mathrm{i}}+S_{\mathrm{ii}}}{2}$? In the text discussed immediately below, a still more spectacular detour (as viewed from the standpoint of abstract algebra) will turn up. Finally, all problems from the group to which the present as well as the following text belongs can be followed in detail on the level of concrete thought. Even before we take the plausible use of the term *takkirtum* into account there seems to be little doubt that the real procedure is close to the one exhibited in the marginal commentary. If a collation confirms the possibility of the new reading, we can presumably regard the interpretation as fully confirmed, since no other replacement of the impossible *takīltum* seems at hand.

If we accept this conclusion, a number of features can be observed in the text. We observe that all intermediate quantities can be given a concrete meaning, either directly or, more significantly, with regard to a hypothetical situation. The "false grain" can be understood as "false" if we see it as that amount of grain which could be collected from the field in question had it been of area 1 sar; and the 2ˋ 30° (sila) of obv. I, 22 can be interpreted as the difference in rents had the two fields been of equal magnitude.

The problem is of a type which in the Islamic Middle Ages might have been solved by a "double false position".[126] The present text avoids the technicalization inherent in this procedure and sticks to steps which can be intuitively and directly justified. The text keeps far from understanding via abstract arithmetical relationships; but it keeps equally far from the use of schemata learned by heart, and close to procedures which can be understood and explained.

Evidently, the problem is artifical. None the less, it appears to reflect the procedures of practical calculation very precisely. In order to see this we shall take note of some characteristics of Babylonian metrology. No metrological series were completely sexagesimal, and only weight measures approached sexagesimality. In order to make use of their tables of fixed constants and of the tables of multiples and reciprocals the scribes therefore had to convert the measures of practical life into sexagesimal multiples of a set of basic units (the nindan, the sar, the sila, etc.), which can be considered "mathematical" in the sense that they formed the basis for computation as performed by the scribes (but which were of course also practical units for measurements of a certain order of magnitude). In order to facilitate the conversions the scribes would make use of tables. This is precisely what happens in the present text. Areas and rents are given in the customary units bur and gur, which are of the relevant order of magnitude. In obv. 6 and 7 the scribe reads from his table that the bur is 30 (i.e., 30ˋ sar), and that 4 and 3 gur are, respectively, 20 (i.e., 20ˋ sila) and 15 (15ˋ sila)—this is the reason that the last numbers can be stated directly. Double conversions, on the other hand (bur per gur into sila per sar) were not tabulated; therefore the specific rents (the "false grains") must be computed, as done in obv. 14–16 and 18–20,

[126] Then the difference in rent would have been calculated e.g. under the two different suppositions that $S_{\mathrm{i}}=S_{\mathrm{ii}}$ and $S_{\mathrm{ii}}=0$, and the real values of S_{i} and S_{ii} would have been derived by "inverse linear interpolation". Cf. Tropfke – Vogel 1980: 371f.

The difference between the procedure of the present problem and that of a "double false position" was already pointed out by Vogel 1960: 90ff.

and without preliminary conversion into "mathematical" units this could not be done by means of the table of reciprocals.

The closeness of the text to practical computation makes its treatment of the bur important. Both in the beginning, in obv. I, 17–21, and again in the proof, "the bur" and "the second bur" are distinguished. This implies that the value of the bur is not just taken note of as a number when it is "posed" in the beginning. It must be written down or represented in some other way in two different calculation schemes or concrete representation of the two fields.

We may compare this use of "posing" with that of obv. II, 6–9, the division of 5` 50° by 1° 10′. The double construction of line 8 shows that "posing" is different from the process of arithmetical multiplication, the "raising", but at the same time part of or presupposition for the performance of the computation—again, "posing" stands for the insertion into a computational scheme or other fixed procedure[127]—but not precisely the scheme in which the bur was posed.

A third function of the term is found in obv. I, 9f.: When $S_{\text{i}}+S_{\text{ii}}$ and $R_{\text{i}}-R_{\text{ii}}$ are posed, it can have nothing to do with fixed procedures—the entities $S_{\text{i}}\pm S_{\text{ii}}$ and $R_{\text{i}}\sim R_{\text{ii}}$ are dealt with differently in the set of related problems. Apparently, these fundamental entities are simply taken note of, presumably in writing, in any case by some material means. It is a fair guess that the way it is done is somehow analogous to the manner in which burs and reciprocals are "posed" in computational schemes or fixed representations.

Our guarantee that "posing" of a given quantity uses some material means is provided by obv. I, 17 and II, 3. In both places, intermediate results are to be "kept in mind", literally to be "held by the head". This is an expression which is only used for intermediate results, never when given quantities or quantities found by naive-geometric manipulations are taken note of. "Keeping-in-mind" appears to concern the recording of intermediate results which fall outside fixed procedures and computational schemes.

VII.2. VAT 8391 N° 3 (MKT I, 321f., improvements from Thureau-Dangin 1936: 58)

The two tablets VAT 8389 and VAT 8391 belong together, and contain a number of problems dealing with the same two fields. In the present problem, $S_{\text{i}}-S_{\text{ii}}$ und $R_{\text{i}}+R_{\text{ii}}$ are given, together with the values of the specific rents, which are common to all problems.

Reverse I

Given are again $r_{\text{i}}=4$ gur/bur, and $r_{\text{ii}}=3$ gur/bur	3. *If from* 1 bur of surface 4 gur of grain *I have collected*, *šum-ma i-na* bùr^{gán} a-[šà] 4 še-gur [*am-ku-us*]

[127] In those rather few cases which go "What shall I pose to Y which gives me X? Pose Z, X it gives you", the "raising" of Z to Y must then be considered as implied by the "posing" as an automatic consequence (cf. section IV.6.).

4. *from* 1 bur of surface 3 gur of grain *I have collected,*
i-na bùr^gán a-šà 3 še-gur *am-[ku-us]*

Further $S_{i} - S_{ii} = 10'$ (sar)

5. *now* 2 meadows. Meadow over meadow 10' *goes beyond,*
i-na-an-na 2 garim garim u-gù garim 10 *i-tir*

$R_{i} + R_{ii} = 18'\ 20°$ (sila)

6. *their grain* I have accumulated: 18' 20°.
še-e-ši-na gar-gar-*ma* 18, 20

7. *My* meadows what?
garim-*ú-a* en-nam

The bur is "posed" (in sar) once for each meadow, and so are r_{i} and r_{ii} (in sila/bur)

8. 30' the *bur* pose. 20' the *grain which he has collected* pose.
30 *bu-ra-am* gar-ra 20 *še-am ša im-ku-sú* gar-ra

9. 30' the *second bur* pose. 15' the *grain which he has collected*
30 *bu-ra-am ša-ni-am* gar-ra 15 *še-am ša im-ku-sú*

9a. pose.
gar-ra

$S_{i} - S_{ii}$ is "posed" (the entity will be designated S' in the following)

10. 10' *which* meadow over meadow *goes beyond* pose. 1[0 *š*]*a* garim u-gù garim *i-te-ru* gar-ra

$R_{i} + R_{ii}$ is "posed"

11. 18' 20° the *accumulation of* the *grain* pose.
[18, 20 *ku-*]*mur-ri še-im* gar-ra

The *waṣûm* is "posed"

12. 1 the *wāṣûm*[a] pose.
[1 *wa-ṣi*]*-am* pose

13. the igi *of* 30', the *bur, detach:* 2″; *to* 20', the *grain which he has collected*
igi 3[0 *bu-ri-im pu-ṭur-m*]*a* 2 *a-na* 20 *še-im ša im-ku-sú*

The specific rent of the first meadow is recalculated in sila/sar

14. raise, 40′, the false *grain; to* 10' *which* meadow over meadow *goes beyond*
íl 40 *še-um* l[ul *a-na* 1]0 *š*[*a* garim] u[-gù garim *i-te-r*]*u*

The rent R' of that part S' of the first meadow which exceeds the second meadow is found to be $R' = 6'\ 40°$. The remainder R'' of the total rent, $R'' = R_{i} + R_{ii} - R' = 11'\ 40°$, must then come from equal areas of the two meadows. Hence, a unit area is regarded; it is seen as composed of

15. raise, 6' 40°; *from* 18' 20°, the *accumulation of* the *grain*
íl 6, 40 *i-na* 18, 20 *ku-mur-ri še-im*

16. *tear out:* 11' 40° *you leave.*
ú-sú-uḫ-ma 11, 40 *te-zi-ib*

17. 11' 40° *which you have left, may your head retain.*
11, 40 *ša te-zi-bu re-eš-ka li-ki-il*

18. 1 the *wāṣûm to two break:* 30′.
1 *wa-ṣi-am a-na ši-na ḫi-pí-ma* 30

19. 30′ *and* 30′ *until twice* pose:
30 *ù* 30 *a-di ši-ni-šu* gar-ra-ma

$^1/_2$ sar from each meadow. These parts are "posed"

The specific rents r_{i} and r_{ii} are recalculated (r_{i} a second time) in sila/sar. The rents of the two halves of the unit sar are found, the first to be 20′

20. The igi *of* 30ʹ, the *bur, detach:* 2″; *to* 20ʹ, the *grain which he has collected*
igi 30 *bu-ri-im pu-ṭur-ma* 2 *a-na* 20 *še-im ša im-ku-sú*
21. raise, 40′; *to* 30′ *which until twice you have posed*
íl 40 *a-na* 30 *ša a-di ši-ni-šu ta-aš-ku-nu*
22. raise, 20′; *may your head retain.*
íl 20 *re-eš-ka li-ki-il*
23. The igi *of* 30ʹ, the *second bur, detach:* 2″.
igi 30 *bu-ri-im ša-ni-im pu-ṭur-ma* 2
24. 2″ *to* 15ʹ, the *grain which he has collected*
2 *a-na* 15 *še-im ša im-ku-sú*

the second to be 15′

25. raise, 30′; *to* the *second* 30′ *which you have posed* raise, 15′.
íl 30 *a-na* 30 *ša-ni-[i]m ša ta-aš-ku-nu* íl 15
26. 15′ *and* 20′ *which your head retains*
15 *ù* 20 *ša re-eš-ka ú-ka-lu*

Hence, the rent of the average unit sar is 35′. Since the total rent of the area $(S_{\text{i}}-S')+S_{\text{ii}}$ can be taken to come from such average sars $(S_{\text{i}}-S'=S_{\text{ii}})$, and since it is known to be $R''=11$ʹ $40°$, $(S_{\text{i}}-S')+S_{\text{ii}}$ can be found through division by 35′ to be 20ʹ.

27. accumulate: 35′; the *igûm I know not.*
gar-gar-*ma* 35 *i-gi-am ú-ul i-di*
28. *What to* 35′ *shall I pose*
mi-nam a-na 35 *lu-uš-ku-un*
29. *which* 11ʹ 40° *which your head retains gives me?*
ša 11, 40 *ša r[e-e]š-ka ú-ka-lu i-na-di-nam*
30. 20ʹ pose, 20ʹ *to* 35′ raise, 11ʹ 40° *will it give you.*
20 gar-ra 20 *a-[na]* 35 íl 11, 40 *it-ta-di-kum*

By error, this area 20ʹ is not bisected, which would give $S_{\text{i}}-S'$ and S_{ii}. Instead, it is confused with the area of the first meadow (which is indeed known in advance to be 20ʹ). S_{ii} is then found through the subtraction of $10' = S_{\text{i}}-S_{\text{ii}}$

31. 20ʹ *which you have posed is* the surface *of* the *first* meadow.
20 *ša ta-aš-ka-[nu* a-]šà garim *iš-te-at*
32. *From* 20ʹ the surface of the meadow, 10ʹ *which* meadow over meadow *goes beyond*
i-na 20 a-šà garim 1[0 *ša*] garim u-gù garim *i-t[e]-ru*
33. *tear out:* 10ʹ the surface *you leave.*
ú-sú-uḫ-ma 10 [a-šà *te-*]*zi-ib*

Reverse II

1–9 [contains a proof of no specific interest]

[a] *wāṣûm* is closely related to the *wāṣītum* of BM 13901, Nos 1, 2, 3 and 23 (cf. above, chapter V).

The basic conclusions could be repeated here: Once more, all more complicated steps in the calculation are chosen such that their results can be given a concrete meaning (and as before, simple transformations like that of bur/gur to sila/bur are performed without commentary). This time, however, there is direct and undamaged textual evidence for the correctness of the concrete interpretation given in the marginal commentary.[128] Firstly, of course, the 35′ of rev. I,27 must necessarily be the rent of an average sar; secondly, the rent of 20′ which corresponds to the semi-sar belonging to the first field is calculated with reference to "the bur", while the 15′ corresponding to the second field is calculated with explicit reference (in rev. I,23) to "the second bur", which all the way through belongs with the second field. The 35′ is clearly not the rent of an abstract average sar but that of a sar composed half from one and half from the other field.

This confronts us with a terminological problem: It appears that the bisection of rev. I,18 does not affect an area but instead a width of 1. Indeed, the *wāṣûm* which is already posed in rev. I,12, and which is later bisected, is nothing but the masculine form of the *wāṣītum* known from BM 13901, the width of 1 which transforms a length into an area of equal magnitude.

Evidently, the term is supposed by our author to refer to a familiar quantity. Like the bur, it is "posed" (in rev. I,12) for use in the calculation without being mentioned before among the given quantities.

The most obvious assumption is that the term means the same thing here as in the quadratic equations. If it does, we are provided with a clear exposition of the conceptualization of the calculation. The unknown area $(S_{\mathrm{I}}-S')+S_{\mathrm{II}}=s$ must be thought of as a rectangle of length s and width *1*. Half of it, of length s and width $^1/_2$ belongs to the first field, and the other half, of equal length and width, belongs to the second field. The 35′ should not then be thought of strictly as the rent of 1 average sar, but as the rent of 1 unit length (1 nindan) of the rectangle; similarly, the division of rev. II, 28–30 does not give us directly the area s, but instead the length s of the rectangle, and thereby implicitly its area.

The idea may seem strange to us. But a related conceptualization appears to lie behind the area unit eše (1 $\text{eše}_2 = 10^{\backprime}$ sar). It corresponds to a field of width "1 rope" (1 $\text{eše}_3 = 10$ nindan) and length 1` nindan; another unit, the "(area) nindan", has the same length but only the width 1 nindan.[129] Similar ideas are also found in Egyptian area metrology (1 "cubit of land" being a rectangle of width 1 cubit and length 100 cubit = 1 "reel of chord", while 1 "thousand of land" had the same length and a width of 1000 cubits[130]) and in Babylonian measures of volume (identifying units of area and volume by means of a standard height equal to 1 cubit). So, the whole idea may have been most concrete to a Babylonian scribe, and hence the identification of *wāṣûm* and *wāṣītum* can be considered reasonable.[131]

[128] An explanation of the procedure which as far as I know has been overlooked by all previous investigators of the text.

[129] See Powell 1972: 185 and passim.

[130] See Peet 1923: 24f.

[131] It can be observed that the length of the bur when applied to the width *waṣûm* = 1 nindan equals the largest Babylonian length measure, the danna (≈10.8 km), as it was pointed out independently of the present analysis by M. Powell at the Third Workshop on Concept Development in Mesopotamian Mathematics, Berlin, December 1985.

We remember that it is precisely the idea that a linear extension possesses a "standard width" of *1* nindan which permits us to see an area calculation as an operation of proportionality or scaling, and which thus gives conceptual unity to all applications of the term "raising" (cf. Fig. 3 and the discussion of the meaning of the term in section IV.3).

VII.3. TMS XVI, parts A and B (TMS, 92, cf. von Soden 1964)

The two preceding texts treated seemingly concrete (if surely not practical) problems of the first degree. The present texts are very different. They deal with the basic abstract length-width-representation, and they solve no problems[132]; instead, they present us with a didactical discussion of the meaning and the transformations of simple "equations of the first degree". They have been excavated in Susa (late Old Babylonian epoch), and they belong to a type not known from Babylonia itself. Maybe the need to fix didactical explanations in writing have to do with the fact that the texts represent a cultural import, no continuous autochthonous tradition; maybe the Susa excavators have simply had good luck where those working on (or looting!) Babylonian sites have not.

Although the two texts are mutually independent, they are so close to each other that both translations are best given together, before the commentary.

Part A

$(x=30,\ y=20)$ $x+y-{}^1/_4 y=45$	1. The *4th of* the width *from* the length and width to tear out, 45. You, 45 [4-*at* sag *i-na*] uš *ù* sag zi 45 za-e 45
$4\cdot(-''-)=3‘$	2. *to* 4 *raise,* 3‘ *you see.* 3‘, *what* is *that?* 4 *and* 1 pose. [*a-na* 4 *i-ší* 3 *ta*]-*mar* 3 *mi-nu šu-ma* 4 *ù* 1 gar
$x+y=50,\ {}^1/_4 y=5$ $4\cdot 5=[4\cdot {}^1/_4 y=]\ 1\cdot y$	3. 50 *and* 5, to tear out[a], pose. 5 *to* 4 *raise,* 1 width. 20 *to* 4 *raise* [50 *ù*] 5 zi ⌜gar⌝ 5 *a-na* 4 *i-ší* 1 sag 20 *a-na* 4 *i-ší*
$4\cdot 20=1‘\,20^\circ=4\cdot \mathrm{y}$ $4\cdot 30=2‘=4\cdot \mathrm{x}$ $1‘\,20^\circ-20=4\cdot\mathrm{y}-1\cdot\mathrm{y}$ $=1‘$	4. 1‘ 20° *you see,* 4 widths. 30 *to* 4 *raise,* 2‘ *you see,* 4 lengths. 20, 1 width to tear out, 1, 20 *ta-⟨mar⟩* 4 sag 30 *a-na* 4 *i-ší* 2 *ta-⟨mar⟩* 4 uš 20 1 sag zi
$2‘+1‘=[4\cdot]x+3\cdot y=3‘$	5. *from* 1‘ 20°, 4 widths, tear out, 1‘ *you see.* 2‘, lengths, *and* 1‘, 3 widths, ACCUMULATE, 3‘ *you see.* *i-na* 1, 20 4 sag zi 1 *ta-mar* 2 uš *ù* 1 3 sag UL.GAR 3 *ta-mar*
$4^{-1}=15'$ ${}^1/_4\cdot 2‘={}^1/_4\cdot([4\cdot]x)=30$ $=1\cdot x$	6. The igi of 4 *detach,* 15′ *you see.* 15′ *to* 2‘, lengths, *raise,* 30 *you see,* 30 the length igi 4 *pu-*[*ṭú-ú*]*r* 15 *ta-mar* 15 *a-na* 2 uš *i-ší* 3[0] *ta-⟨mar⟩* 30 uš

132 True enough, the mathematical commentary in TMS claims that they do, and even tries to make them do it, though with considerable violence to the texts.

$^1/_4 \cdot 1` = 15 = [^3/_4 \cdot]y$ $[1 \cdot x + ^3/_4 \cdot y =]\ 30 + 15$	7. 15′ *to* 1\` *raise*, 15 the *contribution*[b] *of* the width. 30 *and* 15 *retain*[c] (?). 15 *a-na* 1 *i-ši* [1]5 *ma-na-at* sag 30 *ù* 15 *ki-il*
The coefficient to y is found by an argument of type "single false position" to be $(4-1)/4 = 3/4 = ^1/_4 \cdot 3 = 15' \cdot 3 = 45'$	8. *Since* "the 4*th of* the width *to tear out*", *it has been said to you*[d], *from* 4, 1 tear out, 3 *you see.* *aš-šum* 4-*at* sag *na-sà-ḫu qa-bu-ku i-na* 4 1 zi 3 *ta-mar*
	9. The igi of 4 *detach*, 15′ *you see.* 15′ *to* 3 *raise*, 45′ *you see*, 45′ *as much as* (there is) of widths. igi 4 *pu-⟨ṭú-úr⟩* 15 *ta-mar* 15 *a-na* 3 *i-ši* 45 *ta-⟨mar⟩* 45 *ki-ma* [sag]
The coefficient to x is 1 (from line 6) The "width" y of the calculation is known to be 1 times the "true width" (of a figure?); hence $y = 1 \cdot 20 = 20$, and $45' \cdot y = 45' \cdot 20 = 15$, which when subtracted from $45 = 30 + 15$ leaves $30 = 1 \cdot x$	10. 1 *as much as* of lengths pose. 20 the true[e] width *take.* 20 *to* 1 *raise*, 20 *you see.* 1 *ki-ma* uš gar 20 gi-na sag *le-qé* 20 *a-na* 1 *i-ši* 20 *ta-mar*
	11. 20 *to* 45′ *raize*, 15 *you see.* 15 *from* $^{30}15$ *tear out*, 20 *a-na* 45 *i-ši* 15 *ta-mar* 15 *i-na* $^{30}15$ [zi]
	12. 30 *you see*, 30 the length. 30 *ta-mar* 30 uš

[a] TMS transcribes the beginning of this line as [50 ù] 5 ZI.A(!) ⟨GAR⟩ and interpretes ZI as a (phonetically motivated) writing error for SI, which would give the passage the meaning "50 *and* 5 which go beyond ⟨pose⟩". The supposed A is, however, damaged and clearly separated from the ZI. As far as I can see from the autography, the traces might as well represent the lacking GAR, which would give the reading [50 *ù*] 5 zi gar, "50 and 5, to tear out, pose". Not only is this in harmony with the actual text, it also has the clear advantage over the reading of TMS to be in agreement with the zi, "to tear out", of line 4, as well as with those of lines 1, 5 and 8. The latter of these, which is an explicit quotation of line 1, is written in syllabic Akkadian, excluding any error. It is also this quotation which shows that the zi is thought of as an infinitive, not as a finite form (cf. below, note d).

[b] "Contribution" translates *manātum*, an abstract noun derived from *manûm*, "to count". Etymologically, the meaning would be "the count"/"the counting". However, the term is found only here and in two other Susa texts (TMS XII and XXIV). In one of these, its use is unclear, in the other the term is isolated by a break. AHw suggests hypothetically an identification with Hebrew and Aramaic *m*ᵉ*nat*, which in HAHw (pp. 438 2–439 1) is exemplified by "Anteil der Priester und Leviten" and "d. Teil (Beitrag) des Königs". The ensuing "share/contribution of the widths" fits the present text excellently, and it is not contradicted by the other two occurrences.

[c] "Retain" is a conjecture (*ki*!-*il*!) due to von Soden (1964: 49). TMS has *ḫulum*, Assyrian for "way", interpreted as "method" by the editors.

[d] This quotation is very remarkable, since the ideographic zi is rendered syllabically by an indubitable infinitive, *na-sà-ḫu.*

TMS claims that an indubitable gi-na, "true", must be a writing error for *ki-ma, "as much as"*. If this were the case, *kīma* sag, "*as much as* of widths", would represent both the coefficient to the width (45′, in line 9) and the value of the width (20, in line 10)!

Part B

$(x=30,\ y=20)$ $(x-y)+{}^1/_4 y=15$	13. The *4th of* the width *to* that *which* length over width *goes beyond* to append 4-*at* sag *a-na ša* uš ugu sag *i-te-ru* daḫ
$4\cdot(-{}''-)=1`$	14. 15. *You*, 15 *to* 4 *raise*, 1 *you see*, *what* is *that?* 15 za-e 15 *a-na* 4 *i-ší* 1 *ta-mar mi-nu-šu-*[*ú*]
	15. 4 *and* 1 pose. {. . .} 4 *ù* 1 gar {15 *a-na* 4 *i-ší* 1 *ta-mar mi-*[*nu-šu-ú*]}
$x-y=10,\ {}^1/_4 y=5$	16. 15 *scatter*[a]. 10 the going-beyond *and* 5 the appended pose. 20 the width 15 *sú-pi-iḫ* 10 dirig *ù* 5 daḫ gar 20 sag
$(x-y)+y=10+20$ $=30=x$	17. *to* the going-beyond append; 30 the length, 20 to tear out pose.
$4\cdot{}^1/_4 y=4\cdot 5=20=y$	5 *to* 4 *raise*, *a-na* 10 dirig daḫ 30 uš 20 zi gar 5 *a-na* 4 *i-ší*
$4\cdot y=4\cdot 20=1`\ 20°$	18. 20 *you see*; 20, the width, *to* 4 *raise*, 1` 20° *you see.* 20 *ta-mar* 20 sag *a-na* 4 *i-ší* 1, 20 *t*[*a-mar*]
$4\cdot x=4\cdot 30=2`$	19. 30, the length, *to* 4 *raise*, 2` *you see*. 20, the width, 30 uš *a-na* 4 *i-ší* 2 *ta-mar* 20 sag
$4\cdot y-y=1`\ [=3\cdot y]$	20. *from* 1` 20° tear out, 1` [. . .] 1` *you see* [(3 widths(?)); 1`] *i-na* 1, 20 zi 1 [. . .] 1 *ta-mar* [. . . 1]
$4\cdot x-(4\cdot y-y)=$ $2`-1`=1`$	21. *from* 2`, lengths, tear out, 1` *you see*, *what is that?* [. . .] *i-na* 2 uš zi 1 *ta-mar mi-nu šu-ú* [. . .{1(?) *ta*} . . .]
The coefficient to y is found by an argument of type "single false position" to be $(4-1)/4=3/4={}^1/_4\cdot 3$ $=15'\cdot 3=45'$, the "negative" (i.e. subtractive) type of which is noted.	22. *From* 4, *of* the *fourth*, 1 tear out, 3 *you see*. The igi of 4 *detach*, 15′ *you see.* *i-na* 4 *ri-ba-ti* 1 zi 3 *ta-mar* igi 4 *pu*⟨*-ṭú-úr*⟩ 15 *ta-*[*mar* 23. 15′ *to* 3 *raise*, 45′ *you see*, *as much as* of widths pose. Pose to tear out[b]: [15′] *a-na* 3 *i-ší* 45′ *ta*⟨*-mar*⟩ *ki-ma* sag gaṛ gar[!] zi-*ma*
The coefficent to x is 1, $1\cdot x=1\cdot 30=30$	24. 1 *as much as* of lengths pose [. . .] 1 *take*, *to* 1 length 1 *ki-ma* u[š gar . . .] 1 *le-qé a-na* 1 uš
$45'\cdot y=45'\cdot 20=15$	25. [*raise*, 30 *you see* (. . .)][c]. 20 the width, 20 *to* 45′, widths, *raise*, [*i-ší* 30 *ta-mar* (. . .)] 20 sag 20 *a-na* 45 sag *i-ší*

$15+45' \cdot y=15+15$ $=30=x$	26. 15 *you see.* 15 *to* 15 append, 30 *you see,* 30 the length. [15 *ta-mar* 15] *a-na* 15 daḫ 30 *ta*⟨*-mar*⟩ 30 uš

[a] "To *scatter*" translates *sapāḫum,* "auflösen, zerstreuen" (the reading is due to von Soden–private communication, cf. 1964: 49). In fact, 15 is "scattered", i.e. analyzed into its constituent components 10 $(=x-y)$ and $5(={}^1/_4 y)$.

[b] TMS reads "4 zi-*ma*" and neglects the "4" in the translation, since this number gives no sense. Often GAR (=gar, "to pose") and 4 cannot be distinguished; so, we seem to be left with the choice between a formulation which makes no sense in its context, but which could have crept in by a copying error (the reading of TMS) and a reading which makes sense, and which possesses a parallel in line 17 (the present reading). However, close inspection of the autography shows an outspoken tendency to write GAR symmetrically, while 4 is normally written asymmetrically (as and , respectively). Only collation could decide whether the few exceptions are due to the scribe or the copying, and whether the difference reflects a different sequence of impression of the wedges. In any case, the problematic sign is as much a GAR as its left neighbour. So, the reading gar zi-*ma* appears to be established beyond reasonable doubt. Cf. also part A, line 3.

[c] TMS makes a different restitution, which presupposes that *laqûm,* "to take", is used synonymously with *našûm,* "to raise" as a term for multiplication. This presupposition is totally unsupported, and clearly contradicted by part A, line 10.

The present restitution is conjectural–only the "raise" required by the "to" seems secure. Possibly the restitution fills out the entire lacuna, possibly a few more signs can have found their place.

Both parts deal with a length of 30 and a width of 20, and this is supposed by the text to be known in advance[133], as are the sum of length and width, the excess of length over width, and the fourth of the width.

Part A leads off with an equation which in symbolic translation runs $x+y-{}^1/_4 y = 45$ and asks for the meaning of the $3^{\backprime}$ which result when the right-hand-side is multiplied by *4*. It then looks at the single members of the left-hand-side, multiplying each with *4*, explaining $4 \cdot 20 = 1^{\backprime}\, 20^{\circ}$ to be $4y$, $4 \cdot 30 = 2^{\backprime}$ to be $4x$, and $4 \cdot$ ([subtractive] $5) = 20$ to be a subtractive y (cf. below on this indication of sign). The result is $2^{\backprime} + (1^{\backprime}\, 20^{\circ} - 20^{\circ}) =$ (the required) $3^{\backprime}$.

Then, from line 6 onwards, the reverse operation is performed, but this time on the sum of $2^{\backprime} = 4x$ and $1^{\backprime} = 3y$. ${}^1/_4 \cdot 2^{\backprime} = 30$ is told to be simply x, while ${}^1/_4 \cdot 1^{\backprime} = 15$ is told to be the "contribution of y". In line 8f., the coefficient of y is calculated to be ${}^1/_4 \cdot (4-1) = 45'$, and it is given the name "as much as" (*kīma*) (there is) of widths. In line 10, the coefficient of x is stated to be *1*. Finally, the product of y

[133] According to TMS, only the width is known. Had this been the case, the operations of line A.3 would have proceeded conversely: From a width of 20 to its fourth (5), whence from 45 to 50. In part B, similar disagreements between the text and E. M. Bruins's assumption that the length be unknown can be pointed out.

and its coefficient is calculated and subtracted from the *45* of the right hand side (written as it was already analyzed in lines 6f.), and the remainder is seen to be equal to the length, as required.

Part B runs along similar lines, the main difference being perhaps that this time the analysis of the right hand side appears to be made verbally explicit as a "scattering" in line 16. "Contributions" and "coefficients" recur–the former, it is true, without the explicit label *manātum*.

For the sake of clarity, the operations can be organized schematically, as it is shown on the following page.[134] We observe that there is a close analogy between the Babylonian text and our own treatment of the corresponding equation. Not only the coefficients and the contributions but also the multipliers *1* and *4* of the left margin are stated explicitly. It seems, however, that most of the operations are supposed to be followed mentally: in part A, only the multipliers and the numbers *50* and *5* of line 8 are "posed", in a way which suggests written representation; all the rest is done rhetorically, or followed without notation on a graphic representation.

In the previous texts the concrete pattern of thought was noticed. A similar observation can be made here, both on the terminology used for contributions and coefficients and for the way the coefficients are calculated. In both parts, the coefficient of y is found by an argument of type "single false position" and not through the arithmetically simpler but more abstract calculation $1-{}^{1}/_{4}=$ $=1^{\circ}-15'=45'$. Similar patterns are found elsewhere in the material, e.g. in VAT 7532, rev. 6f. (MKT I, 295).

Even if concrete, the designation of the coefficient by a special expression can be considered a formalization of the "accounting technique" which was discussed above (section V.6). Another formalization of something which was done currently with or without formalization is the designation of certain numbers or entities as "subtractive", "to tear out" (in lines 3, 4, 17 and 23), written by the sumerogram zi. That we are really confronted with sort of sign is most clearly demonstrated by lines 4 to 5, where "*20, 1* width", is firstly given the epithet "to tear out", and afterwards really torn out.

zi is not only used to indicate subtractiveness but also for the subtractive operation ("tearing-out") itself, e.g. in line 1, as it is indicated by the preposition "from" (*ina*). It is an old issue whether such occurrences should be Akkadianized in transliterations. F. Thureau-Dangin did so without hesitation, regarding the sumerograms as pure logograms which were read by the scribes as grammatical Akkadian and which should hence be read so by us. He was so confident about this that he did not indicate the sumerogram parenthetically, as it is done in e.g. TMS. O. Neugebauer, on the other hand, claimed that the ideograms functioned as mathematical operators, not as words belonging to current language (see e.g. MKT I, viii). Line 8 of part A shows that O. Neugebauer was at least partly right: The statement is quoted, but the ideographic writing zi is rendered in phonetic writing as an infinitive, *na-sà-ḫu* (the text is written without "mimation", the final *m* of nouns and nominal verbal forms which was gradually dropped).

[134] The symbolic schematization of part A was proposed to me by P. Damerow at the First Workshop on Concept Development in Mesopotamian Mathematics, Berlin 1983, where I first presented my interpretation of the text.

At least the term zi must, at least in the Susa school, have been regarded as an ideogram for an abstract mathematical operation, not as a logogram to be provided with correct grammatical pre- and suffixes when read.

Indications exist that the restrictions to zi and to the Susa school are superfluous. Indeed, if íb-si$_8$ were read *mitḫartum* (as claimed by F. Thureau-Dangin), how are we to understand changes in the ideographic expression following Sumerian homophonic patterns (íb to ib, si$_8$ to si)? How are we to explain the use in certain texts (among which IM 52301, see below, section X.1) of a term *basûm*, evidently an Akkadianized pronunciation of ba-si$_8$? What are we, finally, to do about the distinction between the Akkadianization *igûm*, the table value, and igi, the abstract reciprocal number? It appears that certain Sumerograms were (at least in certain text-types, among which the compactly written series texts must be reckoned) regarded as ideograms, that they were sometimes read in Sumerian and sometimes Akkadianized without proper inflection in person and tense.[135]

A final observation on the text concerns part A, line 10f. Both the formulation and the actual calculation are conspicuous. Why is the width spoken of as a "true width"? And why is 45′ widths calculated not as 20 raised to 45′ but in two steps, the true width being first raised to 1, and the result next raised to 45′?

The immanent analysis of the text provides us with no answer; below we shall see how at least a suggestion can be found in the texts BM 13901 N° 14 and TMS IX (sections VIII.1 and VIII.3, respectively)–a suggestion which appears to be confirmed in TMS XIX (cf. below, note 176).

Symbolic and graphic schematization of the operations

$$\begin{array}{lcl}
\alpha & & \left\{\begin{array}{l} 1x + \underbrace{1y - {}^1/_4y} = 45 \\ 1x + 45'y = 45 \\ \underbrace{30 + 20} - 5 = 45 \\ 50 \underbrace{- 5} = 45 \\ 30 + 15 = 45 \end{array}\right. \\
\beta & & \\
\gamma & 1 & \\
\delta & & \\
\delta' & &
\end{array}$$

....

$$\begin{array}{lcl}
\varepsilon & & \left\{\begin{array}{l} 4x + \underbrace{4y - 1y} = 3' \\ 4x + 3y = 3' \\ 2' + \underbrace{1^\circ 20' - 1} = 3' \\ 2' + 1' = 3' \end{array}\right. \\
\zeta & 4 & \\
\eta & & \\
\vartheta & &
\end{array}$$

Apparently, the 1 and 4 posed in line 2 of the text are the factors written to the left of the two groups of equations. The rest of part A discusses the relations between the lines α to ϑ.

It is seen that α represents the original equation of "lengths" and "widths", written symbolically, while ε is obtained from this original equation through multiplication

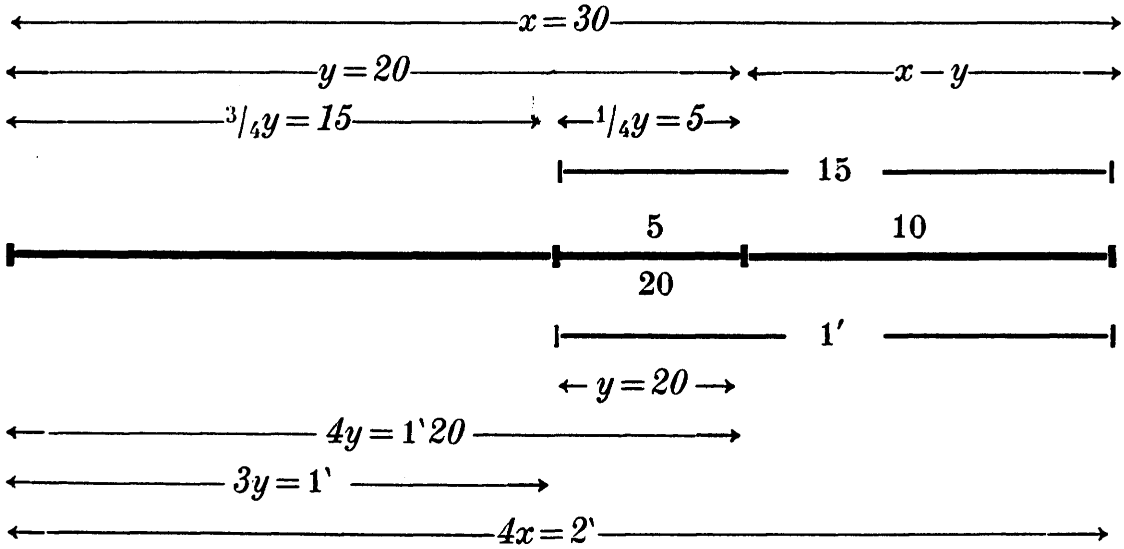
[135] On other occasions we are of course forced to acknowledge some Sumerograms as logograms for proper Akkadian, — viz. when they are provided with Akkadian phonetico-grammatic complements. Cf. note a to BM 13901, No 23 (section V.4).

by 4. γ and η represent the same equations when the known values of length and width are inserted.

In the text, line 3 "poses" the 50 and 5 of γ, representing 5 as "that which is torn out" (from 50). Next (line 3–5), the transformation of γ into η is explained term for term in order to solve the problem raised in line 2: which meaning to ascribe to the 3ˋ which arise when the right-hand side of α is multiplied by 4. This is done with reference to ε, ζ and ϑ.

Line 6f. explains the reverse transformation η to *γ, referring to δ', where the respective contributions of lengths and widths are separated. Line 8–12, finally, explains δ' in terms of β where the coefficients of x and y, i.e. "as much as there is" of lengths and widths, are found and multiplied by the numerical value of these entilies.*

Instead of this symbolic schematization, a graphic scheme could also be used. For the sake of variation we shall apply it to part B, which to a first glance seems somewhat more opaque than part A, but which turns out to be very simple in graphic representation:



Once again, the upper half of the scheme corresponds to the original equation and the lower half to the multiplication by four.

The steps of the text are easily demonstrated at the scheme. Evidently, an oral representation would not need the many lines drawn here. The heavy line in the middle could do, if only the teacher pointed out in each step which segment was spoken of now. While the symbolic scheme is of course anachronistic as a mapping of the text, the graphic representation may thus be close to what actually went on in the Susa school.

A graphic interpretation of part A will be found in my 1989: 24.

VIII. Combined second-degree problems

In chapter V, a number of simple second-degree problem texts were presented and discussed, and in chapter VII we had a look at some very concrete first-degree problems. Together, the two chapters might convey the impression that Babylonian mathematics was not only concrete in its cognitive orientation but also simple, not to say simplistic. In order to counteract at least in part this misleading impression the present chapter shall present a couple of texts which combine the first- and second-degree techniques in various ways, demonstrating a bit of the sophistication to which Babylonian algebra was able to rise while remaining concrete and "naive". The last section of the chapter presents another didactical Susa text, which builds the bridge from simple to more sophisticated second-degree algebra.

VIII.1. BM 13901, Nº 14 (MKT III, 3; cf. TMB, 5)

Several other problems from the same tablet were already presented above in Chapter V. The present problem contains yet another problem of squares, this time in two variables connected through a simple inhomogeneous equation of the first degree. Through substitution and use of the accounting technique, the problem is reduced to that dealt with in section V.5 and solved by the same procedure.

	Obverse II
$x^2+y^2=25^{\backprime}\,25^\circ$	44. The surfaces *of my two confrontations I have accumulated:* 25‘ 25°. a-šà *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* ⸢25,]25
$y={}^2/_3x+5$	45. The *confrontation, two-third of* the *confrontation and* 5 nindan *mi-it-ḫar-tum ši-ni-pa-at mi-it-ḫar-tim* [*ù* 5 ninda]n
$x=1\cdot z$ $y=40'\cdot z+5$	46. 1 *and* 40′ *and* 5 *overgoing* the 40′ *you inscribe.* 1 *ù* 40 *ù* 5 [*e-le-nu* 4]0 *ta-la-pa-at*
	47. 5 *and* 5 *you make span,* 25 *inside of* 25‘ 25° *you tear out:*[a]
$x^2=1^2\cdot z^2=1\cdot z^2$	5 *ù* 5 [*tu-uš-ta-kal* 25 *lìb-bi* 25, 25 *ta-na-sà-aḫ-ma*]
	Reverse I
$y^2=(40'\cdot z+5)^2$ $=26'\,40''\cdot z^2$ $+2\cdot 40'\cdot 5\cdot z+25$	1. 25‘ *you inscribe.* 1 *and* 1 *you make span,* 1. 40′ *and* 40′ *you make span,* [25 *ta-la-pa-at* 1 *ù* 1 *tu-uš-ta-kal* 1 40 *ù* 40 *tu-uš-ta-kal*]
$1^\circ\,26'\,40''\cdot z^2$ $+2\cdot 5\cdot 40'\cdot z=25^{\backprime}$ Putting $Z=1^\circ\,26'\,40''\cdot z$ we get when multiplying by $1^\circ\,26'\,40''$	2. 26′ 40″ *to* 1 *you append:* 1° 26′ 40″ *to* 25‘ *you raise:* [26, 40 *a-na* 1 *tu-ṣa-ab-ma* 1, 26, 40 *a-na* 25 *ta-na-ši-ma*]
$Z^2+2\cdot 5\cdot 40'\cdot Z$	3. 36‘ 6° 40′ *you inscribe.* 5 *to* 40′ *you raise:* 3° 20′ [36, 6, 40 *ta-la-pa-at* 5 *a-na* 4]0 *t*[*a-na-ši-ma* 3, 20]
$=Z^2+2\cdot 3^\circ\,20'\cdot Z$ $=1^\circ\,26'\,40''\cdot 25^{\backprime}$ $=36^{\backprime}\,6^\circ\,40'$ $(Z+3^\circ\,20')^2=36^{\backprime}\,6^\circ\,40'$	4. *and* 3° 20′ *you make span,* 11° 6′ 40″; *to* 36‘ 6° 40″ *you append:* [*ù* 3, 20 *tu-uš-ta-kal* 11, 6, 40] *a-na* 3[6], 6, 40 [*tu-ṣa-ab-ma*]

$+(3°\,30')^2$ $=36`\,17`\,46'\,40''$	5. 36` 17° 46′ 40″ makes 46° 40′ equilateral. 3° 20′ *which you have made span* [36, 17, 46, 40-e 46, 40 íb-si₈ 3,]20 *ša tu-uš-ta-ki*[*-lu*]
$Z+3°\,20'=\sqrt{36`\,17°\,46'\,40''}$ $=46°\,40'$	
$Z=46°\,40'-3°\,20'$ $=43°\,20'$	6. *inside of* 46° 40′ *you tear out:* 43° 20′ *you inscribe* [*lìb-bi* 46, 40 *ta-na-sà-aḫ-*]*ma* 43, 20 *ta-la-pa-a*[*t*]
$1°\,26'\,40''\cdot z=43°\,20'$,	7. The igi of 1° 26′ 40″ *is not detached. What to* 1° 26′ 40″ [igi 1, 26, 40 *ú-la ip-pa-ṭ*]*a-ar mi-nam a-na* 1,2[6, 40]
$1°\,26'\,40''\cdot 30=43°\,20'$ whence $z=30$	8. *shall I pose which* 43° 20′ *gives me?* 30 *its bandûm*[b]. [*lu-uš-ku-un ša* 43, 20 *i-n*]*a-di-nam* 30 *ba-an-da-šu*
$x=1\cdot z=1\cdot 30=30$	9. 30 *to* 1 *you raise:* 30 the *first confrontation.* [30 *a-na* 1 *ta-na-ši-ma* 30] *mi-it-ḫar-tum iš-ti-a-at*
$y=40'\cdot z+5$ $=40'\cdot 30+5$ $=20+5=25$	10. 30 *to* 40′ *you raise:* 20; *and*[c] 5 *you append:* [30 *a-na* 40 *ta-na-ši-ma* 20] *ú* 5 *tu-ṣa-ab-ma* 11. 25 the *second confrontation.* [25 *mi-it-ḫar-t*]*um ša-ni-tum*

[a] From obv. II, 47 to rev. I, 5, only a few signs are preserved; from rev. I, 6 to 11, c. half of each line is preserved. In spite of this, the reconstruction (due to Thureau-Dangin 1936a, taken over in MKT III, 3) appears to be subject to very little doubt, thanks to the closely related No 24 of the same tablet.

[b] Probably a Sumerian loanword (cf. AHw, 102); is it also found in rev. I, 35 of the same tablet, where the numerical value of the entity is $^1/_4$. The mathematical function of the term is obvious, the factor to be multiplied unto 1° 26′ 40″ if we are to obtain the product 43° 20′. The general meaning of the term is unclear, but could perhaps be "that which is to be given together with" (ba, "to allot" etc.; -da, comitative suffix < "side").

[c] Both F. Thureau-Dangin and O. Neugebauer interprete this passage as "20 and 5 you append". Only here, however, and in two strictly parallel passages (rev. II, 31 and 32) is "append" found together with an "and". It is obviously the "and 5 nindan" of obv. II, 45 which gives rise to the present "and" (while corresponding statements in rev. II, 18f. give rise to the other occurrences of the construction). This suggests the interpretation given here. The observation made in note c to VAT 8389 N° 1 (section VII. 1) supports the interpretation, especially because the use of the agentive suffix -e after results in a number of places in the present tablet suggests that results are even here to be understood as nominatives (the natural Akkadian understanding of the Sumerian agentive, the subject case for transitive verbs only).

This calls for various observations. On the one hand the operations correspond precisely to those of a modern solution to the same problem, or to those of a Medieval rhetorical solution. The Babylonians were as fully able to reduce the problem to a basic type as were the Islamic algebrists or their more recent descendants, in spite of their concrete and geometric way of thought. On the other hand, the concrete and geometric method is present all the way through, not only in the final reduction of the basic problem $\alpha x^2+\beta x=\gamma$ (rev. I, 2–9). The squaring of $(40' \cdot x+5)$ appears to be imagined geometrically (cf. Fig. 11): $40' \cdot 40'$ and $5 \cdot 5$ are made by "spanning", while the coefficient $5 \cdot 40'$ (an operation of proportionality, replacing "5 confrontations" by "$(40' \cdot 5)$ confrontations") is performed as a "raising". Great care is taken to take the factor *1* into account and to square it (rev. I, 1 and 9); the reduction to basic type, finally, avoids the unnecessary step to find the total number of "confrontations", which anyhow would have to be bisected.

If we go a bit closer to the text, we notice that the problem is reduced to the basic type of BM 13901 No 3 (section V.5); but the unknown "confrontation" of this reduced problem is not identical with the greater "confrontation" of the problem. Instead, the two confrontations of the problem are 1 times this unknown and 40′ times the unknown plus 5, respectively (this is why the symbolic trans-

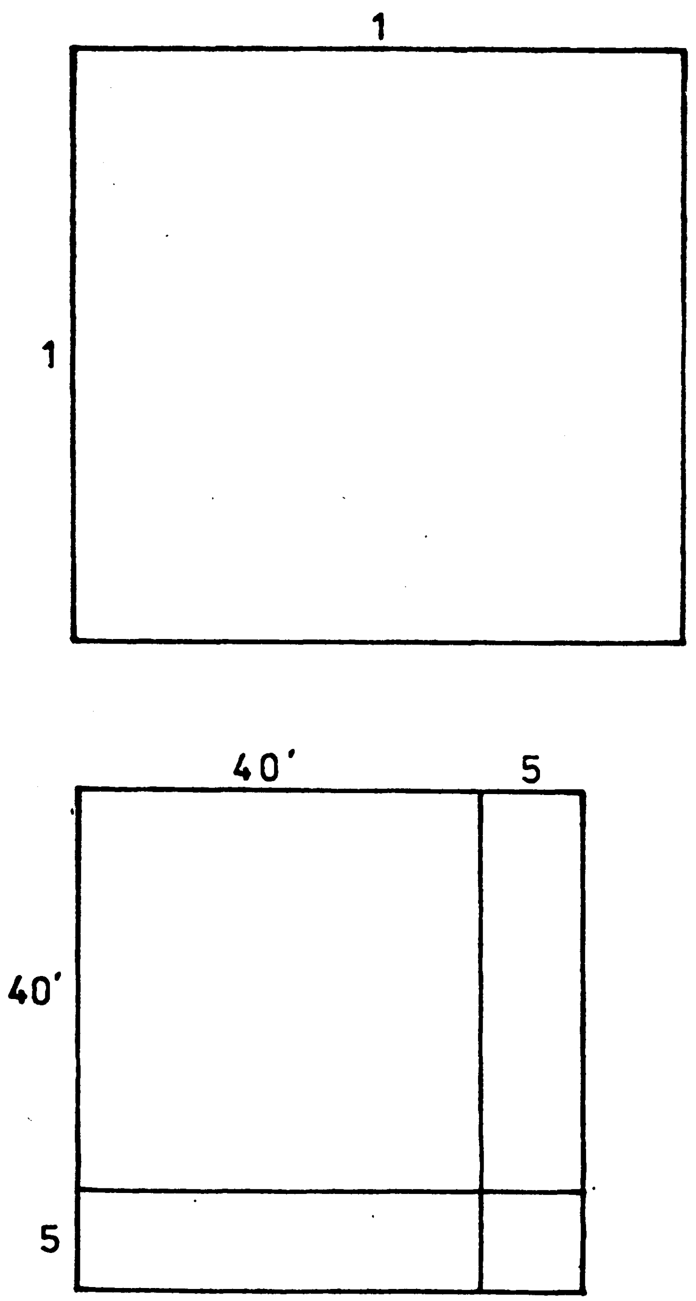


Figure 11. The two "confrontations" of BM 13901 Nº 14, with 1, 40′ and 5 "inscribed", as stated in obv. II, 6.

lation in the left margin introduces a variable z). An analogous distinction between a "true width" and a "width" obtained through a multiplication by *1* could be found in TMS XVI A, line 10. In both cases, the distinction can be said to be a distinction between an original problem and its "basic representation". In the present case, as mostly when concrete entities are represented, the representing entities are not mentioned by any name; we can only see from the calculational steps that a specific basic type is dealt with (here that of N° 3 of the same tablet; cf. section V.5).

VIII.2. AO 8862 N^{os} 1–3 (MKT I, 108–111)

Like BM 13901, this tablet belongs to the earliest documented phase of Old Babylonian algebra. The first three sections deal with problems of essentially the same structure ($x+y=S$, $xy+\alpha x+\beta y=A$) and might have been solved slavishly by the same procedure. Instead, however, N^{os} 1 and 2 make use of the same principle but apply it differently, while N° 3 goes quite different ways. The three problems taken together thus constitute a fine demonstration of the flexibility of Babylonian algebraic procedures.–Had Babylonian mathematics been nothing but a collection of standardized recipes, everything on the tablet had looked differently.

N° 1 was also the first Babylonian algebraic text for which a geometrical explanation was given, viz. by K. Vogel as early as 1933.[136] Finally, the problems are interesting because of various details in the formulations. As these details can all be demonstrated on N^{os} 1–2, I restrict the translation to these two problems, and explain N° 3 only in symbolic and geometric interpretation.

N° 1	I
	1. Length, width[a]. Length *and* width *I have made span:* uš sag uš *ù* sag *uš-ta-ki-il*$_5$*-ma*
	2. a sur*face I have built* a-šà^{lam} *ab-ni-i*
	3. *I went around* (it). *So much as* length *over* width *as-sà-ḫi-ir ma-la* uš *e-li* sag
	4. *goes beyond* *i-te-ru-ú*
$x\cdot y+(x-y)=3'\ 3°$	5. *to* the *inside of* the sur*face I have appended* *a-na li-ib-bi* a-šà^{lim} *ú-ṣi-ib-ma*
	6. 3' 3°. *I turned back.* Length *and* width 3, 3 *a-tu-úr* uš *ù* sag
$x+y=27$	7. I have accumulated: 27. Length, width *and* surface *what?* gar-gar-*ma* 27 uš sag *ù* a-šà *mi-n[u-u]m*

[136] Vogel 1933: 79, in a comment upon Neugebauer 1932a.

b

27	3\` 3°	*things accumulated*
15	length	3\` surface
12	width	
27	3, 3	*ki-im-ra-tu-ú*
15	uš	3 a-šà
12	sag	

	8. *You, by your making,* *at-ta i-na e-pe-ši-i-ka*
	9. 27, the *things accumulated of* length *and* width 27 *ki-im-ra-at* uš *ù* sag
$xy+(x-y)+(x+y)$ $=xy+2x=x\cdot(y+2)$	10. *to* the *inside of* 3\` 3° *append:* *a-na li-bi* [3, 3] *ṣi-ib-ma*
$=3$\` $30°$ $x+(y+2)=29$	11. 3\` 30°. 2 *to* 27 *append:* 3, 30 2 *a-na* 27 *ṣi-ib-ma*
Putting $Y=y+2$: $xY=3$\` $30°$, $x+Y=29$	12. 29. *Its* MOIETY, *that of* 29, *you break:* 29 BA.A-*šu* *š*[*a*] 29 *te-ḫe-ep-pe-e-ma*
$\left(\frac{x+Y}{2}\right)^2=14°\,30'^2$ $=3$\` $30°\,15'$	13. 14° 30′ steps of 14° 30′, 3\` 30° 15′. 14, 30 a-rá 14, 30 3, 30, 15
$\left(\frac{x-Y}{2}\right)^2=\left(\frac{X+Y}{2}\right)^2-xY$ $=15'$	14. *From* the *inside of* 3\` 30° 15′ *i-na li-bi* 3, 30, 15
	15. 3\` 30° *you tear out:* 3, 30 *ta-na-sà-aḫ-ma*
$\frac{x-Y}{2}=\sqrt{15'}=30'$	16. 15′ the *remainder.* 15′ makes 30′ equilateral 15 *ša-pi-il₅-tum* 15-e 30 íb-[si₈]
$x=\frac{x+Y}{2}+\frac{x-Y}{2}$ $=14°\,30'+30'=15$	17. 30′ *to* the *first* 14° 30′ 30 *a-na* 14, 30 *iš-te-en*
	18. *append:* 15 the length. *ṣi-ib-ma* 15 uš
$Y=\frac{x+Y}{2}-\frac{x-Y}{2}$ $=14°\,30'-30'=14$	19. 30′ *from* the *second* 14° 30′ 30 *i-na* 14, 30 *ša-ni-i*
	20. *you cut off:* 14 the width *ta-ḫa-ra-aṣ-ma* 14 sag
	21. 2 *which to* 27 *you have appended* 2 *ša a-na* 27 *tu-uṣ₄-bu*
$y=Y-2=12$	22. *from* 14, the width, *you tear out:* *i-na* 14 sag *ta-na-sà-aḫ-ma*
	23. 12 the true width. 12 sag gi-na
Proof: $x\cdot y=15\cdot 12=3$\`	24. 15, the length, 12 the width, *I have made span:* 15 uš 12 sag *uš-ta-ki-il₅-ma*
	25. 15 steps of 12, 3\` the surface. 15 a-rá 12 3 a-šà

26\. 15, the length, *over* 12, the width,
15 uš *e-li* 12 sag

$x - y = 3$

27\. by *what goes beyond*?
mi-na wa-ta-ar

$xy + (x - y) = 3' + 3$
$= 3'\ 3°$

28\. 3 *it goes beyond*; 3 *to* the *inside of* 3', the surface, *append*,
3 *i-te-er* 3 *a-na li-bi* 3 a-šà *ṣi-ib*

29\. 3' 3° the surface.
3, 3 a-šà

[a] F. Thureau-Dangin translated "length, width" (uš sag) simply as "rectangle" (e.g. TMB, 64). That this is indeed the correct interpretation of the composite expression is confirmed by the Susa table of constants (TMS III, 32), which speaks of the "diagonal of length and width", meaning the diagonal of a standard rectangle of sides 45′ and 1.

[b] This arrangement of the statement between lines 7 and 8 follows the autography (MKT II, plate 35).

N° 2

I

30\. Length, width. Length *and* width
uš sag uš *ù* sag

31\. *I have made span:* A surf*ace I have built.*
uš-ta-ki-il₅-ma a-šà*lam* *ab-ni*

32\. *I went around* (it). The *half of* the length
a-sà-ḫi-ir mi-ši-il₅ uš

33\. *and* the *third of* the width
ù ša-lu-uš-ti sag

34\. *to* the *inside of my* surface
a-na li-bi a-šà*-ia*

$xy + {}^1/_2x + {}^1/_3y = 15$

35\. *I have appended:* 15.
[*ú-*]*ṣi-ib-ma* 15

36\. *I turned back.* Length *and* width
[*a-t*]*u-úr* uš *ù* sag

$x + y = 7$

37\. *I have accumulated:* 7.
[*ak-*]*mu-ur-ma* 7

II

1\. Length *and* width *what*?
uš *ù* sag *mi-nu-um*

2\. *You, by your making,*
at-ta i-na e-pe-ši-i-ka

3\. 2 (as) *inscription of* the *half*
[2 *n*]*a-al-p*[*a*]*-at-ti mi-iš-li-im*

4\. *and* 3 (as) *inscription*
[*ù*] 3 *na-al-pa-ti*

	5. *of the third you inscribe:* *[ša-]lu-uš-ti ta-l[a]-pa-at-ma*
	6. The igi of 2, 30′, *you detach:* igi 2-bi 30 *ta-pa-ṭar-ma*
$^1/_2 \cdot (x+y) = 3° 30'$	7. 30′ steps of 7, 3° 30′; *to* (the place of) 7, 30 a-rá 7 3, 30 *a-na* 7
	8. (of) the *things accumulated*[a], length *and* width, *ki-im-ra-tim* uš *ù* sag
	9. *I bring:* *ub-ba-a[l]-ma*
$xy + {}^1/_2 x + {}^1/_3 y - {}^1/_2(x+y)$ $= xy - ({}^1/_2 - {}^1/_3)\, y$ $= 11° 30'$	10. 3° 30′ *from* 15, *my things accumulated* 3, 30 *i-na* 15 *ki-i[m]-ra-ti-i-a*
	11. *cut off:* *ḫu-ru-uṣ₄-ma*
	12. 11° 30′ the *remainder.* 11, 30 *ša-pi-il₅-tum*
$^1/_2 - {}^1/_3 = 1/(2 \cdot 3)$ $= 1/6 = 10'$	13. *Go not beyond.* 2 *and* 3 *I make span:* *l[a] wa-t[ar]* 2 *ù* 3 *uš-ta-kal-ma*
	14. 3 steps of 2, 6. 3 a-rá 2 6
Putting $X = x - 10'$ we have	15. The igi of 6, 10′ *it gives you.* igi 6 gál 10 *i-na-di-kum*
$Xy \quad = 11° 30'$ $X + y = 7 - 10' = 6° 50'$	16. 10′ *from* 7, *your things accumulated*[b] 10 *i-na* 7 *ki-im-ra-ti-i-ka*
	17. of length *and* width *I tear out:* uš *ù* sag *a-na-sà-aḫ-ma*
	18. 6° 50′ the *remainder.* 6, 50 *ša-pi-il₅-tum*
$\frac{X+y}{2} = 3° 25'$	19. *Its* MOIETY, *that of* 6° 50′, *I break:* BA.A-*š[u] ša* 6.50 *e-ḫe-pe-e-ma*
	20. 3° 25′ *it gives you.* 3, 25 *i-na-di-ku*
	21. 3° 25′ *until twice* 3, 25 *a-di ši-ni-šu*
$\left(\frac{X+y}{2}\right)^2 = 11° 40' 25''$	22. *you inscribe:* 3° 25′ steps of 3° 25′, *ta-la-pa-at-ma* 3.25 a-rá 3, 25
$\left(\frac{X-y}{2}\right)^2 = \left(\frac{X+y}{2}\right)^2 - Xy$ $= 10' 25''$	23. 11° 40′ 25″; *from* the *inside* 11, 40, [25] *i-na li-bi*
	24. 11° 30′ *I tear out* 11, 30 *a-na-sà-aḫ-ma*
$\frac{X-y}{2} = \sqrt{10' 25''} = 25'$	25. 10′ 25″ the *remainder.* ⟨10′ 25″ makes 25′ equilateral⟩ 10, 25 *ša-pi-il₅-tum* ⟨10′ 25″-e 25′ íb-si₈⟩

$X=\frac{X+y}{2}+\frac{X-y}{2}$	26. *To* the *first* 3° 25′ *a-na* 3, 25 *iš-te-en*
$= 3° 25' + 25' = 3° 50'$	
	27. 25′ *you append:* 3° 50′, 25 *tu-ṣa-am-ma* 3, 50
$x = X + 10'$ $= 3° 50' + 10' = 4$	28. *and* (that) *which from* the *things accumulated of* *ù ša i-na ki-im-ra-at*
	29. length *and* width *I have torn out* uš *ù* sag *a[s]-sà-aḫ-ma*
	30. *to* 3° 50′ *you append:* *a-na* 3, 50 *tu-ṣa-am-ma*
$y=\frac{X+y}{2}+\frac{X-y}{2}$	31. 4 the length. *From* the *second* 3° 25′ 4 uš *i-na* 3, 25 *ša-ni-im*
$= 3° 25' - 25' = 3$	
	32. 25′ *I tear out:* 3 the width. 25 *a-na-sà-aḫ-ma* 3 sag
	32a.[c] 7 the *things accumulated* 7 *ki-im-ra-tu-ú*

32b.	4	length	12	surface
	3	width		
	4	uš	12	a-šà
	3	sag		

[a] Since *kimrātum* is written in the status rectus (*ki-im-ra-tim*) and not in status constructus, "length and width" must stand (in this single case) as an apposition, not as the second member of a genitive construction. Hence the translation.

[b] In most of its occurrences, *kimrātum* stands so that it cannot be decided whether a (most peculiar) singular feminine *kimratum* or a plural *kimrātum* is meant. The indubitable plural of II, 32a could at a pinch be explained away (F. Thureau-Dangin, TMB, 67, does so, translating "7 ⟨et 15⟩, les sommes"). In II, 16, however, there can be no doubt that a single sum is spoken of in the plural, as *ki-i[m]-ra-ti-i-ka*. The *ki-i[m]-ra-ti-i-a* of II, 10 is also a most certain plural.
It is noteworthy that the singular form to be expected from the plural (*kimirtum*) is completely absent from the texts. It appears to be established beyond reasonable doubt that the single sum is designated by the plural form (and hence to the plurality of addends), as presupposed in my standard translation.

[c] This ordering follows the autography (MKT II, plate 36). There is no doubt that 32a is meant as a separate line, while the rest (32b) stands as a tabulation.

Designating as usual the length as x and the width as y we can finally transcribe problem 3 as follows:

$$xy + (x-y)\,(x+y) = 1'' \; 13' \; 20° \qquad x+y = 1' \; 40°$$

and from the way the solution is formulated is is clear that the author was aware that this was equivalent to

$$x+y=1'\ 40^\circ \qquad xy+1'\ 40^\circ \cdot (x-y)=1''\ 13'\ 20^\circ$$

which could easily be reduced to a standard problem $Xy=A$, $X+y=B$ by the method already known from N^{os} 1–2. Instead, however, the following steps occur:

$$(x+y)^2=2''\ 46'\ 40^\circ$$
$$(x+y)^2-xy-(x+y)\,(x-y)=1''\ 23'\ 20^\circ$$

which, putting $x+y=1'\ 40^\circ=a$, reduces to

$$y^2+ay=1''\ 33'\ 20^\circ, \text{ whence}$$
$$\left(y+\frac{a}{2}\right)^2=1''\ 33'\ 20^\circ+(1'\ 40^\circ/2)^2=2''\ 15'$$
$$y+\frac{a}{2}=y+\frac{x+y}{2}=\sqrt{2''\ 15'}=1'\ 30^\circ$$
$$\frac{x-y}{2}=(x+y)-\left(y+\frac{x+y}{2}\right)=1'\ 40^\circ-1'\ 30^\circ=10$$

and so finally

$$x=\frac{x+y}{2}+\frac{x-y}{2}=50+10=1'$$
$$y=\frac{x+y}{2}-\frac{x-y}{2}=50-10=40$$

– all of it formulated of course the usual way. The procedure is fully correct, but it looks rather queer in the above symbolic transcription.

First of all the construction of the three problems should be noted. Invariably, a surface is "built", after which the teacher "goes around". As A. Westenholz first suggested to me the text looks like a tale about real surveying: The teacher-surveyor marks out a field (the everyday meaning of a-šà and *eqlum*, we remember) in the terrain, after which he goes around it, pacing off its measures. Only after this walk, indeed, do numbers enter the text, as if, e.g., the excess of length over width is only known now. Using his newly acquired knowledge, the surveyor joins some extra areas to the field–"appending", we observe, not "accumulating" as when measures of sides and surfaces were added in BM 13901. This must of course be done in the terrain, from which he then turns back in order to state the sum ("accumulation") of length and width.

After this observation we shall look at the procedures which appear to be used to solve the three problems. The steps of problem 1 can be easily followed on Fig. 12. The simple addition of one length and one width (regarded as rectangles of width *1*, which is not said explicitly) transforms the irregular surface of area 3' 3° into a rectangle of which the area and the sum of length (x) and width (Y) are known. A bisection of this known length $x+Y=29$, to which the rectangle $x \cdot Y$ is "applied with defect", allows us to reconstruct the rectangular area as a gnomon. The area and hence the side of the small square enclosed by this gnomon are found, and the original dimensions of the rectangle $x \cdot Y$ follow as usual. In this way, everything labelled "length", "width" or "surface" is indeed a length, a width or a surface.

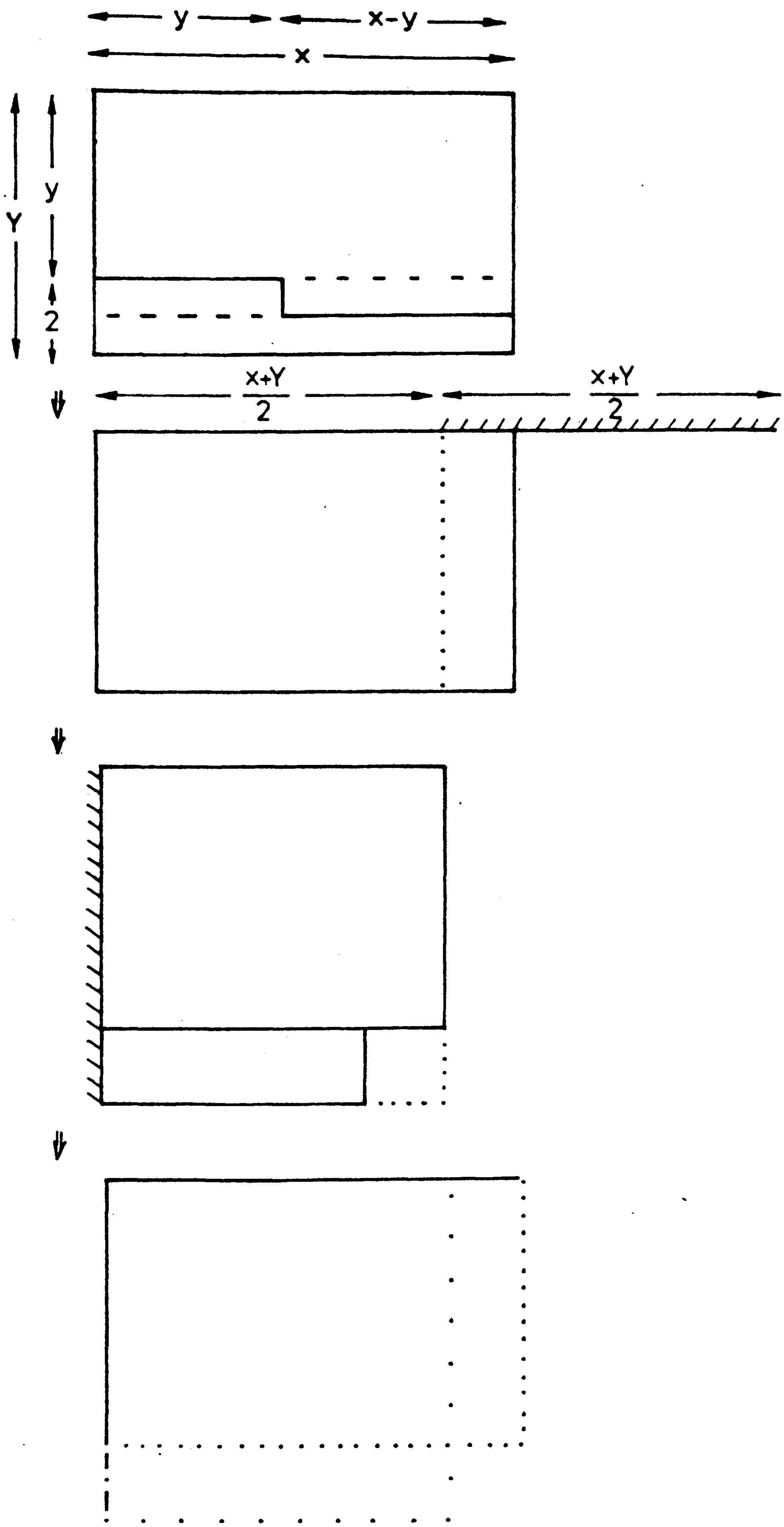


Figure 12. The geometrical interpretation of AO 8862 Nº 1. Distorted proportions.

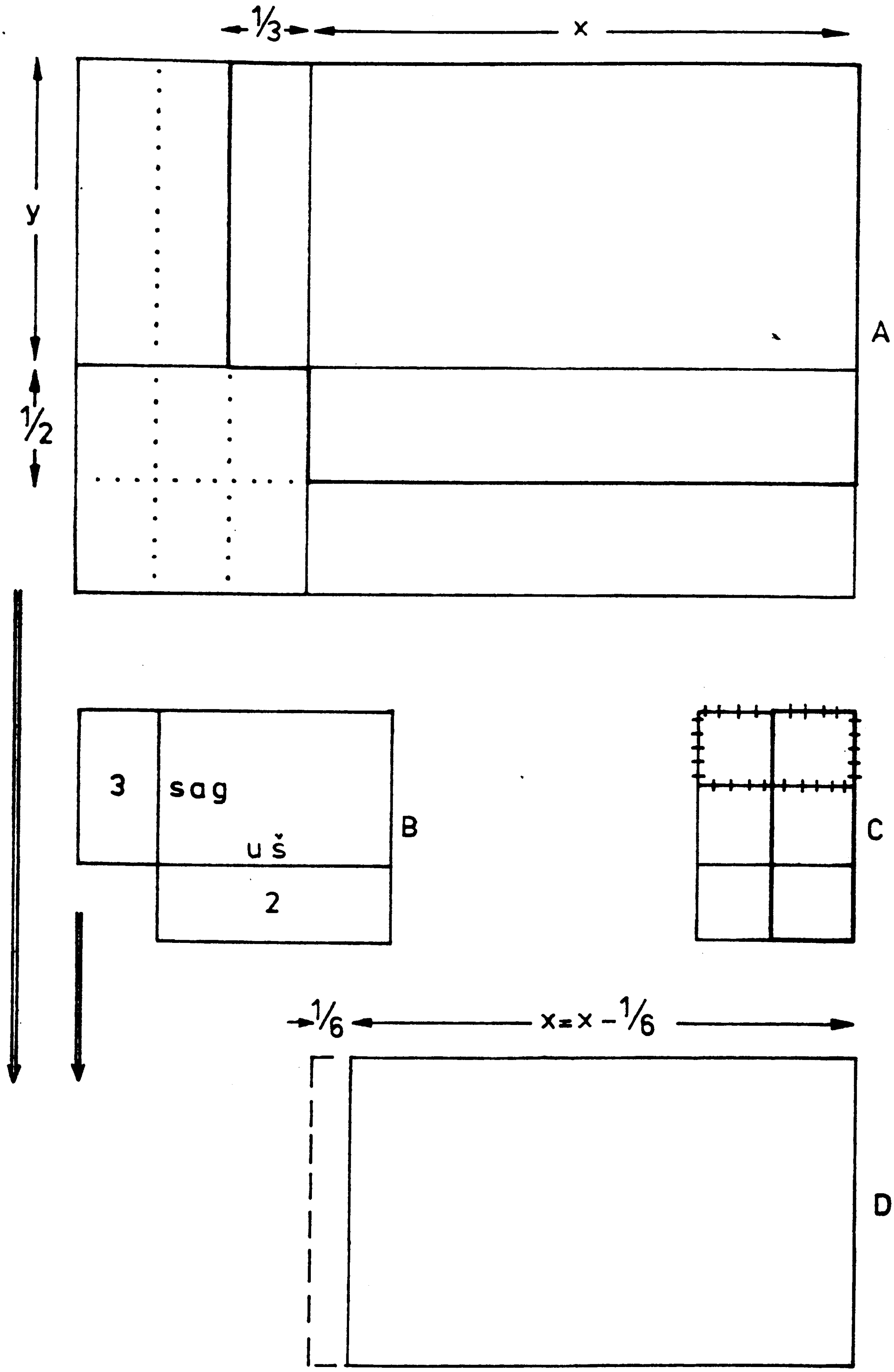


Figure 13. The geometrical interpretation of AO 8862 N° 2. Distorted proportions.

We observe that the procedure is different from the one shown on Figures 4–6, which corresponded to "application with excess". The corresponding problem in one variable is the type $\alpha x - x^2 = \beta$–to give it a formulation which could be formulated inside the Babylonian framework: "from α confrontations I have torn out the surface: γ". This is the type which has two positive solutions; it seems to be completely absent from the Babylonian material[137] even though the corresponding problem in two variables is very common.

The reduction of N° 2 is somewhat more complex, but follows the same pattern, see Fig. 13. Fig. 13 A shows the configuration as we would imagine the geometric situation described, while Fig. 13 B describes what appears to correspond more or less to the Babylonian understanding, as described in the text. The numbers *2* and *3* are "inscribed as inscriptions" of $^1/_2$ and $^1/_3$, probably along the edges of the rectangle, to remind that the widths of these edges are to be understood, not as *1* but as stated; and when $^1/_2 x + {}^1/_2 y$ is to be subtracted from the aggregated surface it is "brought to" the place of "length and width", viz. to those entities which were accumulated. It is indeed clear from the text that the 3° 30′ is not brought to an abstract s u m (which would also be mathematically meaningless) but to the collection of added yet still separate entities–a point where the plural and hence concrete character of *kimrātum* is of importance.

When the half-sum of length and width is brought to the place of length and width, i.e. to the edges of the rectangle, it is obvious and not commented upon that the $^1/_2$-length is eliminated; but more than $^1/_3$-width goes away, and a curious calculation in II.13–15 finds the resulting defect to be 10′ (width). The process of "making 2 and 3 span" can be imagined as in the lower left corner of Fig. 13 A; but an independent procedure as shown in Fig. 13 C seems more plausible, among other things because of the explicit order to stop the ongoing procedure and because Fig. 13 A is described as a real field in the terrain. In sort of parenthesis, an entity is "built" of which both $^1/_2$ and $^1/_3$ are easily taken, to allow for a two-dimensional variant of the "single false position"" (cf. below).

From here on, everything runs as in No 1.

The geometrical reading of N° 3 is shown in Fig. 14. It turns out that the squaring of $x+y$ gives us a figure from which the given surface $xy + (x-y)(x+y)$ can easily be torn out. The figure is seen to be of precisely the same structure as that shown in Fig. 2, and other texts suggest that it was familiar in the Old Babylonian period too.[138] What remains is a square of side y and a rectangle of sides y and $x+y$. This remainder is easily rearranged as a gnomon, as done in Fig. 14 B. The usual quadratic completion yields a side of the completed square equal to 1ˋ 30°.

If the rearrangement had been thought of as a problem in y (the s a g), $y^2 + 50 \cdot y = 1$ˋˋ 33ˋ 20°, then it might have been natural to subtract 50 from this 1ˋ 30° ($= y + 50$). Instead, however, 1ˋ 30° is subtracted from the side of

[137] Absent, that is, in explicit formulation. Indications exist, indeed, that the problem BM 85194 rev. II 7–21 was solved as a problem in one variable and not in two, as it was once proposed by Vogel 1936: 710. See my 1985: 59f.

[138] So YBC 6504 No 2 (MKT III 22, interpretation in my 1989: 28–31) and BM 13901 No 19 (MKT III 4). In both cases, the linear dimensions of the figure are half of those of the present problem (30 and 20, against 1ˋ and 40°).

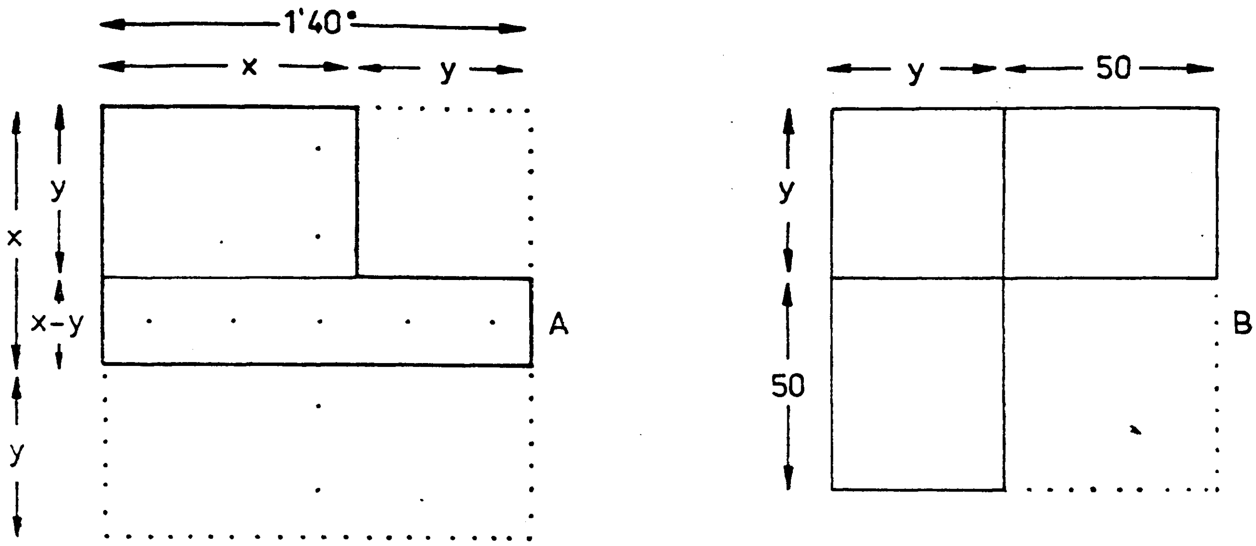


Figure 14. The geometrical interpretation of AO 8862 No 3.

the square of Fig. 14A. If we look at the subdivision of this square through the quartering lines it is indeed evident that the difference between the two entities is the half-difference between the length and the width of the original rectangle. It seems thus as if the steps shown in Fig. 14B shall not be apprehended as a change of problem; instead, everything is to be understood all the way through in terms of the constituent parts of Fig. 14A. By extension, we may surmise that the "changes of variable" to Y and X in Nos 1 and 2 are not really to be understood as explicit changes of the unknown. That is indeed a comprehension inspired by rhetorical or symbolic algebra where certain entities are distinguished by having a name of their own and are hence regarded as fundamental unknowns. Instead, all entities in a figure which are not known are unknown on an equal footing as far as the solving procedure is concerned. Only as far as certain entities are asked for initially can they be considered privileged (and relatively privileged only, as the entities asked for in the beginning and those found in the end need not coincide[139]). This corresponds to our own comprehension of problems of geometrical analysis–the phrase to be understood in its Greek sense.

A number of features of the texts call for separate discussion. Most important among these is the occurrence of the term a-rá, "steps of", the multiplicative term of the multiplication tables. In some places it stands alone, but time after other it is found in double constructions that show the isolated occurrences to be ellipses. Other texts state that a rectangle is to be built from a length and a width, and leave the numerical multiplication implicit, giving directly its result.[140] In the present double constructions, both steps are spelled out explicitly, the multiplication apparently through reference to the auxiliary tables, and in I, 13 and in two places in No 3, it is the building process which is left implicit.[141]

[139] Such a discrepancy is found, e.g., in BM 85194 rev. II 7–21.

[140] Similarly, we remember, the "raising" was sometimes left implicit in the "posing" of one number to another (above, section IV.6).

[141] It may be significant that two of the three ellipses occur after the "breaking" of a "moiety", which already may imply the construction process; similarly, indeed, in II 21, the moieties are not "made span" but instead "inscribed until twice". The third

Another terminological peculiarity of the text is the use of the subtractive term *ḫarāṣum*, "to cut off", along with the more current *nasāḫum*, "to tear out". Already from the metaphorical contents of the two terms we migth expect that the latter would be preferred for identity-conserving subtraction from surfaces and the former for the shortening of one-dimensional entities, if a distinction were to be made. This is, indeed, precisely the main tendency of this as well as all other texts where the terms are found together. But it is only a tendency, in the sense that *nasāḫum* may be used for one-dimensional entities too; most clearly this is seen in I, 19–22: First 30′ is "cut off" from 14° 30′, and next 2 is "torn out" from the resulting 14.[142] It is thus excluded to regard the two terms as names for distinct operations. At the same time the tendential distinction prevents us from seeing the terms as connotationally neutral technical terms, whose metaphorical basis had been completely worn off. They constitute instances of mathematical terms which must be "regarded as open-ended expressions which in certain standardized situations are used in a standardized way" (as formulated above, note 29).

A third formulation of interest is the recurrent BA.A-*šu ša*, "its moiety, that of", which is found in all three problems at the point where a rectangle is bisected in order to allow a gnomonic reorganization (I, 12; II, 19; III, 13). The use of the determinative pronoun *ša* shows that the quantity pointed at, the one which is to be bisected, must have some independent existence, mental or physical, which allows us to think of or point at a definite entity. I, 12, for instance, cannot be read as the bisection of an abstract number *29*; it must by necessity deal with something definite–another confirmation of the concreteness inherent in the naive-geometric interpretation.

A final terminological point to be observed is the distinction which is maintained between *mišlum*, "half", and *bāmtum*, "moiety", and the corresponding distinction between multiplication by igi 2-bi = 30′ (N° 2, II, 6) and "breaking". Once more "breaking" is seen to be reserved to describe bisection into natural "wings" (cf. section IV.5, and note b to BM 13901 N° 1, section V.2).

As concerns the mathematical aspect of the texts, the flexible handling of problems and methods was already pointed at in the introductory remarks. It makes clear that the understanding behind the text must have been flexible, too, that it has nothing to do with blind application of fixed rules or algorithms discovered by equally blind luck, as claimed too often in the secondary literature.

Another related implication of the tablet concerns the purpose of such texts. I think of the tabulation between I, 7 and I, 8. Here, before the description of the solving procedure, the whole construction and solution of problem 1 is told

ellipsis, finally, is found when the area of the square in Fig. 17 A is found: If this configuration is well-established beforehand, there is no need to construct it anew (cf. the concluding discussion in section V.8).

142 But if we look at the written numbers, the distinction holds good even in this case, as A. Westenholz has observed: When 30′ is removed from 14° 30′ it is the end of the number (viz. of the sequence 10,4,30) which is "cut off"; but to take away 2 from the sequence 10,4 requires that we remove part of the compact group of wedges making up the 4.

In one text, viz. YBC 4675 obv. 14, is *ḫarāṣum* used to designate a subtraction from a surface (4ˋ 49°–2ˋ). That text, however, avoids *nasāḫum* altogether.

*22

in advance. The subsequent procedural prescriptions can therefore hardly be seen as an attempt to find the unknown dimensions of the rectangle. The aim is not really to solve the problem and find the solution; it is to demonstrate h o w to solve the problem, to present an argued solution.

The calculation in No 2, II, 13–15, finally, is remarkable, though belonging more on the level of details. The Babylonian predilection for argumentation by means of a "single false position" was pointed out repeatedly above in sections V.6 and especially VII.3, where a representation by countable units was also suggested. Here, however, the trick is extended into two dimensions, as revealed by the term "making span" (extension apart, its relation to the calculation of $1-{}^1/_4=45'$ in TMS XVI is obvious). Since ${}^1/_6$ is stated directly to be 10′, the identities ${}^1/_2=30'$ and ${}^1/_3=20'$ can hardly have been considered a secret. The computation of their difference by way of a geometrical subtlety must therefore be seen as a didactical nicety, as a means to demonstrate the extension of the simple argument.

VIII.3. TMS IX (TMS, 63f.; cf. von Soden 1964)

Such didactical concerns are even more obvious in the Susa text TMS IX, which approaches the style of TMS XVI (above, section VII.3). In this case, however, the text goes from simplest ($xy+x=40'$) to less simple ($xy+x+y=1$) fundamental equation, ending with a fairly complex application of the fundamental principle.

Unfortunately, the transcription in TMS is not very precise, the restitution of damaged lines and the translation are worse, and the mathematical commentary is at times nonsensical. Had it not been for these circumstances, the text would probably have changed much conventional wisdom in the understanding of Babylonian mathematics 25 years ago.

PART A $(x=30', y=20')$ $x\cdot y+1\cdot x=40'$	1. The surface *and* 1 length ACCUMULATED, 40′. [(30′ the length 20′ the width)[b]][a] a-šà *ú* 1 uš UL.GAR 4[0 (30 uš 20 sag)]
Alternative approaches to an understanding:	2. *As* 1 length *to* 10′, the surface [has been appended][a] *i-nu-ma* 1 uš *a-na* 10 ⌜a-šà daḫ]
$Y=y+1=20'+1°$ $=1°\,20'$ or	3. *Either* 1 as BASE(?)[c] *to* 20′, the width, [append] *ú-ul* 1 KI.GUB.GUB *a-na* 20 [sag daḫ]
$x\cdot 1°\,20'=40'$	4. *or* 1° 20′ *to* the width *which* 40′ *together with* [the length (SURROUNDS pose)][a] *ú-ul* 1, 20 *a-na* sag *šà* 40 *it-*⌜*ti* uš (NIGIN gar)]
or $1°\,20'\cdot 30'=40'$	5. *or* 1° 20′ *together with* 30′ the length MAKE SURROUND, 40′ *its name* *ú-ul* 1, 20 *it-⟨ti⟩* 30 uš NIG[IN] 40 *šum-[šu]*
Implicit conclusion: $x\cdot y+1\cdot x=x\cdot(y+1)$	6. *Since so, to* 20′, the width, *which has been said to you* *aš-šum ki-a-am a-na* 20 sag *šà qa-bu-ku*

7. 1 is appended: 1° 20′[d] *you see. Out from here*
1 daḫ-*ma* 1, 20 *ta-mar iš-tu an-ni-ki-a-am*
8. *you ask.* 40′ the surface, 1° 20′ the width, the length *what?*
ta-šà-al 40 a-šà 1, 20 sag uš *mi-nu*
9. [30 the length][a]. *So* the *having-been-made*
[30 uš *k*]*i-a-am ne-pé-šum*

PART B

$(x=30', y=20')$

$x \cdot y+x+y=1$

10. [Surface, length *and* width AC][a] CUMULATED, 1. *By* the *Akkadian*
[a-šà uš *ù* sag U]L.GAR 1 *i-na ak-ka-di-i*

$(x+1) \cdot (y+1)$
$=x \cdot y+1 \cdot x+1 \cdot y+1 \cdot 1$

11. [1 *to* the length append.][a] 1 *to* the width append. *Since* 1 *to* the length is appended,
[1 *a-na* uš daḫ] 1 *a-na* sag daḫ *aš-šum* 1 *a-na* uš daḫ

$1 \cdot 1=1$, and so

12. [1 *to* the width is app][a]ended, 1 *and* 1 MAKE SURROUND, 1 *you see.*
[1 *a-na* sag d]aḫ 1 *ù* 1 NIGIN 1 *ta-mar*

$(x+1) \cdot (y+1)$
$=(x \cdot y+x+y)+1$
$=1+1=2$

13. [1 *to* the ACCUMULATION of length,][a] width *and* surface append, 2 *you see*
[1 *a-na* UL.GAR uš] sag *ù* a-šà daḫ 2 *ta-mar*

$Y=y+1=1° 20'$
$X=x+1=1° 30'$

14. [(*To* 20′ the width 1 appe)][a]nd, 1° 20′. *To* 30′ the length 1 append, 1° 30′.
[(*a-na* 20 sag 1 da)]ḫ! 1, 20 *a-na* 30 uš 1 daḫ 1, 30

$X \cdot Y=1° 30' \cdot 1° 20'$

15. [(*Since* a surfa)][a]ce, *that of* 1° 20′ the width, *that of* 1° 30′ the length
[(*aš-šum* a-š)]à *šà*! 1, 20 sag *šà* 1, 30 uš
16. [(Length *together with* wid)][a]th *is made span*[e], *what* is *its name?*
[(uš *it-ti* sa)]g! *šu-ta-ku-lu mi-nu šum-šu*

$X \cdot Y=2$

16a. 2 the surface
2 a-šà
17. *So* the *Akkadian*
ki-a-am ak-ka-du-ú

$x \cdot y+x+y=1$ PART C
$y+\frac{1}{17}(3x+4y)=30'$

19. Surface, length *and* width ACCUMULATED, 1 the surface. 3 lengths, 4 widths ACCUMULATED,
a-šà uš *ù* sag UL.GAR 1 a-šà 3 uš 4 sag UL.GAR
20. *its* 17*th to* the width appended, 30′.
[17]-*ti-šu a-na* sag daḫ 30

$17y+3x+4y=17 \cdot 30'$
$=8° 30'$

21. You, 30′ *to* 17 *go:* 8° 30′ *you see*
[za-]e 30 *a-na* 17 *a-li-ik-ma* 8, 30 [*t*]*a-mar*

$17y+4y=21y$

22. *To* 17 widths, 4 widths append: 21 *you see,*
[*a-na* 17 sag] 4 sag daḫ-*ma* 21 *ta-mar*

The coefficient of y is 21,

23. 21 *as much as* of widths, pose. 3 *of three of* lengths,
[21 *ki-*]*ma* sag gar 3 *ša-la-aš-ti* uš

that of x is 3	24. 3 *as much as* of lengths, pose. 8° 30′ *what* is *its name*? [3 *ki*]-*ma* uš gar 8, 30 *mi-nu šum-šu*
$3\cdot x+21\cdot y=8^\circ\,30'$	25. 3 lengths *and* 21 widths ACCUMULATED [3] uš *ù* 2[1 sa]g UL.[GAR]
	26. 8° 30′ *you see*[f] 8, 30 *ta-mar*
	27. 3 lengths *and* 21 widths ACCUMULATED [3] uš *ù* 21 sag UL.[GAR]
$x+1=X$ $y+1=Y$	28. 1 *to* the length append *and* 1 *to* the width append, MAKE SURROUND: [1 *a-na*] uš daḫ [*ù* 1 *a*]-*na* sag daḫ NIGIN-*ma*
$X\cdot Y=(xy+x+y)+1$ $=2$	29. 1 *to* the ACCUMULATION of surface, length *and* width append, 2 *you see*, 1 *a-na* UL.GAR a-šà uš *ù* sag daḫ 2 *ta*-⟨*mar*⟩
	30. [2 the sur][a]face. *Since* length *and* width, *those* of 2 the surface, [2 a-]šà *aš-šum* uš *ù* sag *šà* 2 a-šà
$X\cdot Y=1^\circ\,30'\cdot 1^\circ\,20'$ (identifications)	31. [1° 30′ the length *toge*][a]*ther with* 1° 20′ the width *is made span* [1, 30 uš *it*[!]$_{\mathrm{J}}$-*ti* 1, 20 sag *šu-ta-ku-lu*
$1\cdot 1=1$	32. 1 the *appended*[g] *of* the length *and* 1 the *appended of* the width [1 *wu-ṣú-*]*bi* uš *ù* 1 *wu-ṣú-bi* sag
$1+(xy+x+y)=2$	33. [MAKE SURROUND, (1 *you see*). 1 *and* (. . .?)][a] the various (things)[h] ACCUMULATE, 2 *you see*. [NIGIN (1 *ta-mar*?) 1 *ù* (. . .?)] ḪI.A UL.GAR 2 *ta-mar*
$3X+21Y$ $=3+21+(3x+21y)$ $=3+21+8^\circ\,30'=32^\circ\,30'$	34. [(3, 21 *and* 8° 30′ ACCUMULATE)][a], 32° 30′ *you see*. [(3 (. . .?) 21 (. . .?) *ù* 8, 30 (. . .?) UL.GAR] 32, 30 *ta-mar*
	35. *So you ask* [*ki-a*]-*am ta-šà-al*
$\tilde{y}=21Y$	36. [. . .] *of* the width *to* 21 ACCUMULAT(E/ION):[i] [. . .].TI sag *a-na* 21 UL.GAR-*ma*
$\tilde{x}=3X$	37. . . .[j] *to* 3, the lengths, *raise*, [. . .]ḪI(?) .A *a-na* 3 uš *i-ší*
$\tilde{x}\cdot\tilde{y}=3\cdot 21\cdot XY$ $=1^{\backprime}\,3^\circ\cdot XY$ $=1^{\backprime}\,3^\circ\cdot 2=2^{\backprime}\,6^\circ$	38. [1` 3° *you see*. 1` 3° *t*][a]*o* 2, the surface, *raise*: [1, 3 *ta-mar* 1, 3 *a*]-*na* 2 a-šà *i-ší-ma*
$\tilde{x}\cdot\tilde{y}=2^{\backprime}\,6^\circ$ $\tilde{x}+\tilde{y}=32^\circ\,30'$ $\frac{\tilde{x}+\tilde{y}}{2}=16^\circ 15'$	39. [2` 6° *you see* (2` 6° the surface?)][a] 32° 30′ the ACCUMULATION *break*, 16° 15′ *you see*. [2, 6 *ta-mar* (2, 6 a-šà?)] 32, 30 UL.GAR *ḫi-pí* 16, 15 *ta*-⟨*mar*⟩
$\left(\frac{\tilde{x}+\tilde{y}}{2}\right)^2=(16^\circ\,15')^2$ $=4^{\backprime}\,24^\circ\,3'\,45''$	40. {1[6° 15′ *you*][a] *see*}[k] 16° 15′ the counterpart pose; MAKE SURROUND, {16, 15 *ta-mar*} 16, 15 gaba gar NIGIN

	41. 4` 24° 3′ 45″ *you see*. 2` 6° *xxx*! 4, [24,]3, 45 *ta-mar* 2, 6 [. . .]
$\left(\frac{\tilde{y}-\tilde{x}}{2}\right)^2=\left(\frac{\tilde{y}+\tilde{x}}{2}\right)^2-\tilde{x}\tilde{y}$ $=2`\,18^\circ\,3'\,45''$	42. *from* 4` 24° 3′ 45″ tear out, 2` 18° 3′ 45′ *you see*. *i-na* 4, [2]4, 3, 45 zi 2, 18, 3, 45 *ta-mar*
$\frac{\tilde{y}-\tilde{x}}{2}=\sqrt{2`\,18^\circ\,3'\,45''}$ $=11^\circ\,45'$	43. *What* it makes equilateral? 11° 45′ it makes equilateral. 11° 45′ *to* 16° 15′ append, *mi-na* íb-si 11, 45 íb-si 11, 45 *a-na* 16, 15 daḫ
$\tilde{y}=\frac{\tilde{y}+\tilde{x}}{2}+\frac{\tilde{y}-\tilde{x}}{2}$	44. 28 *you see*; *from* the 2nd tear out, 4° 30′ *you see*. 28 *ta-mar i-na* 2-kam zi 4, 30 *ta-mar*
$=16^\circ\,15'+11^\circ\,45'=28$ $\tilde{x}=\frac{\tilde{y}+\tilde{x}}{2}-\frac{\tilde{y}-\tilde{x}}{2}$	45. The igi *of* 3, the lengths, *detach*, 20′ *you see*. 20′ *to* 4° 30′ igi 3-*ti* uš *pu-ṭúr* 20 *ta-mar* 20 *a-na* 4, [30]
$=16^\circ\,15'-11^\circ 45'=4^\circ 30'$ $X=3^{-1}\cdot\tilde{x}$ $=20'\cdot 4^\circ\,30'$ $=1^\circ\,30'$	46. {20′ *to* 4° 30′} *raise*: 1° 30′ *you see*. {20 *a-na* 4, 30} *i-ší-ma* 1, 30 *ta-mar*
$X=1^\circ\,30'$ $\tilde{y}=28=21\cdot Y,\ Y?$	47. 1° 30′ the length, *that of* 2 the surface. [*What*][a] *to* 21, the widths, [*shall I pose*][a] 1, 30 uš *šà* 2 a-š[à *mi-na*] *a-na* 21 sag [*lu-uš-ku-un*]
$1^\circ\,20'\cdot 21=28$ $Y=1^\circ\,20'$	48. *which* 28 *give*[*s me*? 1° 20′ p][a]ose, 1° 20′ the width *šà* 28 *i-na-di*![-*na* 1, 20 g]ar 1, 20 sag
$x=X-1=1^\circ\,30'-1$ $=30'$	49. *that of* 2 the surface. *Turn back*. 1 *from* 1° 30′ tear out *šà* 2 a-šà *tu-úr* 1 *i-na* 1, [30 zi]
$y=Y-1=1^\circ\,20'-1$ $=20'$	50. 30′ *you see*. 1 *from* 1° 20′ tear out, 30 *ta-mar* 1 *i-na* 1,20 z[i]
	51. 20′ *you see*. 20 *ta*-[*mar*]

[a] All these restitutions are mine. Restitutions in simple [ ] can be regarded as fairly well established, those in [( )] are reasoned guesses at a formulation, the factual contents of which can be relied upon.

[b] Line 6 quotes the value of the width in a way which would usually refer back to the statement, but which might of course refer to line 3; in any case, line 3 presupposes knowledge of the width, and line 5 refers to the length as a known quantity.

[c] BASE is a conjectural translation of the logogram KI.GUB.GUB (the testified Late Babylonian reading ki-du-du ~ *kidudûm*, "rites", makes no sense). GUB has two different Sumerian meanings, "to go" (readings du etc., cf. SLa § 268; used logographically for *alākum*) and "to stand, to erect" (gub, cf. SLa § 267; used logographically for *izuzzum* and *zaqāpum*). To judge from the logographic occurrences, the reduplication is used to indicate iterative and durative aspects.

ki can function as a virtual locativic verbal prefix, "on the ground" (cf. SLa, 306). A possible reading of KI.GUB.GUB is thus ki-gub-gub, "to stand/that which stands erected constantly on the ground".

[d] The transliteration in TMS writes 1. Still, the autography writes a sign after 1 which looks like 20 (and a damage to the tablet which has been read as an extra wedge). That is also the correct result, which is in fact used in line 8.

[e] The exact reconstructions of lines 14–16 are rather tentative, although the mathematical substance is fairly well-established thanks to the parallel of lines 28–31. It should be observed that even the extant signs until 1,20 *a* in line 14, and the š)]à and sa)]g of the following lines, are heavily damaged. The remaining traces may but need not correspond to my readings (according to autography and photo). The *aš-šum* of line 15 is needed, if not necessarily in that place, by the *šu-ta-ku-lu* of line 16, if I am right when reading it as the subjunctive mode of a stative (cf. lines 30f., and the subjunctive stative *qa-bu-ku* in line 6).

[f] The transliteration in TMS supposes that something is missing in the beginning of the line. The autography indicates that the line is simply written with indention.

[g] "Zu WA-ZU-*bi* im math. Susatect Nr. IX: Ich hatte mich für die Rezension von MDP 34 (=von Soden 1964 – JH) ziemlich gründlich damit beschäftigt und als mögliche Lesung *wu-ṣú-bi* als St. constr. eines sonst nicht bekannten *wuṣubbûm* notiert, diese Lesung aber dann als zu wenig gesichert nicht veröffentlicht." (Von Soden, private communication).

[h] "the various (things)" translates ḪI.A. This presupposed the assumption that the Sumerian suffix ḫi.a (designating a plurality of different entities) is used as a pseudo-Sumerogram in a nominal function (as a collective name for the collection of surface, length and width). It is also possible that ḫi-a stands as a pseudo-grammatical complement to a noun which was lost with the first part of the line.

TMS restitutes [. . .]-*ti* sag as *ša-la-aš-ti* sag and mistranslates the whole line as "]3 ⟨fois⟩ la longueur à 21 fois ⟨la largeur⟩ additionne" in order to get some apparent sense of the restitution. Apart from the mistake of "length" for "width" this mixes up "appending" and "accumulation". Only the first of these carries a "to" (*ana*) between the addends. A possible restitution which accepts the (somewhat dubious) *-ti* in the beginning of the line, which makes mathematical sense, which is as grammatically correct as can be expected in a text loaded with sumerograms, and which finally is in reasonable harmony with current usage, would be "17 (. . .?) and 4, of the four (*er-bet-ti*), widths, to 21, the ACCUMULATION" or ". . . to 21 ACCUMULATE". In lack of related passages I have, however, preferred to leave the question open.

[j] The transliteration in TMS renders the signs before *a-na* as ḪI.A. The A is in agreement with the autography, but the preceding sign looks very different from the ḪI of line 33. I have not been able to propose any better reading.

[k] The initial "10" is fully and the final *-mar* almost fully to be read on the auto-

graphy, although they are left out in the transliteration. So, a repetition of the previous phrase appears to be the only possible restitution. Cf. also lines 45f.

l The lacuna consists of 1 or 2 signs, probably an epithet to the number 2' 6°. According to the autography, the first sign begins . This could belong to a TA, but such a restitution seems to make no sense. It could also belong to a TAG used logographically for *lapātum*, "to inscribe", and its derivations. This might make sense but would be without parallel ("2' 6° the inscribed").

The purely explanatory character of part A is revealed already in line 2, as the surface (which was never given) is referred to as known ("since . . .") (cf. also the restitution of the last part of line 1). Clearly, we are dealing with one equation in two (known) unknowns, uš = 30′, sag = 20′, and we are taught the way to transform it (in fact the same transformation as that of AO 8862 N^os 1–2: $xy+\alpha x \rightarrow xY$, $Y=y+\alpha$). In this way one can make sense of the "either . . . or . . . or" of lines 3–5 (Ú.UL . . . Ú.UL . . . Ú.UL), which governs three alternative ways to explain the transformation, but which has no place in an interpretation of the text as progressive argumentation (since the 1° 20′ created in line 3 is used in line 4, and line 5 repeats the contents of line 4), and which has therefore puzzled all commentators to the text.

If one follows the text step by step, it turns out that all of it can be read as an explanation of Fig. 15A, up to the end that explains that this is the point out from which problems containing such equations are to be solved, and finally sums up the main argument.

Part B deals with the same rectangle, but with a somewhat more complicated equation, $xy+x+y=1$, and demonstrates how it is to be simplified "by the Ak-

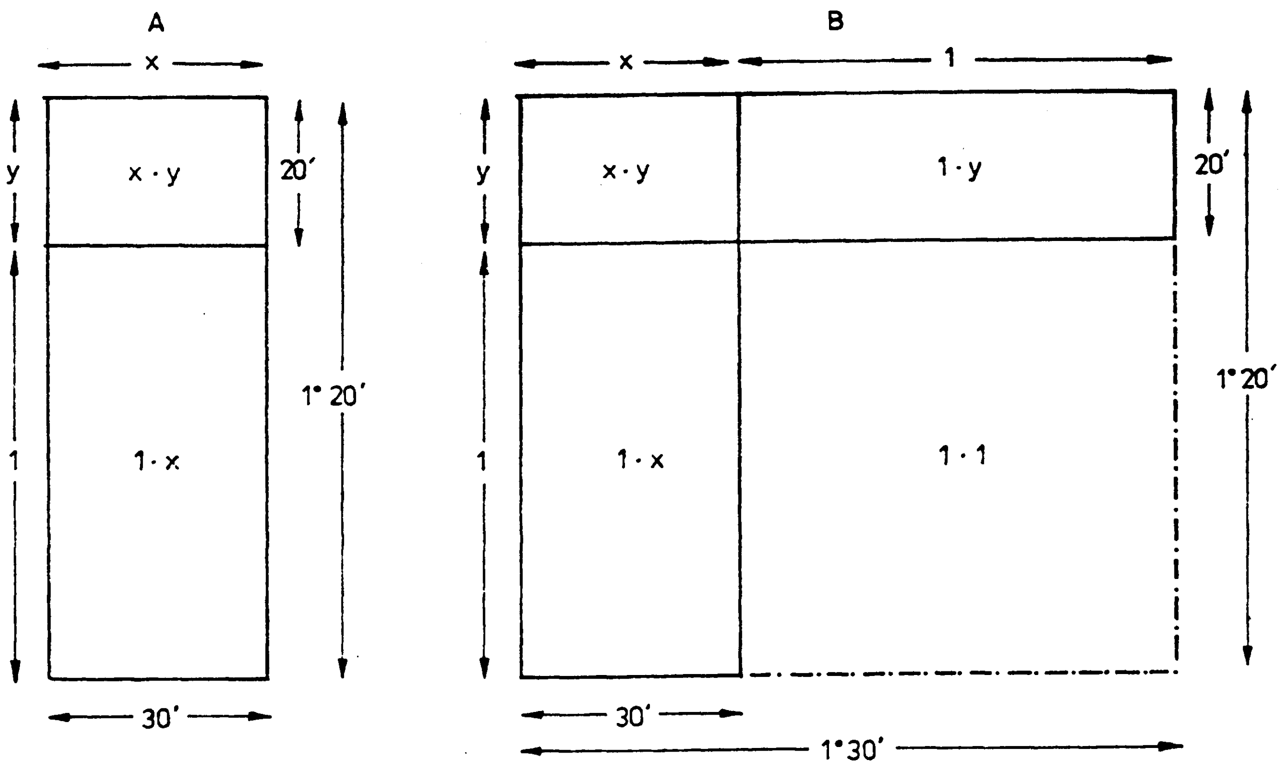


Figure 15. The geometrical configurations and operations described in TMS, parts A and B.

kadian (method)".[143] It can be followed on Fig. 15B. The method consists in completing the quasi-gnomon $xy+1\cdot x+1\cdot y$ into a rectangle XY, $X=x+1$, $Y=y+1$. X and Y are spoken of as "length" and "width" of "2 the surface" ($=XY$), in agreement with the figure.

Denominations of methods are rare in Mesopotamian mathematical texts, and one may wonder what makes the method of part B specifically "Akkadian". Which part of the procedure is it, furthermore, which deserves the label? My guess is that the term characterizes the quadratic completion in general, the basic trick needed to solve mixed second-degree equations. If anything, indeed, distinguishes the Old Babylonian "Akkadian" mathematical tradition from e.g. third millenium Sumerian mathematics, it will be its interest in second-degree algebra. Which more adequate name than the "Akkadian method" could then have been chosen for a trick which, simple as it may look once it is found, was perhaps the starting point for the whole fabulous development of "Akkadian" mathematics; a trick which, when it was first found, will certainly have been noticed as a novelty?[144]

It will be seen from line 14 that the values of both length and width are as-

[143] Truly, E. M. Bruins claims in the commentary in TMS (p. 67, and announced already pp. xi and 2) that the two parts deal with the same equation, and that part A expounds the master's own method and part B the alternative used by the Akkadians. For a number of reasons this is an impossible idea:

1) If the equation $xy+x=40'$ is to be equivalent with $xy+x+y=1$, one must presuppose $y=20'$. On the faith of line 6 E. M. Bruins claims (rightly, I suppose) that this value will have been given before (cf. my restituted line 1), from which he concludes that the text deals with a normal, complete set of two equations. Line 2, however, presupposes implicitly that the length is equally known ($10'$ the surface), while the value is stated explicitly in line 5 still without being calculated.

2) If the first half of line 10 were to be the result of a transformation belonging with the "Akkadian method", it could not precede the announcement of that method in the second half of the line.

3) In any case, the first half of line 10 is clearly in the style of statements; transformed equations are never restated in a similar form. Cf., e. g., the contrast with the formulation in lines 25f.

4) Finally. E. M. Bruins overlooks the identical statement in part C, as well as the fact that the procedure taught in part B is precisely the one used in part C.

It may be observed that the presumed "Susian" method is used in the Babylonian ("Akkadian") AO 8862 N^os^ 1–2, although N^o^ 2 would have been greatly simplified had the "Akkadian method" been used.

[144] In this connection, the over-all character of Old Babylonian scribe school mathematics is worth reflecting upon. Greek mathematics, that other prototype of Ancient non-utilitarian mathematics, can be claimed to be essentially determined by its central problems (squaring the circle, doubling the cube, properties of conics, classification of irrationals, etc.). The great methodological innovations of Greek mathematics were made in order to solve (in a philosophically satisfactory manner!) these great problems. Old Babylonian scribal mathematics was, in as far as we concentrate upon its non-utilitarian aspect, determined by the methods at hand, and problems were chosen that would permit a brilliant display of the methods known to "the learned scribe", which makes scribe school mathematics a perfect parallel to other aspects of Old Babylonian scribal culture as presented, e. g., in the "examination texts". See my 1985a: 10–16 and passim, which discusses the difference between the two mathematical styles less coarsely than enforced by the limited space of a foot-note, and connects the different attitudes to their institutional and cultural context.

sumed to be known (though not given in the statement), and that they are used in the didactical exposition.

Part C contains a complete mathematical problem, a normal set of two equations in two unknown quantities "length" and "width". One of them is precisely the second-degree equation whose transformation was taught in part B, while the other (which can be transcribed $y+\frac{1}{17}(3x+4y)=30'$) is of the type whose transformation was explained in detail in TMS XVI (above, section VII.3). The values of length and width are still referred to during the solution (line 31), but only for identification, no longer as part of the argument. The identification must refer to something outside the written text[145], which can hardly be but a material representation more or less similar to Fig. 15B.

Lines 21 to 26, the transformation of the first-degree equation into $3x+21y=8°\,30'$, must be presumed to follow the pattern from TMS XVI, and hence to be understood as an arithmetical transformation (we observe that the term for a coefficient, "as much as", recurs). Lines 28 to 33 appear to go by "naive geometry". For the next steps, lines 34 to 39, we are unfortunately not in possession of a didactical explanation. But some argumentation from Fig. 15B but similar to the accounting and scaling arithmetic of TMS XVI would at least be adequate, and is perhaps called for in line 27, which appears to connect to the following rather than the preceding section.[146] In any case, lines 39–44 solve the standard problem of a rectangle for which the area and the sum of length and width are known, the "false" length of which is $X=3\,(x+1)$, and the "false" width of which is $Y=21\,(y+1)$. The method is unfortunately not commented upon. Like the transformation of the linear equation the didactical explanation appears to have been given at an earlier stage, and the understanding now inherent in the vocabulary. Afterwards, the extended "real" length and width (those of "2 the surface") and finally the "real" length and width without extension are calculated (lines 45–51).

The whole tablet reflects a mathematics lesson. While part C represents a refined version of a standard problem known from elsewhere (VAT 8520, N^os^ 1–2, cf. note 146), parts A and B are didactical steps toward a particular aspect of the procedure needed to solve the complex standard problem. The other,

[145] The meticulous repetition of all steps appears to exclude a simple reference back to the known entities from section B.

[146] The argument can be imagined in the style of "false assumptions": If the length of the upper left rectangle in Fig. 13B is to represent 3 "true" lengths, the length of the upper right rectangle is 3 instead of 1. Similarly, if the upper left width represents 21 "true" widths, its extension will have to be 21 instead of 1. The sum of length and width of the total figure will then be $3+21+8°\,30'$, cf. line 34. Furthermore, the total scaling factor for the area will be $21 \cdot 3 = 1`\,3°$, and the area of the assumed surface will hence be $1'\,3° \cdot 2 = 2`\,6°$ (lines 36–39).

The last part of the interpretation seems to be confirmed by VAT 8520 N° 1 (MKT I 346f.). Here, an *igûm-igibûm* problem (translatable into $xy=1$, $x-{}^6/_{13}\,(x+y)=30'$) is solved in a similar way (extensions apart). The linear equation is transformed, it appears, into $7x-6y=8°\,30'$, and a scaling factor of $7 \cdot 6 = 42$ is applied to "1 the surface". As the numbers 7 and 6 are to be retained by head, the transformation can be assumed to be performed mentally, not by means of any material representation beyond the changed conceptualization of the basic rectangle.

more general aspects of the procedure are supposed to be known from earlier lessons, and one of them was in fact explained in TMS XVI, as we have seen.

It has often been assumed that the Babylonian mathematical texts should be seen only as supplementary support for an oral tradition, and that the texts could only be understood by a person who knew beforehand what the whole thing was about.[147] The present investigation shows that the latter formulation is not as absolutely true as hitherto assumed, if only one knows the concrete meaning of the terminology. But still, the normal texts give the impression that they are a support for a teaching tradition making use of material representations outside the texts themselves, and referring to methods which had to be known beforehand. The material representations have still not been unearthed, and may be irretrievably lost (cf. above, chapter VI). The two Susa tablets, however, show us how the standard methods were taught, and the one just presented appears to refer more clearly perhaps than any other text to the naive-geometric representation.

IX. Summing up the evidence

The investigation has now arrived at a point where a summary of the results can reasonably be made. How far have we come in our understanding of the procedures, techniques and patterns of thought behind the Old Babylonian "algebraic" texts?

Chapters IV to VIII have by necessity been overloaded with details. If all conclusions were to be referred precisely to the single relevant pieces of evidence, the present chapter would make still heavier reading. As the conclusions to be drawn from the material have, however, been presented in scattered form all the way through, I hope that detailed references to the primary material can now be dispensed with.

On the negative side it will be remembered that the traditional arithmetico-algebraic interpretation left so many unexplainable points in the textual discourse that it can be safely dismissed (cf. most of the texts presented in chapter V). The possibility to make it work by minor corrections and ad hoc assumptions can also be disregarded, because no fundamentally arithmetical interpretation can map the structural distinctions within the vocabulary. Babylonian "algebra" was not a science about pure numbers and the ways in which they can be put into mutual relation, be it understood in analogy with Medieval rhetorical algebra as with F. Thureau-Dangin, O. Neugebauer and B. L. van der Waerden, or through that first-level criticism of the received interpretation which has been expressed by M. Mahoney.[148,149]

[147] This supplementary role is no distinctive characteristic of the mathematical texts. Similar claims could be made for most branches of Babylonian literature.

[148] Mahoney 1971.

[149] It should perhaps be emphasized once more that these remarks, as the whole of my investigation, regard the "algebraic" texts. They have no implications for those texts which are directly concerned with the properties of numbers, e.g. concerning inversion or continued multiplication; they, of course, cannot be denied the label "arithmetical".

Positively, the use of some sort of naive-geometric technique can be regarded as well-established. It fits all details of the textual discourse; it distinguishes operations which have to be distinguished according to the structure of the terminology; it agrees with the apparent metaphorical implications of many terms, including the puzzling *wāṣītum,* the "projection". The exact nature of the geometric representation is, however, open to doubt. We do not know to which extent the texts refer to a purely mental representation, though, truly, common pedagogical experience tells that mental geometry presupposes anterior intercourse with manifest geometry. We do not know the means (clay, dust, wax, or possibly sticks?) which were used to represent geometrical structures, relationships, and transformations manifestly, nor whether such representations should be thought of in analogy with modern geometrical drawings or as mere structural diagrams. These questions were discussed in further detail in chapter VI.

Apart from a two-dimensional extension of the "single false position", the naive-geometrical techniques were only used for problems involving a "surface", i.e. for problems of the second degree.[150] We can list these techniques as follows:

Firstly, there is the partition and rejoining of figures ("cut-and-paste"), which in ordinary "length-width" and "confrontation" problems is represented by the bisection and rearrangement of excessive or defective rectangles. In other, genuinely geometrical problems it is used more creatively[151], and as we shall mention in section X.4 there is evidence for continuity to later interests in the partition of figures.

Secondly, we have the completion technique, the supplementation of a gnomon or a quasi-gnomon into a square or a rectangle. This may be the technique which was spoken of as "the Akkadian (method)" in TMS IX.

Thirdly, we have the "scaling" technique, used e.g. when a non-normalized problem ($\alpha x^2+\beta x=\gamma$) is transformed into a normalized problem (in $z=\alpha x$), and to be understood perhaps as a change of measuring scale in one direction[152], perhaps as a proportional change of linear extensions in that direction.

The "accounting" technique may be claimed to have nothing specifically geometric about itself, and it was indeed set forth most clearly in the Susa text explaining the arithmetical transformations of a linear equation. Nonetheless, the counting of a specific entity (or the measurement of one entity in terms of another entity) is a necessary supplement to the specifically geometric techni-

[150] Inclusion of certain further texts would have forced us to modify this statement as well as the automatic identification of "surface"-problems with problems of the second degree. So, the "surface" problem Str. 367 (MKT I 259f.) is in reality of the first degree, but makes use of certain naive-geometric techniques all the same; other exceptions of various sorts could be mentioned. Already the first-degree "meadow" problems of VAT 8389 and 8391 could indeed be claimed to be exceptions; all of them are of the first degree, but formally they are of course concerned with surfaces, and part of the reasoning is made through imagined partition of a geometrical surface.

Problems "representing" prices, *igûm-igibûm* pairs etc. by dimensions of surfaces are not to be understood as exceptions but as "surface"-problems (cf. the use of the term "surface" in YBC 6967, above, section V.1).

[151] A very beautiful example is VAT 8512 (MKT I 341f.); see Gandz's deciphering of the procedure (1948: 36), or the more detailed analysis of the text in my 1985: 105.15ff.

[152] This would hardly bother the Babylonians, who appear to treat a rectangle of length 45 nindan and height 45 cubit as they treat "any other" square (see my 1985: 53–63).

ques, without which no "analysis" by means of geometry (be it naive or based on Euclidean demonstrations) can reproduce the results of arithmetico-rhetorical algebra. The "accounting" and "scaling" techniques are of course closely related.

Hardly to be counted as regular "techniques" but still parts of Old Babylonian naive-geometric methodology are the reasoning by various "false" assumptions and the ability to take any adequate entity of a geometric configuration as that "basic" entity which is to be submitted to the habitual standard operations.

The global picture arising from the use of these techniques and quasi-techniques is the predominance of constructive procedures; only a single pre-established, fixed geometrical standard configuration–the one presented in Fig. 2, and visible as a basic grid in Fig. 14A–has suggested itself during the investigation.

The investigation was only peripherally concerned with first-degree techniques. Even on the basis of the restricted material presented here can it be seen, however, that most reasoning about first-degree problems is verbal and basically arithmetical in character. Like second-degree problems, however, problems of the first degree are dealt with by means of "accounting" and various "false" assumptions. Like the second-degree "algebra" the reasoning on questions of the first degree is also concrete, bound to representations of manifest entities (mental representations in most cases, I guess). Hence of course the predilection for "false assumptions", which consist precisely in taking one entity, real or imagined, as a representative for another, normally unknown quantity.

It was recognized already in the early 1930es that Babylonian "algebra" problems were constructed from known solutions. In the case of the "series texts", where often large numbers of problems deal with the same figure it is also obvious that the user of the texts would know the solution beforehand. The didactical Susa texts have now shown us (as it was also apparent from the tabulation in AO 8862 N° 1) that even the student would, at least in certain cases, have been told the solution beforehand, which would permit an identification of the entities involved in the procedure and also an explanation of the way it works.

The backward construction has traditionally been taken as evidence that the aim of the mathematical texts was the teaching of procedures and techniques.[153] The insights gained from the improved understanding of the vocabulary, regarding the use of naive-geometric justifications, and from the didactical Susa texts show us that the aim was not only technical know-how but also understanding, "know-why". This helps us grasp how Babylonian mathematics was at all possible at its actual level. If its sole social justification had been a teaching enterprise dominated by empty rote learning, from where should it then have got the necessary intellectual inspiration and surplus?

A summary of the results concerning the details of terminology would mainly become a repetition of chapter IV, which was in fact an anticipation of the results established in later chapters. I shall therefore only refer to Table 1 as the briefest possible summary of terminological details. On the general level, however,

[153] Since our texts are school-texts and not practitioners' notebooks this may seem their only possible aim. The occurrence of problems of the third degree for which the Babylonians knew no general solution, and which are therefore treated by non-generalizable tricks, show that another aim was possible and in fact also present at least occasionally: That of demonstrating the mock ability of the teacher. Cf. also above, note 144.

the somewhat floating character of the terminology should be remembered. Only as a first approximation can it be called "technical". It appears not to have been stripped completely of the connotations of everyday language, nor does it possess that stiffness which distinguishes a real technical terminology. We should rather comprehend the discourse of the mathematical texts as a highly standardized description in everyday language of standardized problem situations and procedures, and we should notice that the discourse is never more, but sometimes less standardized than the situation described.[154] As everyday life contained no second-degree problems (be it the life of a professional scribal surveyor or accountant), terms taken from everyday language would of course have to be applied differently when describing procedures of second-degree "algebra" than in other texts. In as far as the use in such other texts is taken to represent the "basic meaning", the terms of the "algebra" texts will appear in the quality of standardized metaphors,–whence that impression of a technical terminology which is conveyed by standard problems.

The Sumerographic writings inside the otherwise Akkadian mathematical texts presents us with a special interpretative problem. Are they not to be interpreted as technical terminology?

In order to answer this question we have to distinguish different sorts of Sumerographic writing. On the one hand we have a restricted number of terms which are invariably written in Sumerian: uš, sag, a-šà, igi, íb-si$_8$, ba-si$_8$. Even inside this group there is a certain variability, ba-si$_8$ and igi giving rise to Akkadian loanwords and hence spoken with certainty as Sumerian words, and a-šà being often provided with phonetic complements and hence probably spoken in Akkadian. None the less, these terms can be regarded as technical and free of everyday connotations, as it is made especially clear when uš and sag used outside the basic representation are suddenly replaced by corresponding Akkadian words (cf. note 75).

Then we have the large number of pseudo-Sumerian writings, where Sumerograms are used logographically. In as far as the logographic meanings of these Sumerograms are not specifically reserved for mathematical texts they are no more and no less technical than the Akkadian words which they replace, or, alternatively, they are technical with respect to the scribal craft but not with regard to mathematics.

Finally we have a domain of indeterminate extension, that of Sumerograms used as possible alternatives for Akkadian writing but used ideographically. We have met one indubitable instance, viz. zi quoted in Akkadian as an infinitive in TMS XVI, which proves that the category is not empty. But this was an exceptional case, and other instances may be impossible to disclose. Especially the very compact and very ungrammatical Sumerographic writing of the series

[154] Seen in a long-run perspective this is of course also true of modern mathematical terminology. New theoretical developments give rise to new applications of old terms. Just think of a creature like the "infinite-dimensional vector space", in which at most "infinity" can still claim a classical value. Since the time when mathematical terms were given precise definitions, however, every extension by analogy and metaphor constitutes a clear and definite break. This was apparently different in Babylonian mathematics, which saw no absolute conceptual border-line between standard-situation and analogous extension.

texts (ungrammatical both from an Akkadian and from a Sumerian point of view) may be suspected to belong here.

The remainder of the present chapter shall deal with two questions of more general character: The relations of our Old Babylonian discipline to the categories of later mathematical thought, and its relation to the intellectual style of its own age.

Throughout this chapter I have spoken of Old Babylonian "algebra", not algebra. But was Babylonian "algebra" an algebra? Put in this form the question will of course have to be answered by a definition, which is not in itself a very fruitful way. We shall learn more by asking, in which respects Babylonian "algebra" was similar to Medieval or post-Renaissance algebra?

We should start from the outside, observing the uses to which the Babylonian discipline was put–and not put. In later times, algebraic techniques have been used to find the solution to problems which could not be solved by direct computation. We have no Babylonian texts which suggest such uses of the naive-geometric "algebra". On the contrary, the specious problems which had to be constructed in order to give occasion for the display of "algebraic" second-degree techniques suggest that no real uses were known. The abundance of realistic manpower- and brick-problems demonstrate that the Babylonian schoolmasters did nothing to hide a possible real-life importance of their teaching. "Algebra" never served to find a numerical value unknown in advance. In that respect its function was very different from that of algebra.

Recognition of this difference should not force us into the opposite extreme, and should not make us believe that naive-geometric "algebra" was nothing but an investigation of certain numerical properties of squares and rectangles, a peculiar sort of geometry. In chapter I I introduced the concept of a "basic conceptualization". The uš and sag are indeed basic in the sense that they are used to represent other quantities, the arithmetical relations between which can be mapped by the relations between the lengths and widths of rectangles. In YBC 6967 we have seen how a pair of numbers with known product and difference was represented by the dimensions of a rectangle, made visible in the text by the explicit reference to a "surface". Other texts would show a wide variety of quantities being represented as linear quantities, more or less explicitly mentioned. Especially interesting are certain cases where the text appears to distinguish between the linear extensions of a real figure, supposed, we may guess, to be situated in the terrain, and the corresponding extensions of a representing figure (drawn perhaps in the dusty schoolyard), even though the two coincide numerically.[155] Naive-geometric analysis of quadrangles is hence used as a means to

[155] This is the most probable implication of the distinction between "length" and "true length" in TMS XVI (section VII.3). In BM 13901 N° 14, the "confrontation" spoken of in the statement and that inherent in the procedure can also be seen to be kept apart through the multiplication by 1 in rev. I 9 (section VIII.1). Finally, TMS XIX appears to designate a "representing length" 1 as the "counterpart" of the "real length" 1 (cf. below, note 176).

Abstract distinction between a mentally conceived "real entity" and an equally mental "representing entity" may be too abstract to be expected in a Babylonian context. A reasonable guess would be that the traces of an explicitly distinguished representation are also traces of a concrete, material representation.

solve problems from other domains, be they artificial and the solutions known beforehand to exist as regular numbers. Though "algebra" was in all probability not used instrumentally in nonartificial situations, it was obviously taught as a virtual instrument.[156]

In virtual use and scope, "algebra" was hence related to real algebra. Can a similar claim be made for its "essence", its internal structure and characteristics? In a criticism of the unreflected use of the modern term to characterize a Babylonian discipline M. Mahoney has listed three characteristic features of developed algebra[157]: Firstly, the employment of "a symbolism for the purpose of abstracting the structure of a mathematical problem from its non-essential content"; secondly, the search for "the relationships (usually combinatory operations) that characterize or define that structure or link it to other structures"; thirdly, abstractness and absence of all "ontological commitments".

Taken at the letter, and allowing only for divergence "by degree rather than kind", these features are only valid and only meant to be valid for post-Vietan algebra understood as a scientific discipline. Already Medieval or more recent practitioners' algebraic calculation will only deserve the label "algebraic approach". In the same strict language, Old Babylonian "algebra" is algebraic "in approach": It cannot be claimed to possess a real symbolism. Still, even if the uš and sag are no more symbols than the Diophantine ἀριθμός or the Medieval thing, their use as ingredients of a "basic representation" serves precisely if only implicitly "the purpose of abstracting the structure of a mathematical problem from its non-essential content". Secondly, a number of systematic texts (especially among the series texts, but even BM 13901 can be mentioned) are in fact systematic investigations of the relationship characterizing the uš-sag-structure. Only the third criterion is not fulfilled even tendentially–unless we will claim that the use of a common basic representation is already virtual abstraction.

The "essence" of algebra can also be approached in another way, which links the beginnings of scientific algebra more clearly to the Medieval Art of Algebra and to the practitioners' algebra of the Modern era. In his "Introduction to the Analytic Art", in which Vieta aimed at bringing to light the hidden gold of algebra and almuchabala, he found the true essence of that art in the Ancient Method of Analysis, "assuming that which is sought for as if it were admitted [and working] through the consequences [of that assumption] to what is admittedly true".[158] This is exactly what we teach school children to do when solving an equation: "You treat x precisely as if it were an ordinary number". Apart from the known values used for identification purposes during explanations, but

[156] There is no reason to be overly astonished or scandalized on behalf of the poor scribe school students on this account. Apart from a modest (not to say infinitesimal) minority of the school children who have been taught second-degree algebra during the latest $3^1/_2$ millennia, their situation has been exactly the same, when not worse. Unless you make interpolation in trigonometrical or similar tables, physics at least at the level of Galilean ballistics, or something similar, second-degree algebra can only be used to train second-degree algebra.

[157] Mahoney 1971: 372.

[158] Chapter 1, ed. Hofmann 1970: 7; I follow Witmer's translation (1983: 11). Vieta cites Theon's definition of analysis. The gold metaphor is found in the dedicatory letter (ed. Hoffman 1970: xi).

not as steps in the mathematical argument (cf. TMS IX, part C), it is also a precise description of the Old Babylonian procedures. In this respect, too, Old Babylonian "algebra" is therefore algebraic, or at least characterizable as "naive-geometric analysis".[159]

Was "algebra" then an algebra? If we apply M. Mahoney's criteria, it was not. Babylonian mathematics differed more than in degree from the discipline founded by Vieta and continuing through Descartes and Noether. But it was "algebraic in approach", belonging in full right to any family which is able to encompass both al-Khwārizmī, Cardano and Noether. Anybody using confidently the expression "Medieval algebra" can with equal confidence speak of "Babylonian algebra".

Instead of relating our subject to categories of later times we may compare it to the general cognitive style of its own time, thereby regarding it as one aspect of the thought of its times, on an equal footing with others.

In their introduction to a famous "essay on speculative thought in the Ancient Near East"[160], H. and H. A. Frankfort characterize it as "mythopoeic". There are several facets to the concept, but its main implication is that the phenomenal world is no object, no "it": it is a "thou", an animated individual. In as far as this is an adequate description it excludes a scientific cosmology in the modern sense, a cosmology extrapolated under theoretical guidance from rational experimentation and hence in the final instance from technological practice. (I agree with any critical mind who finds this description short-circuited.) In this sense, it is true, we find no scientific cosmology in Ancient Mesopotamia. In the same sense it is indeed difficult to connect a scientific cosmology to any poetical or religious world-view, and so far it is therefore not obvious that the domination of cosmology by myth should imply that Ancient Mesopotamian thought in general be mythopoeic.[161]

Now, not everything in Babylonian thought was speculative; much of it was founded on social practice[162] or on technological practice. In both of these, and especially in the latter, the object-aspect of the external world, which under this view is not just "phenomenal", must be expected to impose itself. It is therefore not astonishing that it seems "difficult to accept [mythopoeiecy] as an adequate characterization" of "the intellectual adventure of ancient man" as "documented in the corpus of administrative, commercial, technical and other genres".[163]

[159] We observe that even the argument by a single false position is a primitive sort of analysis albeit arithmetical. Take e.g. the problem that a "heap" and its fourth is 15. For lack of an x permitting us to rewrite the 15 as $1^1/_4x$ one takes the number to be known, viz. as 4, etc.

[160] H. Frankfort et al. 1946: 3–27.

[161] Precisely this question is raised regarding Babylonian mathematical thought by Mahoney (1971: 370).

[162] That even large parts of mythology were founded on social practice has been argued by Jacobsen (1976; and already in H. Frankfort et al. 1946: 125–219). A proverb like "Workmen without a foreman are waters without a canal inspector" demonstrates clearly that Babylonian overseer-scribes were as able to see their fellow beings under the aspect of objects as their myths were to see nature as a fellow being (H. Frankfort et al. 1946: 203).

[163] Larsen 1987: 205.

Our algebraic texts constitute another exception to the presumed mythopoeic rule. Truly, AO 8862 carries an invocation of the scribal goddess Nisaba on its edge; but this and other similar inscriptions are totally isolated from the rest of the text, which treats its subject not as a "thou" having the "unprecedented, unparalleled, and unpredicatable character of an individual, a presence known only in so far as it reveals itself"[164], but as a fully predictable, manipulable and comprehensible object. No wonder, since Babylonian algebra was definitely not "speculative", i.e "regarding", but active, technical construction. According to the Frankforts' dichotomy it is "modern", dealing with lenghts, widths and surfaces and with its problem-situations as "objects and events [. . .] ruled by universal laws which make their behavior under given circumstances predictable", and which "can always be scientifically related to other objects and appear as part of a group or a series".[165]

This does not mean that Babylonian mathematics and technical thought in general was modern, only that its difference from modernity cannot be grasped by the Frankfort dichotomy. Nor should the secular rationality of Hammurapi's "Code" make us mistake this collection of concrete decisions for an abstract, general law-book in the style of Roman law.[166] A recent investigation of the cognitive character of Babylonian divination science[167] tries to get beyond such mistakes through reference to C. Lévi-Strauss's distinction between "hot" and "cold" societies, between the "savage" and the "domesticated" mind, between "the science of the concrete" and that of "abstract thought", illustrated by the distinction between the "bricoleur" (a cross-breed between the "tinkerer" and the "Jack of all trades") and the engineer.[168]

In the Lévi-Strauss illustration, engineering technology is thought of as developing specialized tools for the job to be done. The bricoleur, on the contrary, takes what happens to be at hand and fits it together as best can be done. "Domesticated" science and thought is seen analogously as building on abstract concepts; the "savage mind", on the other hand, classifies the categories and oppositions of e.g. their social world using pre-existent entities as classifiers and analogies.[169] While concepts are "wholly transparent with respect to reality", meaning *nothing but* their conceptual content, a pre-existent concrete entity used as a symbolizer is a *sign*, preserving to some extent the cultural meaning it possesses in itself and imparting it to those other entities for which it is used as a classifier.[170] (Being a member of the "Arrow Clan" may imply swiftness!)

In his investigation of the Babylonian lexical lists and omen literature, M. T. Larsen comes to the conclusion that many features (the search for classifying

[164] H. Frankfort–H. A. Frankfort, in: H. Frankfort et al. 1946: 5.

[165] Ibid.

[166] See Renger 1976: 229 and passim. The validity of the description is not affected by the discussions whether the decisions were considered paradigmatic or not.

[167] Larsen 1987.

[168] Lévi-Strauss 1972: 16ff.

[169] The opposition between day and night can thus be used as an analogy or "model" for the two moieties of a tribe; clans labeled after animals are part of common lore, not signifying, however, that the clan members assume descent from the real animal in question, but affinity in some higher sense (cf. ibid. 142f. and 149).

[170] Ibid. 20

order and the postulate of direct causation, partly built on recorded experience and partly on analogic thought) can be described as "savage". Other features of the omen literature are, from its Old Babylonian beginnings, better described as "semidomesticated": The intent to engineer the future, the attempt to make exhaustive listings of all possible omina (which presupposes writing, a main domesticator) and the way in which lacunae in the empirical record are filled out by means of abstract, logical rules–rules which are in fact formulated explicitly in a Neo Assyrian compendium. All in all, however, the global logic of the divination prevented the apparent steps toward "domesticated science" from leading to any ultimate breakthrough.

How are we then to regard Old Babylonian mathematics? Is it also "lukewarm", blocked midway between a neolithic "cold" society and our modern "hot" world?

Several features, at least, look "savage". It was claimed time and again in the preceding chapters that a pattern of thought was "concrete", which sounds very much like the classification by means of pre-existent, concrete entities used as signs. But let us look at the "concrete" argument in VAT 8389 No 1. In this case "concreteness" means that the mathematical structure is thought in terms of the real entities involved. There is no distinct, concrete signifier, no sign imparting to the "meadows" any characteristics beyond those of possessing an area and to yield a specified rent per area unit. "Concreteness" simply means "absence of any explicit abstract signifier or abstract calculating scheme" (no x or ἀριθμός, no standardized "double false position").

In second-degree problems like those of BM 13901 or AO 8862 (the "basic representation" itself) we see the same sort of concreteness. "Naive geometry" consists precisely in taking geometrical entities at their phenomenal face value, without submitting them to theoretical reflection through which their properties and mutual relationships might be formulated as abstract principles.[171]

In cases where something else is dealt with by means of a mapping on the basic representation, be it the number pairs of a table of reciprocals, prices, or real linear extensions, we seem to come closer to the use of concrete entities as signs. Even here, however, we should take care. There is no hint that a price represented through a length has anything in common with that line, except, precisely, the relevant characteristic, the measuring number. No text whatever suggests anything similar to the swiftness of the Arrow Clan. On the contrary, the representation is normally only visible through the designations of the operations performed ("breaking", "making span", etc.). Only ocasionally do we find a "surface" or a "true length", etc. In its function, the basic representation can be regarded as an abstract instrument.

[171] The definitions, axioms and postulates of the *Elements* are precisely such a set of abstracted principles, and the deductive build-up of the whole work constitutes a conscious attempt to build the complete argument on these. Truly, the abstracted system is not complete, as it is well known, and at times "naive" knowledge is made use of implicitly; and conversely it is obvious that Old Babylonian "naive" geometry is full of implicit abstraction: assumptions on the calculability of areas as products, knowledge of arithmetical rules, etc. Neither observation affects the fact that we have to do with fundamentally different projects.

Places where the description of "savage thought" is really relevant for Old Babylonian algebra are its terminology, and hence its operations. Like Lévi-Strauss's "concepts", technical terms are "wholly transparent", meaning nothing but their direct technical implication. They have no connotations. Like his "signs", descriptive metaphors, even when used in a standardized way as long as the situation itself is standard, carry a load of everyday connotations, causing e.g. its users to "tear out" rather than "break off" a square from another square. The terminology being only partly technicalized, we might characterize it as "semi-savage".

A second "semi-savage" aspect of Old Babylonian algebraic mathematics is constituted by the series texts. As I have not dealt with them above, I shall only state briefly that the listings of large numbers of variations on the same type of equation is a parallel to the way all possible liver shapes are listed in the omen lists, and to the lexical lists. But it is no perfect parallel. While the lists are first of all additive and aggregative listings, introducing hierarchical ordering only in so far as this reflects "the surrounding highly stratified society"[172], the series texts are constructed in main sections, first order subdivisions, and cartesian products of second-order subdivisions.[173]

In the case of the omen text, the Neo-Assyrian compendium formulating explicit, abstract rules was an unprecedented innovation, at least as far as the written record has been excavated. In mathematics, the corresponding step can be demonstrated to have been taken already by the late old Babylonian period, viz. on the Susa text TMS XVI, which furthermore looks very much as a written documentation of a sort of didactical explanation which would normally be given orally. Didactical explanation does not in itself constitute theoretical reflection on abstract principles, and it was thus no step leading automatically to abstract, deductive mathematics. But it was a starting point from which a critically inquisitive intellectual environment might have been able to proceed indefinitely long. Sticking to the cold-hot metaphor we may say that Old Babylonian algebra was after all not only "lukewarm" but also inflammable. Further development of the discipline was not blocked by any immanent intellectual structure *reflecting* the over-all social and intellectual climate, as was the case of divination science. The blocking factors *resided directly* in global social and intellectual conditions: The scribal school was only moderately inquisitive and definitely not critical; the prime reason for interest in mathematical knowledge beyond the requirements of direct utility was professional pride and social prestige rather than curiosity and openness to the infinite possibilities of an unknown world. Furthermore: By the end of the Old Babylonian era, the scribal environment changed socially and intellectually, cutting off even the supplies for that sort of mathematical research which had been undertaken until then.[174]

172 Larsen 1987: 211.

173 The system is clearly visible in the symbolic transcriptions of three sections of VAT 7537 in MKT I 474f.

174 For the motivations of Old Babylonian non-utilitarian mathematical activities, cf. above, note 144. The changes after the end of the Old Babylonian era are discussed in my 1980: 28f.

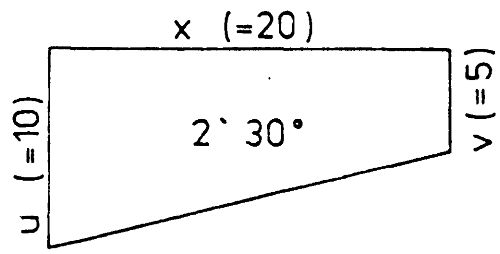
X. *The legacy*

So, after the end of the Old Babylonian era, second-degree algebra vanishes from the documentary horizon for many centuries–as do in fact all specific traces of mathematics teaching. That does not mean, however, that Old Babylonian mathematics was a complete mathematical dead-end without consequences for later mathematical cultures. On the contrary: though rarefied for a millennium below the level of archaeological visibility, the Old Babylonian tradition was to excert its influence on several of the sources of Modern mathematics.

Before looking directly at the evidence for such influence we shall, however, investigate yet another Old Babylonian text, one in which the conceptual dynamics of Old Babylonian algebra can be glimpsed.

X.1. A possible shift in the conceptualization: IM 52301 N° 2 (Baqir 1950a, improved transliteration in Gundlach–von Soden 1963: 252f.)

The text in question is problem N° 2 from IM 52301, perhaps the youngest of the (northern) Tell Ḫarmal mathematical tablets. It deals with a real geometric trapezium[175], and reduces the problem to one of "surface and confrontations equal to number". Besides being a beautiful specimen of "representation", the text is interesting because of its deviations from normal usage, which suggest a tendency toward changing or looser conceptualizations. It runs as follows (the marginal drawing is not in the tablet):



Obverse

16. *If to two-third of* the *accumulation of* the *upper* width
šum-ma a-na ši-ni-ip ku-mu-ri sag *e-li-tim*

$^2/_3 \cdot (u-v)+10=x(=20)$

17. *and* the *lower*, 10, *to my hand*[a] I have appended: 20 the length *I have built*. The width
ù ša-ap-li-tim 10 *a-na qa-ti-ia* daḫ-*ma* 20 uš *ab-ni* sag

$u-v=5$

18. {. . .}[b] the *upper*, *over* the *tower* 5 *goes beyond*.
{e-li} e-li-tum e-li ša-ap-li-tim 5 *i-te-er*

$\frac{u+v}{2} \cdot x=2^{\backprime}\,30^{\circ}$

19. The surface is 2` 30°. *What my* lengths? You, by your saying[b], 5 *which it goes beyond*
a-šà 2, 30 *mi-nu-um* uš-*ia* za-e TUK-zú-dè 5 *ša e-te-ru*

Putting $u+v=Z$:
$x={}^2/_3 \cdot Z+10$
$(Z/2) \cdot ({}^2/_3 \cdot Z+10)$
$=2^{\backprime}\,30^{\circ}$
or, with an adequate choice for α:

20. 10 *which you have appended*; 40′ *of* the *two-third*, *my factors of both*(?)[c]; *inscribe:*
10 *ša tu-iṣ-bu* 40 *ši-ni-pé-tim a-ra-ma-ni-a-ti-a lu-pu-ut-ma*

[175] From the mathematical structure alone, Bruins' interpretation (1966: 207ff.), viz. a triangle cut by a transversal, cannot be excluded. But the expression "upper length" in rev. 17 speaks definitely against it, as does the absence of partial areas from the statement.

$(Z/2)\cdot(Z+2\alpha) = (^2/_3)^{-1}\cdot 2^{\backprime}\,30^\circ = 3^{\backprime}\,45^\circ$

21. The igi of 40′ *of* the *two-third detach:* 1° 30′ *you see.* 1° 30′ {. . .
i-gi 40 *ši-ni-pé-tim pu-ṭú-ur-ma* 1, 30 *ta-mar* 1, 30 {*ḫi-pí*(?)-*ma*

22. . . .}[b] *to* 2` 30°, the surface, *raise:* 3` 45° *you see.*
4⸢5 *t*⸣*a-mar* 45} *a-na* 2, 30 a-šà *i-ši-ma* 3, 45 *ta-mar*

$Z\cdot(Z+2\alpha)=7^{\backprime}\,30^\circ$

23. 3` 45° *repeat:* 7` 30° *you see.* 7` 30° *your head*
3, 45 *e-ṣí-ma* 7, 30 *ta-mar* 7, 30 *ri-iš-ka*

$\alpha=\{(^2/_3)^{-1}\cdot{}^1/_2\}\cdot 10 = 45'\cdot 10 = 7^\circ\,30'$

24. *may retain. Turn back.* The igi of 40′ *of* the *two-third detach*
li-ki-il tu-ur-ma i-gi 40 *ši-ni-pé-tim pu-ṭú-ur*

Reverse

1. 1° 30′ *you see.* 1° 30′ *break:* 45′ *you see; to* 10 *which you have appended*
1, 30 *ta-mar* 1, 30 *ḫi-pí-ma* 45 *ta-mar a-na* 10 *ša tu-iṣ-bu*

2–4. *raise:* 7° 30′ *you see* {. . .}[b]
i-ši-ma 7,30 *ta-mar* {7, 30 *ri-iš-ka li-ki-il tu-ur-ma* i-gi 40 *pu-ṭú-ur-ma* 1, 30 *ta-mar* 1, 40 *ḫi-pí-ma* 45 *ta-mar a-na* 10 *ša tu-iṣ-bu i-ši-ma* 7, 30 *ta-mar*}

$Z^2+2\cdot 7^\circ\,30'\cdot Z = 7^{\backprime}\,30^\circ$

5. 7° 30′ the *counter* {. . .}[b] *part lay down: Make span:*
7, 30 *me-eḫ-*{*ša*}*-ra-am i-di-ma šu-ta-ku-il-ma*

$(Z+7^\circ\,30')^2 = 7^{\backprime}\,30^\circ + 56^\circ\,15' = 8^{\backprime}\,26^\circ\,15'$

6. 56° 15′ *you see.* 56° 15′ *to* 7` 30° *which your head*
56, 15 *ta-mar* 56, 15 *a-na* 7, 30 *ša-ri-iš-ka*

7. *retains append:* 8` 26° 15′ *you see.* The equilateral[d]
ú-ka-lu ṣí-ib-ma 8, 26, 15 *ta-mar* ba-se-e

$Z+7^\circ\,30' = \sqrt{8^{\backprime}\,26^\circ\,15'} = 22^\circ\,30'$

$Z = 22^\circ\,30' - 7^\circ\,30' = 15$

8. of 8` 26° 15′ *make come up:* 22° 30′ *its equilateral*[d]; *from* 22° 30′
8, 26, 15 *šu-li-ma* 22, 30 *ba-su-šu i-na* 22, 30

9. the equilateral[d] 7° 30′, *your takīltum, cut off,*
ba-se-e 7, 30 *ta-ki-il-ta-ka ḫu-ru-uṣ₄*

$\dfrac{u+v}{2} = Z/2 = 7^\circ\,30'$

10. 15 the *left-over.* 15 *break:* 7° 30′ *you see,* 7° 30′ the *counterpart lay down:*
15 *ši-ta-tum* 15 *ḫi-pí-ma* 7, 30 *ta-mar* 7, 30 *me-eḫ-ra-am i-di-ma*

$\dfrac{u-v}{2} = 5/2 = 2^\circ\,30'$

11. 5 *which* width *over* width *goes beyond break:*
5 *ša* sag *e-li* sag *i-te-ru ḫi-pí-ma*

$u = \dfrac{u+v}{2} - \dfrac{u-v}{2} = 7^\circ\,30' + 2^\circ\,30' = 10$

12. 2° 30′ *you see.* 2° 30′ *to* the *first* 7° 30′ *append:*
2, 30 *ta-mar* 2, 30 *a-na* 7, 30 *iš-ti-in ṣí-im-ma*

$v = \dfrac{u+v}{2} + \dfrac{u-v}{2} = 7^\circ\,30' - 2^\circ\,30' = 5$

13. 10 *you see; from* the *second* 7° 30′ *cut off.*
10 *ta-mar i-na* 7, 30 *ša-ni-im ḫu-ru-uṣ₄*

	14. 10 the *upper* width; 5 the *lower* width. 10 sag *e-li-tum* 5 sag *ša-ap-li-tum*
Proof: $u+v=10+5=15$	15. *Turn back:* 10 *and* 5 *accumulate,* 15 *you see.* *tu-ur-ma* 10 *ù* 5 *ku-mu-ur* 15 *ta-mar*
$^2/_3\cdot(u+v)=10$	16. The *two-third of* 15 *take:* 10 *you see, and* 10 *append:* *ši-ni-ip-pé-at* 15 *le-qé-ma* 10 *ta-mar ù* 10 *ṣí-ib-ma*
$x=10+10=20$ $\frac{u+v}{2}=7^\circ\ 30'$	17. 20 *your upper* length. 15 *break:* 7° 30′ *you see.* 20 uš-*ka e-lu-um* 15 *ḫi-pí-ma* 7, 30 *ta-mar*
$\frac{u+v}{2}\cdot x=2^{\backprime}\ 30^\circ$	18. 7° 30′ *to* 20 *raise:* 2‵ 30°, the surface, *you see.* 7, 30 *a-na* 20 *i-ši-ma* 2, 30 a-šà *ta-mar*
	19. *So* the *having-been-made.* *ki-a-am ne-pé-šum*

[a] I.e. a number 10 which is "at my disposition" without being defined in relation to the figure.

[b] The text contains a number of repetitions, other erroneous insertions etc. due to faulty copying. Those of obv. 18 and rev. 5 were already pointed out by T. Baqir. Those of obv. 21f. and rev. 2–4 (the first of which has been induced by the phrase, 1,30 *ta-mar* 1,30 common to obv. 21 and rev. 1, while the second is provoked by the 7,30 *ta-mar* common to obv. 23 and rev. 2) follow from analysis of the procedure.

The reading of zú as a homophonic mistake for zu in obv. 19 was given in von Soden (1952a: 49). That of TUK as dug$_4$ was suggested by Baqir (1950a: 146).

[c] "factors of both" is a tentative translation of *aramaniātum,* a plural form known from nowhere else. The term is an epithet to 40′, which multiplies the sum of the widths. The term thus appears to suggest two (identical) factors multiplying the members of a sum. In agreement with this, von Soden (1952a: 50) suggests conjecturally the word to be a loanword from Sumerian ara-man, "times"-"two", i.e. "factors of both".

[d] The "equilateral" of rev. 7–9 is written in syllabic writing. In rev. 7 and 9, the form is BA.SE.E, indicating that the form normally written ba-si$_8$ (which alternates with íb-si$_8$) was pronounced in Sumerian. (In a similar fashion, the text writes a syllabic i-gi instead of the normal igi.) In rev. 8, the form is a nominative with suffix, *ba-su-šu,* suggesting an Akkadianized form *basûm.* The accusative form in rev. 7 could in principle be a construct state of the same form, but the genitive in rev. 9 cannot, since the rest of the text is written with full mimation. It must render a genuine Sumerian pronunciation of the term.

Both forms confirm, as does the homophonic shift from si$_8$ to si in certain texts, that the term was not read as a logogram for an Akkadian word (*mitḫartum* being the normal assumption), at least not when used for the extraction of a square-root.

In AO 17264 (late Old Babylonian or early Kassite) the forms *ba-si-e-šu* and *ba-si-šu* are found (MKT I, 127). Even here, the equilateral is "asked for" (*šâlum*).

Before drawing any conclusions from the way the text formulates its subject-matter we should of course make sure that this subject-matter is understood correctly. Is the interpretation in the marginal commentary adequate, apart from the anachronism inherent in the use of modern algebraic symbolism? Should we not instead expect that the problem was seen as one in two unknowns (a "length-width"-problem) the product and difference of which are known (Z and $Z+2\alpha$, in the symbolism of the margin)? Or, if it is to be understood in terms of one unknown ("surface and confrontations"), is the average width $\left(\frac{u+v}{2}=Z/2\right)$ not the entity which would normally be chosen by a Babylonian?

Both answers should probably be answered by "yes"; we should perhaps expect the problem to be comprehended in two unknowns, and if not, the average rather than the aggregated width would be a normal Babylonian unknown. But in the first case we would also expect that the difference between the two be really calculated; instead, the scaling factor 1° 30′ is bisected before the multiplication is performed, without any other reason calling for that sequence of operations. In the second case, the operation in obv. 23 would have been a "raising", the normal scaling multiplication (cf. section V.5, BM 13901 No 3), and that of rev. 10 would have been a reverse scaling. Instead, the first is a "repetition" and the second a "breaking", concrete operations which indicate that operations belonging with the standard procedure are only found from obv. 24 to rev. 9, and thus that the sum of the widths, i.e. the *15* found in rev. 9, is the quantity looked for in that procedure. All normal Babylonian habits notwithstanding, the marginal commentary appears to map the original procedure.

If we look at the formulation of the text, it is obviously close to the style known from Old Babylonian algebra in general, so much so, in fact, that only lack of feeling for the stylistic implications of the naive-geometric procedures (most notably the identification of the 7,30 of rev. 9 as a *takīltum*, i.e. as the same as that of rev. 5) has prevented earlier investigators of the text from identifying correctly the dittographies of obv. 21f. and rev. 2–4.

Apart from the erroneous repetitions (which are obviously due to copying errors and which therefore presuppose the existence of a more correct original) and the syllabic writings of Sumerian terms there are, however, certain deviations from normal usage which can hardly be explained unless we assume some slackening of normal conceptual habits.

Firstly, the term "building" is employed in obv. 17 when the length is explained to be equal to the sum of the widths and an extra amount of *10*. It is not excluded that a constructive procedure is still intended, but in that case a mental construction is more plausible than an actual drawing. In any case, the formulation deviates from a normal usage which appears to be strongly bound up with specific procedures.

Secondly, a "counterpart" turns up in rev. 10 in a most unusual function. Normally, it is seen in length-width-problems (cf. YBC 6967, section V.1), when two sides containing a completed square are "laid down", for subtraction and ensuing addition of the *takīltum*.[176] In the present case, addition and subtraction

[176] "Normally", but not exclusively, it is true. In TMS XIX, a number 1 is posed (in a "single false position") for the (real) "lenght" of the problem, and next also for its

of a semi-difference is still meant, but if a geometrical configuration is at all thought of, it is different, the "original" and the "counterpart" being opposing widths of a rectangle, which the addition and subtraction are to transform into a trapezium.

These peculiarities do not prevent a naive-geometric interpretation. Moreover, the "doubling" in obv. 23 suggests the use of a procedure related to a trick used in the two tablets VAT 7532 and VAT 7535 (both in MKT). The suggested procedure is shown in Fig. 16: The step of obv. 21f. corresponds to a scaling in horizontal direction (the first transformation, A→B). The repetition in obv. 23 is a genuine duplication, transforming the trapezium into a real rectangle (B→C), viz. a "surface (of a square) with 15 confrontations". The sequence of operations

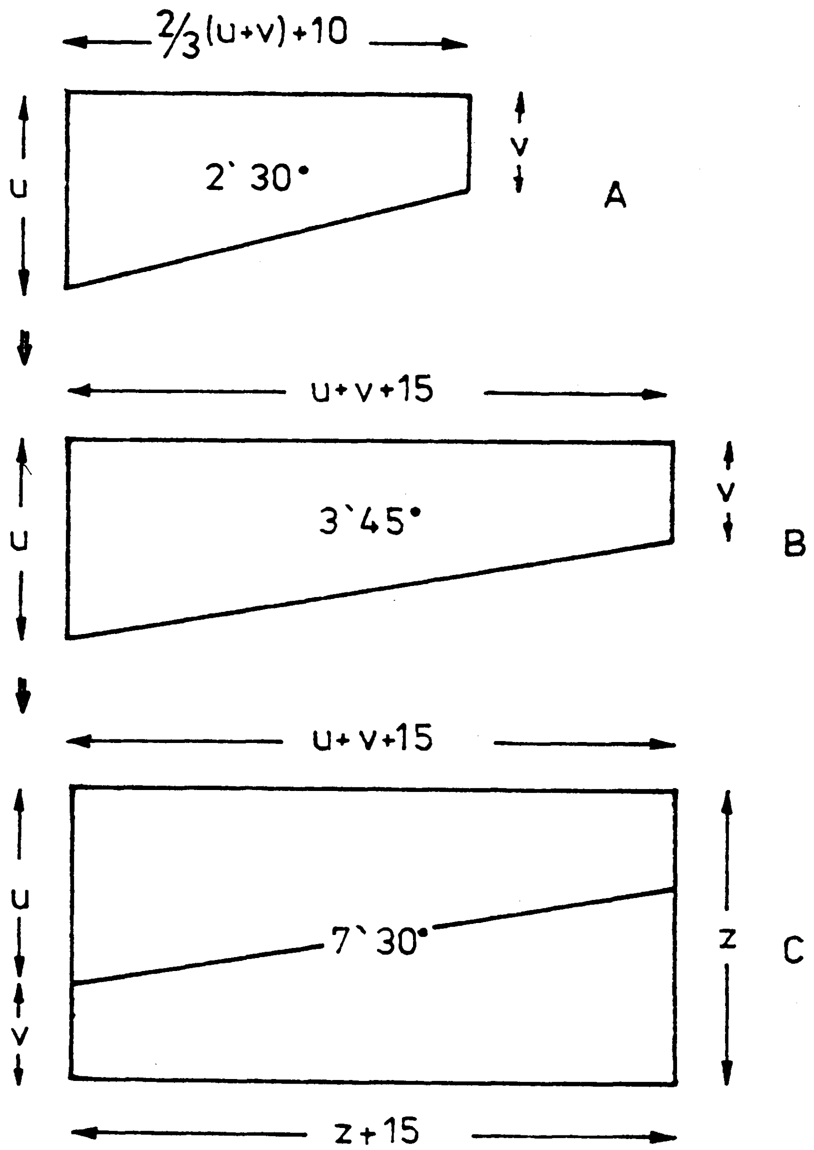


Figure 16. The geometrical interpretation of IM 52301 N° 2 suggested by the parallels in VAT 7532 and VAT 7535.

"counterpart" (TMS, 101, as corrected in von Soden 1964: 49), which in the following turns up to be its "basic representative". In TMS IX 40 (above, section VIII.3) as well as TMS XII 10 (TMS 79, as corrected in von Soden 1964: 49), and rev. 5 of the present text, the "original" and the "counterpart" form the usual geometric configuration, but already at the point where they are "made span" a supplementary square, not when the side of the completed square is found.

An occurrence in IM 55357, 10 (Baqir 1950) is still more deviant but need not occupy us here, as it has to do with a triangle.

is, however, remarkable. If the geometrical procedure had been performed physically, it would have been natural to make the very palpable doubling first, and the scaling afterwards. The actual sequence appears to indicate that a more purely arithmetical understanding of the underlying structure, where the sum of the widths is aimed at as an unknown (in the first transformation) before it is actually produced (in the second transformation).

The deviant use of the term "building" was already mentioned as an indication pointing in the same direction. The implications of the peculiar use of "counterpart" in rev. 10 are more indefinite, and the most that can be said is that an otherwise strict conceptual structure appears to be loosening, especially if we notice that the term is also used in a somewhat more orthodox way in rev. 5. The way the text regards the "equilateral" is, however, yet another indication that an arithmetical conceptualization is present: It is definitely no entity producing a square–it is something which "comes up", i.e. a numerical result.[177]

The awareness of a homomorphism between geometrical and arithmetical procedures need not have been greater with the author of the present text than with the authors of more orthodox, somewhat older texts. The latter, however, formulate themselves strictly within the geometrical conceptualization. This strictness of language has either been regarded as superfluous or has not been understood by the present author. In both cases it is justified to speak of a loosening of the conceptualizations and of an opening toward explicit arithmetical understandings.

X.2. Seleucid arithmetization: BM 34568 N° 9 (MKT III, 15)

Further developments of this opening toward arithmetic are seen in the algebra problems of the Seleucid era. A simple instance is found in BM 34568 N° 9, the very problem which was used in Chapter I to demonstrate the ambiguities of current translations. In transliteration and conformal translation, the text runs like this:

Obverse II

$x+y=14$ $x\cdot y=48$	1. Length *and* width accumulated[a]: 14, *and* 48 the surface. uš *ù* sag gar-[*m*]*a* 14 *ù* 48 a-šà
$(x+y)^2=3'\ 16^\circ$ $4\cdot x\cdot y=3'\ 12^\circ$	2. The NAME[b] I *know* not. 14 steps of 14, 3' 16°. 48 STEPS[c] of 4, 3' 12°. MU nu-zu[ú] 14 a-rá 14 3, 16 48 GAM 4 3, 12
$(x-y)^2=(x+y)^2-4xy$ $=3'\ 16^\circ-3'\ 12^\circ$ $=4$	3. From[d] 3' 12° (to) 3' 16° go up[e]: 4 *remains*[f]. *What* STEPS *of what*[g] 3, 12 -ta 3,[1]6 nim-*ma ri-ḫi* 4 *mi-nu-ú* GAM *mi-ni-i*

177 The same expression is found in the contemporary and equally northern tablet Haddad 104 (al-Rawi – Roaf 1985) and in the late Old Babylonian or perhaps even early Kassite AO 17264 (MKT I 126). Db_2-146 (Baqir 1962), which is also contemporary with the present text, regards the "equilateral" as something which is to be "taken", presumably also as a numerical result.

$x-y=\sqrt{4}=2$	4. *shall I* GO[h] *so that*[i] 4? 2 STEPS of 2, 4. From 2 (to)
$(x+y)-(x-y)=14-2$	14 go up: 12 *remains.*
$=12=2y$	*lu*-rá-*ma lu* 4 2 GAM 2 4 2-ta 14 nim-*ma ri-ḫi* 12
$y=\frac{1}{2}\cdot 12=6$	5. 12 TIMES 30′, 6 the width. To[j] 2 *add*[k] 6: 8, 8 the
$x=(x-y)+y=2+6=8$	length.
	12 GAM 30 6 6 sag 2-še 6 *ta-ṭip-pi-ma* 8 8 uš

[a] "accumulated" translates GAR, which is certainly an abbreviation for gar-gar, not as in Old Babylonian texts a logogram for *šakānum*, "to pose".

[b] NAME translates MU, used logographically for *šūmum*. F. Thureau-Dangin's interpretation as a logogram for *aššum*, "since" (TMB, 59) is possible, but it does not fit the context. O. Neugebauer's interpretation "name" is, on the other hand, confirmed by the Susa text TMS IX.

[c] STEPS translates GAM, which in the contemporary mathematical table text MM 86.11.410 is used as a separation sign (see MCT, 15). In the present tablet, the sign appears to be used as a complete equivalent for a-rá, "steps of" (so also in the contemporary AO 6484 – MKT I, 96–99).

[d] "from" translates the Sumerian ablative-/instrumental suffix -ta.

[e] "go up" translates the Sumerogram nim, which in certain Old Babylonian texts was used as a substitute for íl ~ *našûm*, "to raise", i.e. "to calculate by multiplication". Here the term appears in the original Sumerian meaning, used to describe a subtraction conceptualized as a counting process.

[f] "*remain*" translates *riāḫum*, "übrig bleiben".

[g] The first "how much" (*mīnum*) is a nominative, while the second is a genitive (*mi-ni-i*). So, the two factors in a product by GAM (and, as revealed by obv. I, 16f. of the same tablet, by a-rá) play different roles. It is this construction which has suggested my standard translation for a-rá (cf. section IV.3).

[h] "GO" translates rá, "to go" (TÚM in MKT). This supports the conclusions of notes c and g.

[i] "so that" translates the optative and precative partivle *lū* (also used to denote the precative form of the ideogram rá in the same line, "*shall I* GO").

[j] "to" translates the Sumerian terminative suffix -šè.

[k] "*add*" translates *ṭepûm*, "hinbreiten, auftragen; addieren", which in Late Babylonian had taken the place of *waṣābum*, "to append" (cf. von Soden 1964: 48a). In contradistinction to *waṣābum*, however, *ṭepûm* can be used as a symmetric term, *ṭepûm a together with b*. So, the modernizing connotations of the translation "to add" seem quite to the point.

First of all we observe that certain parts of the vocabulary are continuous with that of our Old Babylonian texts: "length", "width", "surface", "name", "steps of". All except "steps of" belong on the level of algebraic problems, not on that of mere computation. We can therefore be sure that we are really confronted with a descendant of the Old Babylonian algebraic tradition, in spite of the silence of all sources between c. 1600 B.C. and c. 300 B.C.

The next observation will be that of thorough change on all levels, in spite of the continuity. It goes down to the choice of Sumerograms: nim, which in Old Babylonian texts designates a multiplication of the "raising" class, standing presumably for forms of *ullûm* (cf. note 39), is used now for the stepwise counting of a difference, presumably as a logogram for *elûm*. In part, at least, the Sumerianization of mathematical language appears not to have been continuous over the silent millenium.[178]

The discontinuous Sumerianization carries implications for the nature of the transmission, which appears to have taken place in a practitioners' environment rather than a scholarly institution. As far as the conceptual structure of Seleucid algebra concerns it has less to tell. Under the latter aspect, indeed, the absence of all traces of constructive thought and not least the purely arithmetical formulations are the most conspicuous features. Subtraction has become a straight counting process, instead of a concrete process described metaphorically in physical terms ("tearing out", "cutting off", etc.). Only one multiplicative operation is left, described by the term of multiplication tables, i.e., as a repeated counting, when not by the ideogram GAM, the separation sign used apparently as a purely visual symbol. Bisection is no special operation, but only a multiplication by 30′, and the square-root is explicitly asked for as the solution to the problem $x \cdot x = n$. Two additive processes appear to be present, but the one corresponding to "appending" can no longer be identity-conserving, since it is often, though not here, symmetrical with respect to the addends. No doubt, therefore, that the conceptualization of the problem is completely arithmetical.

As discussed at some length in chapter I, an arithmetical conceptualization does not exclude a geometrical method and justification. This combination is precisely what is found in al-Khwārizmī's justifications. A figure which would serve to solve the problem was shown in Fig. 2, and the same figure and a generalized version will in fact explain all problems of the tablet, except one dealing with alloying of metals and one concerned with a rectangle of known proportions (see Fig. 17). Moreover, even the more specious procedures are easily argued from the two all-purpose figures, and in one case, that of N° 13, O. Neugebauer feels obliged to have recourse to Fig. 17B[179] in order to explain why the procedure is at all meaningful. On the other hand, several of the solutions are very difficult to follow unless one uses either geometric support or written, symbolic algebra – purely rhetorical methods will not do. It is therefore reasonable to assume that the method of Seleucid second-degree mathematics remained geo-

178 Another case of re-Sumerianization is that of tab. In Old Babylonian mathematics, it was used as a logogram for *eṣēpum*, "to repeat"; in the present tablet (e.g. obv. I 2) it is used for *ṭepûm*, "to add". Both uses are in agreement with the general meaning of the Sumerian term; in their technical use, however, the two functions of the ideogram cannot be connected in any way, which excludes any continuous existence of tab as a mathematical term.

179 Of course in symbolic transcription (MKT III 21). The important thing is that the entity $(l+w+d)^2$ cannot be avoided in the interpretation of the procedure.

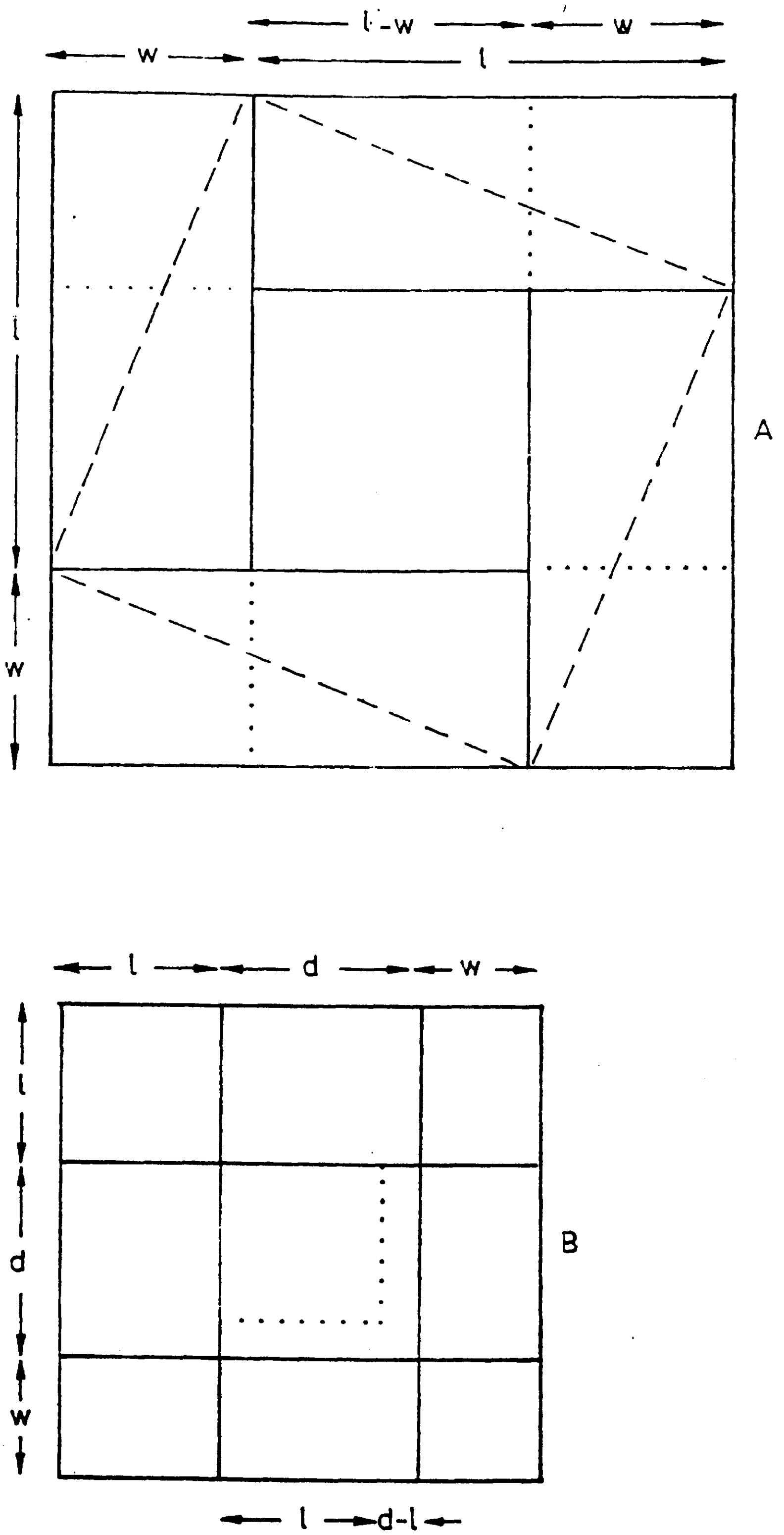


Figure 17. Two all-purpose figures which may support all the second-degree problem solutions of BM 34568. The upper figure will be recognized as a familiar justification of the Pythagorean theorem. For use of the lower figure, where d is the diagonal of a rectangle with length l and width w, one shall remember that the central square equals the sum of the upper left and the lower right square ($d^2 = l^2 = w^2$). In problem 12, the equality of the lower right square and the central gnomon will have to be used explicitly.

The upper figure is seen to contain Figure 14A, the one constructed for AO 8862 N° 3. It will be remembered (see above, note 138) that the same configuration appears to be used in two other Old Babylonian problems.

metric, in spite of the arithmetization of its conceptualization, though probably "synthetic" rather than analytically constructive.

It is tempting to see the arithmetical conceptualization as the final outcome of a natural process already begun during the late Old Babylonian period: Secular use of the same procedures would grind off everything superfluous and leave back only the essential structure, which is indeed arithmetical. Before accepting this as sole and sufficient explanation we should, however, be aware that another factor was also at work, and perhaps even a third circumstance should be taken into account.

The indubitable extra factor is the specific scholarly environment of Seleucid mathematics: The great astronomical centre of Uruk.[180] The enormous numerical calculations performed in this centre may well have made the local scribes more inclined toward arithmetical thought than less specialized practitioners of the algebraic art whoever they may have been. But as we shall see below, such practitioners must have existed.

The possible extra factor is cultural cross-fertilization. Seleucid Uruk was part of the Hellenistic melting-pot, and links back to Old Babylonian traditions should therefore not be taken to exclude combination with other links. In another branch of Seleucid mathematics, viz. mensurational geometry, a definite break with Old Babylonian methods and a striking parallel to Alexandrinian geometry is clearly visible.[181]

In the procedure of our problem there may also be a suggestion of cultural import. All corresponding Old Babylonian problems find the semi-sum and the semi-difference between length and width, even those which appear to make use of the same geometrical configuration. In the present case, the total sum and difference are found. There is no inherent reason for that change. In a group of more orthodox second-degree problems in the Seleucid tablet AO 6484, dealing with *igûm-igibûm*-pairs with known sum[182]–as far as mathematical structure concerns no different from the present problem–, we find indeed the traditional semi-sums and semi-differences, together with a terminology which is about as arithmetical as that of the present problem.[183]

180 AO 6484, the other Seleucid tablet containing second-degree problems, was indeed written by Anu-aba-utēr, an early 2nd-century scribe from Uruk, known as possessor and writer of astronomical and other tablets. See the colophone in MKT I 99, and Hunger 1968: 40 (N° 92) and passim. If the algebraic tradition was really transmitted since the Old Babylonian period in an environment of "higher artisans", as suggested above, the circle of the Uruk astronomer-priests may be the setting where its re-Sumerianization took place.

181 In Old Babylonian mensuration, the area of an irregular quadrangle had been found by the "surveyors' formula", as the product of "average length" with "average width" (see e.g. YBC 4675, in MCT 44f.). In the Seleucid tablet VAT 7848 (MCT 141) the height of a trapezium is calculated by means of the Pythagorean theorem, and everything goes exactly as in Hero's *Geometrica* **16**, 17. New evidence suggests, it should be observed, that the development toward greater precision in mensuration may have taken place before the possible interaction with Greek geometry; indeed, unpublished Late Babylonian tablets contain the explicit calculation and use of the height of a triangle [Friberg (forthcoming) §§ 5.4c and 6.5].

182 Rev. 10–27 (4 problems in total). In MKT I 98f.

183 The subtraction of "surfaces" carries a *libbi*, "inside"; but the subtractive term itself

A purely autochthonous development would probably have affected the method of all isomorphous problems similarly. It is therefore plausible that the specific methods of BM 34568 were introduced together with a specific cluster of length-width-diagonal-problems during the dialogue of scientific cultures.

It is not possible to identify the eventual interlocutor. Similar interest are found in China, in the *Nine Chapters on Arithmetic.*[184] But they are also found in the Graeco-Roman world[185], and in neither case are the similarities complete nor fully convincing. Furthermore, the Hellenistic era was one of wide-range cultural connections, from China to Magna Graecia. The suggestive similarities can at most be taken as indications that mutual inspiration took place, and that Babylonia was probably not the only focal point for "algebraic" investigations of geometric figures.

X.3. Babylonian influence in Greek mathematics?

The hypothetical foreign inspiration of Seleucid algebra is difficult to trace precisely. So are also the possible inspirations flowing the other way during Antiquity and the early Middle Ages. Certain suggestions can be found, however, in Greek sources pointing to inspiration though hardly to direct descendency.

The idea of inspiration from Babylonian algebra to Greek "geometric algebra", i.e. the geometry of "Elements II" etc., is as old as the discovery of Babylonian second-degree algebra. Since the late 1960es it has been submitted to severe criticism[186], mainly because the Greek geometry of areas is a coherent structure of its own which is not adequately explained as a "translation" of an arithmetico-rhetorical algebra, of which it is neither an isomorphic nor a homomorphic mapping.

A naive-geometric reinterpretation of Babylonian algebra changes much of the foundation of the debate.[187] If we recognize further that the structure of Greek geometry is the result of a process and not identical with the structure of its possible inspirations, the question of Babylonian inspiration of Greek mathematics is completely open again.

This is not the place for a thorough investigation of the problem, which I approach elsewhere.[188] I shall just point to the observation which put me on the track. The much-discussed term δύναμις has given rise to precisely the same ambiguities as the Babylonian *mithartum.* In some contexts it seems to mean "square-root" or "side of square", in others it is the square itself. As in the Bab-

is lal, "diminish", and the addition is expressed simply by *u*, "and", and tab, "add". Multiplication is comprehended as "going steps".

184 Translated by Vogel (1968—the relevant problems are found pp. 90—103).

185 One source is a Greek papyrus from the 2nd century A.D. (Rudhardt 1978, cf. Sesiano 1986). Another is a Latin *Liber podismi* (latest edition in Bubnov 1899: 510—516), dating perhaps to the 4th century A.D. and based apparently on Alexandrian sources. One of its problems (ibid. 511f.) deals with a right triangle, for which the hypotenuse and the area are known. The solution is of "Seleucid" type, making use of total sum and total difference.

186 I shall only refer to Szabó 1969; Mahoney 1971; and Unguru—Rowe 1981.

187 See my 1983 (review of Unguru—Rowe 1981).

188 See my 1988.

ylonian case, the apparent ambiguities are eliminated if we read the term as "a square identified by (and hence with) its side". The normal Greek habit is to identify a figure with its area; as with us, a square designated τετράγωνος has a side and is its area. The δύναμις is thus a foreign flower in the Greek conceptual garden.

Investigation of a variety of mostly early sources suggests that the term was not only used in theoretical geometry but also by calculators, seemingly in connection with some sort of algebraic activity, an earlier stage of the tradition behind Diophantos. Links to the theory of figurate numbers are also suggested, and hence to a pebble-abacus-representation of naive-geometric procedures (cf. above, the end of chapter VI).[188a]

Another possible line of transmission of Babylonian influence goes to the pre-Diophantine algebraic tradition. I have already pointed at the similar ways in which the Babylonians and Diophantos deal with non-normalized problems, and other similarities could be found in that tiny part of Diophantos' "Arithmetica" which possesses cuneiform parallels. Such similarities are, however, fairly inconclusive, since the subject-matter itself restricts the range of possible procedures strongly. Supplementary evidence may, however, be hidden in a much-discussed term of the "Arithmetica", the πλασματικός, which occurs in I.xxvii, I.xxviii and I.xxx of the surviving Greek part, and in the Arabic IV.17, V.19 and V.7. In the Greek text, it seems to be the diorism, i.e. the condition for solvability which is called πλασματικόν, while the Arabic passages speak of the whole problem as belonging to the class of *al-muhayya'ah*.[189]

The Greek term derives from πλάσσω, "to form", "to mold", etc., and it is related to πλάσμα, "anything formed or molded, image, figure" etc. (GEL 1412a). Because of this etymology and the Greek passages alone, P. Ver Eecke suggested it to mean that the diorism can be demonstrated geometrically.[190] Since a reference to Euclidean geometry fits badly to the distribution of the term in the Arabic books, both editors of the Arabic text have looked for alternative ways to get a meaning of the term in its actual contexts.[191] Here again, however, the naive-geometric view-point changes the basis of the question. We already know a πλάσμα, a fixed figure or "mold" on which the diorisms of the three Greek passages can be seen immediately; viz. the upper square in Fig. 17 (quartered as in Fig. 14, since Diophantos uses semi-sums and semi-differences). Moreover, the

[188a] In this connection it may be of some interest, but is of course inconclusive, that the method of BM 34568 N° 9 is better suited for treatment by pebbles than the traditional semi-sum/semi-difference procedure, which fails if the sum or, equivalently for integers, the difference is odd.

[189] The first Arabic passage is grammatically impossible as it stands. Rashed (1984: III 27) prefers a minimal correction, which makes the term an epithet to a number. Sesiano (1982: 99 note 48) makes a more radical emendation, in order to obtain agreement with a backward reference in the next passage and with his own interpretation of the term. The first but not the second of these considerations seems compelling to me, which makes me accept that part of Sesiano's correction which makes IV 17 a parallel to IV 19 (whence also to V.7).

[190] Ver Eecke 1926: 36 note 6. There is no reason to go further into the details of his explanation.

[191] Rashed 1984: 133—138; Sesiano 1982: 192f.

diorism of the Arabic V.7 can be seen on the three-dimensional analogue of the same figure.

The diorisms of the Arabic IV 17 and IV 19 are of a different character, involving factorizations of the sides of cubes. There are no direct links to specific Babylonian material. On the other hand, certain techniques used for the computation of large reciprocal tables and the techniques of scaling are akin to the Diophantine procedure. Since at least the Arabic text does not claim that these and none but these problems possess a distinctive mathematical quality but only states that they belong to a certain pre-established bunch of problems possessing the quality, we should perhaps interprete the term as designating problems the feasibility of which is seen by certain naïve-geometric procedures, not necessarily by Diophantos but at least by the people who established the bunch. The interpretation is not compelling, nor is however any rival explanation. A hint of a Babylonian connection may—but need not—hide behind the term and the concept.

X.4. A direct descendant: *Liber mensurationum*

If inspirations from Babylonian algebra to Greek mathematics can only be traced indirectly, through the combination of many sorts of roundabout evidence, influences in Medieval Islamic mathematics are direct and easily verified.

Once more, I shall only sketch the basis of the argument, since I deal with the matter in detail elsewhere.[192] The central source is a Latin translation made by Gherardo di Cremona in the 12th century from an Arabic original due to one otherwise unidentified Abū Bakr, the *Liber mensurationum*.[193] The first parts of the work deal with squares and rectangles (the later parts, related to Alexandrian practical geometry, do not concern us here). It was already noticed by H. L. L. Busard in his edition that the work shares many problem-types and even the coefficients of certain problems with Babylonian algebra (making no distinction between Old Babylonian and Seleucid material). This, however, is not conclusive. Starting from the simplest cases you will necessarily hit upon many of the same problem-types when progressing toward more complex algebraic problems, and if you prefer, e.g., the second-simplest to the simplest Pythagorean triangle, your numbers will be 6, 8 and 10.

The first decisive observation is that many problems are solved twice, first by a method given no specific name and hence to be regarded as the normal, fundamental method, and next by *aliabra*, obviously a term meant to render the Arabic *al-jabr*. In a general sense of the word, both methods are equally algebraic. *Aliabra*, however, refers directly to the fundamental cases known from al-Khwārizmī. It is hence the rhetorical discipline known from al-Khwārizmī and ibn Turk[194] and also referred to by Thābit ibn Qurra in his "Rectification of the cases of *al-jabr*".[195] In several cases, the numerical steps of the fundamental method and the alternative by *aliabra* are identical. The difference between the two must therefore be one of representation and conceptualization.

[192] See my 1986.
[193] Critical edition by Busard (1968).
[194] See Sayılı 1962.
[195] See Luckey 1941.

The next observation is that the discursive organization of the descriptions of the "fundamental" procedures coincides down to the choice of grammatical tense and person and to the use of certain standard phrases ("since he has said"; "may your memory retain") with the familiar structure of Old Babylonian texts. The procedures are also often those known from the Old Babylonian texts, e.g. the "change of variable" of AO 8862 N° 1. The standard length-width-problem is solved by means of semi-sum and semi-difference, showing that the connection of the text is really directly to the Old Babylonian tradition, bypassing the Seleucid astronomical school.

A closer look at the vocabulary shows that the conceptual distinctions known from the classical Old Babylonian tradition are not respected completely. So much remains, however, that we have good reasons to believe that a naive-geometric method is still behind the numerical algorithms described in the text. A final "See" after many procedure-descriptions indicates that the original has indeed contained (naive-)geometric justifications of the methods.[196]

These observations are the main but not the sole reasons to see the fundamental approach of the text as a direct continuation of an Old Babylonian naive-geometric tradition, which must then have been alive until the Arabic original was written, probably not much later than A.D. 800. Even in Abū Kāmil's *Algebra*, dating from c. A.D. 900, an alternative to the normal *al-jabr* procedure is sometimes offered[197] which contains the typical Old Babylonian steps, though in arithmetico-rhetorical disguise. More striking is, however, a passage in Abū'l Wafā''s *Book on What is Necessary from Geometric Construction for the Artisan*, written shortly after A.D. 990. In chapter 10, prop. 13, the author tells that he has taken part in certain discussions between "artisans" and "geometers", apparently regarded as coherent groups. Confronted with the problem of adding three equal geometric squares, the sum also being a square, the artisans proposed a number of solutions, "to some of which were given proofs", proofs which turn out to be of cut-and-paste character. The geometers too had provided a solution in Greek style, but that was not acceptable to the artisans, who claimed a concrete rearrangement of parts into which the original squares could be cut.[198]

[196] One may wonder that so many linguistic observations can be made on a Medieval Latin translation. The reason is that Gherardo's translation appears to be extremely literal, reflecting even some peculiarities in the original usage which could easily have been straightened without loss of mathematical substance.

[197] See Levey 1966: 94, 96.

[198] See Krasnova 1966: 115ff. This Russian translation is the only printed version of the work, although selections and paraphrases from incomplete manuscripts have been published by Woepcke (1855) and Suter (1922: 94—109). Though not algebraic the whole treatise is highly interesting as an eclectic merger between a Near-Eastern naive-geometric tradition and Greek apodictic geometry. Abū'l-Wafā's treatise is a main source for the establishment of a connection between the cut-and-paste technique and the later theory of partition of figures. Another work of possible interest in this connection is a short treatise on the Pythagorean theorem written by Thābit ibn Qurra (description in Sayılı 1960, Arabic text and Turkish translation in Sayılı 1958). The first part of the treatise describes two proofs of the theorem by means of *al-tafṣīl wa'l-waṣl*, "partition and combination" (Sayılı 1958: 535 l. 7). The figures are, however, different from those connected to the Babylonian tradition; they look rather as generalizations of that used by Socrates in Plato's *Meno*, and the method is indeed described

A striking feature of the *Liber mensurationum* is the recurrence of problems adding or subtracing the four sides of a rectangle or square from the area (or reversely); other multiples of the sides do not occur. In a problem collection derived from surveying and surveyors' interest this comes as no great surprise. As in Old Babylonian mathematics, inhomogeneous second-degree-problems could only arise as artificial constructions, and most easily as recreational problems. But a funny problem in surveying is one which adds the area and all four sides of a square field rather than one which (like BM 13901 N° 2) adds $^2/_3$ of the area and $^1/_3$ of the side. Recreational problems in general are not characterized by mere complexity and artificiality but first of all by striking coincidences. This observation is part of the evidence for the above claim that the aberrant problem 23 from BM 13901 (section V.4) was taken over from a surveyors' tradition and adopted into the school tradition, perhaps even as the source for the interest in inhomogeneous second-degree "algebra".[199]

As regards the methods of the *Liber mensurationum*, it is noteworthy that the trick used in AO 8862, problems 1 and 2, is used time and again. These early problems, we remember, were formulated as "surveying anecdotes". Their methodological affinity with the late surveying tradition can thus be regarded as supplementary evidence that Old Babylonian school "algebra" and the *Liber mensurationum* both derive from a common, older mensuration tradition.

In chapter I I used al-Khwārizmī's naive-geometric justifications of his algorithms as a pedagogical device, in order to demonstrate what naive geometry would look like. At the present stage of the investigation it turns out that the old naive-geometric tradition was still alive when al-Khwārizmī wrote his seminal compendium on algebra. We can hardly assume that he invented anew a technique which was widely practiced around him, and we can therefore be confident that his justifications were direct descendants of those of the Old Babylonian calculators. We may guess that even his arithmetico-rhetorical *al-jabr* derives ultimately though highly transformed from the same source, but there we have no direct evidence. Through his justifications, however, we know that the ancient techniques were passed on to Medieval Islam and to the early European Renaissance, and hence to the modern world.

as „Socratic" by Thābit; not being able to follow the text, I am thus not sure about its implications.

[199] If this hypothesis is correct, the tradition will have been carried by Akkadian speakers, according to the explicitly Akkadian *eqlam* introducing BM 13901 N° 23. This fits "the Akkadian" method as a name for the quadratic completion (TMS IX, see section XIII.3). It also agrees with the Akkadian language of the whole Old Babylonian mathematical tradition which, as observed repeatedly above, is visible even in its use of quasi-Sumerian logograms. Old Babylonian school mathematics was—like omen literature which is likewise written in Akkadian—new as a school tradition, but it may well have older oral roots. A Sargonic tablet bisecting a trapezium [Friberg (forthcoming), section 5.4.K] suggests that it goes back at least to the 23d century B.C. The present hypothesis on the relation between Old Babylonian school mathematics and the surveyors' tradition is argued in somewhat more detail in Høyrup 1989a: 28f.

TABLE OF CONTENTS

B

"Zur Frühgeschichte algebraischer Denkweisen". *Mathematische Semesterberichte* **36** (1989), 1–46.

Zur Frühgeschichte algebraischer Denkweisen

Von JENS HØYRUP in Roskilde (Dänemark)

Hans Wüssing zum 60.Geburtstag zugeeignet

Frühes algebraisches Denken: Notwendige Vorbemerkungen zum Begriff

'Geschichte der Algebra' seit der Vorantike: Gab es denn seit 4000 Jahren ein abgegrenztes und als abgegrenzt anerkanntes Ding *Algebra*? Sicherlich nicht. Oder gab es wenigstens etwas, das, obwohl nicht als Algebra und vielleicht nicht mal als 'Ding' anerkannt, sich doch über die Jahrhunderte zum heutigen Algebra entwickelte - also eine 'Geschichte der algebraischen Denkweise'? Auch das nicht, schon weil es kaum heute *eine* einfach abgrenzbare 'algebraische Denkweise' gibt: 'Algebra' deckt die Praxis der Lösung arithmetischer Gleichungen; sie deckt aber vielmehr die *Theorie* solcher Gleichungen und ihrer Lösbarkeit; unter Mathematikern schließlich deckt sie heute hauptsächlich die *Generalisierung* solcher Theorien, d.h. die Gruppentheorie und ihre vielen Extensionen. Diese Bedeutungen sind natürlich (wenigstens paarweise) verwandt, sind aber kaum durch *einen* Begriff zu erklären. Vielmehr muß man von einer Familie gegenseitig abhängiger *algebraischer Denkweisen und Methoden* sprechen. Aus der *Geschichte der Algebra* wird dadurch eine *Geschichte algebraischer Denkweisen*, deren gegenwärtige Verflechtung unseren offenen Algebrabegriff ausmacht.

In dem Maße, wie wir die Geschichte rückwärts verfolgen, löst sich diese Verflechtung allmählich auf. So steht die ganze Klassifizierung irrationaler Größen im 10.Buch der *Elemente* und die Erforschung ihrer Inversion geistig der modernen Gruppentheorie sehr nahe; sie hat aber kaum irgendeine Verbindung mit der *Arithmetica* Diophants, deren *Gleichungslösung mit Elementen impliziter Gleichungstheorie* ebensogern als algebraisch betrachtet werden kann - geschweige denn mit der *al-jabr* des al-Khwārizmī, die uns den Namen der Disziplin gegeben hat[1]).

Wir beschäftigen uns im folgenden mit algebraischen Denkweisen in vorgriechischen Traditionen, besonders im frühen Babylonien, und in den 'subwissenschaftlichen' Praktikertraditionen, die mit der griechischen 'wissenschaftlichen' Mathematik gleichzeitig sind, ohne davon viel geprägt zu sein.

Zuerst aber ein Wort über die Begriffe 'wissenschaftlich' und 'subwissenschaftlich'. Sie beschreiben eine *Orientierung des Wissens* und sind keine Qualitätsurteile. 'Wissenschaftlich' ist eine systematische Verfolgung von Wissen *um des Wissens willen* über die Ebene des Alltagswissens hinaus (inhaltlich oder in seinem inneren Zusammenhang) - 'theoretisches' (d.h. 'betrachtendes') Wissen im griechischen Sinne, *know-why* mit einem Ausdruck aus moderner Zeit; sie wird sich notwendigerweise bemühen, das Wissen so weit wie möglich explizit zu machen. 'Subwissenschaftlich' ist die Erwerbung und Tradierung von Spezialistenwissen *um seiner Brauchbarkeit willen;* die Griechen würden von einer *Techne* sprechen, das 20.Jahrhundert von *know-how*. Im Prin-

1) Die Vielseitigkeit und Undefinierbarkeit der 'algebraischen Denkweisen' und die Auflösung ihrer garantierten Verbindung im Mittelalter und in der Antike erklären die vielen Auseinandersetzungen über den algebraischen Charakter dieses oder jenes Gebietes (sei es babylonische 'Algebra', diophantische 'Arithmetik' oder griechische 'geometrische Algebra'), macht sie aber auf der anderen Seite auch ziemlich sinn- und zwecklos.

zip darf das Wissen einer subwissenschaftlichen Tradition implizit bleiben, obwohl natürlich die Umstände des Anlernens der 'Lehrlinge' wenigstens eine orale Explizitierung von vielem hervorzwingt (umgekehrt mag auch vieles innerhalb einer wissenschaftlichen Tradition in der Praxis implizit, weil völlig unproblematisch, bleiben)[2].

Unmittelbar möchte man annehmen, daß die Unterscheidung mit der Unterscheidung zwischen 'reinem' und 'angewandtem' Wissen identisch sei. Das stimmt nicht ganz. Erstens mag selbstverständlich 'theoretisches' Wissen sehr anwendbar sein, obwohl es um seiner selbst willen erworben wurde - es gibt bekanntlich 'nichts praktischeres als eine Theorie'[3]. Zweitens, und weniger diskutiert, gibt es auch 'reine' Auswüchse der subwissenschaftlichen Traditionen - besonders auffällig im Gebiet der subwissenschaftlichen Mathematik. Sie gehen unter den Namen von 'Scherzaufgaben' oder 'Unterhaltungsmathematik'.

Andererseits sind 'wissenschaftliche reine Mathematik' und 'subwissenschaftliche reine Mathematik' in dem Ursprung ihrer Fragen völlig verschieden. Als Paradigma der ersten kann man nochmals das 10.Buch der *Elemente* in Zusammenhang mit der voreuklidischen Geschichte der griechischen Irrationalitätstheorie erwähnen. Die Entdeckung der Inkommensurabilität führte zur Formulierung bisher ungeahnter Fragen: Wie kann man Strecken konstruieren, die mit einer gegebenen Strecke (oder deren Quadrat mit einem gegebenen

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2) Aus dieser Definition folgt, daß 'wissenschaftliches Wissen' gegebenenfalls weniger inhaltreich und schlechter organisiert sein mag als 'subwissenschaftliches Wissen' - man vergleiche z.B. Nikomachos' ziemlich triviale neupythagoreische Arithmetik mit der babylonischen 'Algebra', die unten beschrieben wird.

3) Nach der Entstehung der Ingenieurwissenschaften im 19.Jahrhundert und der Verwissenschaftlichung vieler sozialer Praktiken im 20.Jahrhundert wird deshalb auch die Distinktion ziemlich sinnlos. Ihre Rolle ist, die Andersartigkeit früherer Wissensorganisationen verständlich zu machen.

Quadrat) inkommensurabel sind? Welche Arten von Größen gibt es denn überhaupt? Und wie verhalten sie sich zueinander? Die ersten zwei Fragen werden schon in Platons *Theaitetos* 4) bearbeitet; alle drei liegen hinter der Euklidischen Irrationalitätstheorie. Die Entwicklung *theoretischer reiner Mathematik* entsteht also als *Antwort auf offene Fragen;* um diese Antwort möglich zu machen, müssen oft (wie auch im eben erwähnten Beispiel) neue Methoden und Begriffe geschaffen werden.

Dieses Verhältnis von Methode und Frage wird in der 'subwissenschaftlichen reinen Mathematik', d.h. in der Unterhaltungsmathematik, umgekehrt. Hier werden *schon vorhandene Methoden* ausgenutzt, um zu zeigen, was man damit alles machen kann - *auch über das Notwendige hinaus.* Der Zweck ist, bei anderen Erstaunen oder Bewunderung oder bei sich das Gefühl eigener Geistesstärke zu wecken; dem Schüler professionelles Selbstbewußtsein einzuflößen; oder seine Fähigkeit in der Ausübung seiner Profession nachzuprüfen: Mit einem Wort, *Virtuosität* vorzuführen5). Um dieses Zwecks willen werden Probleme aufgesucht, die mit den vorhandenen Methoden zwar lösbar sind, aber virtuose Beherrschung dieser Methoden fordern6).

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4) 147c7-148d7 (ed., transl. Fowler 1977). Was Theodoros da macht, ist von der ersten Frage inspiriert, während der junge Theaitetos (als erster, muß man nach dem Text glauben) die zweite angreift.

5) Das wird ganz deutlich von Christoph Rudolff in *Künstliche rechnung mit der ziffer* ... (1540) erklärt. Das Kapitel 'Schimpfrechnung' beginnt mit der Erklärung, wie 'Durch rechnung, *auch nit on sonders auffmercken der unwissenden,* zu ergründen wieuil einer pfenning, creutzer, groschen oder ander müntz vo: im ligen habe ..." (hervorhebung JH).

6) Es ist charakteristisch, daß die Sammlung arithmetischer Epigramme im XIV.Buch der spätantiken *Anthologia graeca* sowohl Aufgaben unterhaltungsmathematischer Art wie eigentliche, nichtmathematische Rätsel einschließt (ed. Paton 1979: V, 25-107).

Da die Aufgaben des praktischen professionellen Alltags dem geübten Praktiker schnell trivial werden, führt der Bedarf an Vorwänden für Virtuosität zur Konstruktion von Problemen jenseits des Praktischen, d.h. zur Entwicklung einer nichtanwendbaren 'reinen' Mathematik, deren Aufbau aber von ihren Methoden und nicht von ihren Problemen bestimmt wird.

Systematisch betriebene 'Wissenschaft' in diesem Sinn ist eine Errungenschaft der griechischen Antike und forderte wohl in ihrem Anfang das besondere geistige Klima der griechischen Kultur mit ihrer Kopplung von Rationalität, Hochachtung für zivilisierte Muße und Verachtung für die niedrigeren Professionen. Die 'Wissenschaften' der Bronzezeitkulturen wurden von den praktischen Professionen getragen und waren daher von subwissenschaftlichem Charakter. Das gilt auch für die Mathematik der babylonischen und ägyptischen Schreiber und Schreiberschulen - und gilt für diese auch im Sinne, daß ihre 'reine' Ebene von ihren vorhandenen Methoden und nicht von theoretischer Erforschung zentraler Probleme bestimmt wurde.

Es versteht sich unter diesen Umständen fast von selbst, daß die einzige in der Bronzezeitmathematik spürbare Art algebraischer Denkweisen die problem*lösende* ist und daß kein Interesse an Lösbarkeits- oder Strukturtheorien sich uns zeigt. Man findet in der babylonischen Mathematik viele Gleichungen, und auch (wie sich unten zeigen wird) viele begründete Auflösungen von Gleichungen - vielleicht auch explizite Erwähnung bestimmter Methoden. Man findet aber weder hier noch in den ägyptischen Quellen oder in der späteren Unterhaltungsmathematik irgendwelche Spur von Gleichungs*theorie*.

Ein Lieblingsverfahren für die Konstruktion komplizierter Aufgaben war die 'Umkehrmethode': Eine Aufgabe aus dem praktischen Gebiet wurde genommen; statt einer der im Alltag bekannten Größen wurde aber das Resultat als bekannt angenommen - im einfachsten Fall waren also Länge und Fläche eines rechteckigen Feldes statt Länge und Breite bekannt. Die Lösung folgte dann (jedenfalls in den komplizierteren Fällen) durch Verwendung einer 'analytischen' Methode[7]: Die unbekannte Größe wurde als ganz normale Größe angesehen, und mit ihr wurde wie mit anderen Größen umgegangen, bis sie aus ihrer Verknüpfung mit anderen Größen herausgelöst war. Das ist genau, was man auch jetzt bei der Auflösung einer arithmetischen Gleichung macht; damit wird wohl mit Recht seit Viète diese analytische Methode als zentrale 'algebraische Denkweise' angesehen.

Altbabylonische 'Algebra'

Die frühesten belegten Beispiele von Umkehr praktischer Aufgaben sind sumerisch und gehen auf die Mitte des 3.Jahrtausends zurück. Sie bauen auf umgekehrten Multiplikationen und berechnen z.B. aus einer Gesamtversorgung von 1152000 *sila* Gerste (1 *sila* ≈ liter) und einer Tagesration eines Arbeiters von 7 *sila* die Anzahl der Arbeitstage (nämlich 164571 Tage mit einem Rest von 3 *sila*), oder aus der bekannten Fläche und Breite die unbekannte Länge eines rechteckigen Feldes. Darin liegt schon ein möglicher rudimentärer Ansatz für die spätere analytische Methode; erstens aber nur ein *rudimentärer* Ansatz und zweitens (weil die Texte nur wenige und zweifelhafte Spuren der Denkart

7) Der Name und die Definition gehen auf die griechische Mathematik zurück. Siehe die Einleitung zum 7.Buch von Pappos' *Collectio* (ed. Hultsch 1876, transl. Ver Eecke 1933).

verraten) höchstens ein *möglicher* Ansatz[8]. Interessant und in Details verfolgbar wird erst die altbabylonische 'Algebra'.

Die altbabylonische Periode geht von etwa 1900 v.u.Z. bis etwa 1600 v.u.Z., und die 'algebraischen' Texte stammen etwa aus der zweiten Hälfte dieser Periode. Sie lösen schon von Anfang an komplizierte Aufgaben zweiten Grades und mögen deshalb auf eine ältere Tradition bauen; eine solche ist jedoch nicht belegt, und ihre Existenz ist durchaus hypothetisch. Wir werden deshalb wie die Quellen *in medias res* anfangen.

Die altbabylonische 'Algebra' wurde seit ihrer Entdeckung um 1930 als 'arithmetisch' angesehen: D.h., es wurde angenommen, daß die darin vorfindlichen Größen als *Zahlen* verstanden wurden; daß ihre Verknüpfungen als arithmetische Additionen, Subtraktionen und Multiplikationen betrachtet wurden; und daß die bei der Auflösung verwendeten Operationen ebenfalls als reine Zahlenoperationen angesehen wurden. Eine genaue inhaltliche, philologische und strukturelle Analyse der Texte zeigt jedoch, daß diese Interpretation nicht korrekt sein kann. Zum Beispiel gibt es mehrere additive Operationen, mehrere subtraktive Operationen und mehrere scheinbar multiplikative Operationen, die streng auseinandergehalten werden, und sich deshalb nicht einfach als synonyme Terme verstehen lassen (Beispiele werden unten gegeben). Die Gesamtargumentation fordert komparative Wort-für-Wort-Analyse vieler Texte und

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8) Die etwa 2500 v.u.Z. datierbare Gersteaufgabe, wo ein Fehler in einem von zwei Parallel-exemplare das Verfahren hervortreten läßt, wird in meinem [1982] nach Methode und Denkart analysiert. In diesem Fall zumindest ist die Methode nur im rudimentärsten Sinn analytisch, eher eine gestufte Abzählung. Andere Aufgaben aus dem 3.Jahrtausend wurden von Powell (1976) übersetzt und inhaltlich diskutiert.

läßt sich nicht in dem hiesigen Rahmen durchführen[9]. Statt dessen müssen wir uns mit einer Übersicht über die Hauptresultate und mit einigen typischen Beispielen begnügen.

Ein einfacher Text[10] läuft wie folgt in wörtlicher Übersetzung:

1. *Die Fläche und das Entgegengestellte habe ich zusammengelegt: 0;45 ist es.*
2. *1 das Herausragende setzt Du.*
3. *Den halben Teil von 1 brichst Du entzwei, 0;30 und 0;30 läßt Du einander [wie zusammenstoßende Seiten eines Rechteckes] halten.*
4. *0;15 fügst Du zu 0;45 hinzu: 1 macht 1 gleichseitig.*
5. *0;30, das Du [einen Rechteck] halten gelassen hast, reißt Du vom Leibe von 1 heraus: 0;30 ist das Entgegengestellte.*

Fast jedes Wort muß hier erklärt werden. Erstens die Zahlen. Die Babylonier verwendeten in ihren mathematischen Texten ein Stellenwertsystem mit Grundzahl 60 (ein 'Sexagesimalsystem') ohne Null und ohne Angabe von absolutem Stellenwert (dem Zahlsystem eines Rechenschiebers in letzterer Hinsicht also ähnlich). Der Verständlichkeit halber werden in der Transkription sowohl die fehlenden Nullen als auch die absolute Größenordnung angegeben; die Notation $a,b;d,e,f,...$ soll demnach $a\cdot 60^2+b\cdot 60^1+c\cdot 60^0+d\cdot 60^{-1}+e\cdot 60^{-2}+f\cdot 60^{-3}+...$ bedeuten; *0;30* steht also für *1/2* und *0;15* für *1/4*.

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9) Ein vorläufiger und wenig überschaubarer Bericht über die Untersuchung ist mein [1985]. Eine bessere Präsentation wird sich in meinem [1987] finden.

10) Aufgabe Nr.1 auf der Tafel BM 13901. Publiziert in MKT III, S.1. Die Übersetzung ist (wie alle folgenden, wo nichts anderes gesagt wird) meine eigene. Die Numerierung ist hinzugefügt; sie ist mit der Linienzählung nicht identisch.

Zweitens die Fachterminologie. Das 'Entgegengestellte' *(mitḫartum)* bedeutet als Figur ein Quadrat; seine Zahlengröße aber ist die Länge der Quadratseite (die ja ihresgleichen entgegengestellt wird). Wir können es als *einen* Begriff verstehen, wenn wir es als *Quadrat, bestimmt von* (und deshalb auch mehr oder weniger *begriffen als) seiner charakteristischen Seite*, auffassen - wie *wir* heute das Quadrat als *Flächenmaß und Fläche zugleich* bestimmen und begreifen.

'Zusammenlegen' *(kamārum)* ist eine symmetrische additive Operation, wo beide Addenden in ihre Summe aufgehen. Vermutlich muß sie öfters als eigentlich arithmetische Addition von Meßzahlen verstanden werden.

'Das Herausragende' *(wāṣītum)* ist ein architektonisch-geometrischer Begriff. In diesem Zusammenhang bezeichnet es eine geometrische Breite von 1, die, wenn sie an eine Strecke x angelegt wird, daraus eine rechteckige Fläche $x \cdot 1 = x$ macht.

Der Sinn von 'setzen' *(šakānum)* ist nicht ganz klar, und das Wort scheint nicht ganz eindeutig zu sein. Das 'Setzen' einer Größe dürfte jedoch immer ein materielles Festhalten sein, u.a. durch Eintragung in Ton (aber auch andere materielle Repräsentation scheint möglich zu sein). Festhalten *im Gedächtnis* wird dagegen als 'den Kopf halten lassen' bezeichnet.

'Entzweibrechen *(ḫīpum)* ist eine Operation, die eine konkrete Größe in zwei gleiche, ebenfalls konkrete 'halbe Teile' *(bāmtum)* teilt oder (wie hier) aus einer Größe einen der 'halben Teile' abtrennt.

'Einander [wie zusammenstoßende Seiten eines Rechteckes] halten lassen' ist eine etymologisch nicht ganz gesicherte Übersetzung von der kausativ-reziproken Verbalform *sutākulum*. Inhaltlich 'halten' jedenfalls a und b einander, wenn ein Rechteck mit zusammenstoßenden Seiten *a* und *b* 'gebaut' *(banûm)*, d.h. konstruiert wird[11].

'Hinzufügen' *(waṣābum)* ist eine asymmetrische additive (oder eher quasi-additive) Operation, wo unter Hinzufügung der konkreten Größe *b* an die ebenfalls konkrete Größe *a* die Identität letzterer bewahrt wird (wie die Identität eines Kapitals trotz Hinzufügung von Zinsen bewahrt wird - Zinsen, die übrigens auf babylonisch genau 'das Hinzugefügte' heißen).

'Gleichseitig machen' (i b - s i $_8$) ist eine Wurzelausziehungsoperation auf geometrischer Grundlage. 'Das Gleichseitige' kann wie 'das Entgegengestellte' das mit seiner Seite identifizierte Quadrat bedeuten. Der Satz '*A* macht *b* gleichseitig' bedeutet also, daß die Fläche *A* als Quadrat vorkommend die Seite *b* hat, d.h. $\sqrt{A} = b$.
'Herausreißen' *(nasāḫum)* ist die subtraktive Umkehroperation von 'Hinzufügen', also ein konkretes Wegnehmen, worunter die Identität der zu vermindernden Größe bewahrt wird. Der Ausdruck 'vom Leibe' *(libba*, eigentlich 'Herz' oder 'Eingeweide'*)* zeigt, daß die zu vermindernde Größe eine sehr konkrete Fülle besitzt und deswegen keine abstrakte Zahl sein kann.

Die Aufgabe handelt also von einem Quadrat, wo die Summe der Meßzahlen von Fläche und Seite gleich *3/4* ist. Die Seite ('das Entgegengestellte') wird zuerst mit einer 'herausragenden' Breite *1* versorgt, so daß sie in ein Rechteck verwandelt wird. Damit ermöglicht sich der Rechner eine geometrische Interpretation des Zusammenlegens

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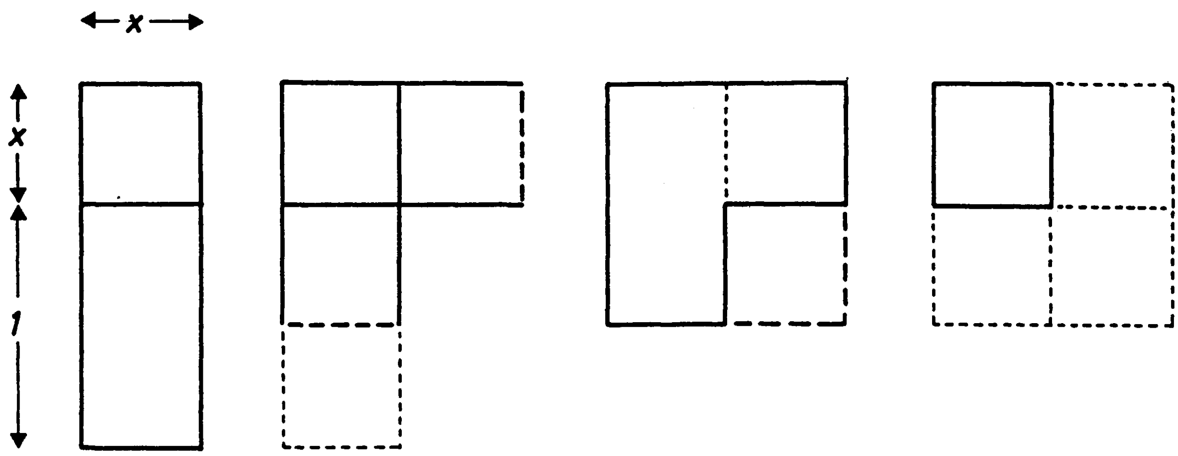
11) Ein irreguläres Viereck kann jedoch auch von zwei *entgegengestellten* Seiten 'gehalten' werden. Das ist geometrisch einleuchtend, obwohl in diesem Fall natürlich die Figur nicht dadurch eindeutig bestimmt wird. Von 'multiplikativem' Prozeß ist selbstverständlich dann überhaupt keine Rede mehr.



Figur 1: Der Lösungsvorgang in BM 13901 Nr.1, 'Quadratfläche + Seite = 0;45'

(s.Figur 1) und schafft sich eine Unterlage für ein analytisches Verfahren.

Die geometrische Summe (= *3/4*) ist nämlich aus einem Quadrat und einem Rechteck zusammengesetzt, wobei die Länge des letzteren als *1* bekannt ist und seine Breite gleich der Seite des Quadrates. Die ganze Figur kann deshalb nach Zerschneidung in einen Gnomon umgelegt werden, wo das fehlende Quadrat von den zwei 'halben Teilen' der 'Herausragenden' *1* gehalten wird und daher als $1/2 \cdot 1/2 = 1/4$ bekannt ist. Wird es hinzugefügt, ist die Fläche des ergänzten Quadrates $3/4 + 1/4 = 1$, und seine Seite also $\sqrt{1} = 1$. Wird der hinzugefügte 'halbe Teil' wieder weggenommen, bleibt die Seite des ursprünglichen Quadrates, das 'Entgegengestellte'.

Das ist genau das Verfahren, das im Text beschrieben wird. Seine Korrektheit ist intuitiv ganz einleuchtend, wird aber nicht ausdrücklich kommentiert. Da die Figur trotz ihrer unbekannten Breite in der ganzen Operation als *ganz konkret vorhanden* behandelt wird, ist der analytische Charakter des Verfahrens außer Zweifel - man könnte mit einem gewissen

Recht von *naiver geometrischer Analyse* sprechen ('naiv', weil die bloße Möglichkeit des Anzweifelns abwesend ist).

Außerdem wird man bemerken, daß der Text zwischen einer geometrischen Größe und ihrer Meßzahl keinen Unterschied macht; besonders auffallend ist die Konstruktion des Ergänzungsquadrates, das dann einfach als *0;15* bezeichnet wird. Andere Texte gehen noch weiter und zeigen uns ganz klar, daß die Meßzahlen im Unterricht als *identifizierende Namen* verwendet wurden - also 'dieses *0;30*', wo wir in derselben Absicht von 'der Strecke *AB*' oder 'der Länge *x*' reden würden. Das ist problemlos, solange nur bekannte Größen so identifiziert werden; in Fällen, wo mehrere Größen numerisch gleich sind, wird weitere Identifikation durch Hinweise auf die frühere Rolle der einzelnen Größe geschaffen (wie das '*0;30*, das Du halten gelassen hast' unseres Textes). Schließlich gibt es sogar noch Fälle, wo eine im Prinzip unbekannte Größe mit ihrer Meßzahl bezeichnet wird, *ohne daß dieser 'unbekannte' Wert jedoch im Argument benutzt wird* (und wo somit die Zahl nur die Funktion eines modernen *x* ausfüllt und ein Unterschied also zwischen der Zahl als *Name* und als *Ergebnis der Rechnungen* gemacht wird). Das wäre in einer praktisch benutzten Kalkulation ein Unding; wir müssen uns aber klar machen, daß alle (oder mindestens alle uns bekannten) 'Algebra'-Texte Schultexte sind. Sie lehren *eine Methode* und lösen keine bisher ungelösten Probleme - im Gegenteil sind die Probleme so zugeschnitten, daß sie mit der zu lehrenden Methode lösbar sind[12]. Übrigens wird die Lösung, auch wenn sie nicht ausdrücklich im Text erwähnt wird, normalerweise vorher den Schülern bekannt gewesen sein -

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12) Das ist bei den Problemen zweiten Grades nicht immer eindeutig aus den Texten zu sehen, da die Babylonier *alle* solchen Probleme lösen konnten. Wenn man zu Problemen dritten oder höheren Grades übergeht, wird es aber unwiderruflich klar - vgl. unten.

öfters wird eine und dieselbe Figur als Unterlage für eine ganze Reihe von Aufgaben benutzt. Auch das kommt uns merkwürdig vor, da wir uns das Problem im Prinzip *primär, weil noch ungelöst*, und die Methode *sekundär und dem Problem zugeschnitten* denken. Kein babylonischer Schreiber war aber je auf ein praktisches Problem zweiten Grades gestoßen; die Lösung solcher Probleme in der Schule sollte, wie oben diskutiert, nicht der kalkulierenden Praxis, sondern dem Vorzeigen oder der Übung von Virtuosität dienen.

Oft spricht man von 'Gleichungen' in Verbindung mit der babylonischen 'Algebra'. Der obige Text zeigt uns, in welchem Sinn das verstanden werden muß: Eine mehr oder weniger komplizierte konkrete Größe wird konstruiert und ihre Meßzahl dann angegeben; zuweilen wird es gesagt, daß eine Größe 'wie' eine andere ist, d.h., daß die Meßzahlen dieselben sind. Wenn wir das Nichtunterscheiden zwischen Größe und 'Zahlenwert' so verstehen, daß Wörter wie 'Länge', 'Breite' usw. als 'Repräsentanten' sowohl für die Größen als auch für ihre unbekannten Werte aufzufassen sind, wird daraus in beiden Fällen eine Gleichung in fast modernem Sinn. Überall, wo im folgenden von babylonischen 'Gleichungen' gesprochen wird, sind solche Größengleichheiten gemeint.

Die Identifizierung von Größe und Meßzahl könnte weiter dazu führen (und hat gewöhnlich dazu geführt), daß der Text als ausschließlich arithmetisch gelesen wird[13]. Tut man das, entsteht der Eindruck von ganz moderner Schulalgebra. Alle Einzelschritte unseres Textes sind ja in moderne Gleichungs-

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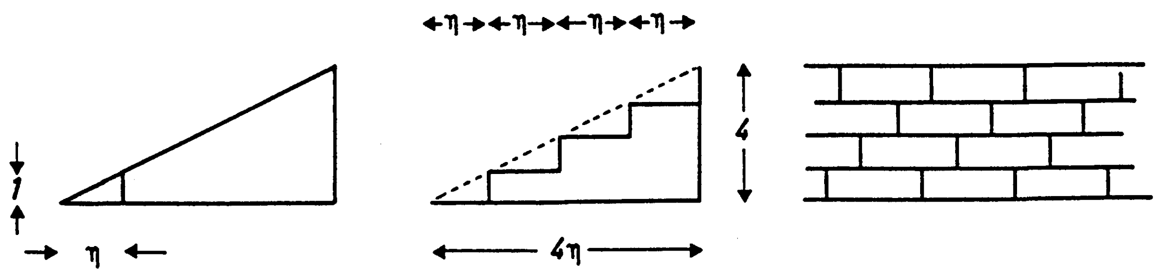
13) Die geometrische Deutung der operativen Termini wird uns nämlich (mit Ausnahme einzelner früher nicht bemerkter bzw. nicht verstandener Worte wie *Leibe* und *Herausragende)* nur durch eine vergleichende Lesung von sehr vielen Texten aufgezwungen; eine unmittelbare Lesung des Einzeltextes könnte ebensogut arithmetisch werden, wenn nur diese isolierten Worte als überflüssig oder unverständlich übersprungen werden.

operationen direkt übersetzbar:

$$x^2 + x = 3/4 \;\Rightarrow\; x^2 + x\cdot 1 = 3/4 \;\Rightarrow$$
$$x^2 + 2\cdot x\cdot 1/2 + (1/2)^2 = 3/4 + 1/4 = 1 \;\Rightarrow$$
$$(x+1/2)^2 = 1 \;\Rightarrow\; x + 1/2 = \sqrt{1} = 1 \;\Rightarrow\; x = 1 - 1/2 = 1/2$$

Dem grundlegenden ontologischen Unterschied zwischen einer Algebra abstrakter Zahlen und einer 'Algebra' gemessener Strecken zum Trotze war also letztere in diesem elementaren Fall der ersten praktisch gleichwertig und pädagogisch vielleicht überlegen. Das gilt auch in vielen komplizierten Fällen. Für solche wird jedoch für alle Koeffizientenreduktionen noch eine Operation benötigt, die sogenannte 'Hebung' (*našûm*). Sie bezeichnet eine eigentliche *Berechnung einer konkreten Größe durch Multiplikation*[14], vermutlich durch irgendeine intuitive Proportionalitätsbetrachtung. Daß 'heben' etwas mit Proportionalität oder Multiplikation zu tun hat, scheint uns wohl kaum einleuchtend; eine mögliche Erklärung finden wir in der Verwendung der Operation unter anderem für die Höheberechnung einer Rampe und die Berechnung der Ziegelanzahl einer Mauer - siehe Figur 2.



Figur 2: Wie das Erheben einer Rampe und einer Mauer als Multiplikativer Prozeß aufgefaßt werden kann

14) Noch eine andere multiplikative Operation wird mit a-rà bezeichnet. *p* a-rà *q* ('*p* schritte von *q*') ist eine numerische Multiplikation der Zahl *p* mit der Zahl *q*; die Operation findet sich deshalb in den Multiplikationstabellen. Gelegentlich findet man sie auch in den 'algebraischen' Texten, z.B. in gewissen seltenen Fällen, wo eine rechteckige Fläche zuerst 'gebaut' und ihre Meßzahl danach in einem zweiten, separaten Schritt durch Multiplikation der Meßzahlen der Seiten berechnet wird.

Auch die Division scheint uns notwendig für die meisten komplizierteren Algebraaufgaben. Eine eigentliche Division gibt es aber trotzdem bei den Babyloniern *nicht*. Statt dessen multiplizieren sie, wenn es möglich ist (d.h. wenn der Divisor eine Potenz von *60* teilen kann - in anderen Worten, wenn er die Form $2^p \cdot 3^q \cdot 5^r$ hat) mit dem reziproken Wert des Divisors (sein i g i genannt; den i g i zu finden wird als 'Abspaltung' [*paṭārum*, nämlich aus der Einheit] aufgefaßt); wenn kein i g i existiert, stellt der Text die Frage, 'was soll ich zu *a* setzen, das mir *b* gibt?' und führt unmittelbar danach die Antwort an - daß eine Antwort immer gegeben werden kann, entweder als ganze Zahl oder in Form eines endlichen Sexagesimalbruchs, folgt aus der Konstruktion der Aufgaben aus bekannten Situationen.

Ein 'praktisches' Problem zweiten Grades

Mit diesen einfachen Techniken und Operationen haben die Babylonier viele verwickelten Aufgaben gelöst. Ein charakteristisches Beispiel (auch charakteristisch, weil scheinbar aus der alltäglichen Praxis des Landmessers genommen, bei genauerem Anblick allerdings ganz künstlich) ist die Aufgabe auf der Tafel VAT 7532 [15] (die Übersetzung ist nur in den mathematischen Teilen völlig wörtlich):

1. *Ein trapezformiges Feld. Ein Schilfrohr habe ich abgeschnitten und als Meßrohr genommen.*
2. *Während es ganz war, bin ich 1 Sechzig Schritte die Länge entlang gegangen.*
3. *Sein 6er Teil ist mir abgebrochen, 1,12 Schritte ließ ich auf der Länge folgen.*

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15) Publiziert in MKT I, S.294 f.

4. *Weiter ist mir 1/3 des Rohres und 1/3 Elle abgebrochen, in 3 Sechzig Schritte bin ich die obere Breite gegangen.*

5. *Mit dem, was mir [zuletzt] abgebrochen ist, habe ich das Rohr wieder vergrößert, und in 36 Schritten habe ich die untere Breite durchgemacht.*

6. *Die Fläche ist 1 bur. Was ist die ursprüngliche Länge des Rohres?*

7. *Du, bei Deinem Verfahren: Das Rohr, das Du nicht kennst, setze als 1.*

8. *Seinen 6en Teil brich ab, dann bleiben Dir 0;50.*

9. *Seinen igi spalte ab, das [darauskommende] 1;12 hebe zu 1 Sechzig.*

10. *Das [darauskommende] 1,12 zu <1,12> füge hinzu: es gibt 2,24, die falsche Länge.*

11. *Das Rohr, das Du nicht kennst, setze als 1.*

12. *Sein 1/3 brich ab, 0;40 zu 3 Sechzig, der oberen Breite, hebe; 2,0 gibt es.*

13. *2,0 und 36, die untere Breite, lege zusammen.*

14. *2,36 zu 2,24, die falsche Länge, hebe; 6,14,24 ist die falsche Fläche.*

15. *Verdopple bis 2-mal die Fläche, das ist 1,0,0. Hebe es an 6,14,24; es gibt 6,14,24,0,0,*

16. *und 1/3 Elle, die Du abgebrochen hast, hebe zu 3 Sechzig.*

17. *Das [darauskommende] 5 hebe an 2,24, die falsche Länge; es ist 12,0.*

18. *Die Hälfte von 12,0 brich entzwei, entgegenstelle.*

19. *Füge das [darauskommende] 36,0,0 zu 6,14,24,0,0; es gibt 6,15,0,0,0.*

20. *6,15,0,0,0 macht 2,30,0 gleichseitig.*

21. *6,0, das Du zurückgelassen hast, zu 2,30,0 füge hinzu; es gibt 2,36,0.*

22. *Der igi von 6,14,24, die falsche Fläche, läßt sich nicht abspalten. Was soll ich zu 6,14,24 setzen, das 2,36,0 gibt?*

23. *Setze 0;25.*

24. *Weil der 6te Teil [...] abgebrochen ist, schreib 6, 1 lasse weggehen; 5 läßt Du zurück.*

25. *<der* i g i *von 5 ist 0;12; 0;12 hebe zu 0;25; 0;05 gibt es>* 16)

26. *0;05 zu 0;25 füge hinzu, 1/2 nindan gibt es Dir als ursprüngliches Rohr.*

Die Verbindung zur (feldmesserischen) Praxis läßt sich nicht nur in der äußeren Einkleidung des Problems sehen, sondern auch in den Zahlenangaben und den Einheiten. Das '*1, Sechzig*' wird im gewöhnlichen, 'absoluten' System geschrieben, nicht im Stellenwertsystem der mathematischen Texte 17). Die Einheit aller Längen ist der nindan, der gleich 12 'Ellen' ist (letztere ist etwa 50 cm, ersterer also etwa 6 m; diese Einheit muß übrigens auch in der vorigen Aufgabe mitgedacht werden - babylonische Geometrie, auch wenn Grundlage für 'Algebra', handelt nie von abstrakten Längen). Der b u r ist eine der gewöhnlichen Feldmessungseinheiten und ist gleich 30,0 nindan^2.

Dieser Verbindung zum Trotze macht der ganze Gang der Aufgabe uns deutlich klar, daß wir weit von der Praxis entfernt sind und uns im Feenland der Rätsel befinden; dort nämlich - und nicht im Alltag - trifft der Feldmesser auf Probleme zweiten Grades.

Versuchen wir, dem Gang der Rechnungen zu folgen (vgl. Fig.3). In 7-9 werden die 60 Schritte mit dem ursprünglichen Rohr in *1,12* Schritte mit dem einmal gekürzten Rohr umgesetzt und die ganze Länge des Feldes demnach als *2,24* Schritte mit diesem angegeben. Das nur halbwegs explizite Argument scheint von dem Typus 'einfacher falscher Ansatz' zu sein (in der Tat findet das Wort 'falsch' (l u l) sich

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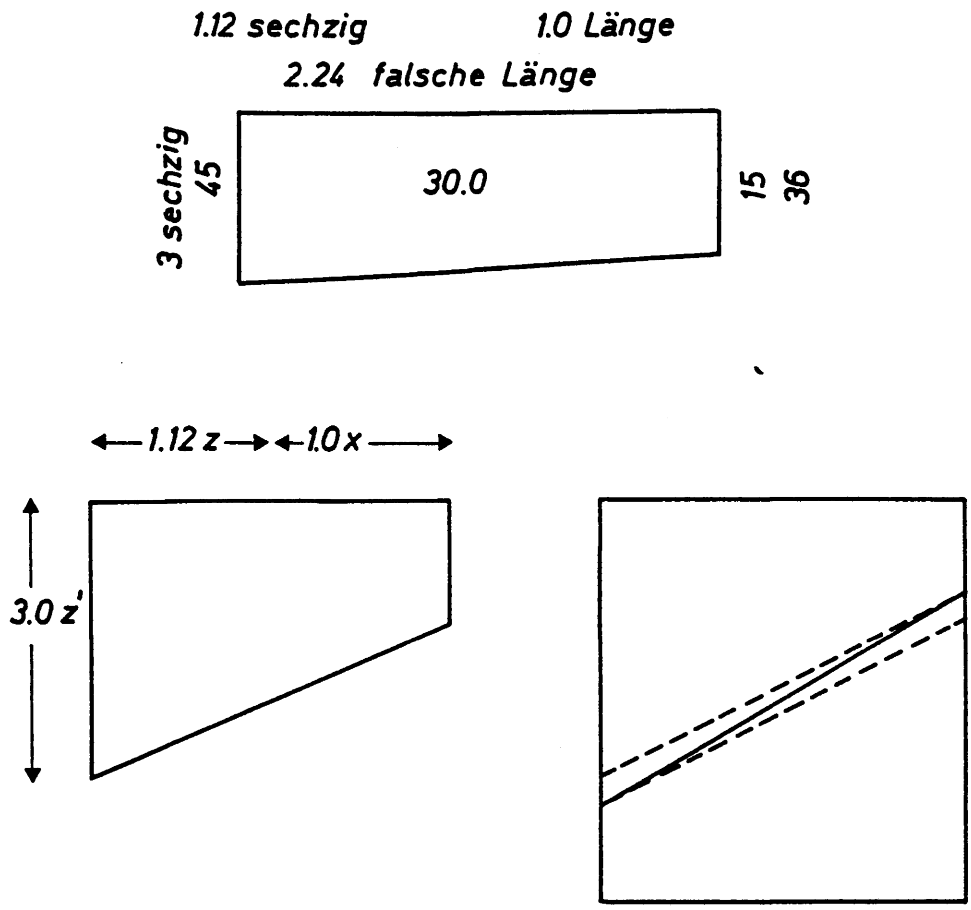
16) Vervollständigung nach Parallelstellen in verwandten Texten.

17) Das ist jedenfalls die natürliche Lesung. Man kann jedoch auch '*1,0, von der Größenordnung Sechzig*' lesen, wo *1,0* eine Stellenwertzahl ist (da diese einfach als *1* mit der *Eins* des absoluten Systems geschrieben wird).



Figur 3: Das trapezformige Feld aus VAT 7532. Oben wie auf der Tafel gegeben (gewisse geschädigte Zahlen sind nach Paralleltexten rekonstruiert). Unten links in korrekten Proportionen, unten rechts verdoppelt.

ja auch im Text): Wenn das Rohr die ursprüngliche Länge *1* gehabt hätte, wäre die gekürzte Länge *1 - 1/6 = 1 - 0;10 = 0;50* . Die ganze mit dem ungekürzten Rohr durchgegangene Strecke wäre *60* , die dann mit dem gekürzten Rohr in *60/0;50 = 1,12* Schritten durchschritten werden könnte [18].

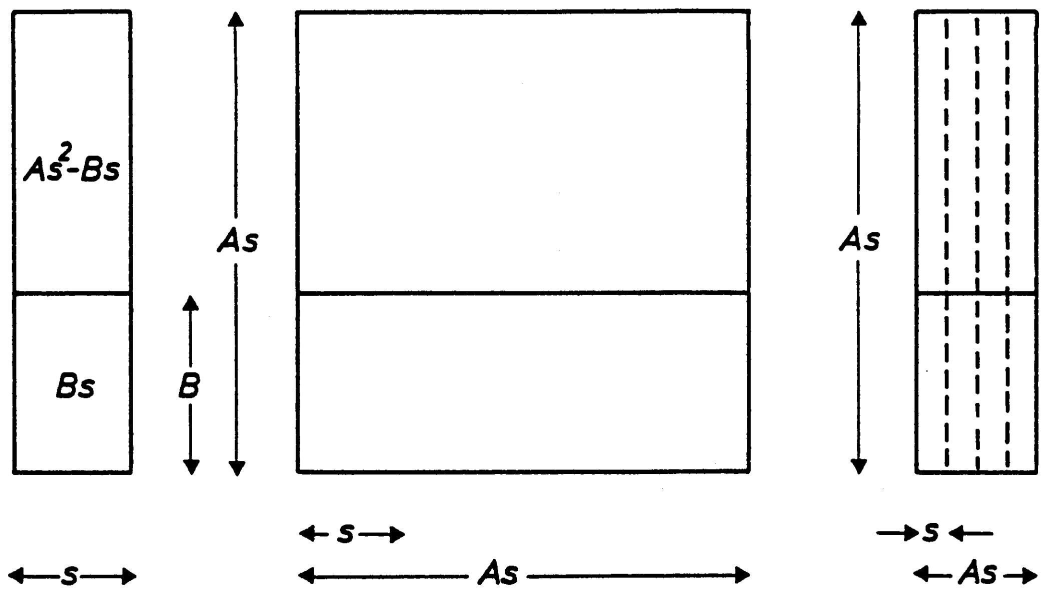
18) Man bemerkt, daß wir hier die Repräsentation einer unbekannten Größe durch eine bekannte Zahl wiedertreffen. In diesem Fall ist allerdings der Repräsentant von dem wahren Wert der Größe verschieden. Die Methode des einfachen falschen Ansatzes kann deshalb benutzt werden, um wirkliche Probleme zu lösen (und wurde auch von praktischen Rechnern so benutzt bis zur Renaissance). Die Rolle der *1* als '*x* des armen Mannes' in einer wahrhaftig analytischen Argumentation ist unverkennbar.

In 11 wird ein neuer Ansatz gemacht. Erstens wird nur die Kürzung von 1/3 in Betracht gezogen, und die *3,0* Schritte mit dem so nochmals gekürzten Rohr werden in 12 in *2,0* Schritte mit dem einmal gekürzten umgesetzt. Eine 'falsche Fläche' von *2,24·(2,0+36)* wird dann in 13-14 gefunden, und zwar das Rechteck, das durch Verdoppelung des Trapezes entstehen würde, *wenn* das einmal gekürzte Rohr die Länge *1* hätte und wir von der abgebrochenen *1/3 Elle* absehen. Daß eine eigentliche physische Verdoppelung stattfindet, wird in 15 klargemacht, wo die 'wahre' Fläche in ähnlicher Weise verdoppelt wird ('bis n-mal verdoppeln' ist eine n-malige konkrete Wiederholung).

Was weiter geschieht, ist nicht direkt Zeile für Zeile verfolgbar, wird aber durch Vergleich mit Standardaufgaben von ähnlicher Struktur verständlich. Es geht etwa so:

Wir betrachten das einmal gekürzte Rohr als 'Entgegengestelltes', d.h. als unbekannte Seite eines Quadrates. Wenn wir von der *1/3* Elle absehen, ist die verdoppelte wahre Fläche gleich *2,24 · (2,0+36) = 6,14,24* mal dieses Quadrat. Nun müssen wir aber die *1/3* Elle mitdenken. Jedesmal, wenn wir mit dem zum zweitenmal gekürzten Rohr einen Schritt gemacht haben, fehlt uns *1/3* Elle, alles in allem *3,0 · 1/3* Ellen = *3,0 · 1/3 · 1/12* nindan = *1,0 · 1/12* nindan = *5* nindan. Insgesamt haben wir also einen Streifen der Breite *5* nindan und der Länge gleich *2,24* mal das einmal gekürzte Rohr, d.h. ein Rechteck von *12,0* nindan mal die Seite des unbekannten Quadrates zu viel genommen; in Wirklichkeit ist deshalb die verdoppelte wahre Fläche nur *6,14,24* mal das unbekannte Quadrat minus *12,0* mal seine Seite (selbstverständlich mit einem 'Herausragenden' von 1 nindan versorgt). Als babylonische Standardaufgabe könnte man das in folgender Weise formulieren: *12,0 mal das Entgegengestellte habe ich aus 6,14,24 mal der Fläche herausgerissen: 1,0,0 ist es.*

Wir würden jetzt die ganze Gleichung mit *6,14,24* teilen, um eine normalisierte Gleichung zu erhalten. Das kann der babylonische Rechner nicht, da *6,14,24* = $2^6 \cdot 3^3 \cdot 13$ keinen igi besitzt. Statt dessen betrachtet er stillschweigend *6,14,24* mal die ursprüngliche Seite als Seite eines neuen Quadrates, das dann *6,14,24* mal so groß wie das Rechteck mit einer Länge gleich *6,14,24* 'Entgegengestellten' und einer Breite gleich dem 'Entgegengestellten' sein muß[19]. Auch *12,0* mal die neue Seite muß *6,14,24* mal so groß sein wie *12,0* mal die ursprüngliche Seite - und also ist das neue Quadrat minus *12,0* mal seine Seite *6,14,24* mal so groß wie die ur-



Figur 4a: Zwei Methoden, das Problem $As^2-Bs=C$ in $(As)^2-B\cdot(As)=AC$ zu verwandeln: Entweder durch Multiplikation der horizontalen Dimension mit A (hier 4), oder durch Multiplikation der horizontalen Meßeinheit mit A^{-1}

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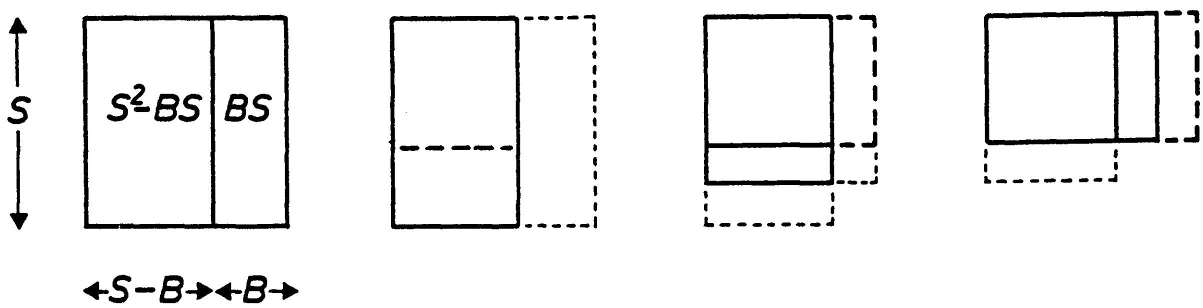
19) Die Vergrößerung des Quadrates könnte eine physische Vergrößerung sein; es könnte sich aber ebensowohl um eine bloße Vergrößerung der Meßzahl handeln, die aus einer Änderung der Meßeinheit (in einer der zwei Dimensionen) folgte. Letztere Möglichkeit (die geometrisch einfacher wäre, da alles dann auf derselben Figur gemacht oder gedacht werden könnte) wäre z.B. der Umrechnung in 7-10 sehr ähnlich. Aus diesem und anderen Gründen finde ich sie plausibler (Die babylonischen Rechner waren daran gewöhnt, verschiedene Dimensionen einer Figur in verschiedenen Einheiten zu messen, weil horizontale Ausdehnungen in nindan gerechnet wurden, vertikale Ausdehnung aber in Ellen).

sprünglichen *1,0,0* [nindan²] und somit *6,14,24,0,0* [nindan²] (vgl. Fig.4a). Damit ist dann die Aufgabe auf folgende normalisierte Standardaufgabe reduziert worden: *12,0 mal das Entgegengestellte habe ich aus der Fläche herausgerissen: 6,14,24,0,0 ist es.* Sie wird mit einem geometrischen Standardverfahren gelöst - siehe Figur 4b: Die ganze Fläche von *6,14,24,0,0* (Quadrat minus *12,0* Seiten) ist ein Rechteck, dessen Länge die Breite mit *12,0* übersteigt. Dieser Überschuß wird halbiert und die äußere Hälfte so umgelegt, daß die zwei Stücke von *6,0* als Ecke eines Ergänzungsquadrates entgegengestellt werden. Das ergänzte Quadrat ist dann *6,14,24,0,0 + 36,0,0 = 15,0,0,0* und seine Seite also *30,0*. Die ursprünglich zurückgelassene Hälfte *6,0* des Überschusses wird (in 21) zur rechten Seite der zuerst umgelegten hinzugefügt, wodurch die volle Breite des verminderten Quadrates restituiert und als *36,0* gefunden wird. Diese Zahl war um einen Faktor *6,14,24* größer als das einmal verminderte Rohr, das dann (22-23) durch Division als *0;25* gefunden wird[20]. Schließlich wird mit einem letzten 'falschen Ansatz' die Länge des ungekürzten Rohres als 1/2 nindan gefunden.



Figur 4b: Die Lösung des Problems $S^2 - BS = C$ wie in VAT 7532 beschrieben

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20) Man kann bemerken, daß die geometrische Situation und die Operationen dieselben sind wie in Figur 1. Nur ist diesmal die gesuchte Größe die Länge und nicht die Breite des Rechtecks.

Das läßt sich natürlich viel einfacher in modernem Buchstabensymbolismus ausdrücken, *wenn man nur im Besitz dieses Symbolismus ist und damit vertraut umgeht:*

$$z = 5/6x \quad \text{und} \quad 1/2 \cdot (1,0 \cdot x + 1,12 \cdot z)((2/3 \cdot z - 1/36) \cdot 3,0 + 36 \cdot z) = 30,0$$
$$\Rightarrow (1,12 + 1,12) \cdot z \cdot ((2,0 + 36) \cdot z - 5) = 1,0,0 \quad \Rightarrow$$
$$2,24 \cdot 2,36z^2 - 2,24 \cdot 5z = 1,0,0 \quad \Rightarrow 6,14,24 \cdot z^2 - 12,0 \cdot z = 1,0,0$$

Setzen wir jetzt $Z = 6,14,26 \cdot z$, kriegen wir

$$Z^2 - 12,0 \cdot Z = 6,14,24 \cdot 1,0,0 \quad \Rightarrow \quad Z^2 - 12,0 \cdot Z = 6,14,24,0,0 \quad \Rightarrow$$
$$Z^2 - 2 \cdot 6,0 \cdot Z + (6,0)^2 = 6,14,24,0,0 + 36,0,0 = 6,15,0,0,0 \quad \Rightarrow$$
$$Z - 6,0 = \sqrt{6,15,0,0,0} = 2,30,0 \quad \Rightarrow \quad Z = 2,30,0 + 6,0 = 2,36,0 \quad \Rightarrow$$
$$6,14,24 \cdot z = 2,36,0 \quad \Rightarrow z = 0;25 \quad \Rightarrow$$
$$x = (6/(6-1))z = ([(6-1)+1]/(6-1))z = z + (1/5)z =$$
$$0;25 + 0,05 = 0;30$$

Das Verwunderliche ist nicht, daß die babylonische Berechnung sich in algebraischen Symbolismus übersetzen läßt. Verwunderlich ist es dagegen, daß diese Übersetzung der modernen Lösungsweise so nahe kommt. In der Tat haben die Babylonier mit ihrem geometrischen und sehr konkreten Gedankengang in fast denselben Schritten gearbeitet, wie wir es tun würden (abgesehen von den besonderen Schritten, die aus ihrem eigenartigen Divisionsverfahren folgen und die nicht in Symbole mitübersetzt wurden).

Der erste Grad

Die Reduktion der 'Standardgleichung' zweiten Grades ist, wie wir sehen, ganz standardisiert. Dagegen scheinen die Operationen ersten Grades in ganz improvisierter Weise behandelt zu werden, meistens mit verschiedenen Varianten des einfachen 'falschen Ansatzes'. Es gibt jedoch auch sehr standardisierte Behandlungen von 'Gleichungen' ersten Grades.

Ein Beispiel, das uns auch einen besseren Einblick in die oralen und didaktischen Aspekte des Unterrichts gibt als die meisten Texte, ist TMS XVI [21]. Der erste von zwei parallelen Teilen läuft wie folgt:

1. *Ein 4tel von der Breite ist von Länge und Breite herauszureißen, 45 .*
2. *Du, hebe 45 zu 4; 3,0 siehst Du. 3,0 , was ist das?*
3. *Setze 4 und 1; setze 50 und 5, das herauszureißen ist.*
4. *5 zu 4 hebe, 1 Breite.*
5. *20 zu 4 hebe; 1,20 siehst Du, 4 Breiten.*
6. *30 zu 4 hebe; 2,0 siehst Du, 4 Längen.*
7. *20, 1 Breite, die herauszureißen ist, von 1,20 , die 4 Breiten, reiße heraus; 1,0 siehst Du.*
8. *2,0 , die Längen, und 1,0 , die 3 Breiten, lege zusammen; 3,0 siehst Du.*
9. *Den igi von 4 spalte ab; 0;15 siehst Du.*
10. *0;15 zu 2,0 , die Längen, hebe; 30 siehst Du, 30 die Länge.*
11. *0;15 zu 1,0 hebe; 15 ist der Beitrag der Breite.*
12. *30 und 15 behalte (?).*
13. *Da er 'ein 4tel der Breite ist herauszureißen' gesagt hat, von 4 reiße 1 heraus; 3 siehst Du.*
14. *Den igi von 4 spalte ab; 0;15 siehst Du.*
15. *0;15 zu 3 hebe; 0;45 siehst Du. 0;45 ist so viel es von Breiten gibt.*
16. *1 ist so viel es von Längen gibt.*
17. *Nimm 20, die wahre Breite. Hebe 20 zu 1 , 20 siehst Du.*
18. *20 zu 0;45 hebe; 15 siehst Du.*
19. *15 von* $^{30}{}_{15}$ *ziehe heraus; 30 siehst Du, 30 die Länge.*

Was uns hier vorgezeigt wird, ist keine Aufgabenlösung, sondern die didaktische Erklärung der Transformation einer 'Gleichung'. Wir können diese Gleichung entweder graphisch

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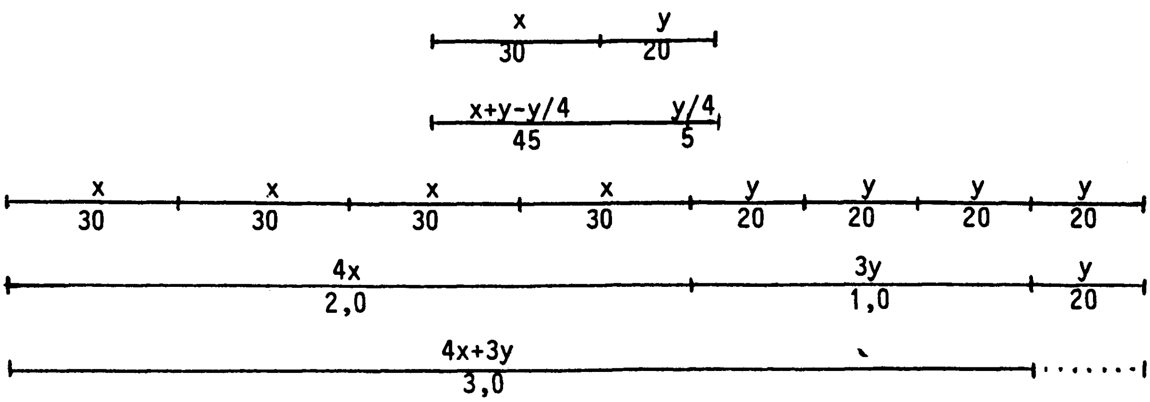
21) Publiziert in Bruins & Rutten 1961; 92. Korrektionen, Übersetzung und Interpretation in meinem [1987, 83 f.].



Figur 5: Graphische Repräsentation der Gleichungstransformation in TMS XVI A

(vgl. Fig.5) oder symbolisch übersetzen - letzteres als

$$(x+y) - 1/4\, y = 45$$

wo x die Länge und y die Breite ist. Wie es klar wird in 3, sind die Länge schon voraus als *30*, die Breite als *20*, ihre Summe als *50* und das 4tel der Breite als *5* bekannt. Die Transformation, die erklärt wird, ist eine Multiplikation mit *4*, die die rechte Seite in *3,0* verwandelt, nach deren konkreter Erklärung dann (in 2) gefragt wird.

Erstens werden in 3 außer den ursprünglichen Größen die Multiplikatoren *1* und *4* (für die ursprüngliche bzw. die multiplizierte 'Gleichung'), die Summe *50* und das herauszureißende *5* 'gesetzt', d.h. irgendwie materiell notiert (im Gedächtnis etwas zu notieren heißt, wie wir uns erinnern, 'seinen Kopf es halten zu lassen').

In 4 bis 6 werden dann das Herauszureißende *5*, die Breite *20* und die Länge *30* mit *4* multipliziert und die herauskommenden Zahlen als jeweils *1* Breite, *4* Breiten und *4* Längen erklärt. In 7 wird das Herauszureißende *20* aus dem *1,20* gerissen, welches *1,0* zurückläßt (in 8 als *3* Breiten erklärt). In 8 sehen wir schließlich die Erklärung des *3,0* als die Summe von *2,0* , das von den *4* Längen herrührt, und *1,0* , das mit *3* Breiten identisch ist.

Die nächste Stufe kehrt jetzt alles um und multipliziert alles mit *1/4 = 0;15* , dem igi von *4* . *1/4* von *2,0* ist *30* , also eine Länge (10); *1/4* von *1,0* ist *15* , das dann als 'Beitrag der Breite' präsentiert wird (11; im selben Sinn ist also der 'Beitrag der Länge' *30* , d.h. eine Länge). Im unklar geschriebenen 12 wird der Schüler vermutlich aufgefordert, diese zwei Beiträge im Kopf zu behalten.

Der letzte Teil des Textes ist dann die Erklärung der zwei 'Beiträge'. Erstens wird in 13-15 der Koeffizient der Breite ('so viel es von Breiten gibt') als *1 - 1/4 = 1 - 0;15 = 0;45* gefunden. Von Anfang an ist es klar, daß der Koeffizient der Länge *1* ist (16) . In 17 wird eine 'wahre' Breite (wohl die Breite einer Figur) in die damit gleich große formal angesetzte 'Breite' der hier durchgegangenen Hilfsberechnung durch Multiplikation mit *1* umgesetzt; in 18 wird diese mit ihrem Koeffizienten 0;45 multipliziert, was natürlich den 'Beitrag der Breite' gibt; wenn er in 19 aus der Summe herausgerissen wird, bleibt *30* , der Beitrag der Länge, d.h. die Länge selbst.

Das kann, in Worten gesagt, ein wenig undurchsichtig vorkommen. Wir müssen es uns aber (nach dem 'setze' in 3) als eine begleitende Erklärung vorstellen, während die Operationen gleichzeitig in irgendeiner materiellen Repräsentation vollzogen wurden, z.B. in einer Zeichnung im Sande des Schulhofes[22] , vgl. Figur 5 . Um uns in die Situation der Schüler zu setzen, kann man versuchen, dem Gang der Operationen auf dieser Figur zu folgen.

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22) Auch im elementaren Schreibunterricht wurde dieses Schreibmaterial vermutlich benutzt - vgl. Tanret 1982.

Kombinierte Probleme zweiten Grades

Eine Aufgabe voll zu übersetzen, wo eine Transformation der eben vorgestellten Art benutzt wird, würde zu weit führen. Statt dessen können wir eine symbolische Übersetzung der ersten Aufgabe der Tafel VAT 8520 [23] betrachten - wobei man allerdings daran erinnern muß, daß der eigentliche Text konkret und geometrisch ist im selben Sinn wie die vorigen (ungeachtet, daß die unbekannten Größen hier als zusammengehörendes Zahlpaar der Reziprokentafel vorgestellt werden). In symbolisch-arithmetischer Übersetzung wird daraus folgendes:

$$x - 6/13 \cdot (x+y) = 0{,}30 \quad \text{und} \quad x \cdot y = 1$$

Die erste Gleichung wird nach den oben gelehrten Prinzipien mit 13 multipliziert und wird so zu

$$7x - 6y = 6;30 \quad , \quad \text{während} \quad 7x \cdot 6y = 42$$

Setzen wir jetzt $X = 7x$, $Y = 6y$, kriegen wir

$$X - Y = 6;30 \quad \text{und} \quad X \cdot Y = 42$$

geometrisch also ein Rechteck mit bekannter Fläche, wo auch der Überschuß der Länge über die Breite bekannt ist. Diese Konfiguration kennen wir schon als Standardkonfiguration aus den ersten zwei Aufgaben (s. Fig.1 und 3), und die Lösung $x = 1;30$, $y = 0;40$ folgt in ähnlicher Weise.

Von Aufgaben dieser Art, wo die Seiten eines Rechteckes mit bekannter Fläche eine (oft ziemlich komplizierte)

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23) In MKT I, 346 f. publiziert.

'Gleichung' ersten Grades befriedigen, gibt es in den sogenannten 'Serientexten' hunderte[24]. Sie macht damit die am besten repräsentierte 'algebraische' Aufgabengattung aus. Die wenigen hier vorgeführten Aufgaben sind damit für die Haupttypen babylonischer 'Algebraaufgaben' zweiten Grades ziemlich repräsentativ[25]. Man soll jedoch wissen, daß es sehr viele Aufgaben gibt, die außerhalb der Haupttypen fallen und recht kompliziert sein können. Oft werden dann für die Lösung Richtwege benutzt, die man bei der Verwendung von symbolischer Algebra kaum entdecken würde, die aber für die ursprüngliche naiv-geometrische Methode einleuchtend sind. In diesen Fällen unterscheidet sich dann die babylonische Lösung - zu ihrem Vorteil - vom modernen Standardverfahren.

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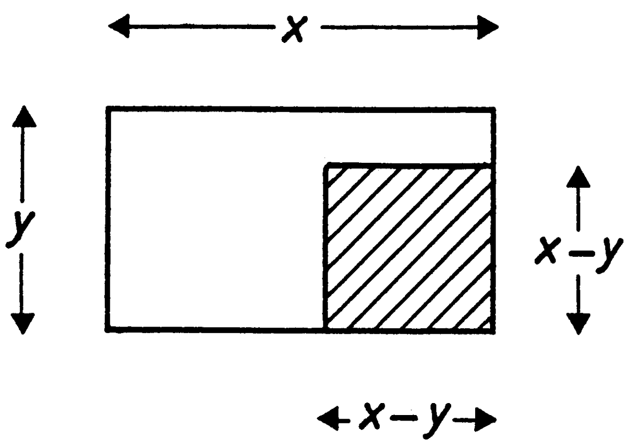
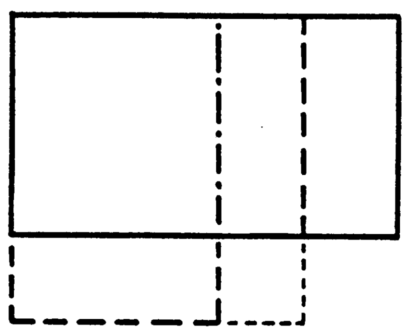
24) Die Länge ist dann immer 30 und die Breite 20, wie in der oben erklärten Gleichung $x + y - 1/4\,y = 45$. Die einzelnen Probleme sind sehr systematisch aufgestellt, und zwar nach einem mehrdimensionalen Schema. So werden in Tafel YBC 4668, Nr.C 38 bis C 53, aus der 'Gleichung' $\alpha + 1/19\,(\alpha - \beta) = A$ durch systematischen und unabhängigen Austausch des Zählers mit 2, des ersten + mit -, des ersten α mit β und des Nenners mit 7 insgesamt $2^4 = 16$ verschiedene 'Gleichungen' hervorgebracht (s. MKT I, p.462). α ist hier $x \cdot x/y$, β ist $y \cdot y/x$ (x steht wie schon vorher für die 'Länge' und y für die 'Breite' eines Rechteckes). Die andere 'Gleichung' ist $x \cdot y = 1{,}0$, weshalb auch $\alpha \cdot \beta = 10{,}0$; das Gesamtsystem ist also genau von der eben diskutierten Art und nur formell vom sechsten Grad (aus α und β sind x und y leicht zu finden, da $\alpha/\beta = (x/y)^3$).

25) Nur fehlt der Aufgabentypus, wo nach Reduktion außer der Fläche eines Rechtecks auch die *Summe* von Länge und Breite gegeben ist. Im Fall, wo es sich um eine Aufgabe mit zwei Unbekannten handelt, bringt sie uns nichts besonderes. In den wenigen Fällen, wo *eine* unbekannte Größe gesucht wird und wo es also im Prinzip eine Doppellösung geben kann, finden die Babylonier nur die eine. Das könnte uns wundern, wenn wir nicht wüßten, daß die Lösung immer von vornherein bekannt ist; die geometrische Analyse wird dann natürlich immer diejenige der zwei Zerschneidungen des Rechtecks benutzen, die mit der bekannten Lösung übereinstimmt.

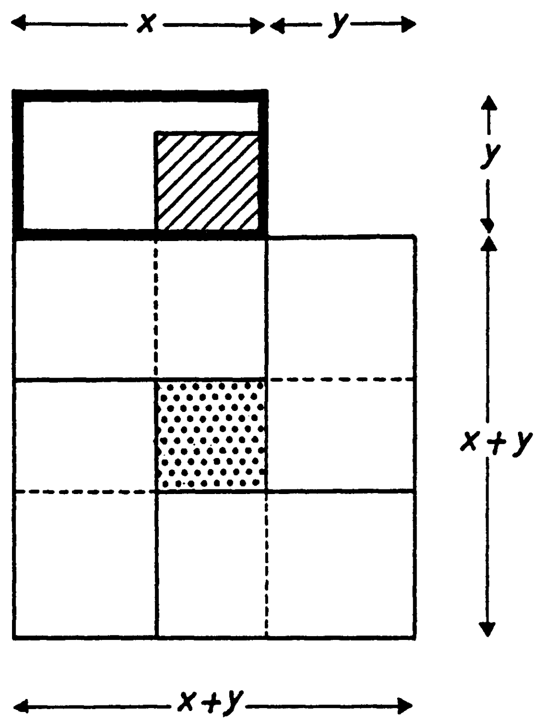
Schlagende Beispiele dafür finden wir in den ersten drei Aufgaben des Textes YBC 6504 [26], während die vierte Aufgabe die Gefahr der naiven Verfahren zeigt. Alle vier Aufgaben handeln von einem Rechteck $x \cdot y$, wo $x \cdot y - (x-y)^2 = 8{,}20$. Außerdem ist gegeben, daß $x - y = 10$ (Aufgabe 1), $x + y = 50$ (Aufgabe 2), $x = 30$ (Aufgabe 3), und $y = 20$ (Aufgabe 4). Die Reduktionsverfahren werden in Figur 6a-d skizziert. In Aufgabe 1 ist ja die Seite des herausgerissenen Quadrates bekannt; seine Fläche ist dann leicht berechenbar, und Seitensumme und Fläche des Rechteckes sind also bekannt. Damit sind wir zurück in der Standardaufgabe $x - y = A$, $x \cdot y = B$. In Aufgabe 2 wird die Summe von Länge und Breite als Seite eines Quadrates genommen. Die Fläche dieses Quadrates ist gleich viermal das Rechteck plus das herausgerissene $(x-y)^2$. Wird es zu dem $8{,}20$ addiert,



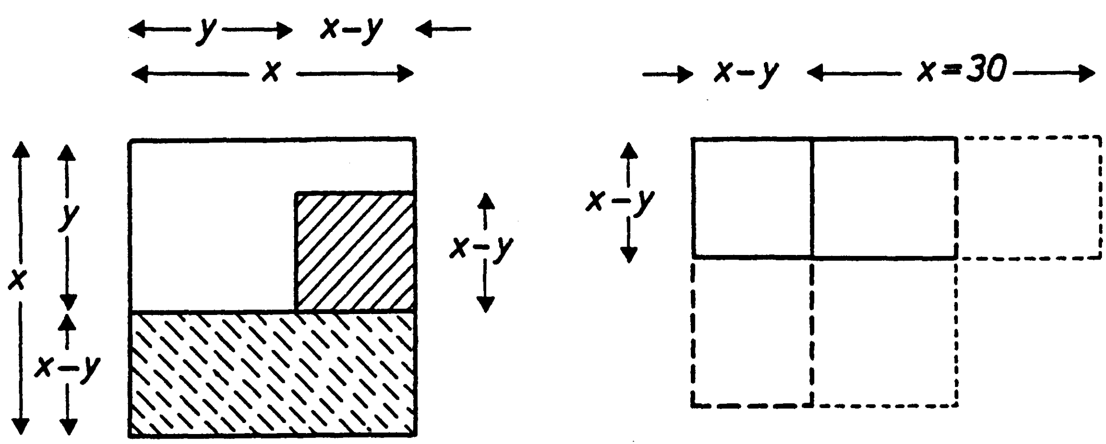


Figur 6a: Die geometrische Repräsentation der Aufgabe YBC 6504 Nr.1. Oben das amputierte Rechteck; unten die Zerschneidung, Gnomon-Umlegung und quadratische Ergänzung des Rechteckes

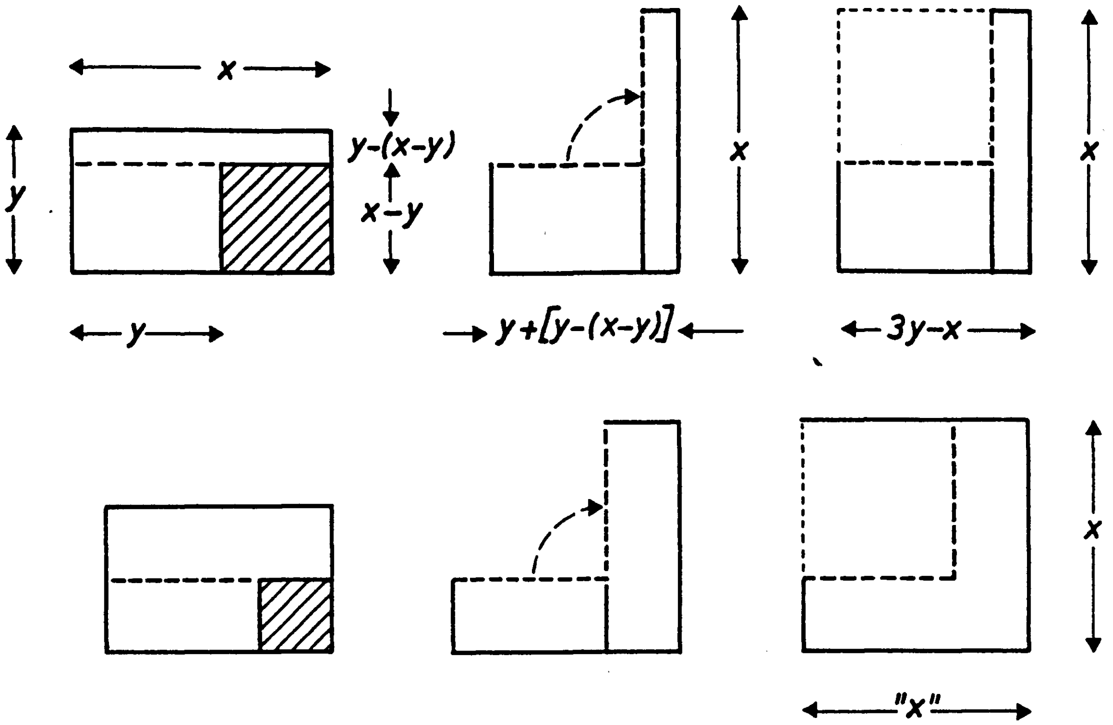
26) Pulbliziert in MKT III, 22 f.; in meinem [1985, 41 ff.] diskutiert. Der Text ist u.a. durch seine Systematik mit den Serientexten verwandt, gibt aber zum Unterschied von diesen die Lösungsverfahren an.



Figur 6b: Die geometrische Repräsentation der Lösung von YBC 6504 Nr.2. Oben in kräftiger Abgrenzung das amputierte Rechteck, unten das Quadrat auf der Summe von Länge und Breite (die zwei Teile der Figur sind nur zufälligerweise hier zusammenstoßend gezeichnet)



Figur 6c: Die geometrische Repräsentation der Lösung von YBC 6504 Nr.3. Links wie die Figursubtraktion das Quadrat auf x - y und ein Rechteck mit Breite x - y und Länge 30 zurückläßt, rechts die Gnomon-Umlegung und die Ergänzung



Figur 6d: Die geometrische Repräsentation der fehlerhaften Lösung von YBC 6504 Nr.4. Oben in verzerrten Proportionen, wo der Fehler deutlich wird, unten in den korrekten Proportionen, wo alles plausibel aussieht

ist die Summe also fünfmal die Fläche des Rechteckes. Nochmals werden wir zu einer Standardaufgabe zurückgebracht, nämlich zu $x + y = A$, $x \cdot y = B$. In Aufgabe 3 wird das Quadrat $x \cdot x$ konstruiert und das amputierte Rechteck herausgerissen. Zurück bleibt dann das Quadrat auf $x - y$ und ein Rechteck mit der Breite $x - y$ und der Länge $[x =]\ 30$. Das wird wie die Standardaufgabe $z^2 + Az = B$ gelöst, mit $z = x - y$. In symbolischer Übersetzung sieht das wie ein eleganter, aber schwerlich durchschaubarer Variablenaustausch aus. Geometrisch kann natürlich die Strecke $x - y$ genauso gut wie die Strecke y gefunden werden; von Austausch ist kaum zu reden.

In Aufgabe 4 wird aber ein anscheinend geometrisch begründeter Fehlschluß gemacht. Weil in diesem konkreten Fall $x - y = y - (x-y)$, kann das amputierte Rechteck in einem Gnomon umgelegt werden, wo das fehlende Quadrat die bekannte Seite y und deshalb auch eine bekannte Fläche besitzt. Der Text glaubt, daß die Ergänzung ein Quadrat der Seite x ergibt. Eine symbolische Durchrechnung (oder eine weniger visuell-naive geometrische Betrachtung) zeigt, daß tatsächlich ein Rechteck mit den Seiten x und $3y-x$ hervorgebracht wird.

Das letzte Beispiel zeigt, daß die babylonischen Methoden im allgemeinen *nicht* besser sind als die ja unendlich flexibleren modernen symbolischen Verfahren - der scheinbare Vorzug des naiv-geometrischen Verfahrens in der zweiten und dritten Aufgabe der Tafel beruht einfach darauf, daß die Babylonier ihre Probleme nach der besonderen Leistungsfähigkeit ihrer vorhandenen Methoden auswählten.

Das sieht man mit außerordentlicher Klarkeit in Fällen, wo ein nichtgeneralisierbares Verfahren verwendet wird - z.B. in der Behandlung von Problemen dritten Grades. Wenn diese nicht homogen sind und sich nicht auf Probleme ersten oder zweiten Grades reduzieren lassen, werden sie durch zufällig mögliche Faktorisierungen oder Übereinstimmung mit einem der verstreuten Tabellenwerte der Tabelle n^3+n^2 gelöst (vgl. auch das in Anm.24 erwähnte Problem sechsten Grades). Wie Thureau-Dangin[27] sagt, muß man 'admirer l'ingéniosité [...] mis en oeuvre', aber auch erkennen, daß dadurch schließlich 'les mathématiciens babylonies avouent [...] leur impuissance à résoudre l'équation du troisième degré'. Endlich kann man bemerken, daß sie anscheinend keinen wesentlichen Unterschied gesehen haben zwischen einer eigent-

27) TMB, p. xxxviii.

lich mathematischen Methode und einem bloßen Trick[28] und also *überhaupt keine Mathematiker im griechischen oder modernen Sinn sind.* Babylonische 'Algebra' zweiten und höheren Grades ist eigentlich nicht als 'reine Mathematik', sondern eher als 'reines, unangewandtes Berechnen' zu charakterisieren.

Als 'algebraisch' kann man sie aber mit gutem Recht verstehen. Sie teilt viele der besonderen Merkmale moderner *praktisch verwendeter* Gleichungsalgebra trotz ihrer naiv-geometrischen Methode. Erstens ist sie ja *analytisch.* Zweitens ist ihre Identifikation von Größe und Meßzahl genau parallel zu jener Art mathematischer Modellierung, die die Grundlage für alle Beschreibung realer Zusammenhänge in Gleichungen ist - vgl. z.B. das Hookesche Gesetz $d = k \cdot B$, wo d als meßbare Dehnung, B als meßbare Belastung und k als meßbare Materialkonstante verstanden werden, die Gleichung aber unter der Voraussetzung gelöst wird, daß sie sämtlich arithmetisch verknüpfte *Zahlen* sind. Drittens, endlich, werden die 'Entgegengestellten', 'Längen' und 'Breiten' im allgemeinen nicht um ihrer selbst willen gefunden (obwohl das natürlich der Fall ist in der Einübung von Standardmethoden); sie stehen als *Repräsentanten* für andere Größen, deren gegenseitige Zusammenhänge mit denjenigen der repräsentierenden geometrischen Größen isomorph sind - seien es Zahlen mit bekannten Produkt und Differenz[29], seien es Kauf- und Verkaufspreise von Fein-

28) Diese Einstellung ist nicht den Babyloniern vorbehalten. Sie folgt der ganzen subwissenschaftlichen Algebratradition bis ins 15. (nachchristliche) Jahrhundert, wo 'the rules given [by Piero della Francesca, abacus master and painter] for solving equations of the third, fourth and fifth degree are valid only for special cases of these equations. The rule for solving the equation of the sixth degree is altogether false' (Jayawardene, 1976, 243).

29) YBC 6967 (MCT, 129) und die oben erwähnte Tafel VAT 8520 (MKT I, 346 f.).

öl [30], seien es wie in der obigen Landvermessungsaufgabe 'falsche' Längen, Breiten und Flächen, seien es endlich wie im erwähnten Serientext die mit ihrem Verhältnis zum Partner multiplizierten Längen und Breiten eines Rechteckes. Wer die äquivoken Gänsefüßchen nicht mag, kann ruhig von altbabylonischer *Protoalgebra* statt von babylonischer 'Algebra' reden.

Die babylonische Spätentwicklung

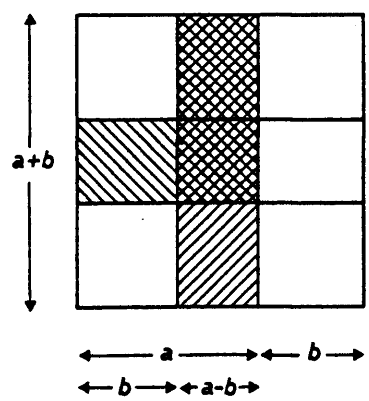
Gewöhnlicherweise sprechen die Mathematikgeschichten nicht von 'altbabylonischer', sondern von 'babylonischer' Mathematik/Algebra, keinen Unterschied machend zwischen der altbabylonischen und der nächsten mit mathematischen Texten gut belegten Periode, d.h. der seleukidischen (etwa 3.Jahrhundert v.Chr.). Solange die altbabylonische Protoalgebra ausschließlich arithmetisch gelesen und das strikte Auseinanderhalten der unterschiedlichen additiven bzw. multiplikativen Operationen nicht bemerkt wurden, waren die Verschiedenheiten der zwei Perioden auch nicht auffallend.

Die naiv-geometrische Deutung der altbabylonischen Texte wälzt das ganz um. Während die Operationen der frühen Texte konkrete (und meistens geometrische) Bedeutung hatten, gehen die Operationen der Spätzeit ausschließlich mit abstrakten Zahlen um. Es gibt nur *eine*, symmetrische Addition; die Subtraktion ist eine Abzählungsoperation '- in wievielen Schritten gehe ich von [der Zahl] *a* bis zur [Zahl] *b* hinauf'; und die einzige Multiplikation ist a - r â (vgl. oben, Anm.14), die Multiplikation von Zahl

30) Der Susatext TMS XIII (Bruins & Rutten 1961, 82, vgl. Korrektionen in Gundlach & von Soden 1983, 261).

mit Zahl (die übrigens auch als 'multiples Gehen' aufzufassen ist).

Diese Arithmetisierung gilt auf der Ebene der mathematischen Operationen. Sie ist nicht in den 'Einkleidungen' der Aufgaben zu sehen, die noch geometrisch sind (dadurch wird die seit der altbabylonischen Zeit stattgefundene Entwicklung noch mehr verschleiert). Sie gilt vielleicht auch nicht auf der Ebene der Methoden. Obwohl *wir* z.B. $(a+b)\cdot(a-b) = a^2 - b^2$ als reine arithmetische Identität auffassen, können wir sie sehr wohl im elementaren Unterricht geometrisch vorführen - siehe Figur 7 . Mehreres spricht dafür, daß die seleukidischen 'Algebraiker' dasselbe gemacht haben: Erstens sind nämlich die schwierigeren ihrer Aufgaben auf geometrische Methoden zugeschnitten; zweitens ist das in ihren Texten gängige Umgehen mit mehreren Unbekannten sehr kompliziert, wenn man nicht entweder eine geometrische oder eine symbolische Repräsentation vor sich hat[31) .



Figur 7: Geometrischer Nachweis, daß das Band $(a+b)\cdot(a-b)$ gleich dem Gnomon $a^2 - b^2$ ist

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31) Aus genau diesem Grund bemüht sich eine 'rhetorische' Algebra auch immer, alle Unbekannten mittels Eliminierung durch eine Einzige zu ersetzen. Wenn bei Diophant die Summe von zwei Zahlen gleich *2A* und ihr Produkt gleich *B* ist, werden sie als *A plus arithmos* und *A minus arithmos* aufgefaßt; in derselben Situation würden die islamischen rhetorischen Algebraiker die eine Zahl als *ein Ding (šay')* auffassen, die andere als *2A minus ein Ding.*

Als Illustration können wir einen kurzen seleukidischen Text anschauen (BM 34568 Nr.18 [32]) :

1. *Länge, Breite und Diagonale addiert ist 1,0. 5,0 die Fläche.*
2. *Länge, Breite und Diagonale mal Länge, Breite und Diagonale nimm.*
3. *Die Fläche mal 2 nimm, von <dem Quadrat von Länge, Breite und Diagonale> ziehst Du (es) ab.*
4. *Was übrig bleibt, mal ein Halb nimm. Länge, Breite und Diagonale mal <was> als Faktor sollst Du nehmen?*
5. *Die Diagonale ist der Faktor.*

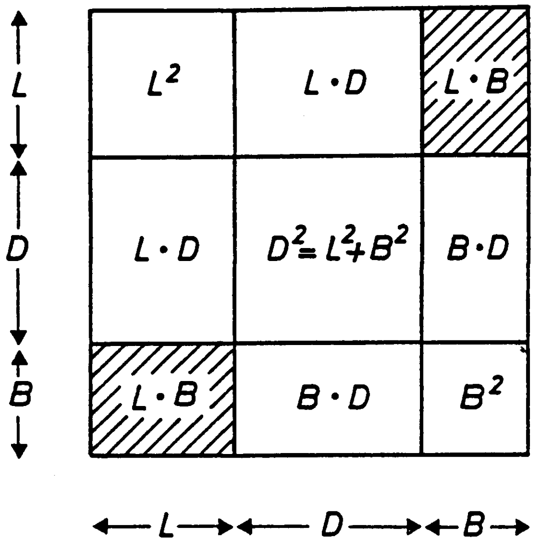
In symbolischer Übersetzung wird das zu einer einfachen *Formel* statt (wie die altbabylonischen Lösungen) zur Beschreibung einer analytischen Prozedur. Ist L die Länge, B die Breite und D die Diagonale, haben wir

$$D = 1/2 \cdot \{(L+B+D)^2 - 2LB\}/(L+B+D) \quad .$$

Die Formel ist richtig und folgt aus $L^2 + B^2 = D^2$, sieht aber kaum einleuchtend aus. Daß sie mit rein rhetorischen Techniken abgeleitet worden sei, kommt unwahrscheinlich vor. Sie ist jedoch aus Figur 8 durch einfache Abzählung und durch Ausnützung des Pythagoreischen Theorems leicht ableitbar (wo übrigens zu bemerken ist, daß diese Figur als Generalisierung von Figur 7 betrachtet werden kann; daß letztere mit Einzeichnung von Diagonalen zum Pythagoreischen Theorem führt; und daß ihre Konstruktion schon aus altbabylonischen Texten hervorgeht, vgl. Figur 6b) .

Auf der Ebene der Methode war die seleukidische 'Algebra' also vermutlich genauso geometrisch wie die altbabylonische. Während letztere aber konstruktiv und analytisch war,

32) Publiziert in MKT III, 16 f.; ich folge diesmal Neugebauers Übersetzung (*ibid.* p.19) mitsamt seiner Korrektion eines Formulierungsfehlers. Das *(es)* in 3 habe ich der Verständlichkeit halber hinzugefügt.



Figur 8: Diagramm, aus dem hervorgeht, daß
$(L+B+D)^2 - 2L \cdot B = 2D \cdot (L+B+D)$, wenn $D^2 = L^2 + B2$

sehen (nicht nur in diesem Fall, sondern allgemein) die seleukidischen Aufgabenlösungen wie Argumentationen auf schon gemachten Figuren aus. Sie sind also *nicht analytisch, sondern eher synthetisch*. Die seleukidische 'Algebra' ist damit, obwohl sie zweifellos von der altbabylonischen abgeleitet ist, mit viel weniger Recht als jene als *Protoalgebra* oder als *algebraisch in ihrer Denkweise* zu charakterisieren.

Das Nachleben der babylonischen Tradition

Daraus ist nicht zu schließen, daß die altbabylonische Protoalgebra als Sackgasse zu betrachten ist. Das wird in einem *Liber mensurationum* bezeugt, den Gherardo di Cremona im späten 12.Jahrhundert ins Lateinische übersetzt hat und dessen verlorenes arabisches Original im frühen 9.Jahrhundert abgefaßt sein mag [33]. Die erste

33) Von Busard (1968) kritisch ediert. Die hypothetische Datierung, der Inhalt und die ganzen Implikationen davon werden in meinem [1986] ausführlich behandelt.

Hälfte des Traktats ist nicht nur eine Weiterführung der altbabylonischen Tradition, sondern dieser auch erstaunlich nahe. Erstens ist die 'rhetorische Struktur' der Aufgaben dieselbe: Die Aufgabe ist in der ersten Person Perfekt formuliert (daß die Länge eines Rechteckes seine Breite mit soundsoviel überschreitet, wird jedoch in Präsens erklärt - wie schon 3000 Jahre früher). Dann kommt der Hinweis auf das Verfahren des Schülers, und schließlich die Vorschriften im Imperativ oder in der 2.Person Präsens (vgl. die oben übersetzten altbabylonischen Aufgaben). Zitate aus der Aufgabenformulierung werden mit den Worten 'Da er gesagt hat' begleitet - und selbst der Hinweis auf den Kopf/das Gedächtnis als Behälter für Resultate gewisser Art wird wiedergefunden.

Zweitens gibt es, obwohl die Figuren selbst in der Übersetzung verlorengegangen sind, ständige Hinweise auf Figuren als Begründung für die Richtigkeit der Lösungsverfahren. Drittens sind die Distinktionen zwischen mehreren additiven, mehreren subtraktiven und mehreren multiplikativen Operationen noch spürbar, wenn auch nicht im lateinischen Text völlig aufrechterhalten. Viertens endlich sind nicht nur wichtige altbabylonische Aufgabentypen, sondern auch (was wesentlicher ist) ihre charakteristischen Lösungsverfahren wiederzufinden.

Der *Liber* ist also viel näher an der altbabylonischen Quelle als die seleukidische Mathematik. Zurselben Zeit findet man im *Liber* Aufgaben, die auch in den seleukidischen Tafeln vorkommen - zum Beispiel die eben erwähnte. Versucht man, einen Stammbaum zu konstruieren, wird die direkte Linie daher von der altbabylonischen Protoalgebra zum *Liber* gehen mit einer kurzen Nebenlinie zu den seleukidischen Tafeln[34]. Die direkte Linie scheint von einer subwissenschaftlichen Tradition unter praktischen Geometern (Landmessern, Architekten, Baumeistern u.ä.) getragen worden zu sein. Die seleukidische Linie dagegen wurde, wie es aus Abschreiber- und Besitzerangaben auf den Tafeln deutlich hervorgeht, von den Priester-Gelehrten am astronomischen Zentrum in Uruk hervorgebracht, deren hochentwickelter arithmetischen Astronomie wir vielleicht die Arithmetisierung auch der 'algebraischen' Operationen zu verdanken haben.

Für viele seiner Aufgaben gibt der *Liber* außer seiner Hauptmethode einen *modus secundum aliabram*, ein 'Verfahren nach *al-jabr*'. Letztere Disziplin ist (etymologisch und genetisch) der Ursprung unserer Algebra und ist uns in seiner Form aus dem frühen 9.Jahrhundert von al-Khwārizmī bekannt. Sie war, bis al-Khwārizmī und andere wissenschaftliche Mathematiker sie bearbeiteten, eine subwissenschaftliche Praktikertradition, anscheinend von Buchhaltern und anderen Rechnern getragen[35]. Ihr *Ansatz* war daher algebraisch im problemlösenden, nicht im theoretischen oder strukturellen Sinn. So weit war sie also der altbabylonischen Protoalgebra (und dadurch auch der Hauptmethode des *Liber*) ähnlich. Ihre *Methode* war aber meistens rhetorische und nie naiv-geometrische Analyse (vgl. Note 31); nur für ihre Lösung der drei normalisierten gemischten 'Gleichungen' zweiten Grades hatte sie weder rhetorische noch geometrische Begründungen, nur feste, unbegründete Standardalgorithmen.

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34) Es scheint, daß die sehr wenigen Gleichungen zweiten Grades, die in der alexandrinischen Tradition (einschließlich der lateinischen Agrimensortradition) zu finden sind, auf diese Nebenlinie zurückzuführen sind. Das ist kaum verwunderlich, da die seleukidischen Tafeln vom hellenisierten Uruk herrühren und die seleukidische praktische Geometrie nicht altbabylonische, sondern griechische/alexandrinische Methoden benutzen.

35) Auch diese Behauptung baut auf die Analysen meines [1986].

Die Vorgeschichte der *al-jabr*-Tradition ist uns unbekannt. Man hat an indische Inspiration gedacht, weil eine hochentwickelte 'synkopierte' Algebra [36] sich schon bei frühen indischen Autoren finden läßt. Leider ist sie so viel höher als die *al-jabr* entwickelt, daß eine Ableitung letzterer von der indischen *wissenschaftlichen* Mathematik kaum denkbar ist. Die frühen islamischen Algebraautoren geben uns keine Hinweise [37] und also auch keine Hinweise auf fremden Ursprung, während der indische Ursprung des dezimalen Stellenwertsystems erklärt wird; es ist daher plausibel, daß sie die Tradition 'zu Hause' gefunden haben, irgendwo zwischen Irak und Turkestan - oder vielleicht überall im ganzen Gebiet von Syrien bis Indien.

Auch die fernere Vorgeschichte der Disziplin ist unklar. Sie mag vielleicht die babylonische Protoalgebra unter ihre Stammütter zählen - wie schon Gandz[38] bemerkt hat, könnte der Name vom babylonischen *gabrum* hergeleitet sein, ein Wort, das in der Tat in den babylonischen Texten eine Rolle spielt (obwohl eine andere als die von Gandz vermutete). Wenn das der Fall ist, ist doch nicht viel von der babylonischen Inspiration unverändert geblieben - eigentlich nur die Vorliebe für wenig nutzbare 'Gleichungen' zweiten Grades und das analytisch-heuristische Vorgehen.

Am besten nehmen wir also einfach die subwissenschaftliche *al-jabr*-Tradition zur Kenntnis, wie wir sie in den früh-

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36) D.h. schematisch-stenographisch geschriebene rhetorische Algebra. Der Ausdruck geht (wie auch die 'rhetorische Algebra') auf Nesselmanns *Algebra der Griechen* (1842) zurück.

37) Am nächsten kommt Thābit ibn Qurra in seiner *Verifizierung der Probleme der al-jabr durch geometrische Beweise*, wenn er von einer offensichtlich subwissenschaftlichen Gruppe von 'Algebraleuten' spricht (ed., transl. Luckey, 1941).

38) Gandz, 1926.

islamischen Quellen treffen. In dieser Form ist sie jedenfalls Hauptgrundlage für die schnelle Entwicklung einer wissenschaftlichen Algebra. Diese Entwicklung beginnt zu dem Zeitpunkt, wo al-Khwārizmī und sein Zeitgenosse ibn Turk Abhandlungen über die bisher ohne Bücher tradierte Disziplin verfassen [39] und in dieser Verbindung auch die naiv-geometrischen 'Beweise' der altbabylonischen (und *Liber mensurationum*-) Tradition als Begründung für die Richtigkeit der bisher unbegründeten Standardalgorithmen der *al-jabr* benutzen. Damit beginnt ein neues Kapitel in der Geschichte der algebraischen Denkweisen, wo auch letztendlich vieles aus den unterhaltungsmathematischen Traditionen einbezogen bzw. überflüssig gemacht wird.

Ägyptische und weitere Traditionen

Im Anfang des Aufsatzes wurden ägyptische und babylonische Mathematik unter demselben Gesichtspunkt diskutiert, und dann wurde plötzlich alles babylonisch. Gibt es denn nichts in Verbindung mit dem alten Ägypten zu sagen?

Ja und nein. Höhere 'Algebra' wie die babylonische gibt es in den ägyptischen Quellen nicht; Umkehraufgaben und elementare analytische Denkweisen gibt es. Alle Beispiele werden mit einer von zwei Varianten des 'einfachen falschen Ansatzes' gelöst - entweder wird für die unbekannte Größe der Wert *1* genommen oder auch ein anderer, bequemer Wert, was dann am Ende der Berechnung eine weitere Proportionalitätsbetrachtung erfordert; und mit Ausnahme von einzelnen homogenen Problemen zweiten Grades sind alle Aufgaben von erstem Grad.

39) Al-Khwārizmī's Werk wurde von F.Rosen (1831) herausgegeben und ins Englische übersetzt. Eine neue russische Übersetzung wurde von B.A.Rozenfeld in S.H.Siraždinov 1983 veröffentlicht. Das überlieferte Fragment von ibn Turks Werk wurde von A.Sayılı (1962) veröffentlicht und übersetzt.

Beide Varianten des 'einfachen falschen Ansatzes' kennen wir schon aus den altbabylonischen Texten, und zwar aus der Aufgabe vom gebrochenen Meßrohr; es ist daher überflüssig, hier nochmals ihre Prinzipien durchzugehen. Nur ist zu bemerken, daß diese wenig technische Betrachtungsweise in der ägyptischen Mathematik bis in die christliche Ära sehr beliebt bleibt und daß sie auch im indischen und islamischen (und davon im lateinischen) Mittelalter verbreitet ist - als leichtere Alternative zur rhetorischen Algebra.

Alternative ist sie freilich nur für homogene Probleme. Für nichthomogene Probleme ersten Grades entwickelte sich jedoch aus ihr die sogenannte *regula falsis* oder 'Regel des doppelten falschen Ansatzes'. Hier werden zwei Ansätze für die Unbekannte gemacht, der eine zu groß und der andere zu klein; der richtige Wert wird dann durch einen fixierten Algorithmus (dessen Prinzip eine lineare Interpolation ist) gefunden. Das ist an sich ganz unalgebraisch und somit ohne Interesse für eine Geschichte der algebraischen Denkweisen, ist aber illustrativ für einen Prozeß, der mehrmals in den subwissenschaftlichen Traditionen stattgefunden hat: Für die Lösung von Umkehrproblemen wird im Anfang eine heuristische, analytische Methode verwendet. Wenn sie vertraut und die erforderlichen Rechenschritte somit völlig bekannt geworden sind, wird sie automatisiert. Statt Analyse bleiben dann Algorithmen, und statt analytischer Denkweise bleibt gedankenloses Auswendiglernen von Regeln.

In den wissenschaftlichen Traditionen geschieht etwas ähnliches, aber trotzdem verschiedenes. Das haben wir vermutlich schon im seleukidischen Fall gesehen. Aus den altbabylonischen konstruktiven geometrischen Verfahren wurde anscheinend, als die für die Beweise erforderlichen Standardkonstruktionen bekannt genug waren, synthetische Argumentation auf schon vorhandenen Figuren. Ähnliches mag auch während einer griechischen Transformation des altbabylonischen Materials stattgefunden haben.

Diese Hypothese muß im Zusammenhang mit der alten Frage der 'geometrischen Algebra' gesehen werden. Vor etwa hundert Jahren bemerkte Zeuthen, daß die Theorie der *Elemente* II als Theorie geometrisch ausgedrückter Lösung von Gleichungen zweiten Grades gelesen werden kann, und vermutete daher, daß sie auch so gelesen werden soll[40]; die Griechen hätten, um um die Probleme der Irrationalität herumzukommen, ihre Algebra in geometrischer Form, in eine 'geometrische Algebra' übersetzt. Als dann um 1930 die 'babylonische (vermutet arithmetische) Algebra' entdeckt wurde, war der Gedanke naheliegend, daß die 'geometrische Algebra' eine geometrische Übersetzung einer übernommenen babylonischen Algebra sei.

Seit 20 Jahren ist diese These unter heftigem Angriff, und es ist wohl jetzt klar, daß die 'geometrische Algebra' nicht als *eine Algebra* aufgefaßt werden darf - sie gehört in eine eigene und ganz andersartige theoretische Struktur. Sie ist kein Glied in irgendwelcher 'Kunst des Auffindens' und also keine Übersetzung arithmetischer Analyse[41].

Das wird nicht an sich anders, wenn der geometrische Charakter der altbabylonischen (und daraus subwissenschaftlich fortgesetzten) Protoalgebra erkannt wird. Auf der anderen Seite sind die in *Elemente* II behandelten Figuren erstaunlich nahe verwandt mit denjenigen, deren Konstruktion in den altbabylonischen Texten beschrieben wird. Es ist dann mindestens eine naheliegende Möglichkeit, daß griechische Mathematiker im 6. oder 5.Jahrhundert angefangen haben, die

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40) Das geschah in Verbindung mit einer Untersuchung der Apollonischen Kegelschnittlehre, wo in der Tat die Theorie als äquivalenter Ersatz für analytische Geometrie fungiert (Zeuthen, 1886, 5 ff.).

41) Das wird nicht dadurch geändert, daß sie analytisch verwendet werden *kann* - wie z.B. in Apollonios' Kegelschnittlehre.

Eigenschaften der aus den Nachbartraditionen bekannten Figuren theoretisch auszuforschen[42] - ganz wie sie in derselben Zeit angefangen haben, die Eigenschaften der auf dem Abakus in Rechensteinen ausgelegten Zahlenmuster zu erforschen (die 'Lehre vom Geraden und Ungeraden')[43].

Aus der Erforschung der babylonischen Figuren und Figurtransformationen und der systematisch-theoretischen Weiterbearbeitung der daraus entstandenen Probleme könnte dann letztendlich die synthetische Theorie der *Elemente* II hervorgegangen sein. Auf halbem Weg im Prozeß fänden sich die *Data* Euklids. Ihre Propositionen beschäftigen sich nicht mit der Lösung generalisierter 'Gleichungen'[44], sondern mit ihrer Lösbarkeit; jedenfalls können sie als *Theorie geometrisch generalisierter Gleichungen* betrachtet werden. Ihre Methode ist manchmal nicht analytisch, sondern synthetisch - aber trotzdem werden sie in den antiken metatheoretischen Kommentaren als analytisch betrachtet[45], als ob eine analytische, 'gleichungslösende' Vorgeschichte noch an ihnen klebte.

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42) Für diese Hypothese sprechen verschiedene Indizien, die ich in meinem [1985a] diskutiere - darunter eine Isomorphie zwischen dem babylonischen 'Entgegengestellten' und dem griechischen geometrischen *dynamis*-Begriff und Andeutungen, daß schon zu Platons Zeit griechische Vorläufer für die von Diophant weiterentwickelte Algebra existierten. Solche Vorläufer wären vermutlich mit den nahöstlichen Algebratraditionen verbunden.

43) Siehe Lefevre, 1981.

44) In der Tat entsprechen, wenn wir von ihrer Identifikation von Größe und Meßzahl absehen, die babylonischen Gleichungen der ersten Definition der *Data:* 'D e r G r ö ß e n a c h g e g e b e n heißen Flächen, Linien und Winkel, zu denen wir uns gleiche verschaffen können.' (Übersetzung Thaer, 1962, 5).

45) Z.B. Pappos, *Collectio* VII - Hultsch, 1876, 636^{18}. Privates Gespräch mit C.M.Taisbak.

Bibliographie und Abkürzungen

BRUINS, E.M. and M.RUTTEN, 1961 *Textes mathématiques de Suse* (Mémoires de la Mission Archéologique en Iran , XXXIV). Paris: Geuthner, 1961.

BUSARD, H.L.L., 1968 "L'algèbre au moyen âge: Le 'Liber mensurationum' d'Abû Bekr". *Journal des Savants,* Avril-Juin 1968, 65-125.

FOWLER, Harold North (ed., transl.), 1977 Plato, *Theaetetus, Sophist* (Loeb Classical Library, 123). Cambridge, Massachusetts: Harvard University Press/London: Heinemann, 1977, 1st ed. 1921.

GANDZ, Solomon, 1926 "The Origin of the Term 'Algebra'". *American Mathematical Monthly* 33(1926), 437-440.

GUNDLACH, Karl-Bernhard und Wolfram von SODEN, 1963 "Einige altbabylonische Texte zur Lösung 'quadratischer Gleichungen'". *Abhandlungen aus dem mathematischen Seminar der Universität Hamburg* 26(1963), 248-263.

HULTSCH, Friedrich (ed., transl.), 1876 Pappi Alexandrini *Collectionis* quae supersunt. E libris manu scriptis edidit et commentariis instruxit Fridericus Hultsch. I-III. Berlin: Weidmann, 1876, 1877, 1878.

HØYRUP, Jens, 1982 "Investigations of an Early Sumerian Division Problem, c. 2500 B.C.". *Historia Mathematica* 9(1982), 19-36.

HØYRUP, Jens, 1985 *Babylonian Algebra from the View-Point of Geometrical Heuristics. An Investigation of Terminology, Methods, and Patterns of Thought.* Second, slightly Corrected Printing. Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science, 1985.

HØYRUP, Jens, 1985a "Dynamis, the Babylonians, and Theaetetus 147c7-148d7". *Preprint,* Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science, March 11, 1985.

HØYRUP, Jens, 1986 "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra". *Erdem* 2:5 (Ankara 1986), 445-484.

HØYRUP, Jens, 1987 "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". *Filosofi og Videnskabsteori på Roskilde Universitetscenter,* 3. Raekke: Preprints og Reprints 1987, Nr.2.

JAYAWARDENE, S.A., 1976 "The 'Trattato d'abaco' of Piero della Francesca", S.229-243 *in* C.H.Clough (ed.), *Cultural Aspects of the Italian Renaissance.* Essays in Honour of Paul Oskar Kristeller. Manchester: Manchester University Press/New York: Zambelli, 1976.

LEFÈVRE, Wolfgang, 1981 "Rechenstein und Sprache', S.115-169 *in* Damerow und Lefèvre (eds.), *Rechenstein, Experiment, Sprache. Historische Fallstudien zur Entstehung der exakten Wissenschaften.* Stuttgart: Klett-Cotta, 1981.

LUCKEY, Paul, 1941 "Iābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen". *Sächsische Akademie der Wissenschaften zu Leipzig. Mathematisch-physische Klasse. Berichte* 93(1941), 93-114.

MCT: O.NEUGEBAUER and A.SACHS *Mathematical Cuneiform Texts* (American Oriental Series, Vol.29). New Haven, Connecticut: American Oriental Society, 1945.

MKT: O.NEUGEBAUER *Mathematische Keilschrift-Texte.* I-III (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3.Band, erster bis dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937. Reprint Berlin etc.: Springer, 1973.

NESSELMANN, G.H.F., 1842 *Versuch einer kritischen Geschichte der Algebra.* Nach den Quellen bearbeitet. Erster Theil, *Die Algebra der Griechen.* Berlin: G.Reimer, 1842.

PATON, W.R. (ed., transl.), 1979 *The Greek Anthology.* In Five Volumes (Loeb Classical Library). Cambridge, Massachusetts: Harvard University Press/London. Heinemann, 1979. 1st ed. 1918.

POWELL, Marvin A., 1976 "The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics". *Historia Mathematica* 3(1976), 417-439.

ROSEN, Frederic (ed., transl.), 1831 The *Algebra* of Muhammad ben Musa. Edited and Translated. London: The Oriental Translation Fund, 1831.

RUDOLFF, Christoff, 1540 *Künstliche rechnung mit der ziffer und mit den zalpfenningen, sampft der Wellischen Practica, und allerley forteyl auff die Regel de tri.* Unpaginiert. O.Ort, 1540. Erstdruck 1532.

SAYILI, Aydın, 1962 *Abdülhamid ibn Türk'ün katısık denklemlerde mantıkî zaruretler adlı yazısı ve zamanın cebri (Logical Necessities in Mixed Equations by Abd al Hamîd ibn Turk and the Algebra of his Time).* (Publications of the Turkish Historical Society, Series VII, No.41). Ankara: Türk Tarih Kurumu Basımevı, 1962. Reprint 1985.

SIRAŽDINOV, C.X. (ed.), 1983 Muxammad ibn Musa al-Xorezmi *Matematičeskie traktaty.* Taškent: Izdatel'stvo 'FAN' Uzbekskoj CCP, 1983.

TANRET, M., 1982 "Les tablettes 'scolaires' découvertes à Tell ed-Der". *Akkadica* 27(1982), 46-49.

THAER, Clemens (ed., transl.), 1962 Die *Data* von Euklid. Nach Menges Text aus dem Griechischen übersetzt und herausgegeben. Berlin etc.: Springer, 1962.

TMB: F.THUREAU-DANGIN, 1938 *Textes mathématiques babyloniens* (Ex Oriente Lux, Deel 1). Leiden: Brill, 1938.

VER EECKE, Paul (ed., transl.), 1933 Pappus d'Alexandrie, *La collection mathématique.* Oeuvre traduite pour la première fois du grec en français, avec une introduction et des notes. Paris & Bruges: Desclée de Brouwer, 1933.

ZEUTHEN, Hieronimus Georg, 1886 *Die Lehre von den Kegelschnitten im Altertum.* København: Höst und Sohn, 1886. Dänische Erstausgabe 1884.

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The Babylonian Cellar Text BM 85200 + VAT 6599 Retranslation and Analysis

Jens Høyrup

Contents

1 Babylonian "algebra"

In a number of earlier publications, I have proposed a new understanding of the Old Babylonian mathematical technique known as "algebra", concentrating on problems of the second and to some extent of the first degree.[1] As a background to the following investigation of a particular text dealing in part with problems of the third degree I shall need a summary of my methods and my main results.

The Babylonian interest in apparently algebraic problems of the second degree was discovered around 1930.[2] As natural, and as a first approximation, the texts were interpreted through the conceptual framework of more recent algebra and arithmetic, with the result that the operations involved were understood as purely arithmetical operations, and the obviously geometrical vocabulary ("length", "width", "area", etc.) was interpreted as nothing but a set of convenient labels (for "the first unknown", "the second unknown", "the product of the unknowns", etc.).

My reinterpretation was based on two methodological principles. One of these may be described as a "structural analysis", the other as "close reading".

It had been observed already at an early moment that the Babylonians employed a whole set of distinct terms for addition, another set for subtraction, and a third for multiplication. *Grosso modo*, the terms from each set were supposed to be synonymous, and no particular effort was spent to find possible differences between them.

This comfortable creed was undermined by the structural analysis. It turned out that two different groups of supposedly additive terms are sharply distinguished. One of them, e.g., will normally not be used when (the measuring numbers of) a length and an area are added; the other is never employed for the operation of quadratic completion. The distinction between the two groups is so sharp that two different operations and two different concepts must be involved. Similarly, two different subtractive operations exist, and no less than four "multiplications".

[1]The most thorough presentation of my methods and results will be found in [HØYRUP 1990]. A more concise exposition has been made in German [HØYRUP 1989], even this dedicated to HANS WUSSING as a delayed homage on the occasion of his 60th anniversary.

[2]A brief history of the historiography of Babylonian mathematics, taking precisely this discovery as its starting point, is [HØYRUP 1991].

These differentiations make no sense within the received arithmetical interpretation, but suggest instead a more literal understanding of the geometrical vocabulary, which they fit. The "close reading" — close observation of the variable contexts in which each term occurs and of the organization of procedures compared to alternative possibilities which are not used[3] — confirms this.

The main outcome of the analysis is that Babylonian "second degree algebra" was organized around a pivotal technique which may be characterized as "cut-and-paste geometry". This geometry is not critically reflective as, e.g., EUCLID's *Data*. Nevertheless, the correctness of its procedures is intuitively obvious to anybody following the transformations (in the same way, say, as the correctness of the transformations of $2x + 4 = 6x - 24$ successively into $6x - 2x - 24 = 4$, $4x = 4 + 24 = 28$, $x = \frac{28}{4} = 7$ is obvious to anybody trained in elementary algebra); we might speak of "naive" geometry.

Together with the geometrical technique goes a geometrical conceptualization. While Medieval and present-day elementary algebra can be understood as the art of finding unknown *numbers* entangled in complex relations, the basic concern of Babylonian "algebra" is the disentanglement of *measurable but unknown lines and areas*. In both cases, this basic conceptualization may serve as a model for other structurally similar problems: The modern abstract numbers may stand for monetary values, geometrical lengths and areas, etc.; the Babylonian lines and areas, on the other hand, are used to represent prices, pure numbers, etc.

Cut-and-paste techniques constitute the pivot of Babylonian second-degree "algebra" but do not carry very far on their own. More complex problems therefore involve two auxiliary techniques, both of which are related to familiar procedures employed since long by Babylonian calculators: An "accounting technique" used, e.g., to find "how much there is of lengths" — in other words, the coefficient of the "length"; and a "scaling technique", which can be assimilated to a change of measuring scale or unit, and which is used to "reduce coefficients to *1*".

2 Essential terms and operations

In the following I shall present and discuss the tablet BM 85200 + VAT 6599 in the light of this reinterpretation of techniques and mode of thought. However, this will be most conveniently done if certain basic aspects of the terminology are presented in advance.

"Algebraic" interest appears to have arisen in a new Akkadian scribe school during the earlier Old Babylonian epoch (which in total covers the period c. 2000 B.C. to 1600 B.C.): "Algebraic" problem texts form, like omen texts, a *new* literary *genre*, and early specimens (18th c. B.C.?) tend

[3] A similar principle has recently been advocated by KARINE CHEMLA [CHEMLA 1991] as a tool for analyzing the methods of Ancient Chinese mathematics.

to be formulated in Akkadian, not in Sumerian, with the exception of some five terms which have their origin in traditional Sumerian mensuration and computation (length, width, area, part/reciprocal and "square root"/side of the square). As time went on, Sumerian word signs were increasingly used as "scholarly" abbreviations for the Akkadian words, but this was clearly a secondary development. As a result, however, each originally Akkadian term will have at least one Sumerographic equivalent[4] — at times several.

The terms and operations used in our tablet can be categorized and explained as follows:

2.1 Additive procedures

Of these there are two of major importance, both of which are used in our tablet. One is represented by the sign group UL.GAR, which is used logographically for *kamārum*, "to amass in a heap". It appears to designate a genuinely numerical addition, for which reason it can also be used to add the measuring numbers of, e.g., lengths and areas. It is symmetrical, connecting its addends by *u*, "and"; concomitantly, it conserves the identity of neither addend. I shall translate it "to accumulate".

The other is designated by the Sumerian word daḫ (Akkadian *waṣābum*), "to append". It operates on concrete entities, for which reason it only joins entities of the same kind and dimension. It is asymmetrical, connecting with the preposition *ana*, "to", and conserves the identity of that entity to which something else is appended while increasing its numerical value (as the identity of *my* bank account is unchanged when interest is added).

The habitual terms for the sum by accumulation are absent from the text dealt with below, except for one plausible occurrence of UL.GAR as a logogram for *kumurrum*, "accumulation"; instead, the term nigin, the "total" of accounting texts, is used on two occasions.

2.2 Subtractive procedures

Even these form a couple. One is a comparison, stating (in its Sumerographic version) that "X u-gù Y D dirig", "X exceeds Y by D" or, in a word for word translation which I shall use in the following, "X over Y D goes beyond". The other is the reversal of appending. Our tablet uses the Sumerogram zi, which corresponds to Akkadian *nasāḫum*, "to tear out".[5]

The palpably-concrete character of the operations of "appending" and "tearing out" is highlighted by the more complete phrase in which they are often embedded: *a* is not simply "appended to" or "torn out from B"

[4]In order to facilitate identification and linguistic classification, syllabically written Akkadian is ordinarily rendered as *italics*, while identified Sumerian words are given in spaced or normal writing. Signs with unidentified reading are written with their sign names (normally one of several possible readings) in small capitals.

[5]A few texts tend to distinguish *nasāḫum*, "to tear out", from *ḫarāṣum*, "to cut off", using the former preferentially when surfaces are involved and the latter for linear entities — cf. [HØYRUP 1992]. Our present tablet, however, exhibits nothing similar.

but "to/from the inside of *B*" (*libbum*, literally "heart" or "bowels", but in mathematical texts apparently worn down to a bare indication that *B* is something possessing bulk or body).

2.3 Multiplicative procedures

Our text only makes use of two of the four multiplications. One of these, furthermore, is only obliquely present. It is referred to by the Sumerogram ì-kú-kú, which normally corresponds to Akkadian *šutākulum*, used when a "length" and a "width" are put in place so as to "span" a rectangle, entailing the creation of an area equal to the product of the two measuring numbers. In the present text, however, it stands as a logographic equivalence for Akkadian *šutamḫurum*, "to make confront its counterpart", i.e., to position a single line together with its equal or "counterpart" as sides of a square.[6]

The other operation is referred to by the Akkadian term *našûm*, "to raise" — the Sumerogram íl is not used in the present text, nor are certain synonymous possibilities. It is used for the "scaling" technique mentioned above, and generally in all cases where considerations of proportionality lead to a multiplication; for the calculation of areas when this calculation is not the tacit by-product of a construction (e.g., for the computation of triangular and trapezoidal areas as average length times average width); for metrological conversions and similar multiplications by technical constants; and when divisions are performed through multiplication by a reciprocal. Below we shall discuss its use in the computation of volumes, which may provide us with the key to the etymology of the term and to the conceptualization of the operation.[7]

Connected in the present text to "raising" is a specific term bal. It designates a transformation factor, necessitated by the discrepancy between horizontal and vertical metrology (see below). THUREAU-DANGIN ([TMB], 232, followed by [MEA], 45) reads the sign as a logogram for *nabalkutum*/*nabalkattum*, "to escalate"/"transgression". However, the alternative connection to the verb *enûm*, "to change", appears much more suggestive; this implies that bal be understood as a "factor of change" or "conversion" (I shall use the latter translation).

2.4 Squaring and square-root

"Squaring" as a specific operation only occurs as a geometric operation, and is then designated by the verb *šutamḫurum* just mentioned. Related

[6]In still another text (YBC 4675, published in [MCT], 44f), the Sumerogram is used where "length and [a different] length" are made span, i.e., where a non-rectangular quadrangle is laid out. The semantic span of the term is obviously large, and based upon the construction of a quadrangle and not upon the computation of the area.

[7]The other two multiplicative operations are designated by a-rá, a term derived from "going" and used in the tables of multiplications; and by *eṣēpum*/tab, which designates the concrete repetition of a palpable entity. None of them is used in the text to be treated below.

to this verb is the verbal noun *mitḫartum*, which designates a situation characterized by a confrontation of equals — i.e., a square geometric configuration. In numerical terms, the *mitḫartum* is equal to the length of the side of the square.[8]

On rare occasions the verb may be written (as in our present text) by means of the Sumerogram ì-kú-kú. Also rare is the use of íb-si$_8$, as is the employment of this term as an abbreviation for *mitḫartum*. Utterly common (unavoidable in fact, apart from minor variations of the expression)[9] is, on the other hand, its use in connections where the arithmetical interpretation sees a square root. Properly speaking, the term is originally a Sumerian verbal form, meaning "makes equal" or (since sides are spoken of) "makes equilateral". That "A makes r equilateral" means that the area A, when laid out as a square, makes r the side of this square figure — to which corresponds of course the numerical relation $r = \sqrt{A}$. Secondarily, the term is used as a noun (which I shall translate "equilateral") designating this side.

The originally geometrical character of the íb-si$_8$ is made clear by texts where the íb-si$_8$ is found and "posed" together with its "counterpart" (*meḫrum*, another verbal noun related to *šutamḫurum*), as two sides forming the angle of a square. But as we shall see below, the term may also be used in more generalized senses.

2.5 Division, parts, and bisection

Evidently, division *as a problem* was encountered regularly by Babylonian calculators. *As a technique*, however, division proper is absent from the texts. Instead, a problem of the type $d = n \cdot x$ is treated in one of two ways. If n is listed in the table of reciprocals, the text will ask that the "igi" of n be "detached" (du$_8$/*paṭārum*) and then "raise" this number to d. The term igi n is derived by abbreviation from the expression for the "n'th part [of something]" (igi n gál-bi), but so clearly kept apart from this original meaning that it must be regarded as a technical term for the reciprocal of n as tabulated in the table of reciprocals.[10] (I shall retain the original

[8]This may seem strange to us, who are accustomed to the idea that a square *is* its area (i.e., is identified by and hence with this characteristic parameter) and *has* a side. *A priori*, however, the Babylonian conception of a square figure as *being* (i.e., being identified by and hence with) its side and *possessing* an area is no worse. The Greek mathematical term *dynamis*, moreover, appears to correspond to a similar conceptual structure (cf. [Høyrup 1990a], as does perhaps an ancient Chinese mathematical term (Jean-Claude Martzloff, private communication).

[9]Some of these consist in homophonous substitution of syllables. More important is, however, the alternative ba-si$_8$. Traditionally it has been suggested that the latter term stood for "cube root" and íb-si$_8$ for square root, and the exceptions to this rule have been regarded as minor anomalies. As the number of exceptions has increased with the publication of further texts, this explanation of the difference between the terms must now be regarded as outdated.

[10]igi n is "detached" because unity is imagined to be split and one n'th part is taken out, as demonstrated by the use of the verb "to tear out" in a variant expression.

term in the translation in order to stress the specifically Sumero-Babylonian character of this concept, connected as it is both to mathematical inversion and to the table). If n is not listed (which of course must happen when n is not sexagesimally regular, i.e., not of the form $2^p \cdot 3^q \cdot 5^r$), the text takes note of this fact and then asks "what shall I pose to n which gives me d?" and give the answer immediately — easily done, in fact, since mathematical problems were constructed backwards and the solution thus both guaranteed and known in advance.

Halving and division by 2 are treated as division by any other regular number, through multiplication by "igi 2". In certain cases, however, where the arithmetical interpretation sees nothing but a halving the Babylonians operated differently. This is the case when a "natural" or "customary" half is to be found, e.g., the radius from the diameter of a circle. Below we shall encounter this specific bisection designated *ḫepûm*, "to break", at the crucial point in the solution of second-degree problems where, in the arithmetical understanding, the coefficient of the first-degree term is halved — indeed a case where *only* exact bisection makes sense.

2.6 Numbers

The Babylonian place value notation for numbers is well-known among historians of mathematics. It was not the only system in use, and could not possibly be, since it did not indicate absolute place. The mathematical texts, however, make use almost exclusively of this system; so does the text to be discussed below. I shall therefore bypass the systems used in economical and administrative texts.

The place value system was sexagesimal, i.e., its base was 60 — or, better perhaps, alternately 10 and 6. "Final zeroes" were never used, nor was any "sexagesimal point". A marker for intermediate empty places was occasionally used in a few texts from the outgoing Old Babylonian period, but mostly these were just indicated by increased distance between surrounding signs or left to contextual understanding.

In my transliteration, I shall render each sexagesimal place by a corresponding Arabic numeral (between 1 and 59); places are separated by commas. The translation introduces an indication of absolute place as derived from context. A number transliterated 21,15,23,6,19 and interpreted as $21 \cdot 60^2 + 15 \cdot 60^1 + 23 \cdot 60^\circ + 6 \cdot 60^{-1} + 19 \cdot 60^{-2}$ I shall thus translate as 21''' 15'' 23°6′19″ (when it is not needed for understanding or as a separator, I shall leave out °).

The distinction between "n'th part" and "reciprocal of n" is normally made as here described; so also in the text under discussion below. Another way to distinguish is that the reciprocal is "detached" while the n'th part is "torn out" — see [Høyrup 1990], 54 n. 69.

2.7 Metrology

The only parts of Babylonian metrology which concern us here are the units for distance, area, and volume.

The fundamental unit for *horizontal* extension is the nindan or "rod", equal to approximately 6 m. It is subdivided into 12 kùš or "cubits", each thus approximately 50 cm (fingertip to elbow), which is the fundamental unit for *vertical* extension (heights and depths). "Fundamental units" are almost invariably left implicit, which makes measures given in fundamental units look like pure numbers. In other cases (e.g., horizontal extensions measured in kùš) the unit will be explicit.[11]

The fundamental [horizontal] area unit is the sar, equal to a square nindan, i.e., a square with the fundamental unit for horizontal extension as its side. The corresponding fundamental volume unit is a block of 1 nindan times 1 nindan times 1 kùš, which, similarly, is called a sar.[12] The standard volume is thus to be understood as a standard area covered to a standard height; as we shall see below, calculating a volume implies "raising" this standard height to the real height.

2.8 "Variables" and metalanguage

Anything somehow "algebraic" in character must possess ways to designate "unknowns" or variables and devices to display the logical organization of problems and procedures. So also Babylonian "algebra".

Designations for variables were more or less standardized, and more or less bound to specific problem types. An example of the highly specific is the "beginning of the reed", the original length of a measuring reed which during a mensuration process looses specified sections; another, used in our present text, is the couple igi and igi-bi, "the reciprocal and its reciprocal", a couple of numbers occurring together in the table of reciprocals and thus with product 1 (or, indeed, 60^n, where $n = 1$ is attested in the tablet YBC 6967, [MCT], p. 129). I shall employ the Akkadian loanwords, *igûm* and *igibûm* in order to emphasize the connection to the term igi.

Of most general use, almost corresponding to our semiautomatic choice of x and y as labels for a pair of unknowns, is the set uš/sag, representing the "length" and the "width" of a rectangle and thus linked to the basic geometrical technique.[13]

[11]Actually, matters are somewhat more complex, which may give (and has given) rise to misreading of texts. A horizontal extension told to be "5 kùš" will often have to be read "5′ [nindan, i.e., a] kùš", alternatively, the expression "5 1 kùš" is used, meaning "5′ [nindan, i.e.,] 1 kùš". Both possibilities are used in the text discussed below.

[12]Often translations refer to this latter unit as a volume-sar, in order to keep the two apart. I shall avoid this convention, because it obscures an important aspect of Babylonian mathematical thought — cf. the discussion of problem N$^{\text{O}}$ 12 below.

[13]The existence of this link is most clearly though paradoxically seen in a case where the mensurational rectangle serving as a pretext for the problem appears to be distinguished from the rectangle serving the procedure ([TMS] XVI A; cf. corrected text in [HØYRUP 1990], 300). In the proof, the "true width", the width of the imaginary *real*

The tablet to be discussed below deals with rectangular parallelepipedal excavations (túl-sag), the horizontal dimensions of which are also designated uš and sag. The depth is designated GAM, a term of more specific but not idiosyncratic use.

The area spanned by "length" and "width" is not designated in the present text by the usual term a-šà, originally "field". Instead, and corresponding to the character of the problems, the sign KI (for Akkadian *qaqqarum*) is used, meaning ground, foundation or (here) "floor". The volume of the excavation is referred to through the [amount of] "earth" (saḫar) which has been excavated.

The various terms indicating the structure of problems and procedure in the text below need not be listed since they are easily understood in context. At this place it should only be pointed out that terms like en-nam "what", *mala* "so much as", *tammar* "you see" etc. are highly standardized, in general or at least in widespread use. The procedure itself has two names, *epēšum* and *nēpešum*, used respectively to announce the procedure and to tell that it has been performed; the first may be translated "the making", the second more clumsily as the "having-been-made", as done below.

3 The actual tablet

The tablet to be scrutinized in the following is BM 85200 + VAT 6599 — which means that one part of it is conserved in the British Museum and another in the Berlin collection of Vorderasiatische Texte. Below, line numbers from the Berlin fragment will be labeled by an asterisk *, while unlabeled numbers refer to the BM fragment.

The exact provenience of the tablet is unknown. Basing himself on the *ductus*, NEUGEBAUER dated the tablet to the late Old Babylonian period, which was confirmed by GOETZE ([MCT], 150f), who showed its spellings to be characteristic of his "6th group", "Northern modernizations of southern (Larsa) originals". Certain writing errors in the tablet demonstrate, furthermore, that the tablet is not the original modernization but a copy (e.g., the characteristic copyist's omissions in obv. II, 14 and rev. II, 4).

The text contains 30 problems, all of which deal with a túl-sag, i.e. (as made clear by the mathematical context), a rectangular parallelepipedal excavation (I shall use the translation "cellar"). Some problems have the mathematical structure of second-degree equations, and are in fact solved by means of the characteristic second-degree cut-and-paste techniques; others are effectively of the third degree, and are correspondingly solved by other means (factorization and recourse to a table, as we shall see). It is thus obvious that the Babylonian calculators knew the practical difference between the two algebraic degrees. It is equally obvious, however, that the characteristic feature shared by all problems of the tablet is the geomet-

rectangle, is multiplied by *1* before it is multiplied by "as much as there is of widths" (i.e., the coefficient of the width).

ric configuration dealt with. As we shall see in chapter 7, this primacy of geometric constitution over algebraic structure holds even on lower levels, which shall provide us with clues to the technique of didactical exposition.

The text was published, translated and discussed by NEUGEBAUER in [MKT] I, 193ff, and [MKT] II, Tafeln 7+39, with corrections [MKT] III, 54f. Other (partial) discussions of interest are [VOGEL 1934, TH.-D. 1937, GANDZ 1937], and [TH.-D. 1940] (where further bibliographic information is found on p. 1). VOGEL's treatment of the cubic problems takes a geometric approach; the others are all based on the customary arithmetical interpretation.

4 The text

The following transliteration builds on NEUGEBAUER's ([MKT] I, 193ff) with corrections suggested by THUREAU-DANGIN and mostly accepted by NEUGEBAUER; many restitutions of damaged passages also go back to [MKT] or to [TMB]. Problems the text of which is too incomplete to allow any meaningful attempt at reconstruction (N^os 1–4, 10–11 and 29) have been omitted.

The translation is my own, building on the results explained in chapters 1–2, and following the principle of "conformal translation" as set forth in [HØYRUP 1990], 60–62 (with the exception that no typographic distinction is made between translations from syllabic Akkadian and Sumerograms, and with the extra feature that italics are used to indicate translation of reconstructed passages). The aim of conformality is to obtain a translation where it is clear what precisely is told in the original text and what not, and where the conceptual distinctions of the original (e.g. between different additive procedures) are still visible. The basic tool is the use of "standard translations", where "all words except a few key terms are rendered by English words; a given expression is in principle always rendered by the same English expression, and different expressions are rendered differently with the only exception that well-established logographic equivalence is rendered by coinciding translation [...], while possibly mere ideographic equivalence is rendered by translational differentiation. Terms of different word class derived from the same root are rendered (when the result is not too awkward) by derivations from the same root [...]. Furthermore, syntactical structure and grammatical forms are rendered as far as possible by corresponding structure and grammatical forms; the simple style of the mathematical texts make this feasible" ([HØYRUP 1990], 61; the standard translations used in the present paper are, a couple of newcomers apart, those of this earlier publication).

As a preliminary philological commentary, two features concerning the way the text is written may be mentioned: Firstly, certain Akkadian words are written occasionally in abbreviated form, e.g. *šu-tam⟨-ḫir⟩*, *ta⟨-mar⟩*, *i⟨-ši⟩* (the same holds for the Sumerian terms ba⟨-zi⟩ and íb⟨-si$_8$⟩ in obv.

II, 30 and rev. I, 6). Secondly, Sumerograms are written either with an Akkadian phonetico-grammatical complement (KI^{ri}) or, more often, with a Sumerian complement (gar-ra, daḫ-ḫa, sum-mu).

Specific commentary is given in footnotes to the text.

Obv. I

N° 5

14*. [túl-sag *ma-la* uš GAM-*ma* saḫar-ḫi-a ba-zi KI^{ri} *ù* saḫar-ḫi-a UL.GAR 1,10]
A cellar. So much as the length: The depth. The earth I have torn out. My floor and the earth I have accumulated, 1°10′

15*. [...]

16*. [...][14]

1. [...uš sa]g en-nam
*...length and wid*th, what?

2. ...⌈3 *ta-mar*⌋ $\frac{1}{2}$ 3 *ḫe-pé* 1,30 *ta-mar*
...3 you see. $\frac{1}{2}$ of 3 break. 1°30′ you see,

3. ...[igi 1,30 du$_8$-a] 40 *ta-mar* bal sag igi 12 bal GAM du$_8$-a
...the igi of 1°30′ detach, 40′ you see, the conversion of the width. The igi of 12, the conversion of the depth, detach;

4. [5 *ta-mar a-na* 1⌉ *i-ši* 5 *ta-mar a-na* 40 *i-ši* 3,20 *ta-mar*
5′ you see. To 1 raise, 5′ you see. To 40′ raise, 3′20″ you see.

5. ⌈3,20⌉ *a-na* 5 *i-ši* 16,40 *ta-mar* igi 16,40 du$_8$-a 3,36 *ta-mar* 3,36
3′20″ to 5′ raise, 16″40‴ you see. The igi of 16″40‴ detach, 3‵ 36 you see. 3‵ 36

6. *a-na* 1,10 *i-ši* 4,12 *ta-mar* 6 íb-si$_8$ 6 *a-na* 5 *i-ši* 30 *ta*⟨*-mar*⟩ 6 *a-na* 3,20 *i*⟨*-ši*⟩
to 1°10′ raise, 4‵ 12 you see, 6 the equilateral. 6 to 5′ raise, 30′ you see. 6 to 3′20″ raise,

7. 20 sag 6 *a-na* 1 *i-ši* 6 *ta-mar* GAM *ki-a-am*
20′, the width. 6 to 1 raise, 6 you see, the depth. So

8. *ne-pé-šum*
the having-been-made.

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[14]It is not quite clear whether problem N° 5 begins in line 14*, as suggested by NEUGEBAUER, or only in line 15* or even 16*, as suggested by THUREAU-DANGIN ([TMB], p. 11). Traces suggesting the end of the term [*ne-pé-šu*]*m* in line 13* support NEUGEBAUER's assumption; no other statements, on the other hand, extend over more than two lines, which supports THUREAU-DANGIN.

N° 6

9. túl-sag *ma-la* uš GAM-*ma* 1 saḫar-ḫi-a ba-zi KI^{ri} *ù* saḫar-ḫi-a UL.GAR 1,10 uš *ù* sag 50[15] uš sag en⟨-nam⟩
A cellar. So much as the length:[16] the depth. 1 the earth[17] I have torn out.[18] My floor and the earth I have accumulated, 1°10′. Length and width, 50′. Length, width, what?

10. za-e 50 *a-na* 1 bal *i-ši* 50 *ta-mar* 50 *a-na* 12 *i-ši* 10 *ta-mar*
You, 50′ to 1, the conversion, raise, 50′ you see. 50′ to 12 raise, 10 you see.

11. 50 *šu-tam⟨-ḫir⟩* 41,40 *ta-mar a-na* 10 *i-ši* 6,56,40 *ta-mar* igi-*šu du*$_8$*-a* 8,38,24 *ta⟨-mar⟩*
Make 50′ confront itself, 41′40″ you see; to 10 raise, 6°56′40″ you see. Its igi detach, 8′38″24‴ you see;

12. *a-na* 1,10 *i-ši* 10,4,48[19] *ta-mar* 36 24 42 íb-si$_8$
to 1°10′ raise, 10′4″48‴ you see, 36′, 24′, 42′ the equilaterals.

13. 36 *a-na* 50 *i-ši* 30 uš 24 *a-na* 50 *i-ši* 20 sag 36 *a-na* 10 6 GAM
36′ to 50′ raise, 30′, the length. 24′ to 50′ raise, 20, the width; 36′ to 10 raise, 6, the depth.

14. [*n*]*e-pé-šum*
*The h*aving-been-made.

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[15]This additive use of a mere "and" is rare but not unprecedented — cf. also rev. I, 1. YBC 4714 ([MKT] I, 487–492) offers a number of analogous examples, together with parallels which suggest that we have to do with an abbreviation "⟨accumulation of⟩ *a* and *b*". The controversy between VAN DER WAERDEN and BRUINS (see [V.D. WAERDEN 1962], 74) over the philological possibility of an interpretation of AO 6770 N° 1 (originally proposed in [GANDZ 1948], 38f, a fact not noticed by any of the contestants) depending on the assumed additive use of *ù* could thus have been settled long before it arose.

[16]I read MA as the Akkadian particle *-ma*. If this reading is correct, the structure of the passage is rendered most clearly when the ":" translating *-ma* is put in this place. It is, however, possible that the sign is simply a phonetic complement indicating that the preceding GAM is to be read gam, not gúr. GAM-*ma* is then to be replaced throughout the tablet by gam-ma, and the translation ": the depth" by ", the depth".

I prefer the first reading because the affix is found invariably when GAM closes an expression beginning with *ma-la*, and never in the final section of the problem when its numerical value is stated, nor in questions for this value. Such systematics is not found in other cases where a Sumerian phonetic indicator is used — compare the use of daḫ-ḫa in obv. II, 6* with that of daḫ in obv. II, 13*.

[17]The volume of earth removed *is* in fact 1 volume sar. The fact, however, is not used to solve the problem, and if it is taken into account, the problem is over-determined. Similarly in N° 7. Cf. below, chapter 7.

[18]A more idiomatic translation would be "removed" or "dug out". It is, however, worthwhile observing that the text uses the same term for digging out earth as for mathematical "subtractions".

[19]Written with a conspicuous space between 10 and 4 to distinguish 10,4 from 14.

N° 7

15. túl-sag *ma-la* uš GAM-*ma* 1 saḫar-ḫi-a ba-zi [K]I^{ri} *ù* saḫar-ḫi-a UL⟨.GAR⟩ 1,10 uš u-gù sag 10 dirig
A cellar. So much as the length: the depth. 1 the earth[20] I have torn out. My *floor* and the earth I have accumulated, 1°10′. Length over width 10′ goes beyond.

16. za-e 1 *ù* 12 [b]al gar-ra 10 ⌈dirig *a-n*⌉*a* 1 *i-ši* 10 *ta-mar a-na* 12 *i-ši* 2 *ta-mar*
You, 1 and 12, the *conv*ersions, pose. 10′ the *going-beyond to* 1 raise, 10′ you see; to 12 raise, 2 you see.

17. 10 *šu-tam*⟨*-ḫir*⟩ 1,40 *ta-mar a-na* 2 *i-ši* 3,2[0 *t*]*a-mar* igi 3,20 du$_8$-a 18 *ta-mar*
10′ make confront itself, 1′40″ you see; to 2 raise, 3′2*0*″ *y*ou see. The igi of 3′20″ detach, 18 you see;

18. *a-na* 1,10 *i-ši* 21 *ta-mar* 3 2 21$^{(sic)}$ íb-si$_8$ [10 *a-na* 3 *i*]*-ši* 30 uš
to 1°10′ raise, 21 you see, 3, 2, 21 ⟨error for 3°30′⟩ the equilaterals. *10′ to 3 r*aise, 30′, the length.

19. 10 *a-na* 2 *i-ši* 20 sag 3 *a-na* 2 *i-ši* ⌈6⌉ *ta-mar* [6] GAM
10′ to 2 raise, 20′, the width. 3 to 2 raise, 6 you see, *6,* the depth.

20. *ne-pé-šum*
The having-been-made.

N° 8

21. túl-sag *ma-la* uš GAM-*ma* saḫar-[ḫi]-a ba-zi KI^{ri} *ù* saḫar-ḫi-a UL.GAR-*ma* 1,10 30 uš sag e[n-nam]
A cellar. So much as the length: The depth. The ear*th* I have torn out. My floor and the earth I have accumulated: 1°10′. 30′, the length. The width, w*hat*?

22. za-e 30 uš *a-na* 12 *i-ši* 6 *ta-mar* GAM 1 *a-na* 6 daḫ-ḫa 7 *ta-mar*
You, 30′, the length, to 12 raise, 6 you see, the depth. 1 to 6 append, 7 you see.

23. igi 7 nu du$_8$-a en-nam *a-na* 7 gar-ra *ša* 1,10 sum-mu 10 gar-ra igi 30 uš du$_8$-a
The igi of 7 is not detached. What to 7 shall I pose which 1°10′ gives me? 10′ pose. The igi of 30′ detach,

24. 2 *ta-mar* 10 *a-na* 2 *i-ši* 20 sag *ta-mar*
2 you see. 10′ to 2 raise, 20′, the width, you see.

25. *ne-pé-šum*
The having-been-made.

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[20] Once more, a value which is correct but not used.

N° 9

26. túl-sag *ma-la* uš GAM-*ma* saḫar-ḫi-a ba-zi KI^{ri} *ù* saḫar-ḫi-a UL.GAR-*ma* 1,10 20 sag uš ⟨en-nam⟩
A cellar. So much as the length: the depth. The earth I have torn out. My floor and the earth I have accumulated: 1°10′. 20′, the width. The length, *what*?

27. za-e 20 *a-na* 12 *i-ši* 4 *ta-mar* 4 *a-na* 1,10 *i-ši* 4,40 *ta-mar*
You, 20′ to 12 raise, 4 you see. 4 to 1°10′ raise, 4°40′ you see.

28. $\frac{1}{2}$ 20 sag *ḫe-pé* 10 *ta-mar* 10 *šu-tam-ḫir* 1,40 *ta-mar a-na* 4,40 daḫ-ḫa
$\frac{1}{2}$ of 20, the width, break, 10′ you see. 10′ make confront itself, 1′40″ you see; to 4°40′ append,

29. 4,41,40 *ta-mar* 2,10 íb-si$_8$ 10 *ša* ì-kú-kú ba-zi-ma
4°41′40″ you see, 2°10′ the equilateral. 10′ which you have made span tear out:

30. 2 *ta-mar* igi 4 du$_8$-a 15 *ta-mar a-na* 2 *i-ši*
2 you see. The igi of 4 detach, 15′ you see; to 2 raise,

31. 30 *ta-mar* {erasure} uš
30′ you see, the length.

32. *ne-pé-šum*
The having-been-made.

Obv. II

N° 12

5*. túl-sag *ma-la* uš GAM-*ma* saḫar-ḫi-a ba-zi KI^{ri} *ù* saḫar-ḫi-a U[L.GAR]
A cellar. So much as the length: The depth. The earth I have torn out. My floor and the earth I have ac*cumulated,*

6*. igi 7 gál *él-qé a-na* KI^{ri} daḫ-ḫa-*ma* 20 *ta*⟨*-mar*⟩ 30 [uš]
the 7th part I have taken, to the floor I have appended: 20′ you see. 30′ *the length.*

7*. za-e 30 *a-na* 12 *i-ši* 6 *ta-mar* GAM 1*a-na* ⌊6 daḫ-ḫa⌋
You, 30′ to 12 raise, 6 you see, the depth. 1 to 6 *append,*

8*. 7 *ta-mar* igi 7 gál *le-qé* 1 *ta-mar* 1 *ù* 1 U[L.GAR]
7 you see. Its 7th part take, 1 you see. 1 and 1 ac*cumulate*,

9*. 2 *ta-mar* igi 2 du$_8$-a 30 *ta-mar* 30 *a-na* 20 UL.GAR-⌊*i*$^!$*-ši*]
2 you see. The igi of 2 detach, 30′ you see, 30′ to 20′ the accumulation r*aise*,

10*. 10 *ta-mar* igi 30 uš du$_8$-a 2 *ta-mar* 2 *a-na* 10 *i-š*[*i* 20 sag]
10′ you see. The igi of 30′, the length, detach, 2 you see. 2 to 10 rai*se, 20′ the width.*

11*. *ne-pé-šum*
The having-been-made.

N° 13

12*. túl-sag *ma-la* uš GAM-*ma* saḫar-ḫi-a ba-zi *qá-qá-ri ù* saḫar-ḫi-a UL.[GAR]
A cellar. So much as the length: the depth. The earth I have torn out. My floor and the earth I have accumulated,

13*. 1,10 igi 7 gál-*šu él-qé a-na* KI^{ri} daḫ 20 20 sag
1°10′.[21] Its 7th part I have taken, to my floor I have appended, 20′. 20′, the width.

14*. za-e 20 *a-na* 7 *i-ši* 2,20 *ta-mar* 20 sag *a-na* 12 *i-ši*
You, 20′ to 7 raise, 2°20′ you see. 20′, the width, to 12 raise,

15*. 4 *ta-mar 4* a-na *2,20 i-ši* 9,20 *ta-mar a-na* 7 1 daḫ-ḫ[a]
4 you see. 4 to 2°20′ raise, 9°20′ you see. To 7, 1 appen*d*,

16*. 8 *ta-mar* 20 *a-na* 8 *i-ši* 2,40 *ta-mar* $\frac{1}{2}$ 2,40 *ḫe-pé* [*šu-tam*⟨*-ḫir*⟩]
8 you see. 20′ to 8 raise, 2°40′ you see. $\frac{1}{2}$ of 2°40′ break, *make confront itself*,

17*. 1,46,40 *ta-mar a-na* 9,20 daḫ-ḫa 11,6,40 *t*[*a-mar*]
1°46′40″ you see, to 9°20′ append, 11°6′40″ yo*u see,*

18*. 3,20 íb-si$_8$ 1,20 *ša* ì-kú-kú ba-zi 2 *ta*[*-mar*]
3°20′ the equilateral. 1°20′ which you have made span tear out, 2 you *see.*

19*. igi 4 du$_8$-a 15 *ta-mar* 15 *a-na* 2 *i-ši* 30 [uš]
The igi of 4 detach, 15′ you see. 15′ to 2 raise, 30 *the length.*

20*. *ne-pé-š*[*um*]
The having-been-ma*de.*

N° 14

1. túl-sag *ma-la* igi uš *ma-la* igi-bi sag *ma-la* igi {u-gù igi-bi dirig}[22]
A cellar. So much as the igûm, the length. So much as the igibûm, the width. So much as the igûm {over the igibûm goes beyond}:

2. GAM-*ma* 1⌊6 saḫar-ḫi-a ba-z⌋i uš sag *ù* GAM en-nam

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[21]This value, again, is correct but not used.

[22]With this emendation, the following calculation (as reconstructed by NEUGEBAUER) is correct. The wrong formulation (which is not solvable in rational numbers, and from which the *ša* of problems 15 and 17 is absent) seems to be a contamination from the following problem.

The depth. 1*6 of earth I have torn* out. Length, width, and depth, what?

3. za-e igi 12 du$_8$-a [5 *ta-*]*mar* 5 [*a-na* 16] *i-š*[*i* 1,2]0 *ta*⟨*-mar*⟩
You, the igi of 12 detach, *5′ you* see. 5′ *to 16* raise, *1°20′* you see,

4. 1,20 igi igi 1,2[0 du$_8$-a 45 *ta-m*]*ar* 4⟨5⟩ igi-bi [16] GAM
1°20′ the igi. The igi of 1°2*0′ detach, 45′ you s*ee, 4*5′* the igibûm. *16* the depth.

5. *ne-*[*pé*]*-šum*
The having-*been*-made.

N° 15

6. túl-sag *ma-la* igi uš *ma-*[*la* igi-bi sa]g *ma-la ša* igi u-gù igi-bi dirig ⟨GAM-*ma*⟩
A cellar. So much as the igûm, the length. So muc*h as the igibûm, the wi*dth. So much as that which the igûm over the igibûm goes beyond: *The depth.*

7. 36 saḫar-ḫi-a ba-zi-*m*[*a* igi igi-bi *ù* GAM] en-nam
36 of earth I have torn out*: Igûm, igibûm and depth*, what?

8. za-e igi 12 du$_8$-a [5 *ta-mar* 36⌋ *a-na* 5 {⌊2 ¿TI?...]} *i*⟨*-ši*⟩
You, the igi of 12 detach, *5′ you see. 36* to 5′ {...!} ra*is*e,

9. 3 *ta-mar* $\frac{1}{2}$ 3 *ḫ*[*e-pé* 1,30 *ta-mar*] 1,30 igi [40 igi-bi 36⌋ GAM
3 you see. $\frac{1}{2}$ of 3 b*reak, 1°30′ you see,* 1°30′ the igûm. *40′ the igibûm, 36* the depth.

10. *ne-*[*p*]*é-š*[*um*]
The having-*been*-m*ade*

N° 16

11. túl-sag *ma-la* igi uš ma-la [igi-bi sag] *ma-la* nigin[23] igi *ù* igi-bi GAM-*ma*
A cellar. So much as the igûm, the length. So much as *the igibûm, the width.* So much as the total of igûm and igibûm: the depth.

12. 26 saḫar-ḫi-a ba-zi igi igi-bi *ù* GAM en-nam
26 of earth I have torn out. Igûm, igibûm, and depth, what?

13. za-e igi 12 du$_8$-a 5 *ta-mar* 5 *a-na* 26 *i-ši*
You, the igi of 12 detach, 5′ you see; 5′ to 26 raise,

14. 2,10 *ta-mar* $\frac{1}{2}$ 2,10 *ḫe-pé šu-tam-*⟨*ḫir*⟩ 1,10,25 *ta-ma*[*r*] ⟨1 *i-na* 1,10,25 ba-zi 10,25 *ta-mar*⟩[24]

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[23] According to its use an abbreviation for šu-nigin, the "total" or "summa summarum" of accounts.

[24] The omission of this passage is one of several indications that the tablet is copied from another tablet, and is neither an original nor the direct reproduction of an oral

2°10′ you see. ½ of 2°10′ break, make confront itself, 1°10′25″ you se*e*.
1 from 1°10′25″ tear out, 10′25″ you see,

15. 25 íb-si$_8$ *a-na* ⟨1,⟩5 daḫ-ḫa *ù* ba-zi 1,30 *ù*[25] 40 *t*[*a-mar*]
25, the equilateral, to *1*°5′ append and tear out. 1°30′ and 40′ yo*u* *see;*

16. 1,30 igi 40 igi-bi 26 GAM
1°30′ the igûm; 40′ the igibûm; 26 the depth.

17. *ne-pé-šum*
The having-been-made.

N° 17

18. túl-sag *ma-la* igi uš *ma-la* igi-bi sag *ma-la ša* igi u-gù igi-bi d[irig]
A cellar. So much as the igûm, the length. So much as the igibûm, the width. So much as that which the igûm over the igibûm go*es beyond*

19. *i-na* igi ba-zi GAM-*ma* 6 saḫar-ḫi-a ba-zi igi *ù* igi-b[i en-nam]
from the igûm I have torn out: the depth. 6 of earth I have torn out. Igûm and igib*ûm, what?*

20. za-e igi 12 du$_8$-a 5 *ta-mar a-na* 6 *i-ši* 30 *ta-mar*
You, the igi of 12 detach, 5′ you see; to 6 raise, 30′ you see.

21. [i]gi 3[0 d]u$_8$-a 2 *ta-mar* 2 igi 30 igi-bi 6 GAM
The igi of *30′ de*tach, 2 you see. 2, the igûm, 30′, the igibûm. 6, the depth.

22. *ne-pé-šum*
The having-been-made.

N° 18

23. túl-sag *ma-la* igi uš *ma-la* igi-bi sag *ma-la* nigin igi igi-b[i GAM-*m*]*a* 30 s[aḫar-ḫi-a ba-zi]
A cellar. So much as the igûm, the length. So much as the igibûm, the width. So much as the total, igûm, igib*ûm: the depth*. 30 of ea*rth* *I have torn out.*

24. za-e igi 12 du$_8$-a 5 *t*[*a-ma*]*r* 5 *a-na* 30 saḫar-ḫi-a *i-ši*
You, the igi of 12 detach, 5′ y*ou s*ee. 5′ to 30, the earth, raise,

25. 2,30 *ta-mar* ½ 2,30 *ḫe-pé šu*[*-tam-ḫir* 1,33,4]5 *ta*[*-mar*]
2°30′ you see. ½ of 2°30′ break, make *confront* itself, *1°33′4*5″ you *see.*

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presentation. Cf. the corresponding omission in rev. II, 4, equally called forth by the presence of two identical sequences of signs close to each other.

[25]With some hesitation, I follow THUREAU-DANGIN's reading of the sign as the first part of an *ù*. The other possibility is a full igi (NEUGEBAUER's reading).

26. 1 *i-na* 1,33,45 ba-zi 3[3,4]5 *ta-mar* ⌊4⌋5 íb-si$_8$
1 from 1°33′45″ tear out, *33′4*5″ you see, *4*5′ the equilateral.
27. *a-na* 1,15 daḫ-ḫa *ù* ba-zi 2 *ù* 30 *ta-m*[*ar*]
To 1°15′ append and tear out, 2 and 30′ you see.
28. *ne-pé-šum*
The having-been-made.

N° 19

29. túl-sag *ma-la* igi uš *ma-la* igi-bi sag *ma-la* {erasure} igi-bi GAM-*ma*
A cellar. So much as the igûm, the length. So much as the igibûm, the width. So much as the igibûm: the depth.
30. 20 saḫar-ḫi-a ba⟨-zi⟩ igi igi-bi *ù* GAM en-nam
20 of earth I have torn out. Igûm igibûm, and depth, what?
31. za-e igi 12 du$_8$-a *a-na* 20 *i-ši* 1,40 *ta-*{*m..*}*mar*
You, the igi of 12 detach, to 20 raise, 1°40′ you see.
32. 1,40 igi 36 igi-bi 20 GAM
1°40′, the igûm. 36′, the igibûm. 20, the depth.
33. *ne-pé-šum*
The having-been-made

Rev. I

N° 20

1. túl-sag *ma-la uš-tam-*⟨*ḫir*⟩ *ù* 7 kùš GAM-*ma* 3,20 saḫar-ḫi-a ba-zi
A cellar. So much as I have made confront itself, and 7 cubit: The depth. 3′20″ of earth I have torn out.
2. uš sag *ù* GAM en-nam
Length, width, and depth, what?
3. za-e igi 7 gál 7 *le-qé* 1 *ta-mar* igi 12 du$_8$-a 5 *ta-mar*
You, the 7th part of 7 take, 1 you see. The igi of 12 detach, 5′ you see.
4. 5 *a-na* 1 *i-ši* 5 *ta-mar* 5 *a-na* 12 *i-ši* 1 *ta-mar*
5′ to 1 raise, 5′ you see. 5′ to 12 raise, 1 you see.
5. 5 *šu-tam*⟨*-ḫir*⟩ 25 *a-na* 1 *i-ši* 25 *ta-mar* igi 25 du$_8$-a 2,24
5′ make confront itself, 25″ to 1 raise, 25″ you see. The igi of 25″ detach, 2′24
6. *ta-mar* 2,24 *a-na* 3,20 saḫar-ḫi-a *i-ši* 8 *ta-mar* en-nam íb⟨-si$_8$⟩
you see. 2′24 to 3′20″, the earth, raise, 8 you see. What the equilaterals?
7. 1 1 8 íb-si$_8$ 5 *a-na* 1 *i-ši* 5 *ta-mar* 5 kùš uš
1, 1, 8, the equilaterals. 5′ to 1 raise, 5′ you see, 5′, a cubit, the length.

8. 8 *a-na* 1 *i-ši* 8 kùš {erasure} GAM
   8 to 1 raise, 8 cubits the depth.
9. *ne-*[*p*]*é-š*[*um*]
   The having-*been*-mad*e*

N° 21[26]

10. túl-sag *ma-la uš-tam-ḫir ù*[27] 7 [kù]š GAM-*ma* 13 ⟨[3,15]⟩ saḫar ba-zi
    A cellar. So much as I have made confront itself, and 7 *cu*b*its*: the depth. *3°15′* of earth I have torn out.
11. uš sag *ù* GAM en-nam
    Length, width, and depth, what?
12. za-e [*ki-m*]*a meḫ-ri-ma e-pu-uš* 4,48 ⟨[7,48]⟩ en-nam íb-si$_8$
    You, *as mu*ch as the counterpart: make,[28] *7*‘ *48*, what the equilaterals?
13. ⌈6⌉ 6 13 íb-si$_8$ 6 *im*⟨*-ta-ḫar*⟩[29] 13 GAM
    6 6 13 the equilaterals. 6 confronts itself, 13 the depth.
14. *ne-pé-šum*
    The having-been-made.

N° 22

15. túl-sag *ma-la uš-tam-ḫir* GAM-*ma* 1,30 saḫar-ḫi-a ba-zi uš sag [*ù*] GAM ⟨en-nam⟩
    A cellar. So much as I have made confront itself, the depth. 1°30′ of earth I have torn out. Length, width, *and* depth, *what?*
16. za-e igi 12 du$_8$-a 5 *ta-mar* 5 *a-na* 1,30 *i-ši* ⌊7,30 *ta-mar*⌋
    You, the igi of 12 detach, 5′ you see. 5′ to 1°30′ raise, 7′*30″ you see*
17. 30 íb-si$_8$ 30 *a-na* 1 *i-ši* 30 *im-ta-ḫar* 30 *a*[*-na* 12] *i*⟨*-ši*⟩ 6 GAM
    30′ the equilateral. 30′ to 1 raise, 30′ confronts itself. 30′ t*o 12* raise, 6 the depth.
18. *ne-pé-šum*
    The having-been-made

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[26]The text as it stands is corrupt. In ⟨[...]⟩ I give THUREAU-DANGIN's corrections as proposed in his [TH.-D. 1936], 181, from where the reading of line 12 is also taken.

[27]In this place, a GAM seems to have been written first. Afterwards, the scribe has discovered the mistake and covered it by the *ù*.

[28]This clumsy phrase results from the use of standard translations. A more idiomatic version would be "proceed as in the corresponding (i.e., the preceding) case".

[29]This change from íb-si$_8$ to *mitḫurum* and the differentiation between the two demonstrates clearly that the former is no logogram for the latter (as claimed consistently by THUREAU-DANGIN, even in his transcription of this passage).

N° 23

19. túl-sag *ma-la uš-tam-ḫir ù* 1 kùš dirig GAM-*ma* 1,45 saḫar-ḫi-a [ba]-zi
A cellar. So much as I have made confront itself, and 1 cubit, going beyond: The depth. 1°45′ of earth *I have* torn out.

20. za-e 5 dirig *a-na* 1 bal *i-ši* 5 *ta-mar a-na* 12 *i-š*[*i* 1⌋ *ta-mar*
You, 5′, going beyond, to 1, the conversion, raise, 5′ you see; to 12 rai*se, 1* you see.

21. 5 *šu-tam*⟨*-ḫir*⟩ 25 *ta-mar* 25 *a-na* 1 *i-ši* 25 *ta-mar* igi ⌊25 du$_8$-a]
5′ make confront itself, 25″ you see. 25″ to 1 raise, 25″ you see. The igi of *25 detach,*

22. 2,24 *ta-mar* 2,24 *a-na* 1,45 *i-ši* 4,12 [*ta-mar*]
2ˋ 24 you see. 2ˋ 24 to 1°45′ raise, 4ˋ 12 *you see.*

23. *i-na* íb-si$_8$ 1 daḫ-ḫa 6 *¿*1*?*[30] íb-s[i$_8$] ⌊6 *a-na 5*⌋ *i*-[*ši* 30] *ta*⟨*-mar*⟩ *im*⟨*-ta-ḫar*⟩ 6^{sic} GAM
from ⟨"in [the table]"? or an error for "to"⟩ the equilateral, 1 append. 6 *¿*1*?* the equil*aterals.* 6 *to 5′* ra*ise, 30′* you see, confronts itself. 6 ⟨error for 7⟩ the depth.

24. *ne-pé-š*[*um*]
The having-been-m*ade.*

N° 24

25. túl-sag 3,20 GAM-*ma* 27,46,40 saḫar-ḫi-a ba-zi uš u-gù sag 50 d[irig]
A cellar. 3°20′: The depth. 27°46′40 of earth I have torn out. The length over the width 50′ go*es beyond.*

26. za-e igi 3,20 GAM du$_8$-a 18 *ta-mar a-na* 27,46,40 saḫar-ḫi⟨-a⟩ *i-ši*
You, the igi of 3°20′, the depth, detach, 18′ you see; to 27°46′40″, the earth, you raise,

27. ⌊8⌋,20 *ta-mar* $\frac{1}{2}$ 50 *ḫe-pé šu-tam*⟨*-ḫir*⟩ 10,25 *ta-mar*
8° 20′ you see. $\frac{1}{2}$ of 50′ break, make confront itself, 10′25″ you see;

28+1*. *a-na* 8,20 daḫ-ḫa [8,3⌋0,25 *ta-mar*
to 8°20′ append, *8°30*′25″ you see,

29+2*. 2,55 íb-si$_8$ *a-di* [2 gar-ra] *a-na* 1 daḫ-ḫa *i-na* 1 ba-zi
2°55′ the equilateral; until *2 pose*, (25′ which you have made span) to 1 append, from 1 tear out.

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[30] As possible alternative readings, NEUGEBAUER suggests "6 1 1" and "6 nindan", none of which make sense. "6 7" seems to be ruled out by the autography. [TMB] appears to regard the traces following "6" as an erasure, neglecting them entirely.

30+3*. 3,20 uš 2,30 sag *ta-mar*
3°20′ the length, 2°30′ the width you see.

31+4*. *ne-p*[*é-*]*šum*
The having-been-made.

N° 25

5*. túl-sag 3,20 GAM-*ma* 27,46,4[0 saḫar-ḫi-a ba-zi uš *ù* sag UL.GA]R 5,⌊50⌋
A cellar. 3°20′: the depth. 27°46′40″ *of earth I have torn out. Length and width I have accumulated*, 5°50′.

6*. za-e igi 3,20 GAM du₈-a 18 *ta-mar* ⌊*a-na* 27,46,40 *i-ši*⌋
You, The igi of 3°20′, the depth, detach, 18′ you see; *to 27°46′40″ raise,*

7*. 8,20 *ta-mar* ½ 5,50 *ḫe-pé šu-tam*⟨*-ḫir*⟩ ⌈8,30,25 *ta-mar*⌉
8°20′ you see. ½ of 5°50′ break, make confront itself, *8°30′25″ you see.*

8*. 8,20 *i-na lìb-ba* ba-zi 10,2[5 *ta-mar* 25 íb-si₈]
8°20′ from the inside tear out, 10′25″ *you see, 25′ the equilateral;*

9*. *a-na* 2,55 daḫ-ḫa *ù* ba-zi 3,20 [uš 2,30 sag]
to 2°55′ append and tear out, 3°20′ *the length, 2°30′ the width.*

10*. *ne-pé*[*-šum*]
The having-been-*made*.

N° 26

11*. túl-sag 3,20 GAM-*ma* 27,46,40 saḫar-ḫi-a [ba-zi *ša* sag u-gù GAM dirig ⅔ uš]
A cellar. 3°20′ the depth. 27°46′40″ of earth *I have torn out. That which the width over the depth goes beyond, ⅔ of the length.*

12*. za-e igi 3,20 du₈-a 18 *ta-mar a-na* 2[7,46,40 *i-ši*]
You, the igi of 3°20′ detach, 18′ you see; to 27°46′40″ *raise,*

13*. 8,20 *ta-mar* 8,20 *a-na* 40 *i-ši* 5,33,[20 *ta*⟨*-mar*⟩ 3,20 GAM *a-na* 5 *i-ši* 16,40]
8°20′ you see. 8°20′ to 40′ raise, 5°33′ *20″ you see. 3°20′, the depth, to 5′ raise, 16′40″.*

14*. nigín-na ½ 16,40 *ḫe-pé* 8,20 *ta-mar šu-tam*⟨*-ḫir*⟩ 1,[9,26,40 *a-na* 5,33,20 daḫ-ḫa]
Go around.[31] ½ of 16′40″ break, 8′20″ you see, make confront itself, 1′ *9″26‴ 40⁗ to 5°33′20″ append.*

[31]Traditionally, this phrase (nigín-na used logographically for the verb *saḫārum* ("to turn/go around") has been understood as indicating a shift from one section of the procedure to the next. As suggested to me by AAGE WESTENHOLZ (private communi-

15*. en-nam íb-si$_8$ 2,31,40$^{\text{sic}}$ *a-di* 2 gar-ra 8⟨,20⟩ da[ḫ-ḫa *ù* ba-zi]
What the equilateral? 2°31′40″ ⟨error for 2°21′40″⟩ until 2 pose; 8′ *20″ append and tear out*

16*. 2,30 sag 2,13,20 *ta-mar* igi 40 du$_8$-a 1,30 *ta-mar* [*a-na* 2,13,20 *i-ši*]
2°30′, the width; (and) 2°13′20″ you see. The igi of 40′ detach, 1°30′ you see. *To 2°13′20″ raise,*

17*. 3,20 uš *ta-m*[*ar*]
3°20′ the length you see.

18*. *ne-pé-šum*
The having-been-made.

N° 27

19*. túl-sag 1,40 uš igi 7 gál *ša* uš u-gù sag dirig GAM-*ma* 1,40 saḫar-ḫ[i-a ba-zi]
A cellar. 1°40′ the length. The 7th part of that which length over width goes beyond: The depth. 1°40′ of eart*h I have torn out;*

20*. uš sag *ù* GAM en[-nam]
Length,[32] width, and depth, wh*at?*

21*. za-e 1,40 uš *a-na* 12 bal GAM *i-ši* 20 *ta*[*-mar*]
You, 1°40′, the length, to 12, the conversion of depth, raise, 20 you *see.*

22*. igi 20 du$_8$-a 3 *ta-mar* 3 *a-na* 1,40 s[aḫar-ḫi-a[33] *i-ši* 5 *ta-mar*]
The igi of 20 detach, 3′ you see. 3′ to 1°40′, the e*arth, raise, 5′ you see.*

23*. 7 *a-na* 5 *i-ši* 35 *ta-m*[*ar* $\frac{1}{2}$ 1,40 *ḫe-pé šu-tam*⟨*-ḫir*⟩ 41,40]
7 to 5′ raise, 35′ you see. $\frac{1}{2}$ *of 1°40′ break, make confront itself, 41′40″.*

24*. 35 ⌊*i-na lib*⌋-[*bi* ba-zi 6,40 *ta-mar* 20 íb-si$_8$]
35′ *from inside tear out, 6′40″ you see, 20′ the equilateral.*

25*. *a-n*[*a* 50 daḫ-ḫa *ù* ba-zi 1,10 *ù* 30 sag ...]
T*o 50′ append and tear out, 1°10′ and 30′, the width.*

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cation), the geometric interpretation allows a much more concrete explanation, *viz* as "going around" a field which is being/has been constructed. Evidently, the traditional reading does not fit the present case, while the concrete understanding seems to give an important hint concerning the procedure — cf. the mathematical commentary.

[32]Already given.

[33]I prefer this reconstruction (proposed in [TH.-D. 1937], 11 and [TMB]) to NEUGEBAUER's, both because its fits the autography best, and because of the parallel to the procedure in N° 29, rev. II, 3–4. (It also happens to make much better sense of the procedure.)

26* [igi 7 gál 1,10 *le-qé* 10 GAM][34]
The 7th part of 1°10′ take, 10′, the depth.

27* [*ne-pé-šum*]
The having-been-made

Rev. II

N° 29

1. túl-sag 1,40 uš igi 7 *ša* uš u-gù sag dirig *ù* 2 kùš GAM-*ma* 3,20 [saḫ]ar-ḫi⟨-a⟩ ⟨ba-zi⟩
A cellar. 1°40′ the length. The 7th part of that which the length over the width goes beyond, and 2 kùš: the depth. 3°20′ of *earth I have torn out.*
2. sag *ù* GAM en-nam
Width and depth, what?
3. za-e 1,40 uš *a-na* 12 bal GAM *i-ši* 20 *ta-mar* igi 20 du$_8$-a 3 *ta-mar*
You, 1°40′, the length, to 12, the conversion of depth, raise, 20 you see. The igi of 20 detach, 3′ you see;
4. 3 *a-na* 3,20 *i-ši* 10 *ta-mar* ⟨10 *a-na* 7 *i-ši* 1,10 *ta-mar*⟩ 10 dirig *a-na* 7 *i-ši* 1[,10 *t*]*a-mar*
3′ to 3°20′ raise, 10′ you see. *10′ to 7 raise, 1°10′ you see*. 10′ going beyond[35] to 7 raise, 1°*10′ you* see.
5. 1,40 uš *a-na* 1,10 daḫ-ḫa 2,50 *ta-mar* $\frac{1}{2}$ 2,[50 *ḫe-pé šu-tam*]-*ḫir*
1°40′, the length, to 1°10′ append, 2°50′ you see. $\frac{1}{2}$ of 2°*50′ break, make con*front itself.
6. 2,25 *ta-mar i-na* 2,25 1,10 ba-zi 50,25 *ta-mar*
2°25″ you see. From 2°25″ 1°10′ tear out, 50′25″ you see,
7. 55 íb-si$_8$ *a-na* 1,25 daḫ-ḫa *ù* ba-zi-ma
55′ the equilateral; to 1°25′ append and tear out:
8. 2,20 *ù* 30 sag *ta-mar* igi 7 gál 2,20 *l*[*e-q*]*é* 20 GAM
2°20′ and 30′, the width, you see. The 7th part of 2°20′ t*ak*e, 20′, the depth.
9. *ne-pé-šum*
The having-been-made.

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[34]Reconstruction suggested by rev. II, 8.
[35]I.e., the 2 kùš of line 1.

N° 30

10. túl-sag 1,40 uš igi 7 gál *ša* uš u-gù sag dirig *ù* 1 kùš ba-l[al] GAM-*ma*
A cellar. 1°40′ the length. The 7th part of that which the length over the width goes beyond, and 1 kùš diminishing: the depth.

11. 50 saḫar-ḫi-a ba-zi sag *ù* GAM en-nam
50′ of earth I have torn out. The width and the depth, what?

12. za-e 1,40 uš *a-na* 12 bal GAM *i-ši* 20 *ta-mar* igi 20 du$_8$-a 3 *ta*⟨*-mar*⟩
You, 1°40′, the length, to 12, the conversion of depth, raise, 20 you see. The igi of 20 detach, 3′ you see;

13. 3 *a-na* 50 *i-ši* 2,30 *ta-mar* 2,30 *a-na* 7 *i-ši* 17,30 *t*[*a-mar*]
3′ to 50′ raise, 2′30″ you see. 2′30″ to 7 raise, 17′30″ you see.

14. 7 *a-na* 5 1 kùš *i-ši* 35 *ta-mar* 35 *i-na* 1,40 uš ba-zi
7 to 5′, 1 kùš, raise, 35′ you see. 35′ from 1°40′, the length, tear out,

15. 1,5 *ta-mar* $\frac{1}{2}$ 1,5 *he-pé* 32,30 *šu-tam*⟨*-ḫir*⟩ 17,36,15 *ta*⟨*-mar*⟩
1°5′ you see. $\frac{1}{2}$ of 1°5′ break, 32′30″ make confront itself, 17′36″15‴ you see,

16. *i-na lib-bi* 17,30 ba-zi 6,15 *ta-mar* 2,30 íb-si$_8$
from the inside 17′30″ tear out, 6″15‴ you see; 2′30″ the equilateral

17. *a-na* 32,30 daḫ-ḫa *ù* ba-zi 35 *ù* 30 sag *ta-mar* 7 35 5 GAM[36]
to 32′30″ append and tear out, 35′ and 30′, the width, you see. (The) 7(th of) 35′, 5′ the depth.

18. *ne-pé-šum*
The having-been-made.

5 The single problem types

All problems of our tablet share the "length", the "width", and the "depth", which determine the "cellar" and are thus silently supposed to be at right angles to each other.[37] The volume of the cellar is represented by the amount of "earth" dug out, while the area of its base is spoken of as the "floor". As always, length and width are supposed to be measured in nindan, depth in kùš, and volume as well as area in sar (nindan^2 and nindan$^2 \cdot$ kùš, respectively). When the depth is stated to be equal to (e.g.) the width, this is meant to concern "real" or "physical" extension, not measuring numbers. This holds even when the depth is equal to a width defined as *igibûm* (N° **19**). Length and width spoken of as *igûm* and *igibûm*, and hence apparently as a pair of numbers from the table of reciprocals, are thus

[36]This sequence of numbers could be filled out as "⟨igi⟩ 7 ⟨gál⟩ 35 ⟨*le-qé*⟩ 5 GAM, but is remarkable enough to stand in its original formulation.

[37]In the sense opposing, so to speak, "right" to "wrong" angles, corresponding to the label "true length" distinguishing the side 1ˋ 20 from the other length (the hypotenuse) in a right triangle 1ˋ – 1ˋ 20 – 1ˋ 40 in the tablet YBC 8633, obv. 8, rev. 2 ([MCT], 53; Neugebauer and Sachs make a mistaken correction in note 150); the width and the "true length" are those sides whose semi-product gives the area.

As often observed (e.g., in [Gandz 1939], 415ff), the Babylonian mathematical texts exhibit no trace of a concept of quantifiable angle.

still to be understood as palpable extensions fulfilling a specific condition concerning the area they span, not as mere numbers.

The use of "length" and "width" as terms for unknowns was almost as standardized in Old Babylonian "algebra" as the use of x and y in modern school algebra. In symbolic representations of the structure of problems it is therefore fitting to make use of these letters, and not of l and w. "Depth" is no similarly standardized unknown, and I shall therefore use d to represent the depth measured in nindan, and G for the depth measured in kùš; if "the depth is as much as the length", we thus have $d = x$, $G = 12d$.

5.1 The third degree

The ordering of problems in the tablet is not derived from principles of mathematical structure, and there is thus no reason to follow it in the discussion. Instead, I shall group problems together which make use of the same characteristic technique; it is evidently no coincidence that this will also be a grouping according to algebraic degree.

Of greatest interest are probably the genuine third-degree problems, characterized by the application of a sophisticated version of the *makṣarum* or "bundling" method spoken of in certain other texts (cf. also [HØYRUP 1985], 105.11f).

In the tablet YBC 8633 ([MCT], 53), a triangle with length 1` 40 and width 1` is regarded as a "bundle" of 3–4–5–triangles, corresponding to a linear scaling factor 20 (1` = 20·3, 1` 40 = 20·5). The other ("true") length is therefore found as the product of "20 the *makṣarum*" and 4.

The tablet YBC 6295 ([MCT], 42) deals with the "*makṣarum* of a [cube] root", actually with the way to find the cube root of a cubic number ($3°22'30''$) not listed in the table of cube roots. The way, again, is to compare to a more familiar standard cube, *viz* with $7'\ 30'' = (30')^3$, finding the ratio to be 27 and the linear scaling factor thus $\sqrt[3]{27} = 3$.

Judging from these examples, "bundling" is nothing but (or at least closely related to) the method of a single false position applied in two or three dimensions. As we shall see, it is also the method used (though in sophisticated versions, and without any reference to the name) for most of the third-degree problems of the present tablet.

Let us first look at N° **6**. We are told that the length equals the depth ($d = x$), that the accumulation of earth and floor equals $1°10'$ ($xyG + xy = 1°\ 10'$), and that length and width equal $50'$ ($x + y = 50'$).

According to the normal conceptualization of 2nd-degree problems of the type "surface + sides", we must expect the sum of "earth and floor" to be imagined as the volume of the cellar prolonged downwards by an extra kùš (cf. Figure 1). This *a priori* expectation is confirmed by N° **8**, which "appends" an extra kùš to the depth (obv. I, 22).

The first step in the procedure is the computation of the volume of a cube. That a volume and no mere product is involved is made clear by the distinction between multiplications: length and width are "confronted" as

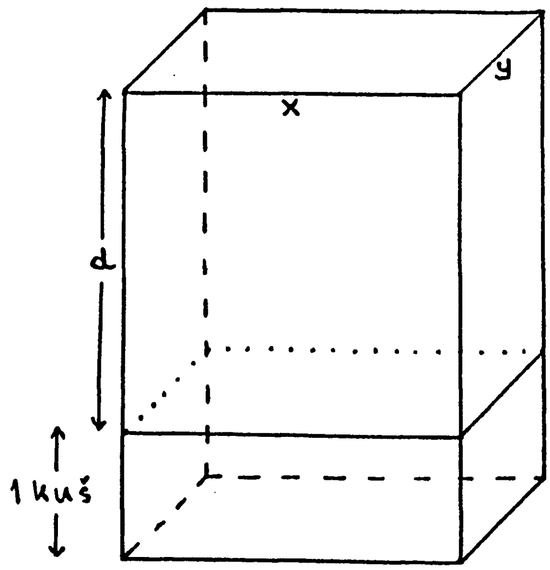

Figure 1

sides of a square, which is then "raised" to the height. The side of the cube is chosen as the sum of the length and the width of the cellar.

The treatment of the three dimensions is remarkably symmetric: all are found by a multiplication by the appropriate conversion factor: 1 for length and width (both thus 50′ [nindan]), and 12 for the depth (thus 10 kùš).

Next, the volume of the extended cellar is found by means of a customary "igi-division" to be $N = 10'4''48'''$ times the reference volume. This ordering of the computational steps is another indication that a concrete reference entity is involved; in the case of a mere normalization,[38] the volume of the cellar would (according to the habits known from other texts) have been divided by 50′, 50′ and 10 one after the other, not once and for all by their product. The "equilaterals" of the "quotient volume" N (actually the "sides" which are *not* equal) are given without explanation to be 36′, 24′ and 42′. What has to be looked for is, indeed, a factorization $N = p \cdot q \cdot r$ where $p + q = 1$, $r = p + 6'$ (6′ represents the extra kùš appended to the depth as measured by 10 kùš, i.e., by the depth of the reference volume).[39]

[38] As proposed by [Th.-D. 1940], 3, in an interpretation which otherwise seems to come close to the one presented here, apart from its arithmetical dressing (the formulation given in [TMB], xxxvff exhibits the difference more clearly). Even [MKT] I, 211 speaks of a "normal form".

[39] In his geometrical interpretation of this and the following problem, Vogel ([Vogel 1934], 91–93) does not build on the actual sequence of operations but rather

The length, finally, is found as 36′ times the length of the reference cube, i.e., as $36' \cdot 50'$ [nindan] = 30′ [nindan]; the width is found to be $24' \cdot 50'$ [nindan] = 20 [nindan], and the depth as $36' \cdot 10$ [kùš] = 6 kùš (while the extended depth would have been $42' \cdot 10$ kùš = 7 kùš).

N° **7** is a close parallel; this time, however, the excess of length over width is given (and equal to 10′ [nindan]). The reference volume is a cube with sides equal to this excess. It is constructed and found to be 3′20″ sar, yielding a quotient volume equal to 21; this is told without explanation to have the "equilaterals" 3, 2 and 21 (mistaken for 3°30′).[40] Since the side of the reference volume equals $x - y$ and is 2 kùs it is indeed required that $p - q = 1$, $r - p = 30'$.

The two factorizations into sets of "equilaterals" may have been found by systematic search — even though the number of possible factorizations is infinite (Babylonian sexagesimals made no distinction between integers and non-integers), start from the simplest possibilities combined with a bit of mathematical reflection would soon lead forward.[41] However, the complete absence of calculation (e.g., of the 6′ and 30′ representing $r - p$ in the two problems) and justification — as compared, e.g., to the careful multiplication with a factor *1* in lines 10 and 16 — suggests that they are drawn from the sleeves. Since the problems have been constructed backwards from known dimensions this will have been quite feasible. On the other hand, the fact that even the factor r for the extended depth is listed — though of no use further on — demonstrates that what may perhaps be drawn from the sleeves is still meant as a solution by factorization.

N° **23** is of a similar though simpler structure. It is told that the depth exceeds "as much as I have made confront itself" by 1 kùš, which means that length and width confront each other as sides of a square; thus $x = y$, $d = x + 1$ kùš. Furthermore, the volume is $xyG = x \cdot y \cdot 12d = 1°45'$; the same structure would have come about if we had added the base and a cubic volume.

This time, the reference volume is a cube with side equal to the excess of depth over length, i.e., to 1 kùš = 5′ [nindan]. Its volume is found to be

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on mathematical feasibility. It is thus not astonishing that his explanation differs from the one given here while being closely related.

The relation between original volume V, reference volume v and quotient volume N may be more clear to the modern reader if made explicit in symbols. In the present case, V represents the prolonged cellar, $V = x \cdot y \cdot d'$, $d' = d + 1$ kùš $= d + 5'$ nindan; $v = a \cdot b \cdot c = 50' \cdot 50' \cdot 50'$ nindan^3 $= 50' \cdot 50' \cdot 10$ nindan^2 $\cdot$ kùš; $N = (\frac{x}{a}) \cdot (\frac{y}{b}) \cdot (\frac{d'}{c}) = p \cdot q \cdot r$. Thus, since $d = x$ and $a = c$, $r = \frac{d'}{c} = (x + 1$ kùš$)/c = p + (1$ kùš$)/(10$ kùš$) = p + 6'$.

[40]While other copyist's mistakes in the tablet (jumps from one occurrence of a sequence of signs to another) could have been made by a scribe who copied word for word without understanding what goes on in the text, this one intimates that the copyist was aware of its mathematical content, and inserted by mistake a 21 which was still on his mind (the same cause seems to have produced the "13" of N° **21**, l. 10).

[41]Cf. VOGEL's tabulations ([VOGEL 1934], 92f).

$5' \cdot 5' \cdot 1$ [nindan^2 · kùš] = 25″ [sar], the quotient volume being hence equal to 4' 12. This must be factorized as $p \cdot p \cdot (p+1)$, and if the habits from N^os^ **6** and **7** had been followed, the listing of three equilaterals 6, 6 and 7 would have been expected. Instead we are told "from ["to"?] the equilateral, 1 append, 6 ¿1? the equilateral[s]", which seems to mean, firstly, that one side should be obtained by adding 1 to the others (which are equal); and secondly that the resulting equilateral is 6.[42] A tabulation of $n^2 \cdot (n+1)$ is actually known (VAT 8492, [MKT] I, 76), which identifies only one number (n) as the equilateral; furthermore, the only other problem of the present tablet which might be solved by means of such a table (N° **5**) also lists only one equilateral, while all others making use of a quotient volume indicate three. It is thus highly plausible that the phrase "from the equilateral, 1 append" refers either to the designation of such a table or to its content, and that a table has indeed been used for the solution of these (and only these) two problems.[43] Since *ina*, beyond "from", also means "in" and "by means of", the phrase should perhaps be interpreted "in/by means of [the table] 'equilaterals, [with] 1 appended', 6 [is found as the] equilateral".

We should now be ready to tackle N° **5**. The beginning is lost, but it is clear from the following that the accumulation of earth and floor will have been given as 1°10′, and that depth equals length. A supplementary condition leads in lines 2–3 to the conclusion that the length is equal to 1°30′ widths, and the width hence equal to 40′ times the length;[44] thus, the "conversion of the width" — the factor converting the measuring number for the length into that of the width, if we are to believe the parallel to the "conversion of the depth" — is 40′.

The total configuration can thus be obtained from that of N° **23** by a simple shrinking of the width by the factor 40′: whether 1 kùš is added to the depth or the "floor" to the "earth" makes no structural difference, and 40′ · 1° 45′ = 1°10′.

It cannot be decided whether the author of the text has noticed this, even though a geometrical interpretation suggests so. In any case, even the reference volume of N° **5** can be obtained from that of N° **23** by a similar

[42]The dubious ¿1? might be another result of the copyist's thinking about the procedure while writing and perhaps attempting to stamp out a number *1* written by mistake — cf. notes 30 and 40.

[43]NEUGEBAUER, who already proposed (in [MKT] I, 210f) that N^os^ **5** and **23** were solved by means of the table $n^2 \cdot (n+1)$, also presumed N^os^ **6–7** to have made use of tables, confessing at the same time, however, that he was unable to imagine their make-up.

[44]The wording of the original condition is not obvious at all, but so much is clear at least that an intermediate step finds 3 widths to be equal to a double length, since 3 follows from a computation and is then "broken", the operation resulting in a "natural" half. One possibility (though unusual — but cf. N° 26) might be that the two lengths are told to exceed the two widths by one width.

shrinking by the "conversion of the width", as 1 kùš length times 40′ kùš width times 1 kùš depth.[45]

Then everything goes as usually, and the quotient between the volumes is found again as 4` 12, which is said to have the (single) equilateral 6, corresponding to a factorization $p \cdot p \cdot (p+1)$.

Other variations on N° **23** might have been produced where the excess of depth over length was a regular number. Arithmetically speaking, the system

$$x \cdot x \cdot (12x + a) = b$$

may be reduced to

$$\left(\frac{12}{a}x\right)^2 \cdot \left(\frac{12}{a}x + 1\right) = \left(\frac{12}{a}\right)^2 \cdot \frac{b}{a}\,.$$

Such problems, however, are not to be found in the conserved parts of the tablet. Instead, N°s **20** and (presumably) **21** demonstrate how to proceed if a is irregular (and its third power does not divide b).

In N° **20** it is first observed that the 7th part of 7 is one, i.e., that a reference cube 1 kùš high divides the excess height 7 times. Next the reference volume is constructed and computed in painstaking detail: its height, 1 kùš and thus 5′ nindan, is reconverted into 1 kùš. The quotient volume is found to be 8, which has to be factorized as $p \cdot p \cdot (p+7)$, and which is indeed told to have the (three) equilaterals 1, 1, and 8.

N° **21** as it stands is corrupt, but so much sense remains that THUREAU-DANGIN's emendations can probably be relied upon. It is then a close parallel to N° **20**, jumping with the (most unusual) phrase "proceed as in the corresponding case" directly to the value of the quotient volume, and factorizing it into the equilaterals 6, 6 and 13. At this point is stops, having shown the essential step and omitting the conversions of the length and width from 6 lengths/widths of the reference volume into 30′ nindan.

The final third degree problem is N° **22**, which is homogeneous and quite simple. All three dimensions of the cellar are told to be equal, and the method seems to be a simple conversion of the volume 1°30 [sar] into 7′30″ [nindan^3]. 7′30″ is found in the standard table of cube roots, and its

[45]So at least it looks. The multiplication by the 1 kùš depth, however, goes unmentioned, and that of length by width is an unexpected "raising", contrary to all other problems where a reference volume is used. The explanation might perhaps be that N° 5 closes a sequence of gradually more complex problems (the tablet contains several series of that kind), and that an explicitly geometric technique introduced in the preceding problems is reduced here to the arithmetical essentials required for reducing the present problem to a preceding one (as happens in other places of the tablet, cf. below). Still, the absence of concrete information on the preceding problems and on the beginning of N° 5 prevents us from knowing.

(single, and true) equilateral is told in agreement with this table to be 30′. Raising this number to 1, the "conversion" of horizontal extension, yields 30′ [nindan] as sides of the square base of the cellar; raising it to 12, the "conversion of depth", gives 6 [kùš] as the depth.

5.2 The second degree: Length-width, depth-width and length-depth

The tablet contains several groups of second-degree problems, which coincidentally and for convenience can be grouped according to their dress. Of greatest interest are the two sequences **24–25–26** and **27–[28?]–29–30.**

In N° **24**, the volume of the cellar (27°46′40″), the depth (3°20′) and the excess of length over width (50′) are given. Elimination of the depth leaves us with a problem which can be translated

$$x \cdot y = 8° \, 20' \qquad x - y = 50' \,,$$

and which is solved by ordinary cut-and-paste methods (cf. Figure 2), transforming the rectangle into a gnomon of the same area, completing it as a square, finding the "equilateral" of this square and posing it twice (along the directions of length and width), finally appending and tearing out that half-excess which was cut and pasted in these two directions in order to form the gnomon.

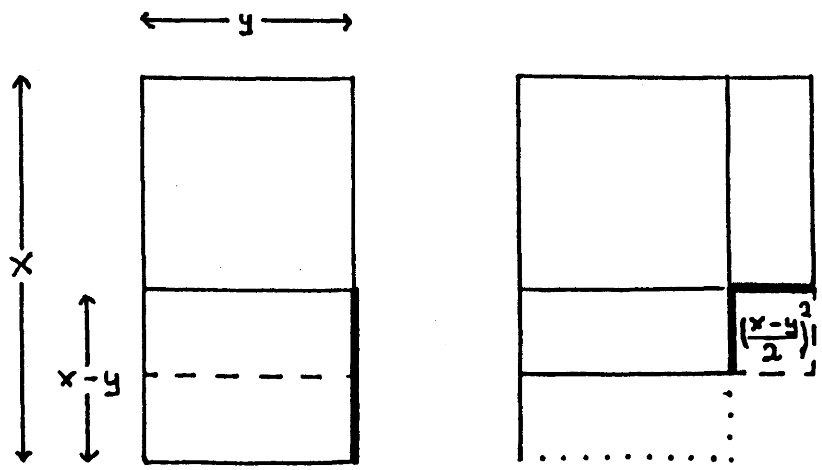

Figure 2

N° **25** is the usual companion-piece, giving the sum instead of the difference between length and width, and presents no noteworthy features apart from a more concise formulation, evidently a recurrent feature of our text when minor variations on already known patterns are presented. N°

26, however, though from the viewpoint of mathematical structure nothing but a slightly more complex variant, provides important information.

The depth is still 3°20′ [kùš] (transformed in line 13* into 16′40″ [nindan]), and the volume is given again as 27°46′40″ (all three problems have the same solution). We are informed, finally, that the excess of the width over the depth equals $\frac{2}{3}$ of the length. Division by the depth thus transforms the problem into one which in symbols (remembering that $x \cdot y$ represents a rectangle and no mere number) can be expressed

$$x \cdot y = A \qquad y = \frac{2}{3}x + D$$

(A = 8°20′, D = 16′40″)
or as

$$\frac{2}{3}x^2 + Dx = A\,.$$

A is multiplied by 40′, corresponding either to

$$\frac{2}{3}x \cdot y = 40' \cdot A \qquad y = \frac{2}{3}x + D$$

or to

$$\left(\frac{2}{3}x\right)^2 + D \cdot \left(\frac{2}{3}x\right) = 40' \cdot A\,.$$

Both of these versions are Babylonian standard problems: the former is similar to N° 24 (rectangle with known area and excess of length over width); the second is an instance of the problem "sides added to square area", and both follow the same cut-and-paste procedure. If the latter interpretation of the procedure was correct, however, we should expect the solution to tell only the side ($\frac{2}{3}x$) of the square, and to find from there first x and next y. Instead, the text "appends and tears out" precisely as N° 24, and presents immediately the larger resulting number (2°30′) as the width, finding the length as $(40')^{-1}$ times the smaller resulting number. It has thus been kept in mind throughout that the longer side of the rectangle $40' \cdot A$ coincides with the original width, while the shorter side is 40′ times the original length. The nigín-na, "go around", appearing at the moment where both sides ($\frac{2}{3}x$) and y are ready for further operations, seems to tell that they should now be marked out "in the terrain". The details of the procedure hence leave no doubt that the transformed problem was thought of in terms of a "rectangle with known excess length" and not as a square with appended sides.[46]

[46] NEUGEBAUER's interpretation of the procedure ([MKT] I, 217) refers to a square area and sides (actually to a quadratic equation in one variable), while THUREAU-DANGIN supports the two-variable option. None of them give arguments for their choice.

The observation is interesting, not because it has general value for Old Babylonian mathematics but rather because it shows that even the opposite observation (following from similar close reading of other late Old Babylonian texts[47]) cannot be generalized (cf. also below on N^os **9** and **13**). Depending on expediency or personal preference, Babylonian calculators might conceptualize problems of this type one way or the other.

N^os **24–26** can be characterized as *length-width*-problems. Correspondingly, N^os **27–30** (with a proviso concerning the missing N° **28**) can be seen as *depth-width*-problems. Their particular interest lies in their relation to the previous group.

In N° **27**, the length is given to be 1°40′, the volume equally 1°40′, and the depth to equal $\frac{1}{7}$ of the excess of length over width.

The first step in the procedure is now to tip the cellar around mentally, putting the length in vertical position: The length is raised to 12, identified as "conversion of depth", and thus converted into 20 kùš.[48] It is then eliminated, and the rectangle spanned by the width y and the depth d is seen to be 5′ [*nindan*^{2}].

Since $d = \frac{1}{7}(1° \, 40' - y)$, the next step is to find the area $7 \cdot 5' = 35'$ of another rectangle with sides $7d$ and y. In this rectangle, the sum of length and width is indeed known, and we are thus brought back to the situation known from N° **25**. The procedure is the same in the part of the text which is conserved, and according to the parallel passages in N° **29** and **30** throughout. According to the parallels, one resulting side is identified immediately as the width, while the other side is divided by 7, and the outcome 10′ [nindan] stated to be the depth without being converted into kùš, in agreement with the reconceptualization of the depth as a horizontal dimension.

N° **29** is strictly similar, containing the slight complication that $d = \frac{1}{7} \cdot (1° \, 40' - y) + 10'$ [nindan], and thus $7d = 1° \, 40' - y + 7 \cdot 10' = 2° \, 50' - y$. Apart from that everything is analogous. The same holds for N° **30**, where the complication is a subtraction of 1 kùš. Together the three problems (and, we may suspect, N° **28**) appear to present an attempt at systematic training of the mutual conversion between horizontal and vertical dimensions.

Comparison with another group of closely related problems (N^os **9** and **13**, *lenght-depth*-problems) shows that a particular and not the normal procedure is thought of in the sequence **27–30**. In N° **9**, the accumulation

[47]See [Høyrup 1990], 341, concerning IM 52301 N° 2, and [Høyrup 1985], 58 concerning BM 85194, rev. II, 7–21.

[48]Once again, that this is what goes on is demonstrated not only by the identification of the factor 12 but also by the exact ordering of steps. A mere elimination of x and a conversion of the resulting area from nindan · kùš into nindan^2 would indeed, according to Babylonian customs, have been performed through successive "divisions" by 1°40′ and 12, not through a single "division" by their product.

of floor and earth is given as 1°10′, the width is 20′, and the depth equals the length. In symbolic translation,

$$xyG + xy = 1^\circ\, 10' \qquad d = x \qquad y = 20' \,.$$

The first step in the procedure is to multiply 20′ with the number 12, which is *not* presented as the conversion of depth or in any similar way. A simple arithmetical recalculation of $x \cdot y \cdot G$ as a certain number of squares with side x (*viz* 4 such squares) appears to be the best interpretation,

$$\begin{aligned} xyG + xy = xy \cdot 12d + xy = x \cdot 20' \cdot 12x + 20'x &= 12 \cdot 20' \cdot x^2 + 20'x \\ = \quad 4x^2 + 20'x \quad &= \quad 1^\circ\, 10' \,. \end{aligned}$$

In the next step, this is transformed into a genuine square-area-and-sides problem with the side equal to $4x$,

$$(4x)^2 + 20' \cdot (4x) = 4 \cdot 1^\circ\, 10' = 4^\circ\, 40' \,,$$

which is solved by the usual cut-and-paste technique, giving $4x = 2$, and hence $x = 15' \cdot 2 = 30'$ (the depth is not spoken about).

N° 13 is similar but more sophisticated. In symbolic translation

$$\frac{1}{7} \cdot (x \cdot y \cdot G + x \cdot y) + x \cdot y = 20' \qquad d = x \qquad y = 20' \,.$$

Once again, the initial steps may be explained in symbols (remembering, as always, that the "products" are areas and volumes, and not mere numbers):

$$(x \cdot y \cdot G + x \cdot y) + 7 \cdot x \cdot y = 7 \cdot 20' = 2^\circ\, 20' \,,$$

whence

$$12 \cdot 20'x^2 + x \cdot y + 7 \cdot x \cdot y = 4x^2 + x \cdot y + 7 \cdot x \cdot y = 2^\circ\, 20'$$

and thus

$$(4x)^2 + (4x) \cdot y + 7 \cdot (4x) \cdot y = 4 \cdot 2^\circ\, 20' = 9^\circ\, 20' \,.$$

It is only at this point, when the problem has been transformed into one concerning $4x$, that the total number of sides to be added to the square area $(4x)^2$ is found, as $1 + 7$ raised to $y = 20'$, i.e., as $8 \cdot 20' = 2^\circ\, 40'$. Then finally everything can go by cut-and-paste geometry, and $4x$ and eventually x be found. Once again, the depth goes unmentioned.

The delayed computation of the number of sides is a recurrent feature in similar problems.[49] It seems as if the primary aim is to reduce in principle to a configuration of square area plus sides, which geometrically is represented by a rectangle; only when this has been achieved is the question about

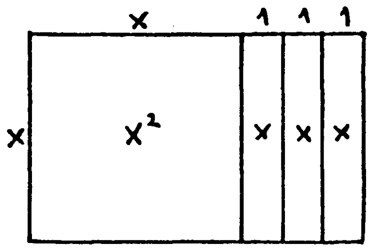

Figure 3

excess of length over width raised — i.e., about the number of sides to be added, cf. Figure 3.

The depth is not only absent from the answer but also from the question of both problems (while e.g. N° **22** asks for and gives the depth, even though it is told to equal the length). Inspection of the steps of the procedure show, furthermore, that they obliterate the very possibility of referring one side of the rectangle which is cut and pasted to the depth (while the other is easily identified as 4 lengths). We can thus be fairly confident, firstly, that even these two problems should be understood as training *a specific* technique; and secondly, that this technique is the use of the "square area and sides" model, as presupposed in my symbolic translation.

5.3 Second-degree igûm-igibûm-problems

A final cluster of second-degree problems (N^{os} **15, 16** and **18**) determine the length and the width as *igûm* and *igibûm*, i.e., as a pair of numbers from the table of reciprocals. In all cases, the volume is also given, implying that the depth follows trivially ($V = x \cdot y \cdot G = x \cdot x^{-1} \cdot G = G = 12d$). N^{os} **16** and **18** furthermore identify the depth with the total of *igûm* and *igibûm*, which leads to a problem of the same type as N° **25**: A rectangle with known area ($x \cdot y = 1$) and known sum of length and width ($x + y = d$). Their only specific interest lies in their use of the elliptic formula "append and tear out" which is shared by N^{os} **25, 26, 27, 29** and **30** and appears nowhere else in the tablet. Since N^{os} **16, 18, 25** and **26** are indubitable "rectangle-" and not "square-problems", this provides us with corroborative evidence that N^{os} **27, 29** and **30** should be understood in the same way.

Rectangle-problems with known sum of length and width normally go together with problems where the excess of length over width is given. So also here: N° **15** tells the depth to be equal to the excess of *igûm* over *igibûm*, while the volume is 36 and *d* thus 3 nindan. The interesting feature is that this problem has no rational (and hence no *Babylonian*) solution. None the less the text proceeds in a way which demonstrates that 36 is no

[49]E.g. BM 13901 N° 14 ([MKT] III, 3, cf. [Høyrup 1990], 306).

writing error, and proceeds as done in all similar problems until the point where the excess is bisected. Then suddenly it breaks off and states the result of the bisection to be the *igûm*, which is impossible whatever the area of the rectangle, as long as this area exceeds 0.

Evidently, either the text or the procedure of the problem is somehow corrupt. On the other hand the presence of a companion piece to N^{os} **16** and **18** with given excess instead of total is next to compulsory. A textual mixup which could produce as much sense as actually present is not very likely; it seems rather as if somebody (not necessarily, and probably not, the mathematically gifted author of the first version of the text[50]) has inserted a problem which for once was *not* constructed backwards from given results, and has then broken off and cheated at the point where the insolubility became evident: when 1°30′ follows from the bisection, even a moderately trained calculator will immediately know its square (2°15′) as well as the result of the quadratic completion (3°15′), and hence that this latter number does not appear in the table of square roots.

5.4 First-degree problems

The tablet contains two groups of first-degree problems, N^{os} **8+12** and N^{os} **14+17+19**, respectively. Both are quite simple as far as mathematical substance is concerned.

In N^{os} **8** and **12**, the length is given as 30′, and the depth is told to equal the length. In N^{o} **8**, furthermore, the accumulation of earth and floor is told to be 1°10′, while N^{o} **12** tells that $\frac{1}{7}$ of this accumulation appended to the floor gives 20′. The procedures are quite similar, and we shall only follow that of N^{o} **12** in the geometrical diagram (Figure 4) which is suggested by the "appending" in obv. II, 6.

As a first step, the length is multiplied by 12, resulting in "6 [kùš], the depth". To this 1 [kùš] is *appended*, giving 7 [kùš] — the depth of the figure representing "earth plus floor". Its 7th is found as 1 [kùš] — implying that the corresponding volume coincides with the floor. That this observation is in fact tacitly made is suggested by the next step: 1 and 1 are *accumulated*, i.e., the number 20′ is understood as two times the floor (which is probably the reason that it is regarded as an "accumulation", in spite of its origin in an appending process[51]), not as a volume 2 kùš high and with base equal

[50]Firstly, as noted, the present tablet is a somewhat error-ridden copy of an original; secondly we may remember GOETZE's statement that the tablet belongs to the group of "Northern modernizations of southern (Larsa) originals".

[51]NEUGEBAUER ([MKT] I, 196) as well as THUREAU-DANGIN ([TMB], 13) read the logogram UL.GAR in its function as a verb, "accumulate", take it to be an error for *i-ši*, "raise", and read traces of the ensuing sign as the beginning of a *-ma* ("and then"/"thus"; in mathematical texts to be translated simply as ":"). According to the autography (in particular the way *i* is written elsewhere), however, the reading ⌊*i-ši*⌋ is just as plausible while avoiding the (always unpleasant) hypothesis of a scribal blunder.

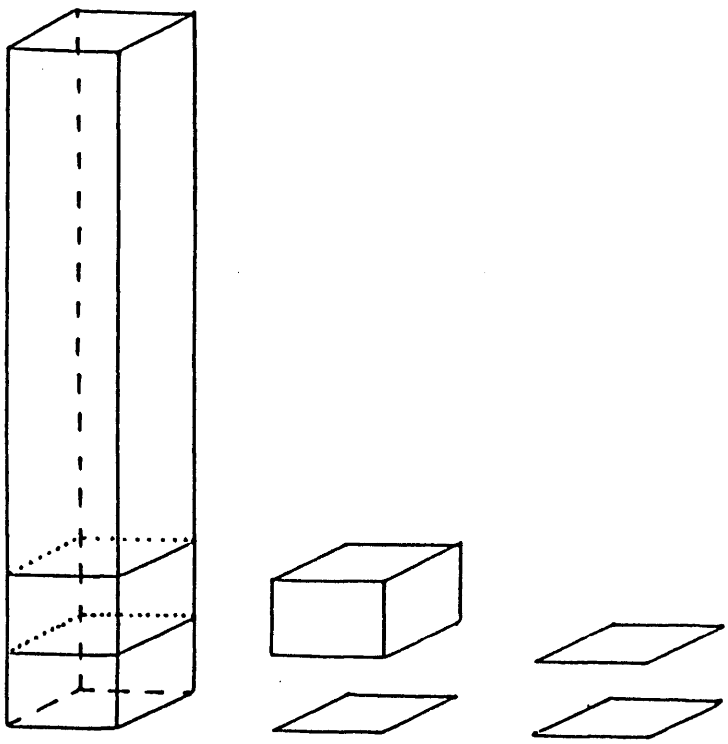

Figure 4

to the floor. 20′ is thus multiplied by the igi of 2, resulting in 10′ [the floor]. Division by "30′ the length" yields 20′, the width.

The shift between the two "additions" thus reveals something about the pattern of thought involved: Accumulation of earth and floor automatically produces a geometric interpretation, so that another "floor" can be *appended*. On the other hand, a height equal to one kùš calls forth an immediate identification of surface and volume (in perfect agreement, of course, with the coinciding metrologies and the coinciding values of the two in sar).

The other group of first-degree problems determine length and width as *igûm* and *igibûm* (whence G = volume). Furthermore, the volume is given. In **N° 19**, the depth is told to be identical with the *igibûm* (even though the results makes it coincide with the *igûm* instead); in **N° 17**, the depth results if the excess of *igûm* over *igibûm* is torn out from the *igûm* — a trivially complicated way to tell that it equals the *igibûm*; in **N° 14**, finally, the procedure and the solution forces us to believe that the depth should have been told to coincide with the *igûm*, even though the statement contains some extra words which might make us expect another companion piece to **N^{os} 16** and **18** in spite of a certain grammatical clumsiness. In all cases, the solution follows from a simple division of the volume (and thus of G) by 12, which yields either *igûm* or *igibûm*. In **N° 17**, no word is wasted upon the identification of $x - (x - y)$ with y; the problem looks more like a challenge or a puzzle than as a step in a didactical sequence.

6 Further observations on mathematical terminology and techniques

6.1 The third-degree technique

The genuine third-degree-problems made use, as we saw, of the *makṣarum* or "bundling" method. This is no staple method for the treatment of second-degree problems. Nor could it reasonably be, since the method of quadratic completion made both factorizations and tabulations of (e.g.) $n \cdot (n+1)$ superfluous as techniques for solving mixed second-degree equations. In a few homogeneous problems, however, related ideas turn up. The triangle of YBC 8633 was mentioned above. In VAT 8390 and in BM 13901 N^os^ 10–11, moreover, a rectangle and two squares, respectively, are cut into smaller "reference squares" (cf. [HØYRUP 1990], 279–284). Even factorization was a familiar technique, as we know from various tablets (e.g., YBC 4704 and VAT 5457, in [MCT], 16). While it remains true that the Babylonians were unable to treat problems of the third degree in general (as already stated by THUREAU-DANGIN in his commentary to the third-degree problems from our present tablet im [TMB], xxxviii), the techniques displayed here must be recognized as not merely ingenious artifices but the very best that could be done by means of the mathematical techniques at hand.

6.2 Raising

"Raising" (*našûm*/íl) was presented in chapter 2 as one of the multiplicative operations. In the text we have encountered it in several functions: In connection with multiplication by "conversion" factors and with reciprocals, etc. Most striking was its role in the construction of reference volumes: Here, length and width were "confronted", a constructive procedure implying but not reducible to the computation of the product; in this context, the ensuing "raising" to the height must therefore also be considered constructive.

In all other connections the term appears to have no connotations beyond the calculation by means of multiplication. The double meaning in the computation of volumes, taken together with the rather obvious metaphor ("raising to n" means "raising from the standard height 1 kùš to the actual height n kùš"), can be taken as evidence if not as fully conclusive proof that the origin meaning of the term is indeed the multiplication by a height in the computation of volumes. Other applications of the term will then have been by analogy; as the period where the extension by analogy has taken place we may point to the Ur III period (21st c. B.C.), where the sexagesimal place value system and tables of reciprocals and metrological and technical constants were apparently introduced.

A look at the order of the factors in the raising multiplications contained in our tablet corroborates the conclusion. In general it is arbitrary, the main rule being the purely stylistic convention that the number which has just been calculated is raised to the other factor. In cases where this stylistic rule does not apply, no constraints can be found. If we compare the various multiplications of the "equilaterals" of quotient volumes by the corresponding side of the reference volume, the former are raised to the latter in N^os^ 5, 6, and 23; but both N^os^ 7 and 20 exhibit alternating orders. Even the stylistic rule is nothing but a non-compulsory habit, as demonstrated by a comparison between rev. I, 23* (7 raised to 5′, against the rule) and the strictly parallel passage in rev. II, 13 (2′30″ raised to 7, in agreement with the rule). Similarly, the stylistic rule implies that the igi of a divisor will have to be raised to the number to be divided, which is indeed normally the case; none the less, obv. II, 8 follows the opposite pattern. In the construction of volumes, however, the base is invariably raised to the height (cf. also the tablet Haddad 104, passim, [RAWI/ROAF 1984]). It seems as if the imagery originally inherent in the term was still felt compulsory by Babylonian calculators.

6.3 "Subtractive numbers"

The question whether the Old Babylonian calculators understood the concept of negative numbers is rather meaningless as long as we have not told *which* concept. What is suggested by two passages of our text is that they possessed an idea not only of "subtraction" (which is evident) but also of "subtractive numbers".

The passages are to be found in the statements of Nos **29** and **30**. The former tells that the depth is "the 7th part of that which the length over the width goes beyond, *and 2 kùš*", the second that it is "the 7th part of that which the length over the width goes beyond, *and 1 kùš* ba-lal". In the first passage, the "and" is clearly additive. The lal of the second passage is certainly used logographically for a derived form of *maṭûm*, "to be(come) small(er)". If the evidence of the two passages is aggregated we may say that the "normal" role for a number brought into play by "and" is accumulative/additive; but an epithet may make the role diminishing or subtractive.

A related phrase can apparently be pointed out in another late Old Babylonian text. TMS XVI, 23 ([TMS], 92, cf. correction and commentary in [HØYRUP 1990], 301f) contains the phrase "45 *ta⟨-mar⟩ ki-ma* sag gar gar zi-*ma*", "45′ you see, as much as of widths pose. Pose to tear out", indicating that this coefficient should somehow be recorded as the number of widths to be subtracted.

The term ba-lal is also familiar from the highly systematic "series texts", long sequences of concisely formulated problems which do not tell the procedure. Its occurrences there have often been quoted (e.g., [MKT] I, 410f, 455f, etc.) as instances of negative numbers; the real function of the term, however, is simply to allow the reversal of the order of two magnitudes which are compared, mostly made for stylistic reasons (cd. [HØYRUP 1992]). Applying OCCAM's razor we should only claim that the Old Babylonian calculators had a categorization of additive and subtractive *roles* of numbers within a computation, perhaps even a way to record these roles; whether they would consider this as a categorization of *numbers* as either "positive" and "negative" is not only subjected to doubt but outright dubious.

6.4 The non-technical character of terminology

At an earlier occasion ([HØYRUP 1990], 331) I have claimed that only as a first approximation can Babylonian mathematical terminology

> be called "technical". It appears not to have been stripped completely of the connotations of everyday language, nor does it possess that stiffness which distinguishes a real technical terminology. We should rather comprehend the discourse of the mathematical texts as a highly standardized description in everyday language of standardized problem situations and procedures, and we should notice that the discourse is never more, but sometimes less standardized than the situation described.

This conclusion is corroborated by two interesting terminological details of the present text. One of them is the use of the term translated here "to tear out". As in so many other mathematical texts it is used for the "identity-conserving" subtractive process. But it is also used to tell how much earth has been dug out from the cellar. Moreover, in *both* functions the same logogram zi (provided with the same Sumerian prefix ba-) and not a syllabic Akkadian *nasāḫum* is employed. Clearly, the author of the text saw no point in distinguishing a technical mathematical terminology from the vocabulary of everyday.

The use of *meḫrum*, "counterpart", in **N° 21** (rev. I, 12) is similar. *meḫrum* is a well-known mathematical term. Where the present text tells (e.g., **N° 24**, rev. I, 29+2*) to "pose the equilateral until 2", i.e., to draw two sides of the square meeting in a corner, others ask us, e. g., to "lay down 8°30′ [the equilateral] and 8°30′ its counterpart" (YBC 6967, obv. 11, [MCT], 129). Once again, there is no clearcut boundary between technical-mathematical and everyday speech. No wonder, then, that a geometrical text concerned with triangles uses the word (written logographically TUḪ = gaba) in still another sense (IM 55357 l. 10, [BAQIR 1950], 42).

At the same time the text gives us a glimpse of what might be a grid of fine terminological distinctions, not as much according to mathematical meaning as depending on problem dress and thus perhaps historical origin. The singular use of the accounting term nigin, "total", in **N°s 16** and **18** was pointed out already. This could of course be another instance of floating terminological boundaries. Both occurrences, however, are found in connection with *igûm-igibûm*-problems, which might be no accident: as mentioned above, *igûm* and *igibûm* refer to tables of reciprocals, and thus to the same sphere of social activity as does nigin: scribal accounting and planning rather than surveying. According to the principle that recreational problems are to be considered as a "non-utilitarian" superstructure on mathematical practice (see [HØYRUP 1989]), this might point to an origin of *igûm-igibûm* problems within this specific orbit[52] and to a tendency to conserve a characteristic vocabulary.

7 Unexpected light on the organization of mathematics teaching

In two respects, our text looks primitive or clumsy from a modern mathematical point of view. At closer inspection, however, both apparent flaws turn out to be sound reflections of the technique of didactical exposition, and thus, reversely, strong supportive evidence for what could be guessed about this technique from weaker data.

7.1 Numbers used for identification

The first apparent weakness is what looks like a tendency to give destructively redundant numerical information. Indeed, N**os** 6, 7, and 13 seem to be overdetermined. In N**os** 6 and 7 the earth is referred to as 1, and in N**o** 13 the accumulation of earth and floor is told to be 1°10′. In neither case are these data used — and the whole point would have been spoiled if they had been taken into account.

Evidently, these numbers were never meant to serve the solution. Nor can they be manifestations of ignorance on the part of the author of the text — everything else in these problems is perfectly clear and points to the goal. Instead, the presence of these numbers can be understood if we think of the purpose and use of the text as a tool for actual teaching. We should imagine the teacher explaining beforehand the total situation: the cellar, its dimensions, the earth and the floor, giving also their numerical values in as far as these may be useful as identifying labels; it is to be observed that N**os** 6 and 7 speak about "1 the earth" and do not use the expression "the earth: 1" found when data for the calculation are told. Afterwards, he shows how to extricate the dimensions from a specific set of data; in the oral exposition of the procedure he will have the possibility to identify, say, the original volume as "1 the earth", in contrast to the extended volume — just as a modern exposition will distinguish V from V'.

In the present case, the written text only conserves traces of this oral exposition technique. A couple of other late Old Babylonian texts, however, are more explicit and exhibit the use of numbers as identifiers beyond doubt.[53] What a modern mathematical reading tends to see as a manifestation of incompetence or deficient understanding is thus a rudiment of

[52]Though certainly *not* to an independent focus for the creation of second-degree "algebra" — as demonstrated by the formulation of YBC 6967, the unknown numbers of *igûm-igibûm*-problems were represented by the geometrical magnitudes of normal "surveying" cut-and-paste geometry ([MCT], 129, cf. [HØYRUP 1990], 263–266).

[53]TMS IX and XVI, cf. translation and interpretation in [HØYRUP 1990], 299ff, 320ff.

an oral technique achieving by other means what *we* are accustomed to achieve in writing by algebraic symbols.

7.2 Ordering determined by configuration

The observations just made on the method of exposition may also serve as a key to the seemingly disorderly arrangement of problems within the tablet. Admittedly, Chapter 5 referred to a number of brief sequences of a similar mathematical structure — yet all problems making use (e.g.) of a reference volume were not collected in one place. Mathematical structure and techniques are thus *not* the primary ordering principle.

Let us look instead at the statements. Firstly, of course, the uniting principle of the tablet as a whole is the cellar, and not the investigation of a specific mathematical structure or training of a particular technique. This was already pointed out in chapter 3. But there is more to it. N^{os} **5–9** all tell the accumulation of earth and floor to be 1°10′. Whatever the mathematical character of the problem, be it of the first, the second or the third degree, it will thus have to be discussed with reference to a cellar prolonged one kùš downwards. N^{os} **10** and **11** are missing. N^{o} **12**, which as far as mathematical substance is concerned is nothing but a slight variation on N^{o} **8**, starts from a corresponding variation of the configuration, as does N^{o} **13**, which regarding mathematical substance has the same relation to N^{o} **9**. Instead of exhausting first the possibilities of the *method* of N^{o} **8**, which would make N^{o} **12** follow it immediately, the possibilities *of the configuration* shown in Figure 1 are exhausted before further training of the various methods is undertaken.

N^{os} **14–19** are then *igûm-igibûm* problems; N^{os} **20–23** deal with cellars with a square floor; N^{os} **24–26** all have the same volume and depth given and a rectangular base; and N^{os} **27–30** all (with a proviso for the missing N^{o} **28**) have the length given as 1°40′ and make use of the entity $\frac{1}{7}(x-y)$.

While a categorization according to mathematical structure and techniques only suggests fragments of local order within a generally chaotic structure, the categorization according to configuration thus uncovers a genuine global order and explains the most striking examples of seeming disorder. There is thus no reasonable doubt that the global order of the tablet is determined by the way didactic exposition was organized, and that this organization was the one imagined above.

Below the level of global order, and subordinated to its principles, we find of course an ordering of shorter sequences according to mathematical principles and progression. Recognition of the important role of didactic exposition should not overshadow the fact that understanding of mathematical principles is *also* demonstrated by the tablet. There is certainly no reason to dismiss it as "merely didactic opportunism and *hence* no testimony of real mathematical thought".

8 Mathematics ?

A widespread joke runs as follows: A physicist and a mathematician are put in front of a cooker with two gas-rings, a match-box and an empty kettle standing on the left gas-ring. Asked how to cook water for tea they both tell that you fill the kettle with water and put it back; you turn on the gas, and then you use a match to light the gas. Asked what is to be done if the kettle is to the right, the physicist says "Act correspondingly". The mathematician has a different solution: You move the kettle to the left, reducing thus the situation to the previous case.

Our tablet shows traces of "the physicist" in Nº **21** — cf. the reference to the "counterpart". This is not astonishing, widespread as this principle is in systematic yet practice-bound discourse. What *is* astonishing is that even "the mathematician" of the joke is visible; reduction to the previous case instead of direct use of the same method *mutatis mutandis* is in fact the principle used in Nºs **27–30**, where the cellar is tipped around, changing the depth into a length.

Traditionally, our tablet has mostly been seen as a high point in Babylonian mathematics because it undertakes an attack on third-degree problems. Since the attack leads to no general breakthrough, the high point may be an illusion seen from this angle. Still, if the gauge is not mathematical subject-matter but rather the organization and progression of thought the tablet may still be closer to modern mathematics than many other Babylonian mathematical text, both according to the "kettle principle" and if the occasional tendency to give only the essentials of parallel cases (Nºs **18, 21**) is taken into account. Both features, indeed, are portends of an incipient break with that casuistic principle which is otherwise so characteristic of Old Babylonian mathematical no less than legal texts.

References

[BAQIR 1950] BAQIR, TAHA: *An Important Mathematical Problem Text from Tell Harmal*. Sumer 6 (1950), 39–54.

[CHEMLA 1991] CHEMLA, KARINE: *Theoretical Aspects of the Chinese Algorithmic Tradition (first to third centuries)*. Preprint, to appear in *Historia Scientiarum*.

[GANDZ 1937] GANDZ, SOLOMON: *The Origin and Development of the Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra*. Osiris 3 (1937), 405–557.

[GANDZ 1939] GANDZ, SOLOMON: *Studies in Babylonian Mathematics II. Conflicting Interpretations of Babylonian Mathematics*. Isis 31 (1939), 405–425.

[GANDZ 1948] GANDZ, SOLOMON: *Studies in Babylonian Mathematics I. Indeterminate Analysis in Babylonian Mathematics*. Osiris 8 (1948), 12–40.

[HØYRUP 1985] HØYRUP, JENS: *Babylonian Algebra from the View-Point of Geometrical Heuristics. An Investigation of Terminology, Methods, and Patterns of Thought*. Second, slightly Corrected Printing. Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science, 1985.

[HØYRUP 1989] HØYRUP, JENS: *Zur Frühgeschichte algebraischer Denkweisen. Mathematische Semesterberichte* 36 (1989), 1–46.

[HØYRUP 1990] HØYRUP, JENS: *Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought*. Altorientalische Forschungen 17 (1990), 27–69, 262–354.

[HØYRUP 1990a] HØYRUP, JENS: *Dynamis, the Babylonians, and Theaetetus 147c7–148d7*. Historia Mathematica 17 (1990), 201–222.

[HØYRUP 1991] HØYRUP, JENS: *Changing Trends in the Historiography of Mesopotamian Mathematics*. Revised Contribution to the Conference *Contemporary Trends in the Historiography of Science*, Corfu, May 27 – June 1, 1991. Filosofi og Videnskabsteori på Roskilde Universitetscenter. 3. Række: Preprints og Reprints 1991 nr. 3.

[HØYRUP 1992] HØYRUP, JENS: *On Subtractive Operations, Subtractive Numbers, and Purportedly Negative Numbers in Old Babylonian Mathematics*. To be published in: Zeitschrift für Assyriologie und Vorderasiatische Archäologie.

[MCT] NEUGEBAUER, O., & A. SACHS: *Mathematical Cuneiform Texts*. (American Oriental Series, vol. 29). New Haven, Connecticut: American Oriental Society, 1945.

[MEA] RENÉ LABAT: *Manuel d'épigraphie akkadienne (signes, syllabaire, idéogrammes)*. 4e édition. Paris: Imprimerie Nationale, 1963.

[MKT] O. NEUGEBAUER: *Mathematische Keilschrift-Texte*. I–III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster–dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937. Reprint Berlin etc.: Springer, 1973.

[RAWI/ROAF 1984] RAWI, FAROUK N. H. AL-, & MICHAEL ROAF: *Ten Old Babylonian Mathematical Problem Texts from Tell Haddad, Himrin*. Sumer 43 (1984, printed 1987), 195–218.

[TH.-D. 1936] THUREAU-DANGIN, F.: *Notes sur la mathématique babylonienne*. Revue d'Assyriologie 33 (1936), 180–184.

[Th.-D. 1937] Thureau-Dangin, F.: *Notes sur la mathématique babylonienne*. Revue d'Assyriologie 34 (1937), 9–28.

[Th.-D. 1940] Thureau-Dangin, F.: *Notes sur la mathématique babylonienne*. Revue d'Assyriologie 37 (1940), 1–10.

[TMB] Thureau-Dangin, F.: *Textes mathématiques babyloniens*. (Ex Oriente Lux, Deel 1). Leiden: Brill, 1938.

[TMS] E. M. Bruins & M. Rutten: *Textes mathématiques de Suse*. (Mémoires de la Mission Archéologique en Iran, XXXIV). Paris: Paul Geuthner, 1961.

[v.d. Waerden 1962] van der Waerden, B. L.: *Science Awakening*. 2nd Edition. Groningen: Noordhoff, 1962.

[Vogel 1934] Vogel, Kurt: *Kubische Gleichungen bei den Babyloniern?* Sitzungsberichte der Bayerischen Akademie der Wissenschaften zu München. Mathematisch-Naturwissenschaftliche Abteilung 1934, 87–94.

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"Mathematical Susa Texts VII and VIII. A Reinterpretation". *Altorientalische Forschungen* **20** (1993), 245–260.

Altorientalische Forschungen	20	1993	2	245–260

Jens Høyrup

Mathematical Susa Texts VII and VIII. A Reinterpretation

Burchard Brentjes und Ernst Werner gewidmet

In an extensive paper published due to the kindness of the editorial staff of the present journal, I have suggested a geometrical reinterpretation of so-called Old Babylonian "algebra".[1] Among the texts analyzed in the paper were the mathematical texts No. XVI and No. IX from Susa, which turned out to contain highly illuminating didactical commentaries of a kind not known from the Babylonian core area – be it because teaching in a peripheral area felt a need to make explicit what could be left to a stable oral tradition in the core, or simply because the Susa teachers had a bent for loquacity.

Text No. XVI turned out not to contain solutions of problems but only a didactical discussion of transformations of linear "equations" of two unknowns (as usually, the uš ("length") and ("width") of a rectangle, with the usual values 30′ and 20′ [nindan][2]). No. IX contained initial didactical discussions of the transformations of complex into simpler second-degree "equations" followed by use of the technique just taught for the solution of a sophisticated set of equations.

These texts are not the only Susa texts to contain illuminating didactical commentaries. In the present paper I shall analyze two further texts, of which one contains an explicit explanatory part, while the other employs some of the concepts introduced in the former.

An extra reason for reanalyzing the two texts is that the treatment given by

[1] J. Høyrup, Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought, in: AoF 17 [1990], 27–69, 262–354. – Abbreviations: MKT – O. Neugebauer, Mathematische Keilschrift-Texte, I–III, Berlin 1935, 1935, 1937 (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Bd., 1.–3. Teil); TMS – E. M. Bruins – M. Rutten, Textes mathématiques de Suse, Paris 1961 (Mémoires de la Mission Archéologique en Iran, XXXIV).

[2] I shall use F. Thureau-Dangin's system for translating sexagesimal place value numbers: \`, \`\`, etc. designate increasing and ′, ″ etc. decreasing sexagesimal orders of magnitude. When needed, ° is used to indicate „order zero" (1° = 1).

E. M. Bruins in the original edition (TMS, 52–62) was highly unsatisfactory even in terms of the received arithmetical interpretation of such texts. In the first text he misread a possessive suffix *-šu* for a Sumerian ŠU, which made him invent a specific "heuristic method of the hand", which has since then spread into the secondary literature[3], and made him mistake one indeterminate equation, something very rare in Babylonian mathematics, for a trivial set of two first-degree equations. In the second text, a number of translations and repairs to the text are overly fanciful, while obvious restitutions suggested by parallel passages are left out.

TMS VII: Indeterminate first-degree problems

Let us first look at Text VII, which runs as follows:[4]

Problem A

1. 4-*at* sag *a-na* uš daḫ 7-⟨*ti*-⟩*šu*[5] *a-na* 10 [*al-li-ik*]
   The 4th of the width to the length I have appended, its 7⟨th⟩, to 10 [I have gone,]

2. *ki*!*-ma* UL.GAR uš *ù* ⟨sag⟩ za-e 4 gar 7 [gar]
   as much as the accumulation of length and ⟨width⟩. You, 4 pose; 7 [pose;]

3. 10 gar 5 *a-na* 7 *i-ši* 35 *ta-mar*

   10 pose; 5′ to 7 raise, 35′ you see.

4. 30 *ù* 5 *be-e-er* 5 a-rá! *a-na* 10 *i-ši*
   30′ and 5′ single out. 5′, the step, to 10 raise,

5. 50 *ta-mar* 30 *ù* 20 gar 5 a-rá! *a-na* 4 *re-*⟨*ba-ti*⟩ sag
   50′ you see. 30′ and 20′, pose. 5′, the step, to 4, of the four⟨th⟩ of the width,

6. *i-ši-ma* 20 *ta-mar* 20 sag 30 *a-na* 4 *re-ba-*⟨*ti*⟩
   raise: 20′ you see, 20′, the width. 30′ to 4, of the fourt⟨h⟩,

[3] So, H. Gericke (Mathematik in Antike und Orient, Berlin etc. 1984) borrows E. M. Bruins' interpretation of problem VII A as his first example of Babylonian algebra (pp. 25–32).

[4] See, apart from the transliteration in TMS, 52ff., the autography (plates 14f.), and W. von Soden's review of TMS, in: BiOr 21 [1964] 44–50, here 48.

[5] This *-šu* is what becomes a „hand" in E. M. Bruins' interpretation. That a *-ti-* has simply been omitted and a 7th is thus meant can be seen from a number of parallel passages (so line 17 of the same tablet; TMS IX, line 20; VAT 8520, obv. 1, rev. 4, in MKT I, 346f.). The reading is confirmed by the consistency of the text which is obtained.

7. *i-ši* 2 *ta-mar* 2 gar uš 20 *i-na* 20 zi
raise, 2 you see. 2 pose, lengths. 20′ from 20′ tear out,

8. *ù i-na* 2 30[6] zi 1,30 *ta-mar*
and from 2, 30′ tear out, 1°30′ you see.

9. *i-na* 4 *re-ba-ti* 1 zi 3,20[7] *ta-mar*
From 4, of the fourth, 1 tear out, 3 {+ 20′} you see.

10. igi 3 *pu-ṭú-⟨úr⟩* 20 *ta-mar* 20 *a-na* 1,30 *i-ši-ma*
The igi of 3 deta⟨ch⟩, 20′ you see. 20′ to 1°30′ raise:

11. 30 *ta-mar* 30 uš 30 *i-na* 50 zi 20 *ta-mar* 20 sag
30′ you see, 30′ the length. 30′ from 50′ tear out, 20′ you see, 20′ the width.

12. *tu-úr* 7 *a-na* 4 *re-ba-⟨ti⟩ i-ši* 28 *ta-mar*
Turn back. 7 to 4, of the fourt⟨h⟩, raise, 28 you see.

13. 10 *i-na* 28 zi 18 *ta-mar* igi 3 *pu-⟨ṭú-úr⟩*
10 from 28 tear out, 18 you see. The igi of 3 det⟨ach⟩,

14. 20 *ta-⟨mar⟩* 20 *a-na* 18 *i-ši* 6 *ta-mar* 6 uš
20′ you s⟨ee⟩. 20′ to 18 raise, 6 you see, 6 (for) the length.

15. 6 *i-na* 10 zi 4 sag 5 *a-na* 6 [*i-š*]*i*
6 from 10 tear out, 4 (for) the width. 5′ to 6 [raise,]

16. 30 uš 5 *a-na* 4 *i-ši* 20 *ta-⟨mar⟩*20 ⟨sag⟩
30′ the length. 5′ to 4 raise, 20′ you s⟨ee⟩, 20′ the ⟨width⟩.

Problem B

17. 4-*at* sag *a-na* uš daḫ 7-*ti*[-*šu*]
The fourth of the width to the length I have appended,
[its] 7th

18. *a-di* 11 *al-li-ik* ugu! [UL.GAR]
until 11 I have gone, over the [accumulation]

6 It appears from the autography, either that the two numbers were at first written together but 30 then deleted and rewritten with distance; or that some small wedges marking a separation are written between the two numbers.

7 As observed by E. M. Bruins, the scribe has tried to correct this number (which should be 3) but has done so incorrectly. 3°20′ will have been on the scribe's mind as 4. (length + width).

19. uš *ù* sag 5 dirig za-e [4 gar]
of length and width 5′ it goes beyond. You, [4 pose;]

20. 7 gar 11 gar *ù* 5 gar
7 pose; 11 pose; and 5′ pose.

21. 5 *a-na* 7 *i-ší* 3[5 *ta-mar*]
5′ to 7 raise, 3[5′ you see.]

22. 30 *ù* 5 gar 5 *a-na* 1[1 *i-ši* 55 *ta-mar*]
30′ and 5′ pose. 5′ to 1[1 raise, 55′ you see.]

23. 30 20 *ù* 5 zi gar 5 [*a-n*]*a* 4
30′, 20′ and 5′, to tear out, pose. 5′ [t]o 4

24. *i-ši* 20 *ta-*⟨*mar*⟩ 20 sag 30 *a-na* 4 *i-ši-ma*
raise, 20′ you s⟨ee⟩, 20′ the width. 30′ to 4 raise,

25. 2 *ta-mar* 2 uš 20 *i-na* 20 zi
2 you see, 2, lengths. 20′ from 20′ tear out.

26. 30 *i-na* 2 zi 1,30 gar *ù* 5 *a-*[*na* ...]
30′ from 2 tear out, 1°30′ pose, and 5′ t[o ...]

27. 7 *a-na* 4 *re-*⟨*ba-ti*⟩ *i-ši-ma* 28 *ta-mar*
7 to 4, of the four⟨th⟩, raise, 28 you see.

28. 11 UL.GAR *i-na* 28 zi 17 *ta-mar*
11, the accumulations, from 28 tear out, 17 you see.

29. *i-na* 4 *re-*⟨*ba-ti*⟩ 1 zi 3 [*ta*]*-mar*
From 4, of the four⟨th⟩, 1 tear out, 3 y[ou] see.

30. igi 3 *pu-ṭú-*⟨*úr*⟩ 20 *ta-*⟨*mar*⟩ 20 [*a-na*] 17 *i-*⟨*ší*⟩
The igi of 3 deta⟨ch⟩, 20′ you s⟨ee⟩. 20′ t[o] 17 ra⟨ise⟩,

31. 5,40 *ta-*⟨*mar*⟩ 5,40 [u]š 20 *a-na* 5 dirig *i-ší*
5°40′ you s⟨ee⟩, 5°40′, (for) the [le]ngth. 20′ to 5′, the going-beyond, raise,

32. 1,40 *ta-*⟨*mar*⟩ 1,40 *wa-ṣí-ib* uš 5,40 uš
1′40″ you s⟨ee⟩, 1′40″, the appending of the length. 5°40′, (for) the length,

33. *i-na* 11 UL.GAR zi 5,20 *ta-mar*
from 11, accumulations, tear out, 5°20′ you see.

34. 1,40 *a-na* 5 dirig daḫ 6,40 *ta-mar*
1′40″ to 5′, the going-beyond, append, 6′40″ you see.

35. 6,40 *n*[*a*]-*sí-iḫ* sag 5 a-rá
6′40″, the t[ea]ring-out of the width. 5′, the step,

36. *a-na* 5,40 uš *i-ši* 28,20 *ta-mar*
to 5°40′, lengths, raise, 28′20″ you see.

37. 1,40 *wa-sí-ib* uš *a-na* 28,20 [daḫ]
1′40″, the appending of the length, to 28′20″ [appe]nd,

38. 30 *ta-mar* 30 uš 5 *a*-[*na* 5,20]
30′ you see, 30′ the length. 5′ t[o 5°20′]

39. *i-ši-ma* 26,40 *t*[*a-mar* 6,40]
raise, 26′40″ yo[u see. 6′40″,]

40. *na-sí-iḫ* sag *i-na* [26,40 zi]
the tearing-out of the width, from [26′40″ you tear out,]

41. 20 *ta-mar* 20 sa[g] (...?)
20′ you see, 20′ the wid[th.]

First of all, the terminology must be explained briefly (for more detailed discussions, see Høyrup, AoF 17,45–65):
1) *waṣābum*/daḫ (translation "to append"[8]) is an asymmetric additive process, in which one quantity is joined to another of the same kind. The latter can thus be said to conserve its identity (and, in geometric manipulation, its place) while being enlarged, while the former is absorbed (and moved if necessary in cases of geometric manipulation). Since all concrete entities are assumed to possess measuring numbers, the operation entails an arithmetical addition (similarly for all concrete processes discussed in what follows). From *waṣābum* originates the term *waṣībum* used here and elsewhere in the Susa corpus, "that which is to be appended", for convenience translated "the appending".
2) *nasāḫum*/zi ("to tear out.") is the corresponding subtractive process, by which a part of an entity is removed. Whence *nasīḫum*, "that which is to be torn out", for convenience "the tearing-out".

[8] I follow a principle of „conformal translation", where the same term is always translated in the same way, different terms (unless logographic equivalence is established beyond doubt within the text group) translated differently, and terms derived from the same root as far as possible rendered by similarly related translations. The result is clumsy but allows that the actual operations of the texts can be discussed in (some kind of) English.
An extensive list of suggested standard translations is given in Høyrup, AoF 17, 67–69.

3) *kamārum*/gar-gar/UL.GAR ("to accumulate") is a symmetric additive process. It may be a genuine arithmetical operation by which measuring numbers of entities of different kinds (e.g., areas and lengths) are added; but it may probably also be meant as a concrete putting-together. UL.GAR may also disignate the sum by this addition ("the accumulation")
4) The phrase *A eli BD watārum*/*A* ugu *BD* dirig ("*A* over *BD* goes beyond") is an operation by which two entities *A* and *B* are compared. Since $D=A-B$, we may speak of a "subtraction by comparison". The difference *D* may be spoken of as dirig (translated "going-beyond").
5) *našûm*/íl ("to raise") is a multiplicative process, designating the calculation of a concrete magnitude by multiplication.
6) In the present text, *alākum* ("to go") appears as a multiplication; in TMS VIII, however, it stands for repeated appending of the same entity. The basic idea is thus the repetition of a certain step, and context will have to tell the precise meaning. The corresponding verbal noun *tālukum* ("the going") stands for the total distance gone.
7) a-rá ("times") is the term used in multiplication tables. It is thus the arithmetical multiplication of number by number. In the present text it is used as a noun, referring to the line segment which is "gone". Since rá is the Sumerian equivalent of *alākum* it is translated "step".[9]
7) igi *n* is the reciprocal of *n* as found in reciprocal tables. Finding the igi is termed *paṭārum*/du_8 ("to detach").
8) *šakānum*/gar ("to pose") designates (apparently a number of different) processes of material recording, most probably in writing and drawing, perhaps also in a calculational device.

The rest of the terminology with appurtenant translations should be more or less self-explanatory, and we may thus start the analysis of what goes on in problem A.

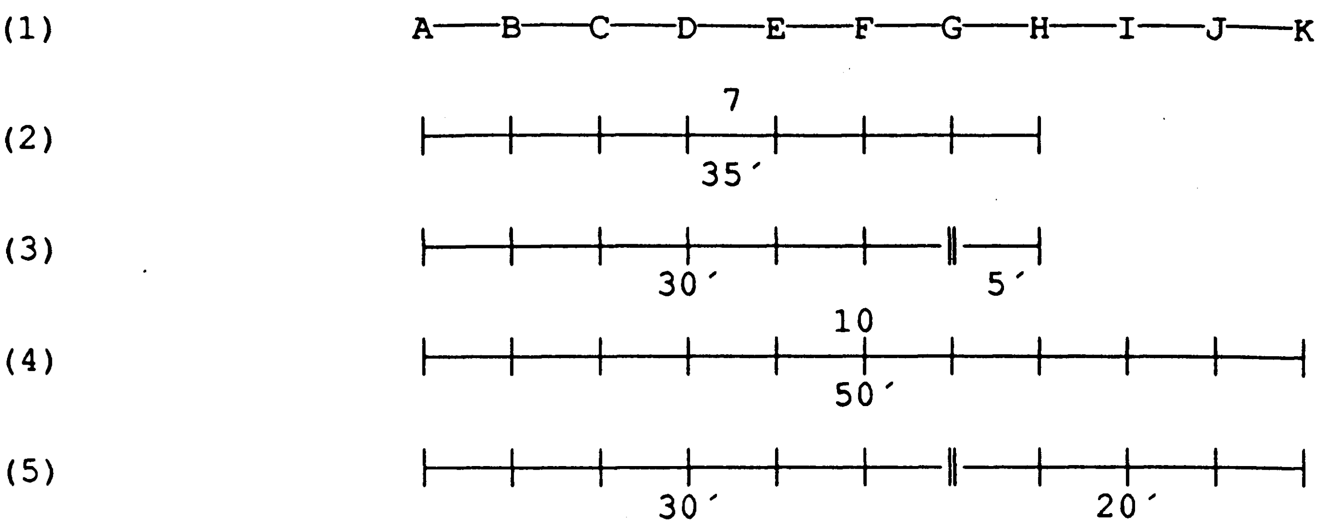
Formally, it deals with the "length" uš and the "width" sag of a rectangle. In the present problem, however, the terms are just labels for two line segments. It may be easier to follow the procedure if we allow ourselves the anachronism of a translation into algebraic symbolism, writing *x* for the length and *y* for the width – but it is important that this translation is only used as a support, since it abolishes, e.g., the distinction between different additive operations.

Lines 1–2 tell that a 4th of the width has been appended to the length, that the 7th of the outcome has been taken 10 times, amounting in total to the accumulation of length and width. In symbols:

$$^1/_7(x+{}^1/_4\,y)\cdot 10 = x+y$$

Lines 2–5 now explain the meaning of this equation by extensive "posing", which we may imagine to take place in a diagram like this:

[9] This corresponds to the usage of the Seleucid text BM 34568–see Høyrup, AoF 17, 343f.



First the numbers 4, 7 and 10 – divisors and multiplier – are recorded. Then the magnitude 5′ (the "step" of line 5 – taken at this point to be known) is raised to 7, giving 35′, which can be decomposed as 30′ + 5′ ($x + {}^1/_4 y$). Next it is raised to 10, which gives 50′, decomposable into 30′ + 20′ ($x + y$). It may seem astonishing to us that the meaning of an equation is discussed in terms of its solution; but the method is attested elsewhere in Old Babylonian mathematical texts (e.g., TMS IX and XVI), and it is a quite effective didactical substitute for algebraic symbols.

So far the text has presented us with a didactical exposition of the numerical foundation of the original equation. Lines 5–11 go on with a similar exposition of the meaning of the first step of the transformation of the equation, which is a multiplication by 4,

$${}^1/_7[4x + y] \cdot 10 = 4 \cdot (x + y).$$

We look at the decomposition 35′ = 30′ + 5′ ((3) in the diagram): multiplying 5′ by 4 gives 20′, the width; multiplying 30′ yields 3, (4) lengths. Now one width and one length are removed. Tearing out a width (20′) from 20′ leaves, literally, nothing worth speaking about: tearing out a length (30′) from the 4 lengths (2) leaves 1°30′; this is found in line 9 to correspond to 3 (*viz* lengths). Multiplying by ${}^1/_3$ = 20′ indeed gives 30′, the length; tearing this out from 50′ leaves 20′, the width.

The ***tu-úr,*** "turn back", of line 13 tells that the explanations are now finished and the procedure is to begin. After the multiplication by 4 which was already explained, the equation is multiplied by 7, which leads from

$${}^1/_7[(4-1)\ x + (x + y)] \cdot 10 = 4 \cdot (x + y)$$

to

$$3x \cdot 10 + (x + y) \cdot 10 = 28 \cdot (x + y)$$

and hence to

$$3x \cdot 10 = 18 \cdot (x+y)$$

Dividing this by 3 (raising to igi $3 = 20'$) yields

$$x \cdot 10 = 6 \cdot (x+y).^{10} \tag{*}$$

The easiest solution to this indeterminate equation is found if we put the first factors equal, $x = 6$, and the second factors equal, $10 = x + y$, from which of course y can be found as $10 - 6$. This seems in fact to be what happens, either on this arithmetical level or perhaps, as it may be intimated by problem B, by imagining the factors as sides of a rectangle (see Figure 1).

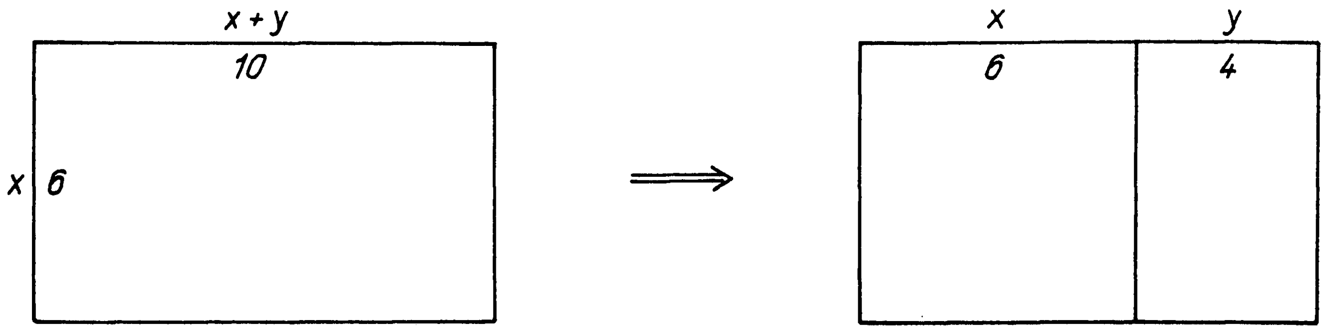


Fig. 1

Evidently, this is not the only set of solutions to the equation (whence the "(for)" in the translation); in fact, every set $6z$, $4z$ will do. By multiplying by $5'$ the text obtains the solution presupposed in the beginning, $x = 30'$, $y = 20'$.

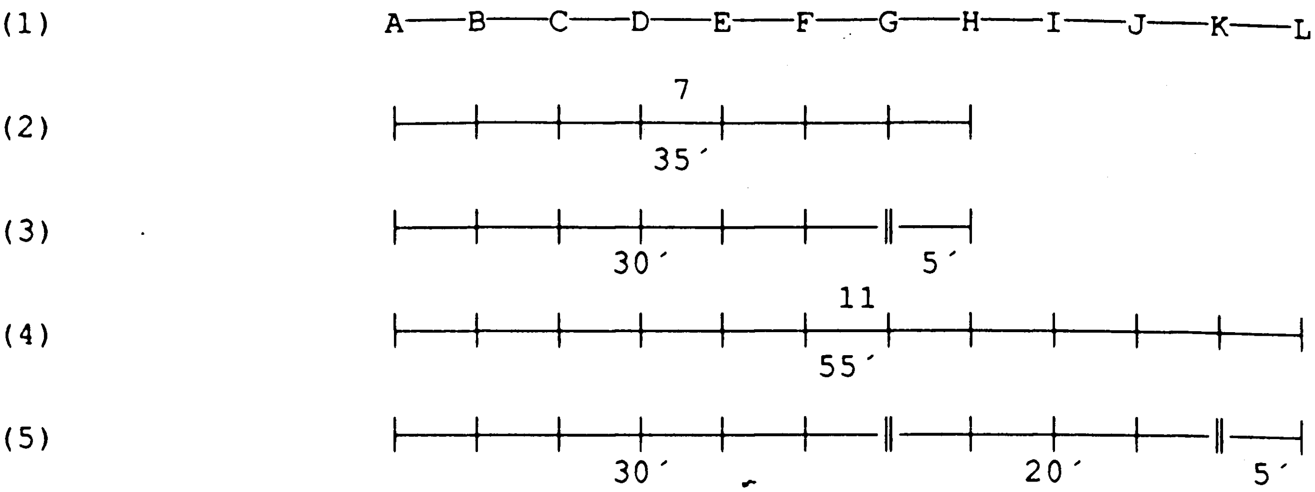
The $5'$ used here is the "step" from line 4, as told in the parallel passage in line 35. There is no compelling reason that precisely this factor should transform a solution obtained from (*) in the way indicated into the set originally thought of – unless the division leading to (*) is chosen so as to give a preliminary value equal to the number of "steps" going into the intended length. We must therefore assume that this is precisely what was done. This would also explain why the equation is not reduced eben further, *viz* to $x \cdot 5 = 3 \cdot (x+y)$.

Problem B, again, deals with length and width of a rectangle, and takes the same 7th. This time, however, 11 steps are made, which makes us go $5'$ beyond the accumulation of length and width:

$$\tfrac{1}{7}(x + \tfrac{1}{4}y) \cdot 11 = x + y + 5'.$$

Again, we may refer the "posings" of lines 19 to 23 to a diagram:

[10] Evidently, the representations of the equations will have been wholly different. They may have been rhetorical (verbal). More likely, however, is something in the likeness of the diagram above, or a kind of number scheme–cf. the discussion of schematic and graphic representations of TMS XVI in Høyrup, AoF 17, 305.



Lines 23 to 26, again, explain the multiplication of $(x + {}^{1}/_{4}y)$ by 4 and the ensuing transformation. The end of line 26 suggests that a further transformation corresponding to

$$ {}^{1}/_{7}[(4-1)\ x - 5' + (x+y+5')]\cdot 11 = 4\cdot(x+y+5') $$

is prepared, but a damage to the tablet prevents us from knowing precisely how.

In line 27, the procedure starts for good. We may follow it in symbols, from

$$ {}^{1}/_{7}[(4-1)x - 5' + (x+y+5')]\cdot 11 = 4\cdot(x+y+5') $$

to

$$ 11\cdot[3(x - {}^{1}/_{3}\cdot 5')] + (x+y+5')\cdot 11 = 28\cdot(x+y+5') $$

and hence to

$$ 11\cdot(x - 1'40'') = 5°40'\cdot(x+y+5'). $$

Already at this stage, 5°40′ is ascribed to the length. The entity to which it corresponds is, however, not the original length x but $x' = x - 1'40''$, where 1′40″ is explicitly introduced as the "appending of the length".[11]

Concomitantly, 11 will have to correspond to $x+y+5'$. Tearing out 5°40′ for x' leaves 5°20′ to correspond to a y', which in line 34 is found to be $(x+y+5) - (x-1'40'') = y + (5'+1'40'') = y + 6'40''$, where 6′40″ is called the "tearing-out of the width". The ease by which this computation is carried out corroborates the conjecture that a geometric or similar intuitively transparent representation is used (cf. Figure 2). In any case, the possible solutions

[11] The construct state *waṣīb* demonstrates that we really have to do with a noun, while its role in the calculation shows that it is of gerundive type. In German it would be called „das Hinzuzufügende".

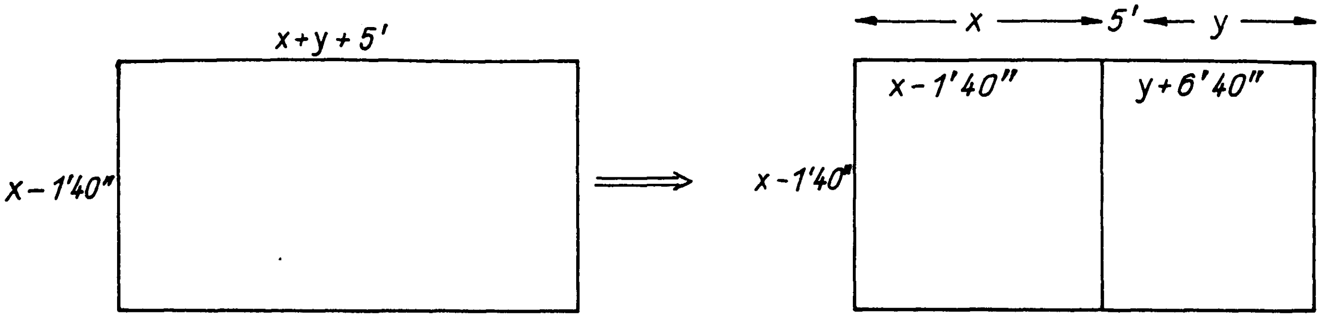


Fig. 2

$x' = 5°40', y' = 5°20'$ are multiplied by the step $5'$, giving the preferred solutions $x' = 28'20'', y' = 26'40''$. Appending and tearing out what according to their names should be appended and torn out finally gives $x = 30', y = 20'$.

After its didactical introduction, problem A thus demonstrates how to solve a homogeneous indeterminate first-degree problem. Problem B, on its part, shows how to reduce an inhomogeneous to a homogeneous problem through a shift of variables made very explicit by the "appending" and "tearing-out".

Obviously, an equation $a \cdot u = b \cdot v + c$ can be brought much more easily on homogeneous form by means of the principles used in our text – *viz* as $a \cdot u = b \cdot (v + c/b)$. The method which is actually followed, which reduces the problem into one in x and $x+y$ (or x' and $x'+y'$), shows that the so-called "accumulation" (UL.GAR, $x+y+5' = x'+y'$) was regarded as an entity of its own; it even seems to betray that this entity was regarded as fundamental, not just introduced *ad hoc* for the construction or solution of certain problems.

TMS VIII: Mixed second-degree problems

The same entities uš, sag, *waṣībum* and *nasīḫum* are also made use of in Text VIII, which otherwise deals with a quite different problem type. The text looks as follows:

Problem A

1. [a-šà 10 4-*at* sag *a-na* sag daḫ] *a-na* 3 *a-li-ik* ... ... [ugu]
   [The surface 10′. The 4th of the width to the width I have appended,] until 3 I have gone ... over

2. [uš 5 dir]ig za.e [4 *r*]*e-ba-ti ki-ma* sag gar *re-b*[*a-at* 4 *le-qé* 1 *ta-mar*]
   [the length 5′ goes] beyond. You, [4, of the f]ourth, as much as width pose. The four[th of 4 take, 1 you see.]

3. [1 *a-na*] 3 *a-li-ik* 3 *ta-mar* 4 *re-ba-at* sag *a-na* 3 d[aḫ 7 *ta-mar*]
   [1 to] 3 go, 3 you see. 4 fourths of the width to 3 ap[pend, 7 you see.]

4. [7] *ki-ma* uš gar 5 dirig *a-na na-sí-iḫ* uš gar 7 uš *a-na* 4 [*i-ší*]
[7] as much as length pose. 5′ the going-beyond to the tearing-out of the length pose. 7, of the length, to 4 [raise,]

5. 28 *ta-mar* 28 a-šà 28 *a-na* 10 a-šà *i-ší* 4,40 *ta-mar*
28 you see. 28, of the surfaces, to 10′ the surface raise, 4°40′ you see.

6. [5] *na-sí-iḫ* uš *a-na* 4 sag *i-ší* 20 *ta-mar* $^1/_2$ *ḫe-pe* 10 *ta-mar* NIGIN
[5′], the tearing-out of the length to four, of the width, raise, 20′ you see. $^1/_2$ break, 10′ you see. Make surround,

7. [1,40] *ta-mar* 1,40 *a-na* 4,40 daḫ 4,41,40 *ta-mar mi-na* íb-si 2,10 *ta-ma*[*r*]
[1′40″] you see. 1′40″ to 4°40′ append, 4°41′40″ you se[e.] How much the equilateral? 2°10′ you see.

8. [10 ¿S]Á!SÁ?*a-na* 2,10 daḫ 2,20 *ta-mar mi-na a-na* 28 a-šà gar *šà* 2,20 *i-na-*[*di-n*]*a*
[10 to the e]qual(?) to 2°10′ append, 2°20′ you see. How much to 28, of the surfaces, shall I pose which 2°20′ gi[ve]s me?

9. [5 gar] 5 *a-na* 7 *i-ší* 35 *ta-mar* 5 *na-sí-iḫ* uš *i-na* 35 zi
[5′ pose.] 5′ to 7 raise, 35′ you see. 5′, the tearing-out of the length from 35′ tear out,

10. [30 *ta-*]*mar* 30 uš 5 uš *a-na* 4 sag *i-ší* 20 *ta-mar* 20 {uš} ⟨sag⟩
[30′ you]see, 30′ the length. 5′ the length[12] to 4 of the width raise, 20′ you see, 20 the length (mistake for width).

Problem B

11. [a-šà 10] 4-*at* sag *a-na* uš daḫ *a-na* 1 *a-li-ik a-na* ¿KI/DI [u]š ugu? sag *¿i-ší-ma*?
[The surface 10′.] The 4th of the width to the length (probably erroneously for width[13]) append, to 1 go, (the outcome falls 5′ short of the length[14])

[12] This probably refers to the „length" of the square $^y/_4 \cdot {^y/_4}$. Several other mathematical Susa texts (N^os V and VI), indeed, speak about the „length" of a square.

[13] Unless the meaning (or the reason for this slip of the stylus) is „append ⟨to the equivalent of the width⟩ along the ⟨direction of the⟩ length.

[14] As pointed out by W. von Soden (BiOr 21, 48a), a better interpretation of this (so far nonsensical) passage can only be attained through collation. Since neither the gar nor the *le-qé* in the next line look as they should on the autography, the first lines of the reverse (line 11–12) must presumably be quite damaged (TMS contains no photo of the tablet, and it has only reappeared quite recently afterhaving been mislaid in the Louvre collection for decades–Jim Ritter, personal communication).

12. [za.]e 4 *re-ba-ti ki-ma* sag gar *re-ba-at* 4 *le-qé* 1 *ta-mar* 1 *a-na* 1 *a-l*[*i-ik*]
[Yo]u, 4, of the fourth, as much as width pose. The fourth of 4 take, 1 you see. To 1 g[o.]

13. [¿1?] 4 gabạ 4 gar 1 *ta-lu-ka a-na* 4 daḫ 5 *ta-*⟨*mar*⟩ *ki-ma* uš gar
[1(?)] 4 the counterpart, 4 pose. 1, the going, to 4 append, 5 you s⟨ee⟩, as much as length pose.

14. [5] *wa-ṣí-ib* uš gar 5 uš *a-na* 4 sag *i-ši* 20 *ta-mar* 20 a-šà
[5′,] the appending of the length, pose. 5, of the length, to 4, of the width, 20 surfaces.

15. [20 *a-na* 10] [*i-ši*] 3,20 *ta-mar* 5 *wa-ṣí-ib* ⟨uš⟩ *a-na* 4 sag *i-ši* 20 *ta-mar*
[20 to 10′ r]a[ise], 3°20′ you see. 5′, the appending of ⟨the length⟩, to 4 raise, 20′ you see.

16. [$^1/_2$ *ḫe-pe* 10 *ta-mar*] NIGIN 1,40 *ta-mar* 1,40 *a-na* 3,20 daḫ 3,21,40 *ta-mar*
[$^1/_2$ break, 10′ you see.] Make surround, 1′40″ you see. 1′40″ to 3°20′ append, 3°21′40″ you see.

17. [*mi-na* íb-si 1,50 *ta-mar* 10] ¿SÁ!SÁ!?[15] *i-na* 1,50 zi 1,40 *ta-mar*
[How much the equilateral? 1°50′ you see. 10′] the equal(?), from 1°50′ tear out, 1°40′ you see.

18. [igi 20 a-šà *pu-ṭú-úr* 3 *ta-mar* 3] *a-na* 1,40 *i-ši* 5 *a-na* 5 uš
[The igi of 20, of the surfaces, detach, 3′ you see. 3′] to 1°40′ raise, 5′ to 5, of the length.

19. [*i-ši* 25 *ta-mar* 5 uš *wa-ṣí-ib a*]*-na* 25 daḫ 30 *ta-mar* 30 uš
[raise, 25′ you see. 5′, the appending of the length, t]o 25′ append, 30′ you see, 30′ the length.

20. ... 20 sag
... 20, the width

Problem C

21. [... ... ...*ki*]*-ma* UL.GAR 3 uš *ù* 4 sag
[... ... ... as m]uch as the accumulation of 3 lengths and 4 widths

[15] Bruins and Rutten read the first signs in line 8 as *a-di*, and the analogous group here as *le-qé*, none of which make sense. The autography agrees acceptably well with the assumption that the same signs are used in both cases. The second sign appears te be a DI; the first is too close to the breaks to be read with certainty, but might be another DI. Though no mathematical standard expression, this makes sense if read SÁ.SÁ ∽ *šāninum*, "that which is equal", the current conceptualization of the side of a square.

22. [... ... ... *t*]*a-mar*
[... ... ... yo]u see

Once again, a few terms have to be explained.
1) *ḫepûm*/gaz ("to break") is the procedure by which (among other things) the "coefficient of the first-degree term" of a second-degree problem is bisected (the real geometrical meaning will be clear below).
2) NIGIN (an approximate ideographic but hardly a logographic equivalent of *šutamḫurum;* here translated "to make surround") is the construction process by which a line and its "counterpart" are made the sides of a square, thus "surrounding" it SÁ.SÁ ("the equal") in lines 8 and 16 appears to designate this line.
3) *meḫrum*/gaba is the "counterpart", i.e., in mathematical texts the "other side" of a square–mostly (but not here) it is used about the squares which result from quadratic completions.
4) a-šà (never fully written in Akkadian but at times with the phonetic complements of *eqlum;* translated "surface") designates primarily the concrete extension of geometric figures; as always, however, these are presupposed to have a measuring number (an area).
Like the previous problems, those of Text VIII deal with the length and width of a rectangle. This time, however, the rectangle is real, since it possesses a surface 10′ (in both problem A and B; problem C will presumably have dealt with the same rectangle, but too little of it is conserved to prove this). This may make us suspect that the sides, once again, are $x = 30'$ and $y = 20'$. However, the present problems contain no introductory didactical explanation, so we may be supposed not to know.

In problem A, apart from the surface, we are told that 3 4ths of the width appended to the width exceeds the length by 5′–in symbolical translation

$$y + 3 \cdot {}^{1}/_{4}\, y = x + 5'.$$

In line 2 we are told to pose 4 "as much as the width". The "4 fourths" (in absolute state) in line 3 shows that this means taking $z = {}^{1}/_{4}\, y$ as a new unit. Returning to line 2, we see that one 4th (of the width) is then 1 [z]. Repeating it 3 times gives 3, for which reason the length including a tearing-out of 5′ will be $7z$ (line 4).

Lines 4–5 calculates the number of surfaces $z.z$ contained by $4z$ times $7z$ to be "28 surfaces". That this, and not "28 the surface", is how the expression is to be read, follows from the multiplication which is used. If "a surface 28" was found, the multiplication would have been a construction, presumably *šutākulum*/ í-kú-kú (but possibly NIGIN, UL.UL or UR.UR.). "Raising" presupposes that a geometrical configuration (*viz* $z.z$) is already there – only the number of times it is present has to be calculated.

What the text is aiming at is thus a problem of the type "square area minus sides equals number" (minus, because the 28 surfaces correspond to a rectangle which is 5′ longer than it should be). So far, we have obtained

$$28\cdot z^2 - n\cdot z = 10'.$$

This will have to be brought to normal form, for which reason it is raised to 28, the "surfaces":

$$(28z)^2 - n\cdot(28z) = 28\cdot 10' = 4°40', \qquad (**)$$

cf. Figure 3A ($S = 28z$). The next step (Figure 3, B–C) consists in "breaking" (bisecting) the rectangular area representing the number of sides $n\cdot 28z$ which is represented by a breaking of n, and in moving it around so as to transform the rectangle into a gnomon and thereby allow a quadratic completion. For this purpose, n has to be found. Since the excess of 28 ($z.z$) over the original rectangle has the sides $4z$ and 5′, its area is $4\cdot 5'\cdot z = 20'\cdot z$; i.e., $n = 20'$ (line 6). 20′ is therefore bisected, "made surround" so as to yield $^n/_2\cdot{}^n/_2 = (10')^2 = 1'40''$ (line 7). Appending this quadratic surface to the gnomon completes the square on $S - {}^n/_2$ as $4°41'40'' = (2°10')^2$. Appending again 10′ where it was broken off gives $C = 2°10' + 10' = 2°20'$ (Figure 3D; line 8).

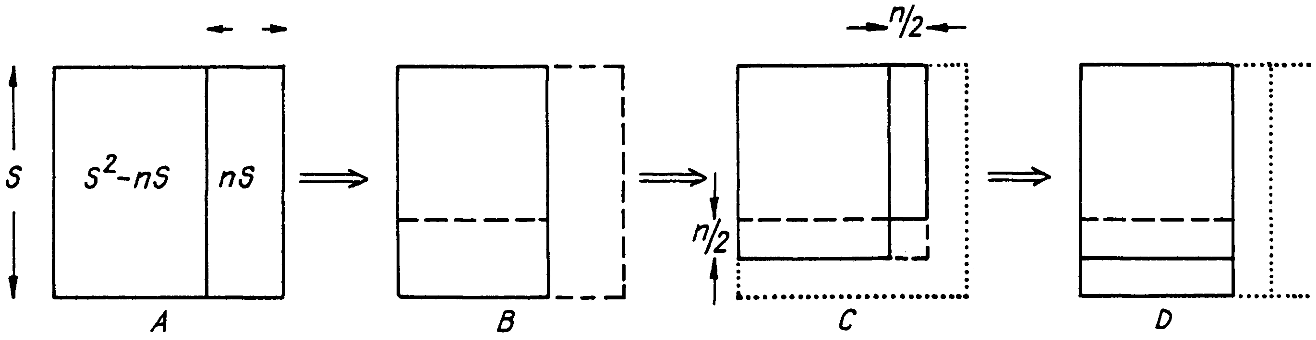


Fig. 3

This is in all respects the standard way to solve a normalized problem "square area minus sides". Next (lines 8–9), z is found as $S/28$ (a division which, again, is performed as all divisions by sexagesimally irregular numbers). x' is found as $7.z$, and the original length x by tearing out the "tearing-out of the length".

It may be of some interest to compare the path which is actually followed to an alternative which was within reach but not used. For brevity, I describe it in algebraic symbols. The original problem is

$$y\cdot({}^7/_4 y - 5') = 10'$$

or

$$1°45'\cdot y^2 - 5'\cdot y = 10'.$$

This could be normalized through multiplication by 1°45′ into

$$(1°45'\cdot y)^2 - 5\cdot(1°45'\cdot y) = 17'30''$$

and then solved as above. In principle this is equivalent to the actual method, since 1°45′ is just as good a number as 28. Furthermore, it bypasses the apparent detour over an intermediate variable z ("the fourth of the width"). That this structurally simpler method is none the less avoided demonstrates that the text, though not provided with a separate didactical explanation, is still meant to function at a level where insight in the meaning of what goes on is important (the alternative way might be followed at the level where training of methods beyond the intuitively meaningful was possible and aimed at–most of the mathematical "series texts" belong here). A rectangle containing 7 times 4 small squares is, after all, easier to grasp visually than one containing 1,45 times 1 (whether this be understood in the proper order of magnitude, as $1\,{}^3/_4$ times 1, or as 105 times 60).

This appeal to visual insight is made more explicit in problem B, which is otherwise a close parallel to problem A. This time, we only go once with the 4th of the width, which makes us fall short of the length by 5′.[16] The area is still 10′. In symbols thus

$$y + 1\cdot{}^1/_4 y = x - 5' \qquad x\cdot y = 10'.$$

Once again, the fourth of the width (z) is taken as the side of an auxiliary square. In spite of some missing signs in the beginning of line 13 the "counterpart 4" seems to tell us that a square of 4 (fourths of the width, as in line 3) times its counterpart 4 is drawn (see Figure 4) and "1 the going" then appended to the length, giving a rectangle of 5.4 small squares, which fills the original rectangle apart from a strip of 5′ times $4z$.

This time, the problem in z is thus of the type "square areas plus sides equals number". *Mutatis mutandis,* everything runs as before from this point onwards.[17]

[16] That this must be the meaning of the final part of line 11 is clear from the following. On the whole, the signs to be read on the autography make no obvious sense, even though it might be attractive to see some of them (sag *i-ši-ma*) as indications that the text introduces a trivial complication, *viz* the width and and ¼ of the width, raised to the width, falls short of the surface 10′ by 5′ widths. Still, a grammatical form *i-ši* seems quite out of place.

[17] Since the number of small squares is now regular (*viz* 20), we must presume the division to be made through raising to the igi. This restitution will also fit into the broken part of line 18, and it is indeed suggested in the autography in TMS. None the less, E. M. Bruins claims in the commentary (p. 62) that „le scribe se demande de nouveau: par quoi faut-il multiplier 20° pour obtenir 1° 40′ et il voit que c'est 5′". Dividing in this manner by a number belonging in the standard table of reciprocals would be totally unprecedented. Comparison with line 8, shows moreover, that the phrase in question could not be fitted into the lacuna.

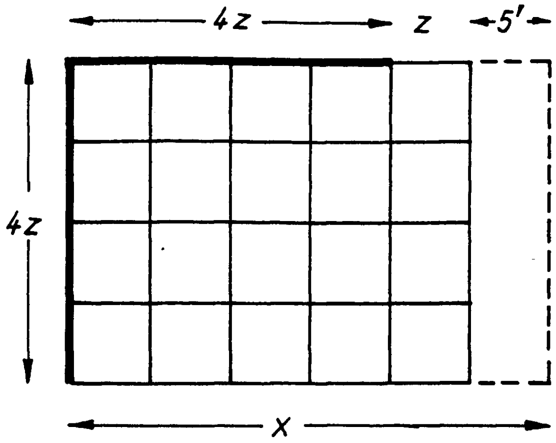


Fig. 4

Subdivision into smaller squares is appealed to in a number of other Old Babylonian mathematical texts:

First of all in BM 8390 (MKT I, 335–337; cf. Høyrup, AoF 17, 281–285, which discusses its first problem in detail). There, the use of the "multiplication" *eṣēpum* makes it clear that an integer number of squares (a concretely repeated square) is thought of, and not a mere numerical multiple; this agrees with the interpretation of the introduction of the auxiliary unit $^1/_4 y$ in the present text as a means to obtain intuitive transparency.

Next also BM 13901 N^os 10 and 11 (MKT III, 2 f.; cf. Høyrup, AoF 17, 278–280, where N^0 10 is discussed). In those problems it is not possible to decide from the text alone whether a subdivision into smaller squares ($x \cdot x = 49 z \cdot z$) or the creation of a new reference square 7·7 is meant. Our Susa text supports the former interpretation, thus contributing hopefully the cover to the coffin of the myth (or, at best, the equivocal statement) that the Babylonians would use the number *1* in the function of an algebraic unknown (say, instead of a modern algebraic x).

Reanalysis of these two Susa texts have thus confirmed the conclusion which was suggested by my earlier analysis of texts IX and XVI: That the mathematical Susa texts, because of their tendency to make things explicit which are tacitly presupposed in texts from the core area, contain important clues to the methods and conceptualizations of Old Babylonian mathematics in general.

E

"On Subtractive Operations, Subtractive Numbers, and Purportedly Negative Numbers in Old Babylonian Mathematics". *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* **83** (1993), 42–60.

On Subtractive Operations, Subtractive Numbers, and Purportedly Negative Numbers in Old Babylonian Mathematics

by **Jens Høyrup** – Roskilde

Aḥmad S. Saidan
in memoriam

The uses of the terms and expressions *nasāḫum*, *eli . . . watārum*, *ḫarāṣum*, *tabālum*, *šutbûm* and LÁ (LAL) in Old Babylonian mathematical texts are investigated. The two first operations turn out to be genuine mathematical terms, designating concrete removal and comparison, respectively; *ḫarāṣum*, *tabālum* and *šutbûm* are terms from everyday life used to formulate "dressed problems" and hence also occasionally more or less metaphorically for subtraction by removal within the description of procedures. LÁ (on one occasion *maṭûm*) is used as a substitute for *eli . . . watārum* when stylistic or similar reasons require that the smaller of two magnitudes to be compared be mentioned first.

The claim sometimes made (going back to misreadings of Neugebauer) that LÁ has to do with a Babylonian concept of negative numbers is thus unfounded.

In a number of earlier studies, some of them as yet unpublished,[1] I have investigated the panoply of operations applied in Old Babylonian so-called "algebra". Among the results is a distinction between two main "additive operations", *waṣābum* (with logogram DAḪ) and *kamārum* (logograms GAR.GAR and UL.GAR), which have quite distinct roles within the texts, and a halfway corresponding distinction between two main "subtractions", *nasāḫum* (ZI) and *eli . . . watārum* (UGU . . . DIRIG). Without pursuing the matter I have also taken note of the apparently distinct use of other subtractive operations (*ḫarāṣum*, *maṭûm*) and of what looks as evidence for a category of "subtractive [role of a] number".

The present paper represents an attempt to pursue these latter questions systematically, and to connect them with a claim which is occasionally made – *viz* that Old Babylonian calculators had a concept of negative number.

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[1] Among the published items, I shall only refer to Høyrup 1990, which contains the most thorough presentation and discussion of evidence and results.

Zeitschr. f. Assyriologie Bd. 83, S. 42–60

As a mathematical term, *waṣābum* designates a concrete process where an entity A is joined or "appended" to another entity C of the same kind – cf. the etymology of Latin *ad-do*. In the process, C conserves its identity, and A is absorbed. For this reason, the sum by this process possesses no particular name of its own. A convenient model for this kind of additive thinking is suggested by the derivative *ṣibtum*, "interest": If interest is added to *my* bank-account the increased balance remains *my account.*

kamārum is the addition where the single contributions are brought together or "accumulated" into a common heap – cf. the etymology of Latin *ac-cumulo.* In this process, the single contributions loose their identity, and the heap (i.e., the sum) therefore has a particular name, the *kimirtum* (the text AO 8862 employs the plural *kimrātum,* referring to the composite nature of the heap).

Those second- and third-degree problems which add entities of different dimension (be it length and area, as, e.g., in BM 13901, or volume and area, as in BM 85200 + VAT 6599), normally do this by "accumulation" (I shall discuss one characteristic exception below). The implication appears to be that this is, or can at least be, a real (i.e., an arithmetical) addition of measuring numbers. "Appending", on the other hand, is additive but not arithmetical, putting together concrete entities; phrases like

30 . . . *a-na* 29,30 *tu-ṣa-ab-ma* 30 *mi-it-ḫar-tum*
30 . . . to 29,30 you append: 30,0 the side of the square

(BM 13901, obv. i 8) should be read as descriptions of a concrete procedure where the concomitant arithmetical operations are *implied.*

nasāḫum, "to tear out", is the reversal of appending. This is made evident, among other things, by numerous texts where they occur in parallel. As pointed out by Vajman (1961: 100), addition and subtraction (by these two operations) of the semi-difference $d = {}^{a-b}/_2$ between two magnitudes a and b to and from their semi-sum $r = {}^{a+b}/_2$ are normally organized in such a way that d is first torn out from one copy of r and next appended to another copy, i.e., *the same* piece d is simply transferred from one to the other.[2]

[2] The geometrical interpretation of the algebra texts makes this transfer even more meaningful than it was to Vajman – cf. Høyrup 1990: 264.
Vajman's rule is not without exceptions. See, e.g., BM 13901 Nos 8, 9 and 12.

All the more strange is the occurrence of *nasāḫum* in the beginning of certain algebra problems as the counterpart of *kamārum.* In order to see what goes on we may look at the statements of the first problems of the mathematical procedure text BM 13901:

No. 1

1. A.ŠÀ$^{\ulcorner lam \urcorner}$ *ù mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45-E 1 *wa-ṣi-tam*
The surface and my confrontation I have accumulated: 45′ is it. 1, the projection,
2. *ta-ša-ka-an* . . .
you pose. . . .
4. . . . 30 *mi-it-ḫar-tum*
. . . 30′ the confrontation.

No. 2

5. *mi-it-ḫar-ti lìb-bi* A.ŠÀ [*a*]*s-sú-uḫ-ma* 14,30-E 1 *wa-ṣi-tam*
My confrontation inside the surface I have torn out: 14, 30 is it. 1, the projection,
6. *ta-ša-ka-an* . . .
you pose. . . .
8. . . . 30 *mi-it-ḫar-tum*
. . . 30′ the confrontation.

No. 3

9. *ša-lu-uš-ti* A.ŠÀ *as-sú*⟨*-uḫ-ma*⟩ *ša-lu-uš-ti mi-it-ḫar-tim a-na lìb-bi*
The third of the surface I have torn out. The third of the confrontation to the inside
10. A.ŠÀ^{lim} *ú-ṣi-ib-ma* 20-E . . .
of the surface I have appended: 20′ is it. . . .
15. . . . [30] *mi-it-ḫar-tum*
. . . [30′], the confrontation.

No. 4

16. *ša-lu-uš*[*-ti* A.ŠÀ *as-sú-uḫ-ma* A.ŠÀ *ù m*]*i-i*[*t-ḫa*]*r-ti*
The third of the surface I have torn out: The surface and my confrontation
17. *ak-mur-ma* ⌈4,46,40-E . . .]
I have accumulated, $4'46^{0}40'$ is it. . . .

23. 20 [*mi-i*]*t-ḫar-tum*
20, the confrontation

First a few words to the translation:

Sexagesimal place value numbers are translated according to Thureau-Dangin's system, `, `` etc. indicating increasing and ′, ″ etc. decreasing orders of magnitude.

"Confrontation" stands for *mitḫartum* and is meant to render the connection between the latter word and *maḫārum*. What should be thought of is a quadratic configuration consisting of four equal lines confronting each other; numerically, the "confrontation" is determined by the length of one side (in other words, the "confrontation"/*mitḫartum* can be imagined as the side of a square which presupposes and implies the presence of the square as inseparably as, to our thinking, a quadratic area presupposes and implies the presence of a quadratic perimeter). The "surface" (A.ŠÀ) is the area of the quadratic figure. (The word "surface" is used instead of "area" to translate A.ŠÀ in order to emphasize that the primary meaning of the term is the geometrical extension – "a field" – and that the number measuring the area of this extension is only a secondary meaning).

The "projection" translates *waṣītum*, and should be understood as a breadth *1* which, when given to a line (in case a "confrontation") of length L transforms it into a rectangle of area $1 \cdot L = L$.

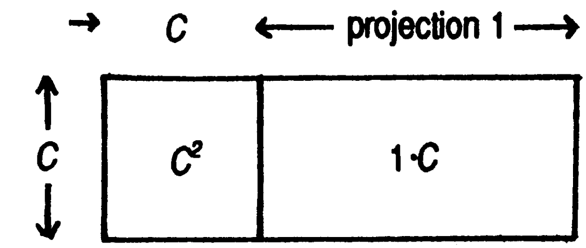
"Inside" is meant to render the use of *libbum* in our text, where it seems to serve as nothing more than an indication that the entity to which something is appended or from which something is torn out possesses bulk or body.

With this in mind we may start by looking at *problem No 1.* At first we are told that the accumulation of [the measuring numbers of] area and side of a square configuration is 45′. In order to make geometrical sense of this, the "projection" 1 is "posed" as in *Figure 1.* It is not said explicitly, but in this way the rectangle $1 \cdot C$ can be "appended" to the surface $C \cdot C$. Cut-and-paste manipulation of the resulting figure (whose area is known to be 45′) allows a final disentanglement of the confrontation.

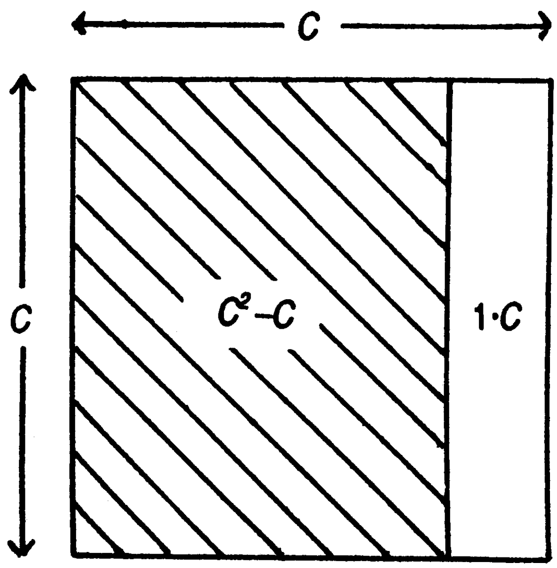
Problem No 2 is similar, but subtracts the side from the area by tearing out the confrontation from the surface. It is only as a first step in the procedure that the projection is posed explicitly, but already in the statement is it implicitly presupposed by the use of the verb *nasāḫum* – cf. *Figure 2,* where the shaded area shows what remains when the confrontation has been torn out.

From *No 2* alone, it is true, we cannot be sure that a "projection" is implicitly presupposed. After all, like *kamārum, nasāḫum* might operate on the measuring numbers. *Nos 3–4,* however, shows us that this is not the case. The statement of *No 3* starts by tearing out a third of the area, before it adds a third of the confrontation. This time, however, the confrontation (actually its third) is *appended,* see *Figure 3.*

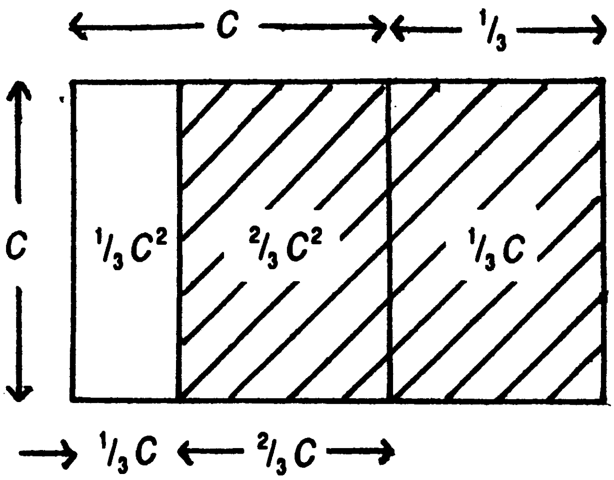
In *No 4*, on the other hand, where a third of the surface is again torn out, the addition of a confrontation is another accumulation. In between, however, an intermediate step has been inserted, *viz* a reference to the reduced surface as a surface of its own. (*Figure 4*).



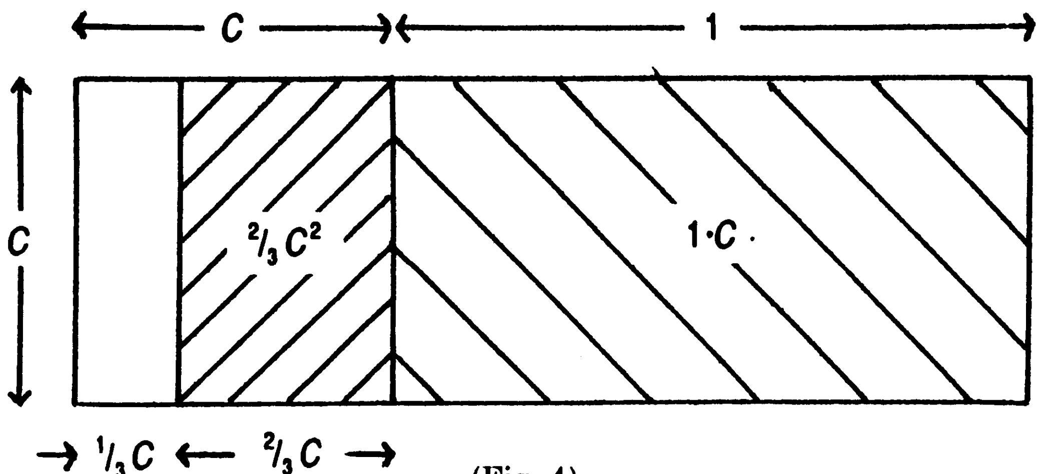
(Fig. 1)



(Fig. 2)



(Fig. 3)



(Fig. 4)

Together, the two formulations suggest the following interpretation: Tearing-out a part of the surface transfers the process to the level of concrete geometric manipulations; once we are there, the only additive operation at our disposal is appending, which presupposes that the confrontation to be added is implicitly provided with a "projection". This is what happens in *No 3*. *No 4*, on its part, by speaking of the result of the tearing as a "surface", takes note of it as an entity of its own, possessing its own measuring number. *This number*, and not the palpable surface resulting from the tearing, can be accumulated with the [measure of the] confrontation, as it happens in *No 4*.

The present interpretation of *No 3* presupposes that the Babylonian calculators were predisposed to think of a line segment as provided automatically with a standard width (a "projection") *1* – an idea which is rather unfamiliar to our post-Euclidean mode of geometrical thought. We are equally unprepared, however, to think of a surface as provided with a standard height. The latter idea, as it is well known, was the very foundation of Babylonian volume metrology, which did not distinguish the area measure sar (NINDAN^2) from the volume measure sar (NINDAN^2 · KÙŠ) – meaning that a volume was measured by the area it would cover if distributed with the standard height 1 cubit[3].

Mathematical texts also tell us that lines were understood as representing the rectangles of which they were the sides – thus, e.g., the confrontation represented the whole quadratic configuration. There is thus nothing strange in the shift from metro-numerical to concrete representation in *No 3*.

This brings us back to *No 2*: if tearing-out in *No 3* enforces a shift to concrete representation in *No 3*, it cannot be the reverse of an accumulation; even when the confrontation is torn out in *No 2* it must thus be provided with a tacit "projection *1*".

The expression *eli . . . watārum*/UGU . . . DIRIG was introduced above as the other main subtractive operation. "*P* UGU *Q R* DIRIG" can be translated *de verbo ad verbum* as "*P* over *Q R* goes beyond" or, if this principle is abandoned, "*P* exceeds *Q* by *R*". Arithmetically, this means that $P–Q = R$. This "subtraction by comparison"

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[3] A less well-known but even more pertinent parallel is provided by the names of the area units NINDAN and EŠE. As pointed out by Powell (1972: 185), "for the definition of both the NÍG and the eše, a rectangle with a fixed side of 1,0 NÍG is assumed as constant. If the base of the rectangle is one NÍG in length, the plot is termed a NÍG; if the base is one eše in length, it is termed an eše".

It may be of interest that Egyptian area metrology refers to similar conceptions: a "cubit of land" is a strip of standard length 100 cubit and one cubit wide; a "thousand of land" equals one thousand such strips. See Peet 1923: 25.

is used in BM 13901 in cases where one confrontation is told to exceed another by a certain amount or fraction: *mitḫartum* UGU *mitḫartim* 10 *itter* (obv. ii 4, rev. i 40) or *mitḫartum* UGU *mitḫartim sebiātim itter* (obv. ii 20). The outcome of the operation can also be used for further operations even if unknown, in which case it is spoken of as *mala mitḫartum* UGU *mitḫartim itteru*, "so much as the confrontation over the confrontation goes beyond", or simply, if the identity of *P* and *Q* goes by itself, as DIRIG, "the excess". In all cases, as we see, entities of the same kind are compared. In contrast to the remainder after a tearing-out, the excess does not take over the identity of *P*.

The relation between accumulating, appending, tearing-out, exceeding and "reclaiming" (another quasi-subtractive operation or term) is highlighted by a problem collection from Susa.[4] The first sequence of problems tells the side of a square – the *confrontation* – and asks for a varying multiple of the *length* (UŠ – numerically the same as the confrontation).[5] The next sequence (*section 3*, an intermediate sequence being broken off) accumulates the confrontation and a multiple of the length, while *section 4* tells how much the confrontation exceeds a multiple of the length. *Section 5* gives the confrontation and asks for a multiple of the surface, while *section 6* gives a multiple of the surface and asks for the confrontation. *Section 7* tells the confrontation and asks for the area of [the square built on] a multiple of the length, while *section 8* accumulates the area and the area of a multiple of the length, and *section 9* tells the excess of the area over the area of a (sub)multiple of the length.

Section 10 introduces mixed second-degree problems, *appending* a multiple of one length to the "surface of the confrontation". *Section 11*, the subtractive counterpart, falls into two subsections. In the first, a multiple of the length is torn out from the surface, leaving a known remainder; in the second, where the multiple of the length is larger than or equal to the area, the multiple of the

[4] TMS V. Bruins' transcriptions, translations and commentaries in the edition of the mathematical Susa texts abound in mistakes, and even if text V has been treated with more attention than most one should still base the discussion upon the transliteration and the autography.

[5] I disregard this other aspect of the text – the principles according to which the multiples are systematically varied and designated – having dealt with the question elsewhere (Høyrup 1990a: 303–305).

length is told to exceed the surface by so and so much, or to be "as much as the surface" (*kīma* A.ŠÀ). *Section 12*, finally, states that a certain part of the surface has been "withdrawn" (*tabālum*) and tells the remainder, asking for the confrontation,[6] after which follow problems of a different type, to which we shall return.

The text tells neither procedures nor solutions, but these can be easily reconstructed from other mathematical texts. It turns out that "accumulation" is used where further operations will take place on the arithmetical level, e.g., by an argument of the type "single false position" (*sections 3* and *8*); in both cases, the entities to be added are of the same kind (a precondition, indeed, for the application of purely arithmetical techniques). The corresponding subtractive sequences (*sections 4* and *9*) go by comparison, corroborating the observation made on BM 13901 that only entities of the same kind can be compared.

Cases where the physical outcome of an additive procedure and not its arithmetical expression will be the basis for further operations (i.e., the additions of area and sides in *section 10*) are made by *appending*. Our present text thus presupposes that sides are already provided with an implicit "projection", carrying hence a surface with them. The corresponding subtractions of *section 11a* are made by *tearing-out*.[7]

This way to subtract sides from the area corresponds to what happens in BM 13901. *Section 11b*, on the other hand, is unusual.

It is unusual already for its mathematical content. No other mixed second-degree problems with a single unknown of this structure are known (the type in question, ax-$x^2 = b$, is the one which possesses two positive roots), even though several complex

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[6] The phrase is, in the first example of this section, "$^1/_3$ A.ŠÀ *it-ba-al* ÍB.SI A.ŠÀ 10 LAGAB *mi-nu*". LAGAB is the standard logogram used in the text for *mitḫartum*, for which ÍB.SI$_8$ (regularly written ÍB.SI in the Susa tablets) is used in a number of other texts, and which in any case has a normal use quite close to that of *mitḫartum:* The standard phrase "*A*-E *r* ÍB.SI$_8$", often translated "r is the square-root of *A*", should rather be "*A* makes *r* equilateral", i.e., when formed as a square, the area *A* will produce the side *r*.

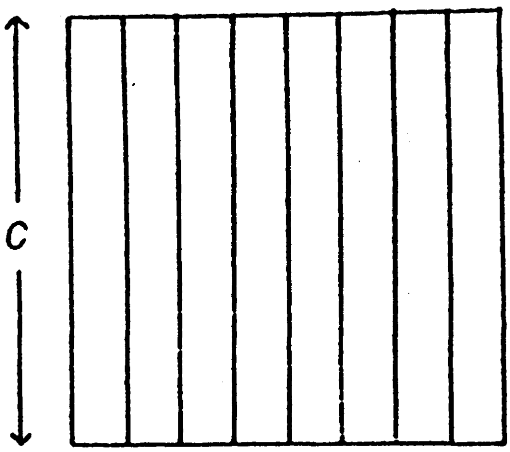
The side of the square is 30 and the corresponding area hence 15′; the most plausible reading of the phrase therefore appears to be "$^1/_3$ of the surface (somebody) has withdrawn (regarding) the surface of the equilateral, 10′. What (is) the confrontation?".

[7] TMS VI contains parallels to sections 10 and 11a from TMS V. Here, similarly, sides are appended to and torn out from the area.

problems are solved in a way which suggests that they were reduced to this type.[8]

But even the formulation is unexpected. What is the reason that sides (provided tacitly with a "projection") can be torn out from an area, but an area apparently not from a multiple of sides? In the absence of parallel texts only a tentative explanation can be given.

We may start from the general semantic of the term *nasāḫum*. In agreement with the translation, one can only "tear out" what is already part of the total: if B is not a part of A one can at best tear out "so much as B" (*mala* B) from A. That n sides of the square on C can be torn out thus appears to imply that the square really consists of C strips each C long and *1* wide – evidently an idea which is close at hand once the side is thought of as a similar strip, cf. *Figure 5*. Since the area cannot be torn out from a multiple $n \cdot C$ of the side ($n > C$), however, it seems that the reverse process (converting an adequate number of strips into a quadratic figure) does not take place automatically. Instead, it is told *by how much it is impossible* to tear out the n sides from the surface, and in one case that tearing-out in the proper sense cannot be performed because everything will be removed.



(Fig. 5)

[8] Thus IM 52301 No 2 (see Høyrup 1990: 341) and BM 85196, rev. ii 7–21 (see Høyrup 1985: 58).

The corresponding problem with two unknowns, $xy = b$, $x + y = a$, is of course well-known. There is thus no doubt that the problem would have to be solved by means of the same geometric cut-and-paste technique as the other mixed second-degree problems of the tablet.

Section 12 confronts us with another puzzle: Why doesn't it make use of the same operation as *section 9*, since the method by which its problems are solved will have been more or less the same? Once again, the absence of parallels in other mathematical texts prevents us from knowing with certainty. However, this absence also suggests the beginning of an answer. A term which is so rare but which is none the less used in a mathematically quite trivial context can hardly represent a genuine mathematical operation. It will rather be an "everyday" term, i.e., a term taken from the extra-mathematical facets of scribal life.

Now, *tabālum*, "to withdraw", is indeed used routinely in connection with fields (the extra-mathematical meaning of A.ŠÀ), in particular when a whole field or a specified part of it is reclaimed from the owner by legal action.[9] Since the term turns up in *section 12*, after the apex of mathematical sophistication represented by second-degree algebra, and since the following part of the tablet deals with squares inscribed concentrically into squares, a subject derived somehow from geometrical practice, the problems about areas with withdrawn parts may plausibly constitute a first section of "dressed problems". The authentic meaning of its first problem (cf. *note 6*) will then be something like this: "from a quadratic field, somebody has reclaimed $1/3$ of the area, and what is left amounts to 10′ sar. What is the side of the square?".[10]

This interpretation explains another enigmatic feature of the text. In ordinary mathematical texts, the statement is made by the teacher in the first person singular preterit, "I have done so and so". This is so much a routine that Bruins overlooked the third person used in the text, translating *it-ba-al* as "j'ai ôté . . .". Only if the problems do not deal with a configuration constructed or

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[9] An Old Babylonian example is found in Walters 1970: 64 (text 45 lines 15 and 18, cf. the note to line 15); a Kassite example specifying that "reduction" (*niširtum*) of the field is involved is in Scheil 1905: 36 (iv 15f.).

[10] The problems of section 12 will thus provide nice examples of what Karen Nemet-Nejat (1993) has spoken of as "mathematical texts as a reflection of everyday life in Mesopotamia".

My choice of order of magnitude has been made so as to fit real-life fields as closely as possible – from the text alone, the area left over might as well be 10′ sar, in which case the square field would be approcimately 3 m by 3 m. Evidently, this choice is arbitrary – neither Babylonian nor modern mathematics teachers care much whether their numerical data are plausible.

prepared by the teacher but with a fictional juridical case, the subject of the action should really be a *somebody*, a third person.

A shift like this within the same tablet, from "pure" calculation to practical computation (real or fictional) is not unprecedented in the Old Babylonian mathematical corpus. One example can be found in BM 13901, where the penultimate problem (No 23) is probably a surveyors' "recreational" puzzle (cf. Høyrup 1990: 275, 352). Other instances are IM 52301, where an excerpt from an IGI.GUB table follows upon two second-degree "algebra" problems, and AO 8862, where second-degree algebra problems are followed by (still artificial) problems on brick-carrying, involving both practical metrology and a "house builder" (*itinnum*). Most relevant of all texts is perhaps IM 52916 (tablet "1" of the "Tell Ḥarmal compendium"), like TMS V a list of problems without solution (actually only *problem types*, since even the given numbers are not stated), which starts out with long sequences of second-degree problems, appending sides to or tearing them out from the area, continues with IGI.GUB-factors for geometric figures and with inscription of geometric figures into other figures, and closes with work norms and other practical computational topics.

In the present text, *tabālum* is thus after all probably not to be read as a mathematical technical term, and still less as the name for a distinct mathematical operation. *nasāḫum* and *eli . . . watārum*, on the other hand, which are used not only in statements but also in the description of mathematical procedures, are technical terms for genuine mathematical operations, or at least as technical as any term in Babylonian mathematics. In two other texts, *tabālum* seems to get closer this role. In YBC 4608, obv. 24 and 27, a line d is "withdrawn" from an entity which is known already to represent the sum $d+b$ of two opposing sides of a quadrangle; the reason for the choice of this specific term may thus be the no less specific situation that d is precisely what can "justly" be withdrawn. In YBC 4662, obv. 9, however, the term is used during the solution of an ordinary second-degree problem in a place where *nasāḫum* would be the standard choice, – and is indeed the actual choice of the parallel passages rev. 9 of the companion text YBC 4663.

tabālum is not the only term which moves imperceptibly between extra-mathematical and mathematical discourse without ever achieving the status of a genuine mathematical term. Similar cases are offered by the verbs *ḫarāṣum*, "to cut off", and *šutbûm* (*tebûm* III), "to make leave", "to remove".

The occurrences of the former in BM 85196 No 18, rev. ii 19,23f. are obvious references to non-mathematical parlance. They simply refer to the cutting-off of parts of silver coils (ḪAR) used for payment. The way the term appears in the fragment VAT 6546 may be inspired from this meaning, since something is cut off from a profit (*nēmelum*); so much is clear, on the other hand, that the term occurs (twice) *inside* the description of the procedure, i.e., that it describes computational steps. This is also the case in AO 6770, No 3. This problem deals with a stone, from which something has been removed and to which something has been added; but *ḫarāṣum* turns up inside the procedure, while the verb of the statement is ZI, "to tear out".

AO 8862, on the other hand, employs the term to describe an indubitable mathematical operation along with *nasāḫum*; so do the twin texts YBC 4663 and 4662. As a general rule, *ḫarāṣum* is used in these three tablets when something is removed from a linear entity; alternatively, *nasāḫum* without *libbi* may be used. In cases where a piece of surface is removed from another surface, the expression used is *ina libbi nasāḫum* (or, as mentioned above, *ina libbi tabālum*).[11]

Only two passages of AO 8862 do not agree with these rules. In iii 11–12, *nasāḫum* without *libbi* is used when a piece of surface is torn out from another. Instead, however, the subtrahend is explicitly told to be a surface (A.ŠÀ). In ii 10–11, finally, *ḫarāṣum* is used even though surfaces (length and width provided with a projection) seem to be involved (see Høyrup 1990: 317).

There is thus no absolute distinction between the two operations. There is, however, an outspoken tendency to keep in mind the concrete character of the process which goes on and to make this visible through the imagery which is inherent in the description – through a distinction between "cutting" and "tearing", between use and non-use of *libbi*, or by an explicit epithet A.ŠÀ.

The merely relative character of the distinction between *nasāḫum* and *ḫarāṣum* in confirmed by a final text where *ḫarāṣum* occurs: in YBC 4675 (and its partial doublet YBC 9852), this term is used for both surfaces and lines; *nasāḫum*, on the other hand, is totally absent.

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[11] It should be observed that this use of *libbi* to distinguish between removal from surfaces and from linear entities does not hold outside the small text group in question. BM 13901, e.g., uses *libbi* (and *libba*) indiscriminately in both cases.

The case of *šutbûm* is somewhat simpler. In extra-mathematical contexts, it is often used when you remove something or make somebody leave that *should* in fact be removed or go away: making workers go out for work; removing guilt, demons, garbage; taking a statue from its pedestal for use in a procession; etc. (see AHw. 1343). Its use as a quasi-mathematical term may derive from the same idea. In one case, the original magnitude A of a measuring reed is to be found when its length after loss of one 5th is 20′ NINDAN: 5 is inscribed (*lapātum*), so to speak as a model of the original reed; 1 (i.e., one fifth) is removed, leaving 4 in the model to correspond to the 20′ of the shortened reed; the IGI of 4 is found to be 15′; "raising" (multiplying) 15′ to 20′ yields 5′ (represented by 1 in the model), which added to 20′ gives the original length 25′ NINDAN (VAT 7535, obv. 25f.; similarly rev., 22–24 and, apart from what looks like a copyists omission of a line, VAT 7532, rev. 7f.).

I have observed no uses of the term in mathematical texts outside of this specific kind of argument by single false position.

The conclusion to draw concerning the relation between *nasāḫum, tabālum, ḫarāṣum* and *šutbûm* is thus that *nasāḫum* is the fundamental term for identity-conserving subtraction. Other terms employed in daily life for processes where something is removed from a concrete totality may be used, firstly, in the formulation of "dressed problems" dealing with precisely such processes; but from there they might also creep into the description of mathematical procedures, in particular into places where the calculation evokes associations related to the everyday connotations of the term – either because of the real-world counterpart of the calculation or because of the structure of the model on which the calculation is based. This observation might hold for other parts of the mathematical vocabulary, too. We might say that the process by which a technical terminology is created was never brought to an end in Old (or, indeed, any) Babylonian mathematics.

The mathematical texts which come closest to revealing a technical vocabulary (for mathematical operations as well as for the real-world problems providing the dress) are the mathematical series texts.

In many respects the terminology used in these texts coincides with what we know from procedure texts. *nasāḫum* is used, while *tabālum, ḫarāṣum* and *šutbûm* are absent. LÁ (l a l in MKT etc.), however, rises to unexpected prominence.

The term (in one case, its Akkadian equivalent *maṭûm*, "be(come) small(er)") only appears in two procedure texts. One of these is BM 13901. Here, No 10 tells that

> *mi-it-ḫar-tum a-na mi-it-ḫar-tim si-bi-a-tim im-ṭi*
> confrontation to confrontation, one seventh is smaller

while No 11 states that

> *mi-it-ḫar-tum* U.GÙ *mi-it-ḫar-tim si-bi-a-tim i-te-er*
> confrontation over confrontation, the seventh goes beyond.

The mathematical structure of the two problems is the same, apart from the fact that No 10 takes the fraction by which the two confrontations differ of the larger and No 11 of the smaller confrontation (in both cases, the larger confrontation is counted as the "first").

The reason for the different constructions is that Babylonian mathematics teachers had their favourite ways when formulating problems. One seventh is taken quite often (as are $^1/_{11}$, $^1/_{13}$, $^1/_{17}$, $^1/_{19}$, and $^1/_4$). One sixth and one eighth, on the other hand, are avoided as uninteresting. By comparing first the smaller to the larger, next the larger to the smaller, the author of the text has managed to use the favourite fraction $^1/_7$ both when the ratio is 7:6 and when it is 8:7.

maṭûm, precisely like *eli . . . watārum*, is thus a "subtraction by comparison", the only difference being the order of the operants. The same holds for the Sumerographic equivalents UGU . . . DIRIG and LÁ in the series texts. In YBC 4714 the reason for the change is precisely as in BM 13901: LÁ (or TUR, which is used synonymously in this tablet and nowhere else) is chosen when this choice makes it possible to refer to one of the favourite fractions while the use of UGU . . . DIRIG would preclude it; in neutral cases (e.g., when the difference is given in absolute value and not in relative terms), UGU . . . DIRIG is preferred (this is also the case in BM 13901).

The other occurrences of LÁ in the series texts have a slightly different explanation. They arise when, for some reason or other, the former of two magnitudes A and B which are compared comes out as the smaller. This can happen for a variety of reasons: A may be complex and B simple, as in YBC 4710, rev. ii 5–15;[12] one or both expressions may be submitted to systematic variation, and A

[12] Similarly also YBC 4673, rev. ii 16.

come out at times smaller, at times larger than B (as in YBC 4668, obv. iii 20–33, where $A > B$ in four cases and $A < B$ in two);[13] or a third entity C may be involved and consecutive lines deal with the amount by which C exceeds B and that by which A falls short of B (YBC 4708, rev. i 16, as corrected in MKT III 61). Since normal mathematical style would require the complex entity to be described first, and the spirit of systematization (as also the compact style of the series texts) would require that the order of entities be conserved in spite of variation of coefficients, all these cases can ultimately be traced back to considerations of style.

If the comparison between two systematically varied expressions A and B is translated into mathematical symbols and the order is made so as to reflect the text precisely, consecutive problems will be represented as in this example (MKT I 455, translating YBC 4668, obv. iii 20–24):

$$6{,}0(x-y)-(x+y)^2 = 18{,}20$$
$$2{,}30(x-y)-(x+y)^2 = -16{,}40$$

In Neugebauer's corresponding verbal translations (p. 440), the right-hand side of these equations become "18,20 geht es hinaus" and "16,40 ist es abgezogen".

When discussing expressions of this kind, Neugebauer would speak of "positivem Überschuß" and "negativem 'Abgezogenen'"; in the subject index he would refer to "Negative Größen" (MKT III 13, 83). What he meant by "negativity" was never more than this. It is thus with good reason that Neugebauer's *Exact Sciences in Antiquity* does not speak of "negative" but only of "subtractive numbers" or "subtractive writing of numbers" (1957: 236, 239).

Speaking of "negative numbers" in Babylonian mathematics is thus, firstly, a misreading of Neugebauer's much more restricted claim; secondly, it is unwarranted, unless one will proceed to claim that terms like "smaller", "below", and "before" also demonstrate knowledge of negative numbers, the "real" dimension to be measured being "larger", "above", and "after". All the more unwarranted, indeed, since the reasons to give the deficiency of A with regard to B instead of the excess the other way round turns out to depend on stylistic considerations or on the aspiration to make use of favourite fractions, and not on any attempt to investigate a particular mathematical conceptual structure or operation.

[13] So also YBC 4695, rev. i 11; YBC 4711, obv. i 13, 20, rev ii 36.

What we do have in a few texts are traces of an explicitly stated idea of "subtractive number" or "subtractive role of a number", in a wider sense than suggested by Neugebauer when he used the former expression.

One of these texts is BM 85200+VAT 6599 (the other procedure text in which LÁ occurs, *viz* in the passage to be quoted here). The statements of problems No 29 and 30 run, respectively,

> TÚL.SAG 1,40 UŠ IGI 7 *ša* UŠ U.GÙ SAG DIRIG *ù* 2 KÙŠ GAM-*ma* 3,20 [SAḪ]AR.ḪI⟨.A BA.ZI⟩
>
> A cellar. $1^0 40'$ the length. The 7th part of that which the length over the width goes beyond, and 2 KÙŠ: the depth. $3^0 20'$ of earth I have torn out

and

> TÚL.SAG 1,40 UŠ IGI 7 GÁL *ša* UŠ U.GÙ SAG DIRIG *ù* 1 KÙŠ BA.L[Á][14] GAM-*ma*
>
> A cellar. $1^0 40'$ the length. The 7th part of that which the length over the width goes beyond, and 1 KÙŠ diminishing: the depth.

In No 29, the words *ù* 2 KÙŠ, "and 2 KÙŠ", and thus the word *ù*, are clearly additive/aggregative. If this understanding is transferred to the parallel formulation in No 30, the aggregation brings into play a *quantity to be subtracted.* Since the expression *ana* GAM 1 KÙŠ BA.LÁ (or 1 KÙŠ *imṭi*, according to BM 13901 No 10) was available, no apparent stylistic reasons enforce the particular construction used, and we may thus think of the two expressions as really reflecting the idea that the "normal role" of a number if aggregated is additive, but that the number may be marked (conceptually or materially) as possessing a subtractive role.

That the marking may indeed have been material is suggested by several passages in the text TMS XVI.[15] Line 8 quotes the statement in the phrase *aš-šum* 4-*at* SAG *na-sà-ḫu qa-bu-ku*, "since 'the fourth, to tear out', he has said to you". This unusual syllabic quotation of a logographically written statement makes it clear that the statement (line 1) [4-*at* SAG *i-na*] UŠ *ù* SAG ZI 45 should be read "the fourth of the width, from length and width to tear out" –

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14 For this crucial correction, see TMB 14.

15 Cf. discussion of revised readings and of the mathematical structure of the text in Høyrup 1990: 299–306.

which again supports the reading of the damaged line 3 [50] *ù* 5 ZI ⌜GAR⌝ . . . as "50 and 5 to tear out pose".

"Posing" (*šakānum*/GAR) is a term which appears to possess several uses in the mathematical texts (cf. Høyrup 1990: 57f.). Common to these seems to be that a numerical value or other entity is taken note of in a calculational scheme or device or written/drawn materially. So, we must presume that the step to "pose" 50 (length+width) and 5 ($^1/_4$ of the width) implies that not only the number 5 but also its subtractive role is recorded.

A similar expression is encountered in line 23, . . . 45 *ta-⟨mar⟩ ki-ma* SAG GAR GAR ZI-*ma*, ". . . 45′ you see, as much as widths pose, pose to tear out" – i.e., the 45′ which result from the preceding calculation is to be recorded as the coefficient of the width together with the subtractive role of the resulting 45′·width.

Considered in isolation, each of these phrases from TMS XVI might be explained away as a stylistic slip or a dittography. Taken together, however, they appear to form a pattern, corroborating the assumption that Babylonian calculators would possess a notion of "numbers with a subtractive role". At the same time, however, they suggest that this role was bound up with material notations, perhaps through the way numbers were inserted into a calculational scheme or represented in a calculational device or similar representation. Nothing suggests that we are confronted with a specific *category* of numbers, say, with an incipient concept of negative numbers.

Instead we are led to the general conclusion that the Babylonian vocabulary for subtraction was somewhat fuzzy, employing a fairly large number of terms to describe only two different operations: Identity-conserving subtraction (*nasāḫum* etc.) and comparison (*eli . . . watārum*, LÁ); but that fixed techniques or calculational schemes were at hand which fully compensated for whatever lack of conceptual precision might follow from the blurred terminology.

Note added in proof: In a recent paper ("The Expressions of Zero and of Squaring in the Babylonian Mathematical Text VAT 7537", Historia Scientarum, 2nd series 1 (1991) 59–62), K. Muroi points to a case of subtraction by tearing-out where diminuend and subtrahend are equal. The result is stated as *ma-ṭi*, stative of *maṭûm*, to be interpreted as "it is missing". If we compare with the expression used in TMS V, section 11 (cf. above), where the subtrahend is told to be "as much as" (*kīma*) the diminuend in a subtraction by comparison, we notice that the ways to indicate a "zero outcome" (certainly not a resulting number zero) agree in semantics with the metaphorical origin of the respective subtractive operations.

Tablets Referred to

AO 6770: MKT II 37 ff.; improved readings MKT III 62 ff.
p. 54.
AO 8862: MKT I 108 ff.
p. 44, 53 f.
BM 13901: MKT III 1 ff.
p. 44; 44, fn. 2, 45 ff., 49 f., 53, 56, 58.
BM 8519: MKT I 142 ff.
p. 51, fn. 8.
BM 85196: MKT II 43 ff.
p. 54.
BM 85200+VAT 6599: MKT I 193 ff.
p. 44, 58.
IM 52301: Baqir 1950.
p. 51, fn. 8; 53.
IM 52916: Goetze 1951.
p. 53.
TMS V: TMS 35 ff.
p. 50, fn. 4, 7; 53, 59.
TMS VI: TMS 49 ff.
p. 50, fn. 7.
TMS XVI: TMS 91 f.
p. 58 f.
VAT 6546: MKT I 268 f.
p. 54.
VAT 7532: MKT I 294 ff.
p. 55.
VAT 7535: MKT I 303 ff.
p. 55.
YBC 4608: MCT 49 ff.
p. 53.
YBC 4662: MCT 71 ff.
p. 53 f.
YBC 4663: MCT 69 ff.
p. 57.
YBC 4668: MKT I 420 ff.; III 26.
p. 56.
YBC 4673: MKT III 29 ff.
p. 56, fn. 12.
YBC 4675: MCT 44 ff.
p. 54.
YBC 4695: MKT III 34.
p. 57, fn. 13.
YBC 4708: MKT I 389 ff.
p. 57.
YBC 4710: MKT I 402 ff.
p. 56.
YBC 4711: MKT III 45 ff.
p. 57, fn. 13.
YBC 4714: MKT I 487.
p. 56.
YBC 9852: MCT 45.
p. 54.

Bibliography

T. Baqir, 1950 = Another Important Mathematical Text from Tell Harmal, in: Sumer 6, 130–148. – J. Høyrup, 1985 = Babylonian Algebra from the View-Point of Geometrical Heuristics. An Investigation of Terminology, Methods, and Patterns of Thought. Second, slightly Corrected Printing, Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science. – Høyrup, 1990 = Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought, in: AoF 17, 27–69, 262–354. – Høyrup, 1990a = On Parts of Parts and Ascending Continued Fractions, in: Centaurus 34, 293–324. – K. R. Nemet-Nejat, 1993 = Cuneiform Math. Texts as a Reflection of Everyday Life in Mes. (= AOS 75). – O. Neugebauer, 1957 = The

Exact Sciences in Antiquity. 2nd Edition. – T. E. Peet, 1923 = The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary. – M. A. Powell, 1972 = Sumerian Area Measures and the Alleged Decimal Substratum, in: ZA 62, 165–221. – V. Scheil, 1905 = Kudurru de l'époque de Marduk Apal Iddin (1129–1117), in: MDP 6,31–39. – TMS = E. M. Bruins/M. Rutten, Textes mathématiques de Suse (= MMAI 34, 1961). – A. A. Vajman, 1961 = Šumero-vavilonskaja matematika. III–I tysjačeletija do n.e. – St. D. Walters, 1970 = Water for Larsa: An Old Babylonian Archive Dealing with Irrigation (= YNER 4).

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"'Remarkable Numbers' in Old Babylonian Mathematical Texts. A Note on the Psychology of Numbers". *Journal of Near Eastern Studies* **52** (1993), 281–286.

"REMARKABLE NUMBERS" IN OLD BABYLONIAN MATHEMATICAL TEXTS: A NOTE ON THE PSYCHOLOGY OF NUMBERS

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Dedicated to Dirk J. Struik,
On the Occasion of His
Hundredth Birthday Anniversary

In two publications written in the last ten years, the authors have investigated Babylonian texts containing pseudo-empirical, yet conspicuously concocted, numerical information: in one, Francesca Rochberg interpreted the "stellar distances" of the Middle Babylonian Hilprecht text HS 229 (copied from an older original) as resulting from a play with the largest entry in the standard table of reciprocals (**1,21**) and with irregular numbers.[1] In the other, Dwight W. Young investigated regnal and dynastic spans of the Sumerian King List and pointed to the apparent importance of square numbers, sums of square numbers, and products involving **7**, **11**, and **13** (viz., **70**, **77**, and **91**).[2] In both cases, the authors referred to the corpus of standard tables and selected mathematical problem texts as evidence for an actual interest in the particular numbers involved.

As a basis for similar investigations, it may be useful to look more systematically at the category of "remarkable numbers" which can be derived from mathematical texts.[3] (The standard table of reciprocals is well known and already organized, so I shall not discuss that.)

Upon closer investigation, the category seems to be made up of several separate divisions.

A Key: BM 13901

Mathematical texts contain many numbers. Most of these are evidently the accidental outcome of calculations or result from systematic variation of the data. For our purpose, these numbers tell us nothing. Only numbers which have come about by deliberate, direct choice do so.

Such numbers are of two types: firstly, those numbers which stand as solutions of mathematical problems. Problems were indeed constructed backwards from known solutions, and these were thus subject to no constraints. Secondly, there are numbers which are used

[1] "Stellar Distances in Early Babylonian Astronomy: A New Perspective on the Hilprecht Text (HS 229)," *JNES* 42 (1983): 209–17.

[2] "A Mathematical Approach to Certain Dynastic Spans in the Sumerian King List," *JNES* 47 (1988): 123–29.

[3] That such an investigation should be undertaken was proposed long ago by Wolfram von Soden in his review of E. M. Bruins and M. Rutten, *Textes mathématiques de Suse,* Mémoires de la Mission Archéologique en Iran, vol. 34 (Paris, 1961), in *BiOr* 21 (1964): 44–50, here p. 47b.

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(XVI A): $(u+s)-\frac{1}{4}s = 45$
(XVI B): $(u-s)+\frac{1}{4}s = 15$
(VII A): $\frac{1}{7}(u+\frac{1}{4}s)\cdot 10 = u+s$
(VII B): $\frac{1}{7}(u+\frac{1}{4}s)\cdot 11 = (u+s)+5$

(u stands for uš, s for sag, i.e., for the length and width of a rectangle; invariably, u is **30** and s is **20**). In all cases, we see, the **4th** part plays a role; in text VII, addition of either $\frac{1}{4}s$ (= **5**) or **5** serves to produce numbers of which $\frac{1}{7}$ or $\frac{1}{11}$ can be taken.

Susa text V[6] presents us with a sequence of composite expressions, either multipliers or parts. With slight variations which demonstrate that repetition is not a mere routine and at the same time with so much uniformity that the numbers involved are seen not to be accidental, the text lists the following factors repeatedly:

2, 3, 4, ⅔, ½, ⅓, ¼, ⅓ of ¼, ⅐, 2 times ⅐, ⅐ of ⅐, 2 times ⅐ of ⅐, 1/11, 2 times 1/11, 1/11 of 1/11, 2 times 1/11 of 1/11, 1/11 of ⅐, 2 times 1/11 of ⅐.

A few times, even **⅔ of ½ of ⅓ of 1/11 of ⅐** and **2 times ⅔ of ½ of ⅓ of 1/11 of ⅐** turn up. The partitive domain, within this text, is thus represented by the "simple fractions" ⅔, ½, ⅓ (which all possess their individual sign), by the higher fractions ¼, ⅐, and 1/11, and by their composites. The possibility to expand 1/11 **of** ⅐ without intermediate steps into ⅔ **of** ½ **of** ⅓ **of** 1/11 **of** ⅐ also demonstrates that ⅔, ½, and ⅓ do form a category of their own.

The numbers **5** and **10**, as we see, are absent from the list of factors and denominators.

Extending the Multiplicative-Partitive Domain

Rochberg's argument presupposes that the list of remarkable numbers does not stop at **7** and **11** but includes also higher irregular numbers. In fact, it does. Higher irregular numbers are favorite tools for creating complex composite second-degree problems.

One example is found in Susa text IX, section C,[7] which can be translated into the following system of symbolic equations (remembering that products between linear entities represent geometrical areas):

$$u\cdot s+u+s = 1 \qquad \tfrac{1}{17}(3u+4s)+s = 0;30$$

In the structurally similar problems from VAT 8520,[8] the linear conditions are, respectively,

19. uš ù sag 5 dirig [. . .]

Alternatively, the sign read *ki* in line 2 could be a defectively writtén *ki* followed by *ma*.

Text VIII, like text VII, involves the **4th** and the **7th** parts.

[6] Bruins and Rutten, *Textes mathématiques de Suse*, pp. 35–51. Bruins claims (p. 36) that the text displays a notation for general fractions. Since the notation becomes highly ambiguous when used generally, this position is untenable; what we see are actually abbreviated writings of composite fractions composed precisely from "remarkable fractions"—and no genuine notation at all. See my article "On Parts of Parts and Ascending Continued Fractions," *Centaurus* 33 (1990): 293–324.

[7] See Bruins and Rutten, *Textes mathématiques de Suse*, p. 64.

[8] *MKT*, vol. 1, pp. 346 f.

in "algebra" problems involving the solutions in complex relationships from which the act of solving the problems must extricate them.

A good starting point is provided by the tablet BM 13901, which is also referred to by Young and which contains a collection of "algebraic" problems concerned with one or more squares.[4] Problem no. 7, for example, adds **7** times the side and **11** times the area of an unknown square; nos. 9–10 make the sides of one unknown square larger or smaller than the side of another by $\frac{1}{7}$. No. 17 has the sides of three squares in continuous proportion $\mathbf{1:(\frac{1}{7}):(\frac{1}{7})^2}$.

A global inspection of the tablet demonstrates, firstly, that occurrences of the numbers **7** and **11** invariably involve them as factors or as denominators of parts. When the amount by which the side of one square exceeds the other is given in linear measure, it is **5** [nindan] (nos. 14, 24), **10** [nindan] (no. 18), or **2;30** [nindan] (no. 24). Secondly, **7** and **11** turn out not to be alone in the multiplicative-partitive role: no. 13 is a two-square parallel to no. 17, in formulation as well as method, but tells that "the side is one-fourth of the (preceding) side" instead of "the side is one-seventh of the (preceding) side."

The numbers **4**, **7**, and **11** are used to produce what we may term "complex variants" of basic problem types. A distinct class within the partitive domain is used to design "simple variants": while no. 1 adds the area and the side of the square and no. 2 subtracts the side from the area, no. 3 removes **one-third** of the area and adds **one-third** of the side, while no. 4 removes **one-third** of the area and adds a side; no. 16, which even for other reasons seems to have been displaced and to have belonged with the early, simple problems, subtracts **one-third** of the side from the area.

"Simple variation" (regarding the multiplicative aspect taken in isolation) is probably also involved when, in no. 14, the second side is **two-thirds** of the first plus **5** [nindan], and when, in no. 24, the second is **two-thirds** of the first plus **5** [nindan], and the third **half** of the second plus **2;30** [nindan]. (Evidently, the combination of simple multiplicative variation with additive variation engenders complexity).

All in all, analysis of the selection of given numbers suggests the existence of two distinct classes of remarkable numbers, the former of which falls into two subclasses: the complex multiplicative-partitive domain, the simple partitive domain, and the linear-additive domain. The existence of these classes and their separate roles is amply confirmed by numerous other texts.

Some Susa Texts

Susa texts XVI and VII[5] are didactical explanations of equations:

[4] See O. Neugebauer, *Mathematische Keilschrift-texte* (Berlin, 1936–37), vol. 3, pp. 1–5 (hereafter *MKT*). A complete analysis appears in my "The Old Babylonian Square Texts BM 13901 and YBC 4714: Retranslation and Analysis" (forthcoming). On the character of Old Babylonian "algebra," see my article "Algebra and Naive Geometry," *Altorientalische Forschungen* (*AoF*) 17 (1990): 27–69, 262–354.

[5] Both texts are transliterated with many misreadings and fanciful misinterpretations in Bruins and Rutten, *Textes mathématiques de Suse*. A revised transliteration and interpretation of text XVI can be found in my article "Algebra and Naive Geometry," pp. 299–306. The statements of text VII should read

1. *rebat*^{at} sag *a-na* uš daḫ 7-⟨*ti-*⟩*šu a-na* 10 [*al-li-ik*]
2. *ki*$^{!}$-⟨*ma*⟩ UL.GAR uš *ù* ⟨sag⟩ [. . .]

and

17. *rebat*^{at} sag *a-na* uš daḫ 7-*t*[*i-šu*]
18. *a-di* 11 *al-li-ik* ugu$^{!}$ [UL.GAR]

$$i-6\cdot\frac{1}{13}(i+j) = 0;30$$

and

$$6\cdot\frac{1}{13}(i+j)-j = 0;20$$

(i and j represent *igûm* and *igibûm,* a pair of numbers belonging together within the table of reciprocals—in the actual case **1;30** and **0;40**).

If we go to the series text YBC 4714,[9] which, like BM 13901, deals with "the ('algebraic') study of square and squares," we find the **7th** part (nos. 4, 7, 12, 13, 17, 18), the **17th** part (no. 6), and the **11th** part (nos. 7, 12).[10] YBC 4695[11] has, apart from the simple parts (and in composite expressions $\frac{1}{5}$ and $\frac{1}{6}$) $\frac{1}{7}$ (nos. 1 ff., 66 ff., 80 ff., 85, 86 ff.), $\frac{1}{11}$ (nos. 26 ff., 51 ff., 80 ff.), $\frac{1}{13}$ (no. 86 ff.), and $\frac{1}{19}$ (90 ff.).[12] The rather damaged series text YBC 4697[13] uses at least $\frac{1}{7}$, $\frac{1}{11}$, $\frac{1}{13}$, and $\frac{1}{19}$, and the equally damaged YBC 4711[14] at least $\frac{1}{11}$ and $\frac{1}{13}$, etc. All in all, the material makes clear that problems concerning the sides of squares and rectangles were constructed in such a way that the nth part of something could be taken (often, this nth part would be **5** or **10**), where n is one of the numbers **7**, **11**, **13**, **17**, and **19**; the number **4** being sexagesimally regular, no special care needed to be taken in order to make possible the formation of $\frac{1}{4}$ of occurring entities, but the parallel formulations and other observations make it clear that **4** belonged to the same group.

BM 13901, nos. 7, 13, and 17, suggests that the numbers which were considered "remarkable" in the complex-partitive domain were regarded in the same way in the multiplicative domain. In the absence of parallels in other texts to precisely these problems, we cannot know whether this observation can be generalized.

None of the texts which make **4**, **7**, **11**, **13**, **17**, and **19** stand out as distinctive in the partitive domain make use of them as addends of subtrahends. Obviously, their distinctiveness was not general but bound to a specific role.

The Linear-Additive Domain

As for the selection of linear-additive contributions, the predilection of BM 13901 for **5** and **10** is indeed amply confirmed. Most probably, however, this should not be interpreted as a direct expression of a particular fondness for these numbers as differences: even problems which do not refer to additive contributions, in fact, presuppose sides of squares to be multiples of **5** (or **0;5**). As far as lengths and widths of rectangles are concerned, these tend to be **30** and **20** (or **0;30** and **0;20**). If this is so, most of the operations which are close at hand will produce excesses, etc., which are **5** or **10** (but they may, as in BM 13901, produce an excess **2;30**). It is thus not in the actually occurring

[9] Ibid., pp. 487–92.

[10] I do not count the occurrences of $\frac{1}{7}$, $\frac{1}{6}$, $\frac{1}{5}$, and $\frac{1}{4}$ within the sequence nos. 21–28, where **5** is expressed systematically as a fraction of the members of the sequence **35**, **30**, **25**, **20**; since they result from application of a different principle, they do not indicate a deliberate choice.

[11] *MKT,* vol. 3, pp. 34–36.

[12] That a specific fraction is represented in nos. n ff. means that the following problems presuppose an expression defined in no. n tacitly.

[13] *MKT,* vol. 3, pp. 40 f.

[14] Ibid., pp. 44–46.

linear-additive contributions that we shall seek the expression of numerical predilections. Instead, we may notice that the authors of Babylonian mathematical texts would take great care to formulate problems so that *two* conditions were respected. Firstly, the resulting linear values should be multiples of **5** or **10** (they are mostly to be found between **15** and **35**); these were the values from which the problems were constructed backwards. Secondly, these entities had to be combined arithmetically in such a way that "remarkable parts" could be taken.

It will be observed that many of the problems which deal with more than two squares have the sides of these in arithmetical progression. This reminds us of another favorite practice of the Old Babylonian mathematicians: to deal with squares inscribed concentrically in squares;[15] even in this case, in fact, the distance (*dikšum* or *messētum*) between the squares tended to be constant if more than two squares were involved and **5** irrespective of the number of squares involved.

Remarkability with a Purpose?

The selectivity of roles demonstrates that **7**, etc., were not simply *interesting numbers per se:* if such had been the case, we would also expect to encounter rectangles with sides **7** and **11**, squares with the side **13**, etc. Number-psychological universals (if such exist) explain nothing.

Precisely in the partitive domain, however, **7**, **11**, **13**, **17**, and **19** are particular. They are not only irregular—**14** is so too—but *irreducible to simpler cases* through elimination of sexagesimally regular factors. They are thus—but only within the multiplicative-partitive domain—numbers which by necessity must be dealt with as they are and not through some indirect scheme; more clearly, that as reducible numbers such as **8**, **9**, and **14**, they represent a *number in general;* **4**, for its part, can be regarded as the first "non-small" number—note that ⅓, ½, and ⅔, but not ¼, are "simple fractions" and that the list of integer factors of TMS V stops at **4**.

Remarkability—Remarked Since When?

We have no texts demonstrating the existence of "algebraic" interests before perhaps the mid-Old Babylonian era, and there are good reasons to believe that this discipline was introduced into the scribal schools only after the end of the Ur III period.[16] Yet, even if the special status of **7**, etc., has been demonstrated on "algebraic" texts, it does not depend on their "algebraic" substance. What must be kept in mind is that these texts served a double purpose. Firstly, as "algebra," they permitted the display of virtuosity in the solution of problems which were sophisticated but of no practical use (which made them an expression of the much-discussed "scribal humanism"); secondly, and in particular through the complex variants where the "remarkable parts" appear, they were

[15] See J. Friberg, "Mathematik," *Reallexikon der Assyriologie und vorderasiatischen Archäologie*, vol. 7, pp. 531–85, here §5.41.

The connection between the "algebraic" problems about sequences of squares and the interest in the concentric configuration is made explicit in the final part of Susa text V (Rev. II, ll. 33–44, III, ll. 1–15); see Bruins and Rutten, *Textes mathématiques de Suse*, pp. 45–47 (ME.SI.TUM should read *me-se-tum*, against p. 46, n. 1).

[16] Discussed, for example, in my "The Old Babylonian Square Texts."

a pretext for the training of practical computational skill, which, of course, was mandatory for the scribal school. It is here—in the practical use of the sexagesimal place-value system and its third-millennium forerunner[17]—that the special character of the irreducible numbers must have been noticed. We may guess that the entire sequence **7–19** was already singled out as remarkable during the Ur III period, when there was a vast amount of keeping accounts and conversion of units. The number **7** was already singled out in the Fara period, as is evident from the existence of two parallel problem texts dividing a large round measure by **7**,[18] whereas the number **4** may only have been adopted into the group at a time when more theoretical interest materialized, i.e., in the earlier Old Babylonian period.

A Methodological Tool?

Both Rochberg and Young argued for the particular status of certain numbers from a quite restricted selection of texts. The present investigation has shown, firstly, that these few texts are indeed representative; secondly, that the number **4** belongs together with **7**, **11**, **13**, **17**, and **19**; thirdly, and finally, that *within mathematics* the particular status was domain-specific.

The latter restriction does not necessarily imply that the numbers would not achieve a more broadly conceived special status if we look at less mathematical thinking about numbers. In particular, it does not preclude that they may have been used additively or multiplicative-additively in the construction of the King List and the list of stellar distances. That **7** did, in fact, acquire a much more widely recognized special status within Babylonian and related cultures is well known.

A particular observation to be made on the list of stellar distances may speak for the approach. The stellar distances of HS 229 are **19**, **17**, **14**, **11**, **9**, **7**, **4**—all to be divided by **1,21**, their sum and indeed the largest number occurring in the table of reciprocals. In order to explain that **13** has been replaced by **9** and **4**, Rochberg has to argue from the visual pattern of the final digits which is brought about by this splitting: **9-7-4-1-9-7-4.** Without denying the relevance of this observation, we may now point out that **4** does, in fact, belong among the "remarkable numbers" and seems to be much more important than **13**, according to the statistics of occurrences. (The presence of **14**, on the other hand, becomes less evident.)

In spite of the all-too-often demonstrated dangers inherent in playing around with numbers, insights into Mesopotamian pseudo-empirical numerology might be gained from what the construction of mathematical texts tells about the status of numbers. We probably possess no better source for the Babylonian psychology of numbers.

[17] See, in particular, M. Powell, "The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics," *Historia Mathematica* 3 (1976): 417–39. Through analysis of computational errors of some Sargonic school tablets, Powell establishes, beyond a doubt, that the scribal school students of around 2200 B.C. were already using the fundamental tricks of the place-value system, yet still without possessing an adequate representation and therefore regularly mixing up different orders of magnitude.

[18] Ibid. and my article "Investigations of an Early Sumerian Division Problem, c. 2500 B.C.," *Historia Mathematica* 9 (1982): 19–36.

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"Mathematics and Early State Formation, or, the Janus Face of Early Mesopotamian Mathematics: Bureaucratic Tool and Expression of Scribal Professional Autonomy". *Filosofi og videnskabsteori på Roskilde Universitetscenter.* 3. Række: *Preprints og Reprints* 1991 nr. 2.

TIMETABLE

3500		
	Uruk V	
3300		
	Uruk IV	
3100		
	Uruk III/Djemdet Nasr	
2900		
	Early Dynastic I	
2750		
	Early Dynastic II	*Gilgameš*
2600		
		Fara
	Early Dynastic III	
2350		
	Akkadian period	
2200		
		Gudea
2100		*Šulgi*
	Ur III	
2000		
	Old Babylonian period	
1750		*Hammurapi*
1600		

MATHEMATICS AND EARLY STATE FORMATION
or
THE JANUS FACE OF EARLY MESOPOTAMIAN MATHEMATICS: BUREAUCRATIC TOOL AND EXPRESSION OF SCRIBAL PROFESSIONAL AUTONOMY

By Jens Høyrup

Kenneth O. May in memoriam

Contents

Preface

The following essay is the result of an invitation to present »something Babylonian« at the symposium »Mathematics and the State« at the XVIIIth International Congress of History of Science, Hamburg/Munich, 1st-9th August 1989. I took advantage of the opportunity to attempt a synthesis of a number of approaches to the »anthropology« of Mesopotamian mathematics, each concentrating on specific aspects, in which I have engaged myself at various occasions during the last decade. Evidently I have made no effort to repeat everything which I have said at these earlier occasions on the subject; on the contrary, the attempt at synthesis has led me to change quite a few formulations and to shift the emphasis at certain points. Furthermore, of course, new epigraphic and archaeological material as well as new interpretations of familiar sources have come up during the 1980es. I will certainly not be aware of everything, especially not outside the domain of mathematical texts; none the less, what has come to my knowledge since 1980 weighs heavily at several points.

Of special importance has been the series of Berlin Workshops on Concept Development in Babylonian Mathematics (four to date). As it will be clear from the references, the synthesis draws extensively on work done by the members of this workshop, in particular on the works of Peter Damerow, Robert Englund, Jöran Friberg, Hans Nissen and Marvin Powell. It is a pleasure for me to express my gratitude to all of them for inspiration, discussions and invaluable information. I am also thankful to Denise Schmandt-Besserat for her constant efforts to keep me oriented on her results by means of offprints; to Michael Boakye-Yeadom, Pernille Jensen, Charlotte Justesen, Lucca Weis Kalckar, Morten Hjort Mikkelsen and Carsten Smith Petersen, who gave me the occasion to supervize a student project on state formation theory and state formation in early Mesopotamia in the Spring term 1989; and (as so often!) to the staff of the interlibrary service of Roskilde University Library, without whose kind and effective assistance I would never have been able to engage in Mesopotamian studies.

Special thanks are due to Herbert Mehrtens and Walter Purkert, organizers of the symposium »Mathematics and the State«. Had it not been for their invitation to the symposium I would certainly not have undertaken anything as venturesome as a global analysis of the relation between Mesopotamian mathematics and the social and cultural forces moulding and moulded by the early Mesopotamian state.

I dedicate the work to the memory of Kenneth O. May, who in 1974 commented upon my first amateurish attempt at broad historical syntheses that although he agreed with my general thesis and found the generalizations plausible, what was needed was specific examples in which the interactions between mathematics and other phases of culture was »*traced out* and *verified in detail*« (his emphasis). I hope the present work would have been to his taste.

I. Mathematics and the early state

In his famous and somewhat notorious book on »Oriental Despotism«, Karl Wittfogel [1957:29f] presented a simple thesis connecting the first development of mathematics and astronomy with the rise of the early »Oriental« state —*viz* that the state was »hydraulic«, i.e., developed in order to plan large-scale irrigation, and that mathematics and mathematical astronomy were created for that purpose:

(A) The need for reallocating the periodically flooded fields and determining the dimension and bulk of hydraulic and other structures provide continual stimulation for developments in geometry and arithmetic. [...] Obviously the pioneers and masters of hydraulic society were singularly well equipped to lay the foundations for two mayor and interrelated sciences: astronomy and mathematics.

As a rule, the operations of time keeping and scientific measuring and counting were performed by official dignitaries or by priestly (or secular) specialists attached to the hydraulic regime. Wrapped in a cloak of magic and astrology and hedged with profound secrecy, these mathematical and astronomical operations became the means both for improving hydraulic production and bulwarking the superior power of hydraulic leaders.

This thesis is in fact widely held, though often in less outspoken and rigid form. As also observed by Wittfogel, it was already proposed by Herodotos to explain the presumed Egyptian origin of geometry. My reason to take Wittfogel's very explicit statement as my starting point is that it exposes the problematic nature of the conventional thesis so clearly. If we concentrate on Mesopotamia, Wittfogel is wrong on all factual accounts (Egypt would come out no better):

— Irrigation systems only became a bureaucratic concern (and then only in certain periods) many centuries after the rise of statal bureaucracy (which took place in the later fourth millennium[1]). No doubt the

[1] The date is B.C., of course, like all dates in the following. And approximate, like

irrigation economy provided the surplus needed to feed the bureaucracy; but it was taken care of locally, and often by kin-based communities (as it often is even in today's Iraq)[2].

— Old Babylonian mathematical texts (c. 1700 B.C.) deal with construction of irrigation works, but only with the need for manpower, the wages to be paid, and the volume of earth involved. The dimensions of the constructions were not determined mathematically.
— Neither the sacred nor the secular calender were ever involved in irrigation planning in Mesopotamia.
— Mathematical astronomy was only created almost 3000 years after the rise of the state, and was concerned with the moon and the planets, i.e., irrelevant for irrigation planning.
— Even astrology is a late invention. Only in the first millennium are bureaucratic computation and occult endeavours of any sort connected through a common group of practitioners.

The easy version of the connection between the rise of the state and the development of mathematics (in Mesopotamia and elsewhere) is thus an illusion. In order to approach the issue in a profitable way we will have to ask some apparently trite questions: what *is* a state, and what *is* mathematics—if we are to discuss the two entities in the perspective of the Bronze rather than the Atomic Age.

II. The early state, and its origin

In his book, Wittfogel points [*ibid.*, 383-386] to two classical approaches to the problem of early state formation—both due to Friedrich Engels. Engels summarizes the thesis of *Die Ursprung der Familie, des Privateigentums und des Staats* as follows (MEW XXI, 166f):

all dates below!

[2] See, e.g., [R. McC. Adams 1982], and [C. C. Lamberg-Karlovsky 1976:62f].

(B) Da der Staat entstanden ist aus dem Bedürfnis, Klassengegensätze im Zaum zu halten, da er aber gleichzeitig mitten im Konflikt dieser Klassen entstanden ist, so ist er in der Regel Staat der mächtigsten, ökonomisch herschenden Klasse, die vermittelst seiner auch politisch herrschenden Klasse wird und so neue Mittel erwirbt zur Niederhaltung und Ausbeutung der unterdrückten Klasse.

In *Anti-Dühring*, on the other hand, he considers the state as »Verselbständigung der gesellschaftlichen Funktion gegenüber der Gesellschaft« which then, as the opportunity presented itself, changed from servant to master, be it »als orientalischer Despot oder Satrap, als griechischer Stammesfürst, als keltischer Clanchef u.s.w.«, but where it shall still be remembered that »der politischen Herrschaft überall eine gesellschaftsliche Amtstätigkeit zugrunde lag« (MEW XX, 166f).

Both points of view are present in the standard references of modern political anthropology. According to Morton Fried's *Evolution of Political Society*, the state arises as »a collection of specialized institutions and agencies, some formal and others informal, that maintain an order of stratification« [Fried 1967:235], where a »stratified society« itself is understood as one »in which members of the same sex and equivalent age status do not have equal access to the basic resources that sustain life« [*ibid.*, 186]—i.e., in a generalized sense, a class society. Elman Service, on the other hand, sees statal organization as the result of a quantitative and often gradual development from »relatively simple hierarchical-bureaucratic chiefdoms, under some unusual conditions, into much larger, more complex bureaucratic empires« [1975:306]. The chiefdom itself is a hierarchical organization legitimized by social *functions* wielded by the chief for common benefit (according to Service mostly functions of a redistributive nature) in a theocratic frame of reference, where »economic and political functions were all overlaid or subsumed by the priestly aspects of the organization« [*ibid.*, 305].

Another oft-quoted contributor to the general debate should be singled out for relevance for the following. Robert Carneiro, arguing (1981:58) that »what a chief gets from redistribution proper is esteem, not power«, observes [*ibid.*, 61] that

(C) As long as a chief merely returns everything he has been handed, he gains nothing in wealth or power. Only when he begins to keep a large part of it, sharing with his retainers and supporters but not beyond that, does his power begin to augment.

> But the power of a chief to appropriate and retain food does not flow automatically from his right to collect and redistribute it. Villagers freely allow a chief to equalize each family's share of meat or fish or crops through redistribution because they benefit from it. But they will not willingly suffer the same chief to keep the lion's share of food for himself. Before doing this, he must acquire additional power, and that power must come from some other source.

Power, then, depends on the ability of the chief to transform redistribution proper (where the chief retains only a small percentage of what passes through his hands) into *tribute* or *taxation*, where he keeps a large part for himself and for the »core of officials, warriors, henchmen, retainers, and the like who will be personally loyal to him and through whom he can issue orders and have them obeyed« [*ibid.*, 61]. The origin of this transformation Carneiro sees in *warfare resulting from population pressure*. Warfare is the reason that early class societies consist of *three* and not just two classes [*ibid.*, 65]:

(D) The two classes that are added to a society as it develops are a lower class and an upper class, and the rise of these two classes is closely interrelated. The lower class [...] consists initially of prisoners who are turned into slaves and servants. At the same time, however, an upper class also emerges, because those who capture and keep slaves, or have slaves bestowed upon them, gain wealth, prestige, leisure and power through being able to command the labor of these slaves.

Even though considering the transition »from autonomous villages, through chiefdoms and states, to empires« as a continuous process [*ibid.*, 67], Carneiro finally finds it useful to distinguish the state [*ibid.*, 69, quoting *idem* 1970:733] as

(E) an autonomous political unit, encompassing many communities within its territory and having a centralized government with the power to draft men for war or work, levy and collect taxes, and decree and enforce laws.

Though illustrated by references to ethnographic and historical material, the theories cited here are *general theories*. During the last 15 to 20 years they have been tried out by specialists on a large number of single cases, which has provided many insights into the applicability of the concepts involved and into the historical variability of the diverse processes to which the theories make appeal. It would lead too far to discuss them in general[3], and I shall only quote two of special relevance for the Mesopotamian case. Firstly, in a discussion of Archaic Greece Runciman [1982:351] distinguishes »the emergence of a state from nonstate or stateless forms of social organization« by »these necessary and jointly sufficient criteria«:

(F) Specialization of governmental roles; centralization of enforceable authority; permanence, or at least more than ephemeral stability, of structure; and emancipation from real or fictive kinship as the basis of relations between the occupants of governmental roles and those whom they govern.

Secondly, working on Mesopotamian and Iranian material Henry T. Wright and Gregory A. Johnson [1975:267] formulate a description focusing »on the total organization of decision-making activities rather than on any list of criteria«, defining a state

(G) as a society with specialized administrative activities. By »administrative« we mean »control«, thus including what is commonly termed »politics« under administration. In states as defined for purposes of this study, decision-making activities are differentiated or specialized in two ways. First, there is a hierarchy of control in which the highest level involves making decisions about other, lower-order decisions rather than about any particular condition or movement of material goods or people. Any society with three or more levels of decision-making hierarchy must necessarily involve such specialization because the lowest or first-order decision-making will be directly involved in productive and transfer activities and second-order decision-making will be coordinating these and correcting their material errors. However, third-order decision-making will be concerned with coordinating and correcting these corrections. Second, the effectiveness of such a hierarchy of control is facilitated by the complementary specialization of information processing activities into observing, summarizing, message-carrying, data-storing, and actual decision-making. This both enables the efficient handling of masses of information and decisions moving through a control hierarchy with three or more levels, and undercuts the independence of subordinates.

Unless »information«, »data-storing« etc. are taken in a rather loose sense, societies traditionally regarded as indubitable states (like Charlemagne's Empire) may well fall outside this definition. But in the Irano-

[3] A large number of case studies and further references will be found in [Claessen & Skalník 1978] and in [Gledhill, Bender & Larsen 1988].

Mesopotamian case the authors succeed in making it operational by means of sophisticated archaeology and through the application of geographical »central place theory«. Furthermore, the specific definition of »control« involved may serve to distinguish the specific character of Irano-Mesopotamian state formation.

Control, indeed, may differ in kind—even control developed to the degree of vertical and horizontal specialization and division of labour described by Wright & Johnson. But if control and decision-making involve intense message-carrying and *data*-storing as the fundament for further decision-making, as was the case in Mesopotamia (cf. below), then some means for accounting and the handling of data must develop together with the state—be it writing and numerical notations, be it something like the Andean *quipu*, be it some third possibility. For this same reason, indeed, »archaeologists like[d] to use "writing" as a criterion of civilization« (roughly synonymous with statal culture), as Gordon Childe pointed out in 1950[4], while at the same time himself pointing to the equally important role of accounting [*ibid.*, 14]. This brings us back to the problem of Mesopotamia.

III. The rise of states in Southern Mesopotamia

The centre of early Mesopotamian state formation was the southernmost part of Mesopotamia (»Sumer«); furthermore, for the whole period which I am going to consider in depth, the essential developments as far as mathematics is concerned took place in the Sumerian and Babylonian South-to-centre—whence the above caption. A description of the pre-historic development, however, cannot be circumscribed meaningfully to this area—already because most of the Sumerian territory was covered by water during the larger part of the prehistoric period, but also because much

[4] [Childe 1950:3]. A recent comprehensive discussion of the connection between state formation, writing and alternatives to writing is [M. Tr. Larsen 1988].

wider areas were involved in parallel developments.

By 8000 B.C., permanent settlements had been established and agriculture and herding had become the principal modes of subsistence, although hunting, food-gathering and fishing remained important subsidiaries well into historical times. Within the single settlements, social stratification may have developed around redistribution—needed precisely because of the combination of several complementary subsistence modes, cf. quotation (C). The single villages, however, were involved in no higher structures of settlement or redistribution—their very ecological localization shows that they were meant to live on their own, apart from participation in long distance trade in obsidian and similar scarce goods. This self-sufficiency holds good even for the rare large settlements like Jericho, level B (7th millennium), with at least 2000 inhabitants, and Çatal Hüyük (6th millennium) with at least 5000, although the internal social organization and stratification will probably have been much more complex here than in smaller settlements[5].

In the sixth to fifth millennium, the paths followed by different parts of the Middle East diverged. In geographically suitable places like the Susiana plain in Khuzestan (Southwestern Iran), larger numbers of settlements can be seen to form interconnected systems, some of them possessing apparently central functions or positions (to judge, *inter alia*, from systematic size differentiations)[6]. In the late fifth millennium, the city Susa had an area of some 10 hectares and was the centre of a system of smaller settlements in Susiana. Central store rooms in what may be a sacred domain have been found in the city, and findings of seals and seal impressions in Susa and a neighbouring small settlement bear witness of controlled delivery of goods from the small settlement to the centre. But no traces of higher level recording or summarizing occur in this archaeological layer [Wright & Johnson 1975:273].

After a setback in population density[7], the »Early Uruk« period (before

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[5] Cf. for this description [Nissen 1983:36-40, 55]; [Mellaart 1978]; and below.

[6] See [Wright & Johnson 1975:269f], and [Nissen 1983:57f].

[7] Disputed by Weiss [1977]. The difference of opinion depends on different estimates for the relative lengths of archaeological periods, again dependent on different

the mid-fourth millennium) brought new expansion. Susa had grown to 13 hectares and was the centre of a three-level settlement system, ceramic ware was distributed from central workshops, and bitumen, chert and alabaster were produced locally for exchange. In the following (»Middle Uruk«) period, the size of Susa doubled to 25 hectares and the city was internally differentiated; the settlement system became four-tiered, there is direct evidence for differentiated levels of administrative control (by means of seals, »tokens« and »bullae«, cf. below), and perhaps already indirect evidence for the distribution of standardized grain rations to institution workers[8]. In the Late Uruk period (3500-3100 B.C.), the trend toward specialization and hierarchical control continued. Now, however, a similar level was reached in Southern Mesopotamia, where Uruk became the dominant centre. Susa, on the other hand, fell behind, and will be less interesting for the arguments of the following[9].

The reason for this development is to be sought in climatic changes, which lowered the water-level in the Gulf by some 3 m after the mid-fourth-millennium and diminished the rainfall in the area [Nissen 1983:58-60]. As a consequence, land which had been covered by salt marshes or had been inundated regularly by the rivers now became available for irrigation agriculture. Until then, settlements in Southern Mesopotamia had been rather few and not part of higher-level systems. Now, however,

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absolute datings. The most recent calibrated radiocarbon datings appear to favour Wright and Johnson [B. D. Hermansen, personal communication].

[8] See [Wright & Johnson 1975:272, 282f] and [Johnson 1975:295-306]. The presumed evidence for ration distribution (the particular »bevelled rim bowl«) has been challenged by Beale [1978]. In proto-literate Uruk (see below), however, the connection between the bowl in question and the delivery of rations is corroborated by its seeming appearance in the pictogram for rations (KU_2). Cf. also [Damerow & Englund 1989:26].

[9] Writing turns up in Susa (and indeed in the Iranian area at large) somewhat after its emergence in Uruk. The idea of writing appears to be borrowed, but the pictographic script itself is independent—while, on the other hand, there is clear kinship but not identity between the »proto-Sumerian« and the Iranian »proto-Elamite« counting and metrological systems. For detailed information I shall only refer to [Damerow & Englund 1989] (including Lamberg-Karlovsky's introduction to that work).

a larger settlement density (larger than anything known before in the Near East) and the creation of a noticeable surplus in agriculture became possible. The city Uruk (as large as 50 hectares in the Late Uruk period, and soon much larger still) became the centre of a 4-tiered settlement structure; the internal productive and administrative organization of the city was highly differentiated, vertically as well as horizontally; huge public works in the form of temple building were performed, workers as well as officials being paid in rations in kind; and outposts in Northern Iraq as well as trading relations to Bahrain were established[10].

The evidence for this development is two-fold. Part of it is made up by the traditional archaeologists' array of settlement and building remains and of other artefacts. Part of it, however, consists of *carriers of meaning:* pictures carved in cylinder seals, on relief vases, etc.; and inscribed clay tablets, first with numbers only (in the pre-literate »Uruk V« stratum, before 3300 B.C.) and then also with pictographic writing (in the »proto-literate« Uruk IV and Uruk III periods, 3300-2900 B.C.).

Even though there is an indubitable continuity from the Late Uruk script to the later Sumerian cuneiform, it is far from completely deciphered; cylinder seals, like all other pictures, are of course always ambiguous and polyvalent. None the less, the combination of these carriers of social and linguistic meaning (and more than that, cf. below) conveys a lot of information not available from earlier periods. Prominent facets of the picture which emerges are these:

— The city (and, as a consequence, the settlement system whose centre it was) was under theocratic control. Its core was made up by a sacred terrain dominated by a number of large temples, which can only have been built because of the existence (and availability to the theocratic rulers) of a large agricultural surplus.
— Part of this surplus was apparently given as tribute—a famous temple vase shows a procession bringing offerings to the city Goddess Inanna (reproduction and discussion in [Nissen 1983:113-115]). But part of it must also have been extracted from labourers working directly on

[10] [Adams & Nissen 1972:17-19]; [Johnson 1975:310-324]; [Nissen 1983:73-116, 132-134; 1986a:330].

Temple domains, many of them most likely enslaved prisoners—apparently the most popular theme of the cylinder seals of high Temple officials shows vanquished and pinioned prisoners watched by a high (supreme?) official and being beaten up more or less explicitly (reproduction of select specimens and discussion in [Nissen 1986:146-148]).

— The ruling group of the city was constituted by the top officials in a hierarchy also encompassing lower officials and craftsmen's and workers' foremen (cf. below, on the »profession list«). All appear to have received rations in kind in some sort of quasi-redistributive system, while at least the highest officials received important allotments of land ([Vaiman 1974:20f]; whether this land was worked by personal servants or slaves or by »public« labourers is unclear).

— Quasi- (or pseudo-)redistributive features were also furthered by the lack of virtually all natural ressources apart from pastures, agricultural land, fish, fowl, reed and clay. All needs apart from these (in particular, i.e., those arising from temple building and equipment and the luxury needs of the governing group) depended on organized import and distribution.

— To keep track of tribute and other deliveries and of the products of public agriculture and herding, and also in order to calculate the rations of officials, workmen and domestic animals, techniques for accounting and computation were developed (details below). In the earliest (»Uruk V«) phase, tablets carrying only numerical/metrological inscriptions and seal impressions of responsible officials were employed. Whether used for accounting, as receipts or as delivery notes they could only be understood by somebody possessing full knowledge of the context of the transaction in question. In the next, terminal phase of the »Late Uruk« period (stratum IV, 3300-3100 B.C.), pictograms are put together with the numbers. Even though there is no full rendition of any spoken language, nor any attempt to render syntax, the tablets could now be used as supports for memory, and to summarize a whole series of transactions while tracking its course—especially because the tablets are written according to a fixed format for single transactions and totals. In the ensuing »Uruk III« or »Jemdet Nasr« phase (c. 3100 to c. 2900

B.C.), these formats grow more complex and more regular[11].

— There is no doubt that the script was developed as an accounting and control device. 85% of all written documents belong to the category of economic tablets. The remaining 15% are made up by »lexical lists«, apparently used for teaching purposes. A »profession list« describing the hierarchy of officials and professions turns up most frequently in the record. Other lists enumerate herbs; trees and wooden objects; dogs; fish; cattle; birds; place-names; vessel-types; and metal objects (see [Nissen 1981]). Literary and religious texts are as absent as monumental inscriptions.

— Nothing in the record suggests that general Temple functions, management of the Temple estate and practical book-keeping were separated. To the contrary, literacy (confined to the sole function of economic control) will probably have been too restricted for any full separation to have taken place (nor has a specific scribal function been identified in the profession list). As to the merging of priestly functions and Temple estate management, precisely the sanctification of originally redistributive functions will have made possible that transformation of redistribution into taxation which might otherwise have been impossible (cf. quotation (C)).

While this much is fairly well-established, other questions remain open—not only because the script is largely undeciphered but also because of the nature of the written evidence. Three open questions are of some relevance for the present study.

First of all, the reach of statal domination is unclear. The profession list as well as the location and immense size of temple buildings tells us that the statal institution *par excellence*, irrespective of our choice of precise

[11] The tablets are never found in the places where they were originally made or used but mostly in rubbish heaps. The relative dating thus relies on paleographic criteria, which, however, seem reliable (see [Nissen 1986a:319-322] for details). Because of the greater complexity and regularity of Uruk III tablets, some of the administrative features ascribed here to the whole proto-literate period may indeed only be fully developed in the later phase.

The organization of text formats and the use of formats as carriers of information are explained and discussed in [Green 1981:348-356].

defining criteria, was the Temple. We know that the Temple bureaucracy had command of a large work force, that these workers as well as a number of officials of varying rank were supplied rations in kind. But we do not know how many of the workers were enslaved, nor whether there existed a stratum of peasants only loosely submitted to the Temple (paying, e.g., a limited tribute in form of temple offerings or perhaps none at all, maybe and maybe not contributing corvée labour)[12]. Temple accountants, after all, recorded transactions which regarded the Temple economy; they were not engaged in social statistics. Evidence from the third millennium suggests that free, kin-based peasant communities will have been an important part of the total social fabric[13].

Secondly, we do not know the real constitution of the bureaucracy. Because we only know it from accounting and glyptics we may be inclined to see it as a suppressive and theocratic yet fundamentally Weberian bureaucracy. Ethnographically, however, this picture is highly improbable, and prosopographic studies of third millennium material has given Marvin Powell [1986:10] the impression that »the entire bureaucracy is knit together by an elaborate system of kinship, i.e., what we would call nepotism and influence«.

Thirdly, the specific organization of urban society, of the total settlement structure (not least concerning outposts like the town Habuba Kabira built in Northern Mesopotamia during Uruk V and then abandoned, and the relations to other administrative centres developing no later than Uruk III) and of most trades and handicrafts is unclear. Were traders Temple officials (in the mid-third millennium, some private venture must be presumed, see [Adams 1974:248])? Were the »chief«, »junior chiefs« and »foremen« of the professions testified in the profession list (see [Nissen

[12] Details of the settlement structure, it is true, suggest that an inner core of settlements (until some 12 km from the city) was bound more strongly to the centre than those farther removed [Nissen 1983:144f]. The outer zone can be surmised not to belong to the Temple estate proper; but we have no means to assure that all land of the inner zone was submitted uniformly to the theocratic system.

[13] For a discussion of the general arguments for the presence of such communities, see [Diakonoff 1975]. [Diakonoff 1969a] is an English summary of his epoch-making investigation of 24th century Lagaš.

1974:12-14]) really members of an all-encompassing hierarchy, or is the organization of the profession list due to the particular and biased perspective of literate Temple bureaucrats? Is the appearance of the »chairman of the assembly« in the profession list an indication that a formerly primitive-democratic assembly of citizens had been subsumed under the Temple hierarchy, or is this just an expression of priestly wishful thinking[14]? Once again, third millennium parallels suggest that the real situation was more intricate than the information which we are able to extract from the written documents.

These conclusions from third millennium parallels may be combined with an observation made by Joan Oates [1960:44-46]: since both the essentials of temple groundplans in Eridu (one of the originally isolated settlements of the extreme south) and many other religious customs exhibit continuity since the fifth millennium, at least the culturally pivotal segment of the Late Uruk state building population appears to be autochthonous. The violent increase in population after the mid-fourth millennium, on the other hand, is probably not to be ascribed to autochthonous breeding alone. Influx of new population segments regimented somehow by the Temple institution (whose organization may have taken over much from the corresponding Susa institution) may have contributed to the creation of the three-class situation described by Carneiro (see quotation (D)): thanks to the surplus extracted from the Temple clients and subjects, the Temple staff could evolve into a new upper class, while the clients and possible enslaved workers made up the new lower class. Non-subject populations (be they autochthonous or immigrants) may have continued a traditional non-state existence with only limited submission to the statal institution[15].

[14] Or ours? Our own bureaucratic conditioning in combination with the internal rationality of the book-keeping records may easily lead us into more Weberian readings of the text than intended by its original authors.

[15] While proto-literate Uruk was a full-fledged state according to Wright and Johnson (quotation G) it is thus far from certain that it would be so according to Carneiro (quotation E) and Runciman (quotation F). From their point of view, the control system will probably have directed not *a state* but only *an estate* immersed into and influencing a pre-state society. Especially for Runciman, who sees early seventh century Athens as a »proto-state« only, the proto-literate Uruk system can have

For the very same reason, however, they will have been out of the administrative focus of the Temple managers. That accounting rationality which, as we shall see, contributed to the formation of mathematics, was only concerned with the relation between the Temple estate and its officials and dependants—and whatever the real complexities of state formation, the written record only reflects the pseudo-redistributive features of the situation.

As long as we restrict ourselves to the proto-literate period alone, however, all talk of the »real complexities« is, firstly, pure speculation, and secondly inane speculation. It is only given sense by the perspective of the following, »Early Dynastic« period (cf. [Diakonoff 1969a:178-180], and [Powell 1978:139]).

IV. City states and centralization

Apart from an initial lacuna of some 200 years in the written record, the source situation improves steadily and significantly during the following millennium. This has several reasons.

— Firstly, the script evolved to the point where it is fairly well understood—both because of changes in the sign repertoire and because of incipient use of syllabic writing. Due to the latter development we even know that the language in use was now Sumerian, while we have no means to decide in which language the pictographs of the proto-literate period were told[16].

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been no more.

[16] Traditionally, it is true, the opposite view has been accepted on preliminary evidence from a single, somewhat ambiguous sign combination in a single text. However, the ongoing progress on a large project on the archaic texts directed by Hans Nissen (see [Nissen 1986b]; the results of the project are reflected in many references in the present paper) has uncovered no supplementary testimony; for this and other reasons discussed by Robert Englund [1988:131-33] in a two-page footnote, we must now opt for a vigorous *nescimus*.

— Secondly, writing was used much more broadly and more systematically. Around the mid-third millennium royal inscriptions, literary texts and political and juridical documents (some of them involving communal and private land) turn up. Even the traditional genre, the economic texts, improves in coherence and systematization.
— thirdly, certain aspects of early third millennium society are reflected in oral epics written down in the second half of the millennium.
— fourthly and finally, on a number of archaeological sites strata from the third millennium cover those from the late proto-literate period, for which reason the latter are poorly known.

The first 500 years after the proto-literate phase are known as the Early Dynastic period (ED). Its first part is characterized by continued population growth—around 2900 B.C. Uruk had grown to 6 km^2, half of Imperial Rome at its culmination—and by further diminishing rainfall and lowering water-level in the Gulf and hence also in the great rivers. Around the mid-third millennium, moreover, a new main branch of the Euphrates was formed. This had decisive consequences, as discussed in some detail by Nissen [1983:141-148]. What is important in the present connection is the development of a system of city states, competing and often at war for the same water resources; and of kingly functions in these city states, formally originating as Temple offices but in reality regents on their own and eager to stand forward in their inscriptions as benefactors and protectors of the temples of their cities and city gods (see the collection of royal inscriptions in [Sollberger & Kupper 1971]).

One of the Sumerian epics offers an interesting insight into the social structure, somewhat at cross purposes with naive identification of State and Temple estate. In *Gilgameš and Agga* (translated in [Römer 1980]) we are told that Agga, son of king Enmebaraggesi, proceeds with his army from Kish to Uruk and delivers an ultimatum. King Gilgameš of Uruk first tries to convince the council of elders of his city to fight back; he fails, and

My present pet hypothesis (the reasons for which I present in [Høyrup 1992]) is that Sumerian shares so many grammatical features with »creole languages« (on which see, e.g., [Romaine 1988]) that it may have originated as a creole at the influx of new population segments in the later fourth millennium.

instead he puts the matter before the council of »men« (capable of bearing arms? or commoners, if the »elders« are elder by status and not by age?), who agree with Gilgameš and entreat the aristocrats and mighty of the city to fight for Eanna, the city's temple established by An the heavenly god and »cared for« by the hero-king.

Most likely, the epic was only committed to writing toward the end of the third millennium; but since Enmebaragesi is a historical person (he has left an inscription, and belongs around the 27th century, in early ED II) the written text must build on fairly stable oral transmissions. Moreover, the conciliar institutions were definitely not as powerful toward the end of the millennium as presupposed by the text. The social situation delineated in the poem must therefore correspond to *some* historical reality.

That, however, is striking. Admittedly, Eanna is mentioned as the pride of the city—but definitely not as supreme owner or overlord. The affairs of the city are taken care of by the king in agreement with the two councils. The whole make-up reminds more of the *Iliad* than of the managerial society intimated by the proto-literate archives. If the higher Temple officials are mentioned (and they probably are!) it is only as rich and powerful »1st class citizens«, i.e., as aristocrats or »elders«.

On the other hand, there is no doubt that the managerial tradition *was* very much alive, as testified by the continued and expanding use of the same script and the same accounting techniques as in Uruk IV-III, and by the persistent use of the familiar lexical lists. We are thus confronted with a truly *dual society*, as suggested above: one aspect can be described with some approximation as that »military democracy« which Engels portrays in *Der Ursprung der Familie*[17]. The other is the formally redistributive, functional state presupposed in his *Anti-Dühring*—and since these two complementary theories anticipate the main approaches of modern political anthropology we may conclude that the disagreements within this field correspond to the dialectic of real state formation[18].

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[17] This aspect has been investigated by Thorkild Jacobsen in several publications [1943; 1957].

[18] Basing himself on other evidence, Nissen [1982] argues for duality of the Sumerian society along several other dimensions.

... At least to *the dialectic of real state formation as it happened in Mesopotamia*. The duality is, indeed, more obvious here than in many other cases (cf. however chapter XII on parallels in Medieval Europe). That is seen, e.g., if one compares the ways in which early Mesopotamian and other ancient monarchs made use of the techniques of literacy, once developed for accounting, to glorify themselves. While most royal inscriptions of the Ancient world boast of prowess and military success, until mid-ED III Sumerian royal inscriptions boast of temple building, of gifts given to the temple, of ceremonies performed, and of canal-building. Early Mesopotamian literacy was thus no transparent medium but a strong ideological filter which would not allow certain utmost important aspects of the kingly function to be seen[19].

Towards the end of the Early Dynastic period the temples and temple estates have come under the sway of the city kings, who treat them as their private property[20]. The existence of communal land is testified by sales contracts, but these always show that the land is sold to private individuals with high social status (high officials, members of the royal family), and often »at a nominal price« ([Diakonoff 1969a:177]; cf. [Powell 1978:136f]). Since peasant clans will in any case only have sold their hereditary land when in distress or when submitted to severe pressure, we may conclude that this was probably the point where a state in Runciman's sense was

[19] In this connection one may also recall the oft-made observation that nobody would have guessed from the written record that Sumerian rulers might be buried with a large retinue of killed servants (as it was actually the case in Ur, during the first phase of ED III).

[20] This is particularly clear in a series of »reform texts« by Uru'inimgina, either elected king of Lagaš by the assembly or usurper in the late 24th century B.C., describing the abuses which had developed and his restoration of good old time, which includes giving back the temple land appropriated by the ruler to the gods (a recent though not fully convincing discussion of the obscure texts and an exhaustive bibliography is [Foster 1981:230-237]; cf. also [Hruska 1973]). But since Uru'inimgina and his consort are to function as stewards of the gods on their reacquired estates, realities did not change at least on this point [Tyumenev 1969a:93f]. Whether his protection of »widows and orphans« fared any better is unclear. In any case, Uru'inimgina was soon brushed aside by Lugalzagesi's conquest and unification of the whole Sumerian region.

establishing itself (cf. above, quotation F, and note 15).

In the mid-24th century, as a next developmental step, the whole Sumerian region was then united into one territorial state by conquering kings, first by Lugalzagesi of Umma and soon afterwards by the Semitic Sargon of Akkad. Powell [1978] sees this as a result of the conflicts arising, *inter alia*, from growing social and political tensions caused by the increase of private large-scale property—tensions which could not be released or held in check within the single city state, in spite of attempts like Uru'inimgina's »social reform«[21]. From now on, the »despotic« territorial state or empire can be regarded as a rule in Mesopotamia and the decentralized phases as interludes.

For reasons of obvious necessity, Sargon and his dynasty introduced more far-ranging social controls than any predecessor, many of them further developments of the traditional accounting controls. Already the Early Dynastic radical transformations of the socio-political structure, however, had led to changes in the domain of written administration. Both phases of the transformation were reflected in the structure and practices of the environment responsible for this administration. The evolution which took place during the Sargonic reign continued trends established during the preceding two centuries while at the same time reshaping them to the advantage of government.

The first step is testified in Fara (Ancient Šuruppak) around the mid-third millennium. Here, for the first time, *the scribes* turn up in the administrative documents as a separate and hierarchically organized group, even provided with overseers and a »senior scribe« [Tyumenev 1969:77]; until then, the very term is absent from the sources—with the exception of one Jemdet-Nasr tablet which shows that the profession is not hidden in one

[21] In fact, the analysis reminds strikingly of Engels' (and Aristotle's) analysis of the Solon reforms in *Die Ursprung der Familie* ... . Even this formation of a mature state in Athens followed upon a phase considered as »military democracy«—and followed shortly after the establishment of a state in Runciman's sense.

That conflicts between the city states became intense in late ED III is obvious from the surviving royal inscriptions. After centuries of mounting city-walls combined with amazing royal taciturnity on warlike matters, proclamations of military triumph and menaces against potential aggressors suddenly abound.

of the uninterpreted lines of the proto-literate profession list.

The reason for the emergence of the profession is probably straightforward: writing itself was used more widely for socially important purposes, apparently in connection with the beginning of the above-mentioned socio-economic transformations of ED III (see [Powell 1978:136f]). It is precisely in Fara that legal contracts, *viz* concerning the sale of land, turn up (see [Krecher 1973, 1974]). In Fara, too, a monetary function becomes visible for the first time (in Fara accomplished by copper, in later ED III by silver). Temple estate accounting, too, grew in extent and systematics. We seem to stand at the threshold dividing »ultra-limited literacy« from »limited literacy«, to use a conceptual distinction proposed by John Baines [1988: 208].

As pointed out by Baines, »limited literacy« is really a *new* situation, with problems and possibilities of its own. First of all this reflects itself in the education of the literate-to-be. Even though the old lexical lists were still in full use (but in decline after Fara), new types of school-texts emerge, as it appears from Deimel's collection ([1923]; on p. 63 we find a student's drawing of the proud teacher); of special importance are the mathematical exercises, to which I shall return below. Finally, the Fara period produced the beginning of *literary texts*, testified by fragments of a temple hymn and by the first proverb collection [Alster 1975:15, 110]. It seems that the scribes, once they had become a profession halfway on their own[22], tried out the possibilities of the professional tools beyond their traditional scope (this will be even more obvious when we come to the mathematical exercises)—and a perusal of the tablets which the Fara scribe students produced suggests that they liked the enterprise: in many of the empty corners of tablets, irrelevant but nice drawings have been made, portraying teachers or deers or featuring complex geometrical patterns. One gets an immediate impression of enthusiasm for the freshness of scholarly activities similar to that reported from Charlemagne's Palace School in Aachen or from Abaelard's and Hugue of Saint Victor's »12th century renaissance«.

[22] Halfway only—many of Deimel's didactical tablets carry names of what seems to be authors, editors or teachers, and many of the persons mentioned carry a priestly title [Deimel 1923:2*f].

The trends beginning in Fara continue during ED III, during the Sargonic era, and during the post-Sargonic decentralized 22nd century interlude. The number of legal contracts of many sorts keeps growing; archives are used on many levels[23]. Systematic school teaching continues, though relatively few records (among which, however, mathematical exercises) survive. Writing becomes more phonetic and orderly already in ED III (Maurice Lambert [1952:76] speaks of an outright *reform of writing* under Eannatum of Lagaš)[24]. Even the creation of literary text continues, though with a change. No longer an expression of semi-autonomous scribal identity, hymns are written in the royal environment where they serve to demonstrate the king's affection for those temple institutions which had been subjected to royal authority, as discussed by William Hallo [1976:184-186]. Sargon's daughter Enḫeduanna may indeed be the first poet in world history known by name. Gudea, the most important ruler of Lagaš during the post-Sargonic decentralized interlude, appears to have had epics composed on command which transposed his own feats into the mythical past. Also in another respect is he seen parading as a culture hero: not only a temple builder in the abstract like the kings of earlier inscriptions, Gudea has drafted the ground-plan himself »in the likeness of Nisaba [the scribal goddess], who knows the essence of counting« (Cylinder A, **19**, 20-21, in [Thureau-Dangin 1907:110]); he has also formed and baked the brick, brought precious materials from foreign countries, and performed all other crucial steps in person. Though the ruler of a city-state similar to those of former times and perhaps conscious of himself as a restorator of the order of old, Gudea no less than the Sargonides represents the tendency to make *inter alia* scribal culture subservient to a fundamentally secular power.

This is no less true in the following centralizing period, the so-called Third Dynasty of Ur or »Ur III« (not to be mixed up with »Uruk III«, a period named after an archaeological stratum in a different city), coinciding

[23] Foster [1982:7-11] distinguishes three Sargonic archive types: family or private; »household« (with a horizon restricted to a single city) and »large household«.

[24] When systematic writing of the Semitic Akkadian began, using the phonetic values of the Sumerian signs, orderly succession of the signs became compulsory.

with the 21st century B.C.[25] The founding king, Urnammu, subdued the whole of Southern Iraq, and undertook large building programs. Since relatively few written documents are known from his time, we have no detailed knowledge of his policies, nor from the first 20 years of his successor Šulgi. At that point, however, Šulgi instituted a military and administrative reform, and from then on huge amounts of administrative tablets exist. They uncover a centralized economy submitted to meticulous control. It is probably *not* true, as has been believed, that *all* land belonged to the state or to temple estates in practice controlled by the state; that *all* industry was governmental; that *all* merchants were *exclusively* government agents; nor that *all* manual work was done by semi-enslaved populations. But the very fact that these theses have been widely held show that royal estates, governmental trade and governmental workshops and even textile factories worked by slaves were all-important[26]. The precise booking of rations, work-days, and of flight, illness and death within the work-force allotted to each overseer also reveals an extremely harsh regime. As pointed out by Robert Englund in his conclusive words [1990:316], the understanding of working conditions conveyed by the administrative texts »kann vielleicht helfen, sich in den historischen Darstellungen des 3. Jahrtausends v. Chr. die Kosten der babylonischen Paläste und Statuen plastischer vorzustellen«.

In this situation, whatever autonomy may have been left to communities and crafts will have been severely restricted. This is demonstrably true for scribal culture. The scribe, of course, was the pivot and, in principle, the hero of an administrative system the precision and scope of which Nikolaus Schneider [1940:4] regarded as »überspitzt« even from his writing perspective within the National Socialist war economy. The scribal title was used as an honorific title of dignitaries in general [Falkenstein 1953:128]. Moreover, in one of the hymns glorifying King Šulgi he also

[25] Brief expositions are given by Nissen [1983:207-213] and Liverani [1988:267-283]. A recent critical survey of the state of the art concerning Ur III administration is given by Robert Englund [1990:1-6].

[26] An overview of the centralized economy as well as the exceptions is given by Hans Neumann [1988]. Cf. also [Neumann 1987:151-154] on non-statal artisanate.

presents himself as »a wise scribe of [the scribal goddess] Nisaba«, a characteristic which stands as the culmination of a long series of images (transl. [Klein 1981:189, 191]):

(H) I, the king, from the womb I am a hero, [...], I am a fierce-faced lion, begotten by a dragon, [...], I am the noble one, the god of all the lands, [...], I am the man whose fate was decreed by Enlil, [...], I am Šulgi who was voluptuously chosen by Inanna [goddess of Uruk], I am a horse, waving its tail on the highway, [...], I am a wise scribe of Nisaba. Like my heroism, like my strength, my wisdom is perfected, its true words I attain, righteousness I cherish, falsehood I do not tolerate, words of fraud I hate!

Looking back at *Gilgameš and Agga* we observe that nothing is left of dual society. The world of kingly prowess and that of scribal administration (identified with wisdom and justice) are united in the same person who boasts on both accounts in the same composition.

The so-called Ur-Nammu law-code, which should in fact carry Šulgi's name ([Kramer 1983]; cf. [Neumann 1989]), shows a similar mixture in its prologue (ed., transl. [Finkelstein 1969:66-68]). At the same time it elucidates the royal idea of justice, which on one hand involves metrological regularization and reform, on the other repeats the nice words (and the details!) of Uru'inimgina and Gudea too much in the manner of a literary *topos* to be really convincing (cf. [Edzard 1974]).

Two other Šulgi hymns [Sjöberg 1976:172f] tell about the king's purported time in the scribal school, and thus make clear which aspects of scribal cunning were central seen from the official perspective (which, we can be fairly sure in a society like Ur III, was also the perspective communicated to the students): addition, subtraction, counting and accounting according to one; writing, field-mensuration and drawing of plans, agriculture, counting and accounting (and a couple of ill-understood subjects) to the other.

Traditional *topoi* and nice hand-writing apart, the idea of *justice* had been reduced to unified metrology and menaces against trespassers of royal regulations, and that of scribal art to functionality within the administrative apparatus. According to all evidence, scribes were taught in school to be proud of their function in the administrative machinery; no more place is left (in the official ideal) to professional autonomy than to communal primitive democracy. The higher level of literary (and, as we shall see below, mathematical) creativity was in all probability the preserve of a »court chancellery« (»Hofkanzlei«, [Kraus 1973:23]) where year names, royal hymns, politically suitable epic poems and royal inscriptions were produced. On all accounts, the scribal art had been harnessed to a no longer dual state—in trite practice in as far as rank-and-file scribes are concerned, as a source for ideology in the case of the elite.

V. Breakdown and apogee

In spite of the immense role played by the scribes in Ur III, the problems associated with »limited literacy« appear to have been solved or suppressed. Scribal autonomous thought, as any autonomy except perhaps nepotism and appropriation of »public« property among the privileged, is absent from the sources. But the cost of bureaucratic control was too high, and the price of extensive building activities and an all-encompassing administrative network was a work-force plagued by illness, death and problems of flight—and even, if we are to believe indirect literary evidence, rebellious strikes[27]. Internal breakdown resulted[28],

[27] After toiling 40 years night and day in the great marsh, the minor gods decide to confront their chamberlain (the god Enlil); they do so, armed with spades and hods to which they have set fire, and claim that the chamberlain call in the collective leadership (consisting of Enlil himself together with the gods An and Enki). When asked for the instigator, the strikers deny the existence of such a person and declare their solidarity—thus begins the plot of the Old Babylonian *Story of Atraḫasīs* (ed., tr. [W. G. Lambert & A. R. Millard 1969]; this passage pp. 45ff). The whole description is too close to the social psychology of real wild cat strikes to have been freely invented, and the setting suggests that the author builds on experience from Ur III estates rather than contemporary events.

In the end, the problem is solved by a »social reform«: *man* is created in order to take over the toil of the gods.

[28] Most likely, ecological reasons were also involved in the breakdown, accentuating the incompatibility between the costs of the state apparatus and the productivity of the work force. In any case, the political centre of Iraq from now on moved

followed by now irresistible barbarian invasions and another interlude of decentralization, the beginning of the »Old Babylonian« period (2000 to 1600 B.C.).

One of the resulting smaller states (Isin) continued the Ur III system as best it could for a century, and has provided us with a school hymn describing the high points of the scribal art as embellished »writing on the tablets« together with use of »the measuring rod, the gleaming surveyor's line, the cubit ruler which gives wisdom«[29], not far from Šulgi's ideals though without his emphasis on accounting. The other main successor state (Larsa) inaugurated a trend which was to culminate during the next phase of centralization, achieved by Hammurapi of Babylon (1793-1750)[30]. On the whole, the system of state-controlled production was abandoned. Royal land was often (though not always) given to tenants instead of being organized as large estates run by servile labour, or it was assigned to officials or soldiers who leased it to farmers. Similarly, land belonging to wealthy city-dwellers was often leased—and in general, private possession of large-scale landed property became common. (The survival of community-owned land is disputed, cf. [Komoróczy 1978] versus [Diakonoff 1971]).

Similarly, public foreign trade was replaced by private trade; at least one major city appears to have been run by the body of merchants with some autonomy [Oppenheim 1967]. Royal workshops had probably been taken over by their managers at the breakdown of the Ur III system, and were now run privately; free labourers working for wage largely replaced the semi-enslaved workers receiving rations in kind. We even observe a kind of banking developing, conducted by members of an institution for unmarried noble-class women using their double kinship affiliation (to the

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northwards.

[29] Lipit-eštar Hymn B, lines 21-23, transl. [Vanstiphout 1978:37].

[30] A very readable narrative not only of Hammurapi's history and policies but also of the socio-political and cultural conditions since early Old Babylonian times is given by Horst Klengel [1980]. Other works to be consulted include [Dandamajev 1971], [Diakonoff 1971], [Gelb 1965], [Jakobson 1971], [Klengel 1974, 1977], [Komoróczy 1978, 1979], [Kraus 1973], [Leemans 1950], [Oppenheim 1967], [Renger 1979], and [Stone 1982].

real kin, and to the pseudo-kin of the institution) to bypass traditional obstacles to free trade in land [Stone 1982].

The activity of the latter institution testifies to the tendency to evade the constraints of communal traditions; it is also, on the other hand, one of many proofs that land—the all-decisive productive asset—was not exchanged on real market conditions (cf. [Jakobson 1971]). Individualism and monetary relations dominated the economy, but capitalism was far away. None the less, the new economic structure caused multiple changes in the socio-cultural sphere.

Firstly, of course, business did not give up accounting and archives just because it was private. On the contrary, these spread to new social circles. Private letter-writing emerged, describing both private business and personal affairs—until then, only official letter-writing had been known. *Seals*, hitherto insignia of officials, became tokens of private identity. And of course, accountants and surveyors in private employment and street scribes writing down the personal letters for pay appeared, as did free-lance priests performing private religious rites.

Secondly, *individualism* itself took shape as a world view, manifesting itself not only in the private seal and the personal letter but also in the religious sphere and in art. While Ur III had consummated the transformation of the ordinary member of the primitive community into a subject of the state[31], the Old Babylonian era made him reappear as a *private man*.

On the other hand, Old Babylonian society was still a royal state. The king was, as during many preceding centuries, the largest estate owner, and directed many affairs while local autonomies when existing were restricted. A new duality had thus evolved, where clearly the »modern« aspect of society was the more vulnerable. Corresponding to the traditional royal aspect of society the ancestral royal ideology also survived, and in fact got its most famous expression precisely in this time: the preface and postface to Hammurapi's »law-code« (translated in [Pritchard (ed.)

[31] This is in fact part of the complaint of the minor rebellious gods in the *Story of Atraḫasīs* (above, note 27). While they were originally the »sons«, i.e. the lower-ranking members of the clan community, and the »chamberlain« thus nothing but the »elder« member governing common affairs, he has now become the master and they the dumb subjects.

1950:164-180]), where the king appears as sort of Bronze Age social democrat, assuring for his country affluence *and* justice. (The details of the text and the king's personality as it can be seen from his letters makes this look somewhat more honest than in Šulgi's comparable text).

The institution which connects this to the development of mathematics is the scribe school[32]. Before discussing the school itself, however, a brief remark should be made about language. Sumerian had been retreating as a spoken language already during Ur III, and maybe centuries before, as can be seen from the increasing dominance of Akkadian names. Official writing, it is true, persisted in Sumerian. In early Old Babylonian times, Sumerian was in all probability a dead language, and all non-scribal business was done in the Babylonian dialect of Akkadian. Official writing, always produced by one scribe and meant to be read by another scribe, was still made with some recourse to Sumerian: at times full and more or (often!) less grammatically correct Sumerian, at times staple Sumerian word signs used as abbreviations within otherwise Akkadian sentences. The Sumerian literary tradition, moreover, was transmitted in the scribal school, though increasingly in bilingual versions.

As to the school itself, its situation reflected that of the general economy. Some schools have been found within palace precincts, and may hence be regarded as official institutions. Others, however, have been located in living areas for scribes; they can hardly have been anything but private enterprise ([Lucas 1979:311f] offers a survey). In both cases, however, the students were trained for similar, »notarial«, accounting and »engineering« functions, i.e., for key positions in general social practice in private or official business[33]. Evidently, the *sine qua non* for any scribe was to master

[32] Two fairly recent presentations are [Sjöberg 1976] and [Lucas 1979]. Older important general discussions are [Falkenstein 1953], [van Dijk 1953:21-27], [Gadd 1956], [Landsberger 1960], and [Kraus 1973:18-45]. Didactical texts illustrating various aspects of the school enterprise have been published and translated by Kramer [1949] and Sjöberg [1972, 1973, 1975].

[33] Mostly in public administration. »Scribes were limited to positions connected with administration or with substantial accumulations of private capital. Perhaps, also, they filled out contracts and legal documents at the gate of the city. If I were to make an intuitive sweeping estimate, I would say that perhaps seventy percent

the practical skills needed to perform these tasks.

Besides these skills, however, future scribes were taught to be proud of their profession. A number of texts have survived which were used in the school to inculcate professional pride. They tell us about the curriculum, but they also tell us which part of the curriculum was central for professional pride. The picture gained from these texts stands in significant contrast to actual scribal functions.

Firstly, indeed, the continuation of the Sumerian tradition beyond Hammurapi's time is, as formulated by Kraus [1973:28], »das größte Rätsel, welches der altmesopotamische Schreiber uns aufgegeben hat«. Scribes had to learn Sumerian because other scribes used Sumerian! Even more paradoxical, scribal school students were expected to speak the dead language with good pronunciation. Tradition alone will not do (though even the survival of traditions requires a motivation on the part of their carriers and hence an explanation), since the scribal school tradition appears to take a fresh start in the early Old Babylonian period (all the texts formulating its ideology belong to the second millennium).

Sumerian simply, however, is not the culmination of the scribal art. According to the »Examination Text A«[34], the accomplished scribe must know everything about bilingual texts; he must know occult writings and occult meanings of signs in Akkadian as well as Sumerian; he must be familiar with the concepts of musical practice, and he must understand the distorted idiom of a variety of crafts and trades. Into the bargain then comes mathematics, to which we shall return. All that, as a totality, has

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of the scribes had administrative positions, twenty percent were privately employed, and the remainder became specialists in the diagnosis of illness, charms, magic, and other activities calling for some knowledge of writing«, as formulated by Landsberger [1960:119] in answer to a question whether the important role played by secret idioms of various crafts in the »Examination Text A« (see below) could correspond to future employment.

Employment outside the »notarial«, accounting and »engineering« sphere was clearly secondary: "A disgraced scribe becomes a man of spells«, we are told by a proverb [Lucas 1979:325].

[34] Ed., transl. [Sjöberg 1975]; cf. [Landsberger 1960:99-101]. Admittedly, the earliest extant copies of the text are quite late (*viz* Neo-Assyrian); as observed by Sjöberg, however, the contents of the text seem to require an Old Babylonian origin.

a name (of course Sumerian): nam-lú-ulù, »humanity« ([van Dijk 1953:23-26]; [Sjöberg 1973:125]).

True enough, the phenomenon has some similarity both to the practice of legalese and to the worst aspects of Modern humanism as a self-aggrandizing device for bureaucrats and court servants. Instead of making analogies, however, we may try to formulate an explanation starting from a more precise analysis of the Babylonian concept itself. We may then notice that *everything has to do with scribal practice,* but scribal practice transposed from the region of practical necessity into that of virtuosity. What appears from other didactical texts is that the scribe is expected to be proud, not of accomplishing his actual tasks but *of his identity and ability as a scribe.*

This connects scribal ideals to both aspects of contemporary general ideology. On one hand, the scribal function *as a whole* was by tradition a public function. If the King was to guarantee affluence and justice, who but the scribe was to do the job? On the other, the scribe was also *an individual,* a *private man.* In order to assure oneself of being something special, a *human being par excellence,* it was of course excellent to stand out as the one who gives the king prudent advice, and this is in fact part of scribal boasting [Landsberger 1960:98]. But there was not much satisfaction in pointing to trite everyday scribal activities, i.e., to the actual ways to »guarantee affluence and justice«. After all, phonetic Akkadian could be written with some 80 cuneiform signs. Everybody would be able to learn that. But everybody would not attain the level of virtuosity. Scribal professional *pride* needed something *really difficult* as its foundation; but the difficulties had to *belong at least formally to the territory of scribal tasks* if it was to serve *professional* pride. This, according to all evidence, is the reason for the specific configuration of Old Babylonian scribal »humanism«, and for its appearance as *art pour l'art.*

Another characteristic of the »examination texts« and related didactical texts should be mentioned before we leave the subject. In contrast to the picture presented by the Fara school texts they always appear to reflect a rather suppressive ambience—ever-recurrent in an early text (known as »Schooldays«) where the school-boy tells his experience of the day are the words »caned me« [Kramer 1949:205]. In »Examination text A« the student stands back as an ignorant dumbfounded by the teacher. Admittedly, it *is* the teacher who speaks through the text. But the double-bind situation which it suggests is still psychologically informative. The message seems to be that the scribe should be proud of being a scribe, but only privately; on service he should be a humble functionary knowing his place. Scribes were to be servants, not rulers and in reality rarely advisors of those in power. The scribe was to keep balance between actual loyalty and personal autonomy. His situation may have been similar to that of a Medieval clerk. Yet Renaissance humanism was as far ahead as capitalism; the Old Babylonian scribe was, after all, closer to the Fara scribe testing for the first time the possibilities of his professional tools than he was to Benvenuto Cellini.

VI. Mathematics

»The state« as a concept turned out to be subject to more dispute than presupposed by Wittfogel, my initial punching ball. What about mathematics?

Nowadays, of course, we know the meaning of the term inside our own world—at least until we are asked about borderline cases like accounting, engineering computation, magic squares or generative grammar. Well within the border we have a cluster of indubitably mathematical practices, disciplines and techniques, cohering through shared use or investigation of abstract, more or less generalized number or space or of other abstract structures.

Many single elements of this cluster can be traced far back in time, and be found in non-literate contexts, often at quite advanced levels. Currently, the term »ethnomathematics« is used about these elements when found in non-literate cultures [M. Ascher & R. Ascher 1986]. It is important to notice, however, that »ethnomathematics«, no less than »mathematics«, is *our* concept. The inhabitants of Malekula in Vanuatu would hardly have recognized the bunch of elements of their culture classified by us as »mathematical« as one entity. Their »kinship group theory« belongs more closely together with the kinship and marriage customs in general than with the drawing of closed patterns, which on the other hand belongs with the relation and passage between life and death[35]. Counting and the geometry of house-building will belong to still other domains.

Non-literate populations visited by modern ethnographers are not identical with the ancestors of Ancient civilizations; but it is a fair assumption that the mathematical techniques and practices of the latter constituted something similar in structure (or rather, lack of *own* structure) to ethnomathematics. Similarities may well have gone much further—as we shall see, graphs similar to those of Malekula were familiar in the Ancient Near East. If we are going to look for *mathematics* as *one* entity we may thus choose between two options: either we define one specific domain (traditionally number and counting) as being *their* mathematics, which will allow us to postulate the existence of mathematics far back into an indefinite past; or we may decide (as I intend to do) that the distinctive characteristic of mathematics as *one* entity is the *coordination* of several abstracting practices.

The choice of coordination as the defining feature does not free us from all arbitrariness. It is still a question, e.g., whether counting and addition are one or two practices; if they are two, the introduction of addition is already mathematics, since it cannot be done in isolation from counting. So, I shall end up by defining the transition to mathematics as the point where *preexistent and previously independent* mathematical practices are coordinated through a minimum of at least intuitively grasped understand-

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[35] The abstract marriage algebra of Malekula is described by M. Ascher & R. Ascher [1986:137-139], the graph-theoretically refined closed patterns by M. Ascher [1988:207-225]. The disconnectedness between the two does not imply, of course, that the intellectual training gained through graphs cannot have made it easier for the informant to formulate the principles of marriage rules explicitly for the benefit of the ethnographer.

Ascher & Ascher [1986:132] make the point that the »category mathematics is our own« but stop short of drawing the same conclusion about ethnomathematics, for fear perhaps of devaluating the non-literate cultures which they discuss. This caution should be superfluous: the elements of ethnomathematical thought are no more random or isolated than our elements of mathematical thought—their connections are different.

ing of formal relations. Remaining ambiguities I shall accept as an unavoidable ingredient of human existence.

VII. From tokens to mathematics

The earliest mathematical technique which can be attested in the Near East is represented by small objects of burnt clay found as far back as the late ninth millennium B.C. and still present in the proto-literate period[36]. From early times, a variety of shapes are found: spheres, rods, cones, circular disks, more rarely other shapes. Many types are found in two sizes, and in certain cases the objects are marked by various incisions. During the fourth millennium, the number of shapes and of extra varieties created through multiple incision proliferates violently.

Because of continuity with later metrological notations (on which below), the objects must be tokens, i.e., *tangible symbols* for other objects—normally goods of economic importance, it appears. Obviously, the tokens constitute a *system* of symbols, used all over Iran, Iraq, Palestine and Turkey.

The emergence of the system appears to coincide with the change to agricultural subsistence [Schmandt-Besserat 1986:254]. Agriculture itself, of course, will have had no need for symbolization, nor will barter of grain for obsidian (or whatever exchange can be imagined). The most plausible suggestion for the function of the token-system is supplied by the excavation of a fifth millennium site (Tell Abada) in east-central Iraq [Jasim & Oates 1986:352]. Tokens are found in several places; yet groups of varied tokens (e.g., 8 spheres, 4 cones, 1 disc, one rod) contained together in

[36] Denise Schmandt-Besserat, who discovered the widespread appearance and high age of a system which until then had only been recognized in the later fourth millennium, has published a long array of papers on the subject, of which I shall only refer to the original publication [1977], an early popularization [1978], and a recent paper [1986] discussing *inter alia* social and cognitive interpretations. Another recent publication on the matter to be mentioned is Jasim & Oates [1986].

vessels are found only in one place, but there repeatedly: in the most important building of the village, which according to a number of infant burials may have had religious functions, but whose many rooms shows it not to be a mere shrine (or »temple«). Most likely, it was also a communal storehouse, the heart of a religiously sanctified redistributive system which was moving toward taxation in favour of responsible personnel, and within which the tokens have served for accounting [Schmandt-Besserat 1986:268f].

This interpretation is supported by other evidence. Tell Abada is not the only place where the tokens turn up in non-residential buildings [*ibid.*, 254]. Moreover, tokens (or, rather, prestige versions of tokens made in stone) are also found as high-status grave goods from the sixth millennium onwards,, e.g. in the fourth millennium site Tepe Gawra (near Ninive)—in the grave apparently possessing the highest status 6 stone spheres constitute the total deposit [Schmandt-Besserat 1986:255]. Admittedly, Jasim and Oates [1986:351f] mention this as an argument for non-accounting functions of the objects; more plausible, however, is Schmandt-Besserat's explanation ([1986:269] and, in more detail, [1988:7f]) that the occurrence of tokens in the deposits of high-status burials reflects a high-status position for those who administered by means of tokens while living; their presence in infant graves in Tepe Gawra and elsewhere, furthermore, suggests that the manipulation of tokens was (or belonged with) a hereditary function (as burial deposits in children's graves are normally taken by archaeologists as evidence for hereditary social ranking)[37].

Due to later continuity the meaning of certain tokens can be interpreted. So, a disk marked with a cross appears to stand for a sheep (and two disks for two sheep). Most, however, are uninterpreted or only tentatively interpreted, while the principles involved are only subject to limited doubt. They can be illustrated by Schmandt-Besserat's suggestion that a small cone

[37] It may be objected that we would not expect so highly developed stratification in the beginning of the Neolithic. Some indications exist, however, that the ecology of the Near East was rich enough to support stratified settlements and to call for organized redistribution as early as the late Mesolithic Natufian, and that ranking and even hereditability of high status had developed by then (see [G. A. Wright 1978:218-221]).

stands for a specific measure of (i.e., a specific type of basket or jar containing) grain, a small sphere for another, larger measure/container, and large cones and spheres for still larger measures [1986:268]. Other types might signify other staple products (dried fruit, oil, wool, ...). We observe that the marked disk stands for both quality (sheep) and quantity (one) at the same time; the same holds for the cone if representing the *grain*-contents of *a specific container*. There is no symbol for *abstract number* or for *volume as such*. Since the containers for grain and for oil were different, »volume concepts« had to be specific. Measure only exists as »natural measure«, and number only as »concrete number«[38].

The fourth millennium proliferation of the number of token types corresponds to the need of the more highly organized economy of social systems like that of the Susiana plain. New commodities had to be handled, and those of old to be followed in more detail (from later evidence we may guess, for instance, that »sheep« would be differentiated into ewes, rams and male and female lambs). In addition, the tokens were now used as »delivery notes« for goods sent from the periphery to Susa, enclosed in sealed containers made of clay (»bullae«)[39].

A disadvantage of the sealed bulla as a bill of lading was that it had to be broken in order to be »read«. A solution, however, was at hand: before the tokens were put into the bulla they were pressed into its surface, each leaving a clearly visible impression. The observation that thereby the

[38] Evidently, this cannot be read out from the tokens themselves. It follows from an agreement between general ethnomathematical experience and the reflection of the token system in proto-literate metrologies.

One question which cannot be solved in this way is whether »bundling« was included into the system. *If*, e.g., a small disk corresponded to an animal, would then a large disk correspond, e.g., to »a hand« (5) or »hands and feet« (20) of animals? Would a »sphere-container« be supposed to contain a fixed number of »cone-containers«? At some point in the development such bundling was introduced, but we have no means to assure that it had already happened in the Neolithic.

[39] At this point we begin to approach hard facts. This last-mentioned use of the tokens follows from the geographical distribution between Susa and lower-ranking settlements of seals, broken sealings, bullae prepared for use but not yet closed, and dispatched bullae (see [Wright & Johnson 1975:271]).

enclosed tokens had become superfluous will have called forth another step: the replacement of the hollow bulla by a flattened lump of clay where the impressions could be made (by tokens or, rather, by styli able to make similar impressions) and over which the cylinder seal could be rolled. These are the first genuine *clay tablets*, normally known as »numerical tablets«; like the bullae, they are found in Susa and the Susian orbit as well as in Habuba Kabira, the Uruk V-outpost (those of Uruk are found in rubbish heaps and cannot be dated)[40].

As carriers of information, the numerical tablets had an important advantage over the bullae: their surface could be structured, first by distinguishing the four edges of an approximately square tablet and next by dividing the surface into compartments through incised lines. Another advantage was discovered in Uruk IV: through pictographs *quality* could be separated from, or added to, *quantity*. A drawn circle with a cross was used to indicate sheepness, and impressions looking like pictures of small and large cones and spheres were used to indicate the number of sheep[41].

The whole development from the introduction of bullae with impressions of tokens and seals to the creation of the pictographic script was evidently coupled to the development of a complex society and to the needs of statal administration for more precise controls, as it was delineated above. It was no consequence of state formation *per se:* as pointed out already, the control involved in state formation need not be bureaucratic control. But the development was a consequence of *state formation as it actually happened in the Sumero-Susian area,* and we may assume that it was

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[40] See [Le Brun & Vallat 1978:47, 57] for Susa and [Jasim & Oates 1986:349] for Habuba Kabira.

[41] Readable expositions of the various facets of the development are given by Nissen [1985] and by Damerow, Nissen & Englund [1988, 1988a].

It should be observed that the sequence bulla—numerical tablet—pictographic tablet is in the main derived from the inner »logic« of the process combined with indirect arguments rather than from direct stratigraphic criteria: because only numerical and no pictographic tablets are found in Habuba Kabira, this setlement must be earlier than Uruk IV, where pictographic writing *is* attested. But then, since bullae and numerical tablets are found in Habuba Kabira, they must be earlier than pictographic writing; etc.

the age-old connection between sanctified unequal redistribution and token accounting which made bureaucratic control a natural corollary of the further change of the redistributive system toward taxation.

Improvement of book-keeping is an improvement of a mathematical technique, which was thus an effect of state formation. But book-keeping alone *does not constitute mathematics.*

On the other hand, mathematics *did* emerge in the process, and even in the form of multiple coordination. Firstly we may look at the metrological sequences and number systems used in the texts. These were first analyzed thoroughly by Jöran Friberg [1978], whose preliminary results have now (on the whole) been confirmed and expanded through computer analysis as part of the Berlin Uruk project [Damerow & Englund 1987].

The first thing to be observed about these systems is that counting is still concrete. In fact, although the basic signs (varied through combination in various ways and addition of strokes) are pictures of the small and large spheres and cones[42], a number of different systems are in use, with different relations between the visually identical signs.

Firstly, there are *two sequences for counting*[43]. One (the »sexagesimal system«) starts by a small cone (»1«), continues by a small circle (»10«), a large cone (»60«), a large cone with an impressed small circle (»600«), a large circle (»3600«), and culminates with a large circle with an impressed small circle (»36 000«). This system, characterized by its systematic shift

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[42] In principle, the appearance of the signs could be an accidental result of the fact that these are the impressions which can be made by vertical and inclined impression of a thin and a thick circular stylus; the existence of bullae where the tokens actually contained are impressed [Schmandt-Besserat 1986:256] suggests, however, that the similarity between tokens and signs is not accidental, and that the circular stylus was chosen precisely because it could so easily produce the desired impressions.

[43] A sequence »for counting« is characterized by a separation of quantity from quality, as, e.g., in our »3 sheep« or »6 m«. A »metrological sequence«, on the other hand, has quality inherent in quantity (as in »mmmm« instead of »4 m«).

Throughout the history of Mesopotamian mathematics this distinction remains less clear than the historian of mathematics might prefer. Instead of our »4 m«, e.g., an Old Babylonian scribe would usually have written »4«, expecting everybody to know that lengths are measured in this unit.

between the factors 10 and 6, is used to count slaves, cattle, tools made from wood or stone, vessels (standing for a specific measure of their customary content), and probably lengths.

The other main counting system (the »bisexagesimal system«, with units in the ratios 1:10:60:120:1200:7200, i.e., successive factors 10, 6, 2, 10, 6) is used to count products related to grain (rations? bread?), and certain other products.

Besides, *three metrological sequences* have been identified. One is used for capacity measures for grain. If the basic unit is *B* (a small cone), the next are 6 *B* (small circle), 60 *B* (large circle), 180 *B* (large cone) and 1800 *B* (large cone with inscribed small circle)—the factor sequence is thus 6, 10, 3, 10. We observe that both order and ratios differ from those of the sexagesimal number system.

Another metrological sequence (testified only in Uruk III/Jemdet Nasr) is used for areas. It was still in use in far later times, which allows us to interpret the small cone as an iku (c. 60m·60m). Then follows a small cone with inscribed small circle (6 iku), a small circle (18 iku), a large circle with inscribed small circle (180 iku) and a large circle (720 iku) (factor sequence 6, 3, 10, 6).

A third metrological sequence is of unidentified use.

Obviously, all sequences are based on the principle of bundling, which demonstrates that principles derived from counting were applied to the regularization of natural measures. Apart from that (admittedly important) step, however, the plurality of sequences and the absence of any system in the succession of the same symbols and in the sequence of ratios is hardly a proof that the career of mathematics had begun.

This beginning, however, is demonstrated by closer investigation of features not yet mentioned. Firstly, what I have just described is just one part of the sequences, from the »basic unit« upwards. This is the part whose signs derive from the old token system, and which may therefore be of indefinitely older age—even though it is not implausible that the counting notations and the area notation were fresh creations, taking over the symbols of the grain system and adapting them to the actual bundling steps of the verbal counting systems and to the area metrology in use (on areas, see below). The other part consists of fractional sub-units, which are positively new. In the counting sequences, the first sub-unit (»$\frac{1}{2}$«, in the sexagesimal system, and in specific contexts perhaps »$\frac{1}{10}$«) is symbolized by the small cone turned 90° clockwise, which would evidently make no sense for freely rattling tokens. In the grain system, a first step is made in a similar fashion[44], producing »$\frac{1}{5}$ *B*« (=»*C*«). In a second step downward, »$\frac{1}{n}$ *C*« (n=2, 3, 4, probably 5 and possibly higher values) is symbolized by n small cones arranged in a rosette. (No area units below the iku are attested, but this may well be because such smaller units do not occur in allocations of land—our only epigraphic evidence for area metrology). This involves an knowledgeable application of »inverse« counting to metrological innovation, and must thus be characterized as *mathematics* as defined above.

Another metrological innovation based on mathematical premeditation pertains to the calendar—more precisely, one of the calendars[45]. Until much later, indeed, the »time-keeping calendar« is a luni-solar calendar, whose months are on the average 29$\frac{1}{2}$ days, shifting between 29 and 30. Of these months there are 12 to a year, and about every three years an intercalary month is inserted in order to adjust the year to the tropical and agricultural year. To the meticulous Ur III administration, months of changing length were unacceptable, as we may easily imagine, and a system was employed where the overseer was responsible for pressing 30 days worth of work out of each worker per month, irrespective of its real length, and got food and fodder rations for his workers and animals according to the same principle. Now, through fastidious analysis of certain proto-literate herding texts Robert Englund has been able, firstly, to confirm an interpretation of the time-keeping notation proposed by Vaiman [1974] on intuitive grounds, and secondly to show that the Ur III administrative calendar was in reality a proto-literate invention and practice.

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[44] The sign itself, it is true, differs from the turned picture of the cone used in the counting sequence: it might look as a picture of the half- or quarter-sphere tokens, and could thus have been present already in the token-system. But like the fractional counting number, it is turned 90° clockwise, indicating that both are conceptualized as belonging to the same (»fractional«) category.

[45] The following description of Sumerian and proto-literate timekeeping is built on Robert Englund's pioneering work on administrative timekeeping [1988].

The notation combines the pictogram showing a sun half raised above the horizon with strokes (counting the years), ordinary sexagesimal numbers (months) and sexagesimal numbers turned 90° clockwise (days). Already for the reason that these distinctions only make sense when the symbols are fixed in clay will this be a fresh invention of the proto-literate period. The free creative manipulation of several sexagesimal counting systems demonstrates mental independence of context-bound counting and ability as well as resolution to combine different elements of mathematical thought in order to create an adequate tool[46].

Similarly, even the creation of a counterfactual calendar in order to attain mathematical regularity can be seen legitimately as evidence of coordination, *viz* between bureaucratic organization and mathematical thought. It will also involve at least an intuitively based decision that the rounding error was not larger that acceptable. On both accounts the administrative calendar thus testifies to the emergence of genuine mathematics.

All this had to do with the complex of counting, metrology and accounting. A final observation involves geometrical practice in the network.

We have as yet no direct proof that the area of a rectangular field was calculated from its length and width—none of the texts which appear to indicate lengths and widths contain area information. But two pieces of indirect evidence can be found. Firstly, the same area system (or at least an area system with the same factor sequence) is known from later times to be strongly geared to the length unit[47]. Thus, the basic area unit is the sar, which is the square of the fundamental unit of length (the nindan or »rod«, equal to c. 6 m), but whose name (presumably meaning a »garden

[46] A similar albeit weaker observation could be made from the existence of »dependent metrological sequences« produced from those described above through addition of strokes and used to count or measure specific varieties of the goods counted or measured by the corresponding fundamental system—for instance, to measure emmer instead of barley. In this case the innovation may go back to the late pre-literate creation of supplementary token types (and token sequences?) by means of incisions.

[47] See Powell [1972], the principal reference for Sumerian area measures.

plot« [Powell 1972:189-193]) suggests an independent origin as a »natural unit«. The iku itself is a square ešé (the ešé, meaning a »rope«, being equal to 10 nindan). Further on in the sequence, the bur (=18 iku), again appears to have originated as a »natural unit«.

This suggests that the system emerged from a process of mathematical normalization, where natural seed or irrigation measures were redefined in terms of length units, thus stabilizing the system, as Powell points out [*ibid.*, 177]—and since the upper end of the sequence is already present in Uruk III (where no area units below the iku are testified but may still have existed), the redefinition must have taken place already then.

The other piece of indirect evidence is a proto-literate tablet referred to by Damerow & Englund [1987:155 n.73]. It deals with a surface of which the two (identical) lengths and the two (slightly different) widths are told. Calculating the area by the »agrimensor formula«[48] one finds a nice round value: 10 times the highest area unit, i.e., c. 40 km^2. The implausibly large value tells us that we have to do with a school exercise, and the improbability to hit upon the round value by accident suggests that the exercise was constructed so as to achieve it, and thus that the area had to be calculated as done in later times.

Area measurement is not the only element of geometrical practice attested in the proto-literate period. Already the ground-plan of the late pre-literate »Limestone Temple« [E. Heinrich 1982:74 and Abb. 114], perhaps even two fifth millennium temples [*ibid.*, 32 and Abb.71, 74], possess a regularity which suggests architectural construction. Remains of a ground-plan left under an early Uruk IV (or possibly late Uruk V) temple, moreover, shows that it was carefully laid out by coloured string ([Heinrich 1938:22], cf. [Heinrich 1982:63, 66]). One of the many different groups of experts present in proto-literate Uruk must hence have been architects skilled in practical geometrical construction[49]—and since only

[48] I. e., average length times average width. This method was used in the computation of the area of not too irregular quadrangles at least from ED III to Old Babylonian times, and even far into the Middle Ages.

[49] Like the idea of writing (but not the script itself), this technique also seems to have been borrowed by the proto-Elamite culture (which had a centre in Susa but had others far into the Iranian East, and which was more or less contemporary

»official« prestige buildings suggest the existence of a geometrical plan, they must have worked exclusively for the Temple.

We can also be reasonably sure that the planning of buildings and of building enterprises will have involved computation of brickwork and manpower requirements. Firstly, a culture which defines a specific administrative month for the sake of fodder calculations would hardly take the enormous costs of prestige building just as they came. Secondly, the evidence for precise geometrical lay-out coupled to the standard brick demonstrates that calculations could be made, as indeed they were in later times; it is plausible that this was even the idea behind the mutual adjustment of standards. If so, however, the computation of areas and volumes from linear dimensions will have arisen already in the architects sphere, and the gearing of area measurement to measures of length will also have involved the architectural branch of practical geometry. Protoliterate mathematics will already have coordinated number and metrical space—one and the other, we may safely assume, as practical concerns and not as abstract fields of interest[50].

The formation of mathematics as a relatively coherent complex was thus concomitant with the unfolding of the specific Uruk state. Is that to say that it was a direct consequence of statal bureaucratic rationality—sort of modified and attuned Wittfogel thesis, mechanistic-functionalist though on revised premisses? Hardly. Other early bureaucratic states have existed

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with Uruk III). This follows from Beale's and Carter's careful analysis [1983] of the geometry of the proto-Elamite architectural complex of Tepe Yahya IVC, in which base-lines separated by integer multiples of a standard measure (equal to 1.5 times the standard brick length) define the exterior edge of outer walls and the mid-lines of inner walls. Apart from a different choice of ratio between the standard measure and the standard brick, moreover, the same code appears, e.g., in buildings from Habuba Kabira (the Uruk V outpost mentioned in chapters III and VII).

[50] One field which was not yet integrated (and which never was until the modern era) was »ethnomathematical graph theory«, cf. [M. Ascher 1988]. That it was none the less present we may infer from somewhat later evidence: in the Fara tablets such »graphs«, complex symmetric patterns drawn by a continuous line, turn up time and again—see the specimens in [Deimel 1923:31] (broken)); [Jestin 1937: CLXXX, #973]; and [Edzard 1980:547].

without producing similar results[51], and bureaucratic management of agriculture would probably have been better served by natural measures (as suggested by the changes in Babylonian metrology after the mid-second millennium). Bureaucracy itself does not demand the type of coherence inherent in the Uruk formation of mathematics. What is involved is, we might say with Weber, a *particular* spirit of bureaucracy, one tempted by intellectual and not by *merely bureaucratic* order. We also find it expressed in the *lexical lists,* which are *more than a means of teaching the script:* they also provide an ordered cosmos, and a cosmos of a specific sort: putting wooden objects together in one category, vessels in another, etc., amounts to what Luria [1976:48ff] labels »categorical classification«, in contra-distinction to his »situational thinking«[52]. Still, the lists *are* a means for teaching, and thus a vehicle not only for literacy but also for the »modern«, abstracting mode of thought—precisely the mode of thought preferring mathematical coherence to situationally adequate seed measures, etc. The latter part of their message will have supported, and have been supported by, the development of the main administrative tool: the clay tablet with

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[51] A beautiful example seems to be presented by the linear B tablets of the Mycenaean palace bureaucracy. Even though Mycenaean art bears witness of a strong and inquisitive interest in geometrical regularity [Høyrup 1983] there is to my knowledge no evidence whatsoever of a transformation of scribal accounting arithmetic into *mathematics.*

[52] Illustrated, e.g., by this dialogue [*ibid.*, 55]:

Luria, explaining a psychological test: »Look, here you have three adults and one child. Now clearly the child doesn't belong in this group".

Rakmat, an illiterate peasant from Central Asia: »Oh, but the boy must stay with the others! All three of them are working, you see, and if they have to keep running out to fetch things, they'll never get the job done, but the boy can do the running for them [...]«.

Situational thinking was found in Luria's investigation of prevailing modes of cognition in Soviet Central Asia to be »*the* controlling factor among uneducated, illiterate subjects«, while both modes were applied (with situational thinking dominating) among »subjects whose activities were still confined primarily to practical work but who had taken some courses or attended school for a short time«. »Young kolkhoz activists with only a year or two of schooling«, on the other hand, employed the principle of categorical classification »as their chief method of grouping objects«.

its ordered formats[53].

In so far as the emergence of mathematics is to be ascribed to a particular Uruk variant of the bureaucratic spirit, this spirit was thus interacting intimately with, and largely a consequence of, the school organization of teaching (whose typical features we already encountered in a mathematical exercise). If a complex process is to be reduced to a simplistic formula, the emergence of mathematics was called forth neither by technical needs nor by the bureaucratic organization or by writing *per se*, but *only through the interaction of these with each other and with that school institution which provided recruits and technical skills to the bureaucracy.*

VIII. Trends in third millennium mathematics

As long as the Sumerian city-states remained dual societies, mathematics was on the same side as writing and bureaucracy. Throughout the third millennium, therefore, the career of mathematics runs parallel to that of expanding bureaucratic systems, spreading literate activities, and improved writing. In so far as all this was a simple continuation of the trends inherent in the proto-literate state, mathematics too was a continuation.

Let us first look at metrology. It may wonder that no metrological sequence for weights has been mentioned above (unless, of course, the unidentified sequence contains weight units)—especially in view of the fact that metal smelting is actually attested in Uruk [Nissen 1974:8-11]. But technical activities of this sort were not the concern of accounting, and whatever the craftsmen have done was not committed to writing and thus subjected to mathematical regularization[54].

[53] This problem of the interplay between tool and mode of thought I shall not pursue any further in the present connection, only refer to its position as the central theme in [Damerow & Lefèvre (eds) 1981].

[54] Import of metals will of course have been a matter of bureaucratic interest. But nothing so far known suggests that archaeologists have come upon tablets from

Later, when copper and silver acquired monetary functions, on the other hand, weight became an accounting concern *par excellence*. In the beginning of ED III, thus, the weight system *is* well attested. A consequence of this late development of weight metrology is a high degree of mathematical systematization (see [Powell 1971:208-211]) in the shape of »sexagesimalization«, adoption of the fixed factor 60 from the principal (in ED III the *only*) counting system, in analogy with what had already happened in the proto-literate creation of the calendar notation. Starting from the top, a »load« (some 30 kg, the Greek »talent«) is divided into 60 mana, each again subdivided into 60 gin (the later šekel). The gin is subdivided into še, »barleycorns«, which in real life weigh much too little to fit another sexagesimal step; but 180=3·60 še to a gin agrees fairly well with real barley.

Sexagesimalization was not the preserve of the weight system. In general, when pre-existent systems were extended, it was done »the sexagesimal way«. So, e.g, 60^3 and 60^4 were added to the counting sequence; the gin was transferred from weights to other systems in the generalized sense of $\frac{1}{60}$; and established systems were expanded upwards through multiplication of the largest traditional unit by sexagesimal counting numbers. This development is most straightforwardly explained as the natural consequence of the situation that mathematics was *already present* as a coherent way of thought, both *in actu* and as impetus and challenge, carried by continuing school teaching.

Another perceptible trend is parallel to that of centralized reforms of writing and bureaucratic procedures (and, though only on the ideological level, to the recurrent idea of a »social reform«): intentional and methodical changes of metrology in order to facilitate bureaucratic procedures. This is of course analogous to the proto-literate introduction of the administrative calendar; the instance which is best certified in the pre-Ur III period is the Sargonic introduction of a new capacity measure in the order of the barrel, the »gur of Akkad« of 30 ban = 300 sila (≈ 300 l) instead of the current gur of 24 ban = 240 sila and the Lagaš gur of 144 sila (see [Powell 1976:423], where the advantages of the new unit in connection with

the archive of trade.

computations of rations are discussed).

A third trend, finally, is akin to the appearance of literary texts, and like literary text it begins in the Fara period, concomitantly with the emergence of the scribal profession as a separate group. We might speak of a first instance of *pure mathematics*, namely, of mathematical activity performed in order to *probe the possibilities* of existing concepts and techniques and neither for immediate use in practice nor for plain training of skills to be used in practice.

The evidence is constituted by the oldest mathematical exercises after those of the proto-literate period, which could only be distinguished from real-world accounting and mensuration by the occurrence of round and implausibly but not impossibly large numbers and by the lack of the name of an official carrying responsibility for the transaction [Friberg 1990:539]. One of the Fara problems ([Jestin 1937, #188]; unpublished analysis by Jöran Friberg) is *almost* of the same type, with the difference that now the area involved is rather *impossibly large*. Two other Fara texts [*ibid.*, #50 and #671] require that the content of a silo containing 2400 »great gur«, each of 480 sila, be distributed in rations of 7 sila per man (the correct result is found in #50: 164 571 men, and a remainder of 3 sila; the solution of the other tablet is wrong or at best uncompleted—analysis of the two texts and of the method used in [Høyrup 1982]). A fourth text (analyzed by Jöran Friberg [1986:16-22]), comes from the Syrian city Ebla (whose mathematics was avowedly taken over from the Sumerians) and is presumably of slightly later date. It deals with the successive division of 100, 1000, 10 000, 100 000 and 260 000 by 33 (concretely: if 33 persons get 1 gubar of barley, how much barley do you count out for 100, 1000, 10 000, 100 000 and 260 000 persons?).

Apart from being division problems and from the »impossibly large« numbers of rations dealt with, the three last problems have one decisive thing in common: the divisors are *irregular*, they fit the metrologies and number systems used *as badly as possible* (Ebla spoke a Semitic language and had decimal number words, but combined these in writing with the Sumerian sexagesimal system; 33, of course, is irregular on both accounts). As Jöran Friberg [1986:22] puts it,

(I) the fact that three of the four oldest known mathematical problem texts[55] were concerned with exactly the same kind of »non-trivial« division problems must be significant: the obvious implication is that the »current fashion« among mathematicians about four and a half millennia ago was to study non-trivial division problems involving large (decimal or sexagesimal) numbers and »non-regular« divisors such as 7 and 33.

A number of school exercises dating between the Fara period and Ur III (mostly Sargonic) have been identified (see [Powell 1976]). Some of them are characterized by the occurrence of »impossibly large« numbers, e.g., a field long enough to stretch from the Gulf to central Anatolia. There is no trace, however, of continued interest in »pure mathematics«—which, in view of the striking statistics cited by Friberg, must be significant. As literary creativity, once a scribal exploration of the possibilities of a professional tool, was expropriated by the royal court as a political device, so also mathematical exploration appears to have vanished from a school more directly submitted to its bureaucratic function in a society loosing its traditional dual character. Two verifiable forces survived as determinants for the development of »school-and-bureaucracy mathematics«: sexagesimalization and systematization governed by the dynamics of internal coherence; and regularization determined by the requirements of bureaucratic efficiency.

A small and isolated tablet found on the floor of a Sargonic temple suggests that a third force *may* possibly have operated outside the school-and-bureaucracy system. More on this below in connection with the Old Babylonian development, to which it is connected (note 69).

[55] Apparently for rhetorical reasons, Friberg discards the proto-literate school exercises which he himself has been the first to identify.

IX. The paramount accomplishment of bureaucracy

Waning duality dwindled further in Ur III, the school-and-bureaucracy complex reached a high point, and so did bureaucratic and accounting rationality. No wonder, then, that Ur III brought about the culmination of the tendencies of late ED and Sargonic mathematics.

We already encountered Šulgi's administrative reform above, and we remember that metrological reform was presented as a cornerstone in his establishment of »justice«. Another, mathematically more decisive part of the administrative revolution was the development of the conceptual and technical tools for the many calculations inherent in the reform.

First of all a new number notation was created as a final outcome the process of sexagesimalization: the sexagesimal *place value system*, which permitted indefinite continuation of numbers into the regions of large and small. The idea had been in the air for several centuries, as demonstrated firstly by the generalized use of the gin in the sense of $\frac{1}{60}$, and next also by the particular idiom of a late Sargonic school exercise discussed by Powell [1976:427], where a »small gin« is introduced for $\frac{1}{60}$ of $\frac{1}{60}$. But precisely the use of *names* for the fractional powers shows that the system was not positional, and was not extendable *ad libitum*. We can thus be fairly sure that the introduction of place value does not antedate Ur III[56].

[56] This conclusion is not changed by the claims and the partially new text material presented by Whiting [1984], who conflates place value notation with what I have here called »sexagesimalization«. But Whiting's evidence underscores how much was in the air in the actual computation techniques in use at least since the Sargonic era, and his explanation of two apparent writing errors in a pre-Sargonic tablet of squares (OIP 14,70, transliterated and translated in [Edzard 1969]) suggests that an idea similar to the gin-tur was used already in the 25th c. B.C.

The errors so abundantly present in the computations on which Whiting bases his argument, on the other hand, make it obvious that the system after which calculators were groping was *not yet at hand* as more than an inherent possibility—similarly, perhaps, to the way the decimal place value system may have been potentially present in the Chinese use of counting rods for perhaps 2000 years before

The Mesopotamian place value notation was a pure floating-point-system, with no indication of absolute place (in the likeness of a slide rule); it could thus only be used for intermediate calculations—in accounting, one sixth of a workday, e.g., would be designated »10 gin« [Powell 1976:421] in order to avoid misunderstandings. For this reason, only very few indubitably Ur III tablets carry indubitable place value numbers[57], though some do (one instance is discussed [*ibid.*, 420]).

The important point about the place value notation is not the possibilities it offers in additive and subtractive accounting, where the disadvantage of a double number system will have outweighed the ease of writing which it brought about. It lies in the multiplicative domain, in the possibilities of the system to surmount the conflict between mathematical and technical rationality (as discussed in connection with the tendency of proto-literate scribes to prefer mathematical coherence to practical orientation), and to do this more radically than could be done by changes in the metrological system. If a platform had to be built to a certain height and covered by bricks and bitumen, e.g., changes in length measures could not be made which at the same time would facilitate manpower calculations for the earth- and brickwork, the computation of the number of bricks to be used, and the consumption of bitumen. But once the place value system was available, tables could do the trick. A »metrological table« could be used to transform the different units of length into sexagesimal multiples of the nindan. A table of »constant factors« would tell the amount of earth carried by a worker in a day, the number of bricks to an area unit, and the volume of bitumen needed per area unit. With these values at hand everything was a question of sexagesimal multiplications and divisions, which again

giving rise to the genesis of a genuine place value notation (see [Martzloff 1988:170f, 181-184]), and to the way it was demonstrably mimicked by the Greek idea of *pythmens* (see Pappos, *Collectio* II.1, in [Hultsch 1876:I,2]).

[57] In the integer range between 1 and 599, place value and »normal« administrative notation cannot be distinguished. Therefore, the scribe did not need to decide whether he used one or the other in such cases, nor can we settle the question.

A few undated tables of reciprocals (see below) probably belong to Ur III, but the paleographic distinction between Ur III and Old Babylonian tablets is not very safe for tablets containing exclusively or predominantly numbers.

were facilitated by recourse to tables, this time tables of multiplication and of reciprocal values. The conflict between »natural« and »mathematical measure« was solved similarly in other domains, and so well solved that supplementary technical measures could be introduced *ad libitum,* as indicated by an apparent proliferation of brick systems. This was the great advantage of the system in a society where the scribes were financially responsible overseers of all sorts of productive activities.

It is a fair guess that the place value system was probably invented with the purpose to solve these problems, but since we do not possess the memoirs of the inventor we cannot know[58]. What can be known is that other highly adequate place value systems are known historically to have spread at a snail's pace, in processes taking hundreds of years or even longer. If the invention was not made in Šulgi's think-tank (something like the administrative department of Kraus' conjectured *Hofkanzlei*), a central decision must at least have been made to propagate the system through the scribe school, which must thus have been under centralized control (as one would guess anyhow, given the character of Ur III society and Šulgi's interest in having the school teach what his scribes needed).

Much the same could be said about other aspects of the administrative system, especially about the introduction of a system of *balanced accounts,* at times with automatic cross-checking[59]. The school provided the administration with accountants and calculators whose collective competence has hardly been equalled by any comparable body before the 18th or 19th century (CE, for once!). Judged on the purely utilitarian premisses inherent in the Šulgi hymns cited above, the Ur III school did everything that could be done.

It is remarkable, then, that no trace whatsoever is left of non-utilitarian

[58] So much was in the air, indeed, that the most difficult step was not to get the idea in itself but to find the courage to do so. For an isolated inventor (be he practical calculator or teacher) the system would be worthless. Only when backed by tables of constants, reciprocals etc., and thus only when large-scale use made it economically feasible to produce these, were place value numbers any good.

[59] See [Høyrup 1980:19f, and 85f notes 39, 42 and 44], which contains cross-cultural comparison, whose references for Ur III book-keeping itself, however, are partly outdated. The most recent treatment of the subject is given by Englund [1990:13-55].

mathematical interests from the period. Not only are texts lacking, which in itself proves nothing, since no school texts at all from the period have been identified. More decisive: an investigation of the mathematical terminology of the subsequent Old Babylonian period shows that terms used for current operations of utilitarian calculation are Sumerian; the key terms of the non-utilitarian branches, on the other hand, are Akkadian, and the oldest non-utilitarian texts formulate even the additive and subtractive operations (for which current Sumerian terms existed of course) in Akkadian—with the exception of the finding of reciprocals and the extraction of square-roots, which referred to tables in the Ur III tradition, and the traditional Sumerian terms for which were even adopted as loanwords and provided with Akkadian declination[60]. According to all evidence, Ur III thus managed to bring its scribes to a high level of mathematical competence without engendering any sort of pure-mathematical interest, i.e., without leading to any intellectually motivated investigation of the possibilities of professional tools beyond the needs of current business—in contrast to the situation in Fara, where much more modest competence did call forth »pure« investigation. Borrowing an expression from a classical discussion of other aspects of the Mesopotamian intellect [von Soden 1936], Ur III demonstrates »Leistung und Grenze« of the early bureaucratic state as a promotor of mathematical development.

X. The Culmination of Babylonian mathematics

The vast majority of Mesopotamian genuine mathematical texts come from the Old Babylonian period. Before Marvin Powell and Jöran Friberg began their work, almost nothing was known from the third and fourth millennia, and no system whatsoever had been noticed in the meagre material (even the connection between the Ur III administration and the

[60] The details of the argument build on my investigation of Old Babylonian »algebra« [Høyrup 1990].

creation of the sexagesimal system was only suggested as a conjecture by Powell in 1976). From the 1300 years separating the Old Babylonian from the Seleucid period, again practically nothing was known (since then, Jöran Friberg has located a few items). Finally, a small number of texts with Seleucid dating had been published. No wonder that the Old Babylonian period was considered the culmination of Babylonian mathematics, which in histories of mathematics was simply identified with this climax.

In part, this is certainly a consequence of the source situation. As there is some though not full continuity from Old Babylonian to Seleucid mathematics, something *must* have existed in the intermediate years. Yet today, when at least a sketchy picture of the state of the mathematical art in the early and the intermediate period can be made, Old Babylonian mathematics is enforcing its particular character upon us in more real terms: never before, and never after, was comparable depth and sophistication achieved in Ancient Mesopotamian mathematics. Even the source situation seems to reflect realities and not merely the random luck of excavators and illegal diggers: after the Old Babylonian period the institutional focus for the production of sophisticated mathematics disappeared.

Why is that? What *was* the make-up of Old Babylonian mathematics? And what was its purpose?

First of all, Old Babylonian, quite as much as third millennium mathematics, spells *computation*. All texts compute something, they *never prove* in Euclidean manner, and they *only explain* through *didactical discussion of specific examples* of computation.

Many computations are purely utilitarian, and for good reasons. The texts are scribe school texts (the teacher's copies, not students' solutions as most of the pre-Ur III texts which have come down to us); and graduate scribes, as we remember, would normally go into notarial jobs, where they needed little but accounting mathematics, or into engineering-like occupations, where a wider range of practical geometry etc. would be required[61].

[61] It seems likely that some specialization was present. According to Landsberger ([1960:97]; cf. [1956:125f]), indeed, the Old Babylonian »lexical lists distinguish, according to degree of erudition and specialization, fifteen varieties of *dubsar* or scribe« which, however, all disappear in the subsequent period, together with the scribe school. The evidence is insufficient, however, to decide to which extent the

Utilitarian mathematics was thus a continuation of Ur III mathematics, involving sexagesimal calculation, the use of the tables of metrological conversion and of »constant factors«, knowledge of accounting and surveying procedures and of computational techniques at the level of the rule of three, familiarity with the computation of areas and (occasionally fairly intricate) volumes[62]. All this, in fact, is found, often in complex combinations as in »real scribal life« where the manpower needed to dig a trench and carry off the dirt was more interesting than its volume.

Just as important in school, however, were non-utilitarian computations, to judge from the statistics of extant texts. Dominating in this field was a domain traditionally denoted »algebra« by historians of mathematics, and which is in fact homomorphic with second- and higher-degree equation algebra of the Medieval and Modern epoch. The designation can be argued to be problematic, both because a literal reading of the terms of the Old Babylonian discipline indicates that it does not deal with number but with areas (quite literally: with fields), and because a close investigation demonstrates that the methods used were indeed sort of »naive« (i.e., reasoned but not explicitly demonstrative) cut-and-paste geometry[63].

Many problems belonging to this category look fairly abstract. For instance, we may be given the sum of the length and the width ($l+w$) of a rectangular field and the sum of the area and the excess of the length over the width ($A+(l-w)$), and then be asked to compute the length and the width (AO 8862, in MKT I, 108f, cf. interpretation in [Høyrup 1990:

job specialization was reflected in specialized school curricula.

It should be observed that dub-sar NIG.ŠID, translated »mathematician« by Landsberger [1956:125], should rather be understood as »accountant«.

[62] Often of course by means of what we would call »approximate formulae«, forgetting in this distinction that even the most exact area formula becomes approximate when the terrain surveyed is hilly and no Euclidean plane.

Karen Rhea Nemet-Nejat (forthcoming, chapter III) presents a survey of practical problem types occurring in the Old Babylonian mathematical texts.

[63] [Høyrup 1990] presents the arguments for this interpretation in philological and mathematical detail, while [Høyrup 1989] presents an overview. [Høyrup 1985] is a fairly complete but preliminary and rather unreadable exposition (»It is difficult to follow the red thread—provided there is any«, as Asger Aaboe put the matter).

309ff]). In this case, only the remark that »I went around it« tells that the person stating the problem speaks of a real field; other problems are even more deprived of the smell of real life. Still others, however, attach themselves directly, e.g., to military engineering practice, as may be illustrated by this example[64]:

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[64] BM 85194, rev. II.7-21—ed. MKT I, 149. The translation is mine, and builds on my reinterpretation of the Old Babylonian mathematical terminology (I have left out indications of restituted damaged passages and corrected a few copyist's errors tacitly). Without going into irrelevant details, the following explanations should in principle make the text comprehensible for those who want to wrestle with a real piece of fairly complex Babylonian mathematics:

1) Numbers are written in a sexagesimal place-value system (Neugebauer's notation).
2) Horizontal extensions (length, breadth) are measured in the unit nindan (≈6 m).
3) Vertical extensions are measured in kùš (cubits), where 1 kùš = $\frac{1}{12}$ nindan ≈ 50 cm.
4) Volumes are measured correspondingly, in the unit sar = nindan^2·kùš, here left implicit (»gán« is not the unit but an indicator of category and loose order of magnitude).
5) To »append« designates a concrete addition, and to »tear out« the corresponding concrete subtraction.
6) To »detach the igi of *n*« means finding its reciprocal ($^1/_n$)—actually looking it up in the table of reciprocals.
7) »To raise« means calculating a concrete entity through multiplication, as done, e.g., in operations involving proportionality.
8) To »double« designates a concrete process—in the actual case the doubling by which a rectangle is produced from a right triangle.
9) To »break« denotes a bisection into »natural« or »customary« halves—as, in the actual case, one side of a triangle is customarily bisected when its area is calculated.
10) To »make *a* surround« means constructing a square with side *a*; if we do not care about the real (geometric) method of the Babylonians we may translate it »to square«.
11) The »equilateral« of an area is the side which it produces when laid out as a square; in numerical interpretation, its square root.

Some hints can be found in MKT I, 186. Those who want to apply the geometrical interpretation (not given in MKT) should be aware that a rectangle *n* [cubit high] by *n* [nindan long] is dealt with as a square; i.e., the units which anyhow are left implicit are disregarded. Cf. Høyrup 1985: 56.

7. Of dirt, 1,30,0 (sar), gán. A city inimical to Marduk I shall seize.
8. 6 (nindan) the (breadth of the) fundament of the dirt. 8 (nindan) should still be made firm before the city-wall is attained
9. 36 (kùš) the peak (so far attained) of the dirt. How great a length
10. must I stamp in order to seize the city? And the length behind
11. the *ḫurḫurum* (the vertical back front reached so far?) is what? You, detach the igi of 6, the fundament of the dirt—0;10 you see. Raise 0;10 to
12. 1,30,0, the dirt—15,0 you see. Detach the igi of 8—0;7,30 you see.
13. Raise 0;7,30 to 15,0—1,52;30 you see. Double 1,52;30—
14. 3,45 you see. Raise 3,45 to 36—2,15,0 you see. 1,52;30
15. make surround—3,30,56;15 you see. 2,15,0 from 3,30,56;15
16. tear out—1,15,56;15. What is the equilateral? 1,7;30 you see.
17. 1,7;30 from 1,52;30 tear out—45 you see, the elevation of the city-wall.
18. ½ of 45 break—22;30 you see. Detach the igi of 22;30—0;2,40.
19. Raise 15,0 to 0;2,40—40, the length. Turn back, see 1,30,0, the dirt. Raise 22;30,
20. ½ of the elevation, to 40, the length—15,0 you see. Raise 15,0 to 6—
21. 1,30,0 you see, 1,30,0 is the dirt. The method.

To a first inspection, this looks like a slightly idealized piece of engineering mathematics: a siege ramp formed like a right triangular prism is to be constructed, and we know certain parameters concerning the structure and have to find the others. (The minor blunder that an already given value is asked for again, instead of another which is actually found, will be due to an editor-copyist's mixup with other problems dealing with the same configuration—one follows on the same tablet).

A second look, however, changes everything. The construction has already started; we already know how much dirt is going to be used for the ramp, as well as the height already reached and the remaining distance. But we do not know the intended total length or final height of the ramp, nor the length of the part built so far! The outcome, after intricate geometrical considerations, is a problem of the second degree.

Evidently, such a problem would never present itself to a surveyor in real life. In fact, no single second-degree (or higher) problem in the texts solves a problem which could be encountered in practice, nor can any be imagined within the Babylonian horizon. And yet, such problems were extremely popular (the same unfinished ramp, for example, turns up in another tablet making use of a somewhat different terminology and thus probably produced in a different school)[65]. Definitely, mathematics needed not be applicable in order to acquire high status within the curriculum—*if only it looked applied,* as the above puzzle from the engineers' wonderland.

This may look as a paradox. Why should evidently »pure« mathematics be disguised as applied? Neugebauer [1954:790], obviously disgusted, speaks of »educational artificiality which fancies it is making simple geometrical problems more appealing by using practical examples containing unreal examples«. Why should pure mathematics be restricted to computation? And why on earth should a school for future clerks, managers and engineers make so much of the training of useless skills?

The answers have to do with the position of the scribal profession and the role of the scribal school. Like the writing of phonetic Akkadian, accounting mathematics and trite computations of prismatic volumes were too uncomplicated to serve as foundation for professional pride. In order to demonstrate *virtuosity,* Akkadian had to be supplemented by Sumerian and secret writing, and the volume computation had to be turned around into a second-degree puzzle. Higher »algebra« was thus the expression of scribal »humanism« corresponding to the numerate aspect of the scribal vocation (and a choice expression), as Sumerian was the expression corresponding to the literate aspect. The important thing about second-degree »algebra« was not that it could not be used; the distinctive characteristic was that is was *complex,* i.e., *non-trivial.* The situation repeats that of the Fara scribes on a higher level, whose investigation of the possibilities of writing produced the first literary texts, and whose comparable experiments with their computational tools produced »pure« division problems.

But virtuosity had to be *scribal* virtuosity in order to serve *professional* pride (which would of course be the only sort of pride at which a scribal school could aim). Therefore, even complex mathematical problems should

[65] More precisely: such problems were popular according to their place in the corpus of texts and thus in the curriculum. There is no particular reason to believe that average students liked them. To the contrary, the generally suppressive character of the examination texts might suggest that mathematics was, within scribal humanism no less than in 19th century (CE) German neohumanism, also accepted because of its disciplining effects.

belong at least in form to the category of scribal problems. Though »pure« *in substance,* scribal mathematics was by necessity *applied in form.*

Strictly speaking, furthermore, the numerate aspect of the scribal venture was not *mathematical* in a general sense but *computational.* The virtuoso scribe had to be a virtuoso in *finding the correct number. Pure mathematics* in the sense which we have derived from the Greeks was not open as an option. Only *pure computation* would make the day[66].

Finally, the scribe was a *practitioner,* no philosopher or teacher. In Babylonia as everywhere else, the main thing for a practitioner is to be able to handle his methods aptly and correctly. In mathematics at the Old Babylonian level, this requires more than a *modicum* of understanding[67]. But in all vocational training then as now, apt and correct handling of methods is learned primarily through *systematic training supported by explanation,* not vice versa—as it was once formulated, you do not extinguish a fire by lecturing on the nature of water. Though transmission of *methods* was the central *aim* of the school, the *solution of* (adequately selected) *problems* was thus necessarily the *central teaching mode*—as, again, real, practical problem solution was the ultimate purpose of the training of *utilitarian* methods.

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[66] The situation was certainly going to be different in the Middle Ages, even for professional groups ressembling the Old Babylonian scribal profession. By then Greek mathematics was *already at hand,* and »scribal« computation could (and would, in the Islamic and Christian worlds) be seen as a special instance of that lofty enterprise. What is at stake here is the option of *inventing* something like Greek mathematics, which was a task quite different from that of assimilating the *Elements*—cf. the analysis of the former process in [Høyrup 1985a:17-30].

[67] Truly, quite a few historians of mathematics have supported the view that it was based on a tool-kit of recipes found empirically and assimilated by the scribes through rote learning—a view mostly based on familiarity with one or two problems quoted in translation in some semi-popular exposition. Scholars really familiar with the sources have always known that Babylonian mathematics could only have been produced by people who understood what they were doing, and they have supposed that oral explanations will have accompanied the terse expositions written in the tablets. During my own investigation of the sources I have located a couple of texts which in fact contain this fuller explanation (see [Høyrup 1989:22-25], and [Høyrup 1990:299-305, 320-328]).

Nothing thus remains of the supposed paradox when it is seen in the light of Old Babylonian scribal humanism. But another important characteristic persists which should be discussed. The »unfinished ramp« illustrated the »humanist« character beautifully, through exaggerating features which are present yet less conspicuous in other problems. But exaggeration is, already by definition, untypical, and so also in this case. The text is so much of a riddle that we can almost hear the real wording of lines 9-10 as »[...] *Tell me, if you are a clever scribe,* how great a length must I stamp in order to seize the city?«. Most texts, however, are much more terse and, more important, the majority of those which contain several problems are fairly or even highly systematic (with the exception of some late Old Babylonian anthology texts—the »unfinished ramp« is known from precisely these). The riddle shows the family likeness between Old Babylonian »pure computation« and »recreational mathematics«, which before it became a column in newspapers and mathematics teachers' journals was a »pure«, virtuoso outgrowth of practitioners' »oral mathematics«[68]. But systematization is of course foreign to any genre of campfire riddles, mathematical as well as non-mathematical. The systematics of the Babylonian mathematical texts reflect that school system in which they were composed, and that tendency to establish »bureaucratic order« even in the intellectual realm which had characterized it since the proto-literate period[69]. If many of the distinctive »humanistic« characteristics of Old

[68] The relation between practitioners' mathematics and recreational problems is discussed in [Høyrup 1987:288-290], and again more fully in [Høyrup 1990a], which also takes up the »scholasticized« character of Old Babylonian »pure« mathematics.

[69] Evidently, this difference in kind between recreational and scribe school mathematics does not preclude that a scribe school in need of non-trivial problems and corresponding methods borrowed them from a non-literate, recreational tradition. Evidence exists that this is precisely what happened:

Firstly, it is characteristic that the key terminology of the early »algebra« texts is Akkadian (as is in principle the whole Old Babylonian mathematical corpus even in texts where Sumerographic shorthand and Sumerian technical terms abound). In one text the quadratic completion, the essential trick in the solution of second-degree equations, even seems to be designated »the Akkadian« (*viz.*, Akkadian method; see [Høyrup 1990:326]). No doubt, thus, that »algebra« was no heritage from the Sumerian school tradition.

Babylonian mathematics must be explained with reference to the particularities of the carrying school institution as a relatively autonomous institution in an individualistic culture, its over-all character of *mathematics* was still guaranteed by the traditional character of the school as developed in interplay with the bureaucratic state.

The dependence of Old Babylonian mathematics on the school-and-bureaucracy-complex and its characteristic double-bind conditioning, on the other hand, was also the factor which effectively inhibited the emergence of *theoretical* mathematics of the Greek kind. As I have formulated it elsewhere [1990:337], the scribal school was »only moderately inquisitive and definitely not critical«. This befitted the education of future »humble officials knowing their place« yet proud of their social status. In later times

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Secondly, at least a cognate of second-degree »algebra« predates Ur III. Another favourite problem, indeed, shares part of the characteristic terminology (and, presumably, the naive-geometric technique) with the »algebra«: the bisection of a trapezium by a parallel transversal. The oldest known specimen of this problem, however, is the tablet mentioned in the end of chapter VIII, which was found on the floor of a Sargonic temple (see [Friberg 1990:541]).

The problem is so specific that independent reinvention is unlikely. But if the school has not transmitted the problem and its solution, who has? My best guess is an Akkadian surveyor's environment, which can quite well have existed in central Mesopotamia in the early Old Babylonian epoch, to the north of that Sumerian core area where graduates from the scribe school may possibly have had a monopoly of surveying.

Interestingly, one Sargonic school exercise (A 5446, see [Whiting 1984:65f]) seems to presuppose knowledge of a basic »algebraic« identity. It asks for the areas of two squares of side R-r, where R is a very large, round measure, and r a very small unit. Without knowledge of the identity $(R-r)^2=R^2-2Rr+r^2$ (which of course follows easily from geometrical considerations), the calculation will be extremely cumbersome.

Even in the Sumerian South, it should be added, scribal monopoly on surveying and geometrical practice is not too firmly established. Krecher [1973:173-176] points out that Fara contracts for purchases of houses involve a »master who has applied the measuring cord to the house« (um-mi(-a) lú-é-éš-gar), while a »scribe of fields« (dub-sar-gána) is involved when land is bought; a Sargonic document groups together the »surveyor« (LU$_2$.EŠ.GID), the »scribe« (dub-sar) and the »chief of the land register« (SA$_{12}$.DU$_5$). Krecher supposes even the »master« and the »surveyor« to be scribes, but in particular concerning the latter we cannot know for sure.

similar institutions would provide fairly suitable vehicles for the transmission of the ultimate outcome of the Greek mathematical endeavour; yet the dawn of the Greek endeavour itself was too dependent on non-suppressive critical discussion to be within the reach of a scribal school culture.

XI. Devolution

I shall finish my discussion of Mesopotamia by some cursory and undocumented remarks on the state, the scribal profession and the development of mathematics after the end of the Old Babylonian period, before passing to some even more brief comparative observations.

The end of the Old Babylonian epoch inaugurated the dissolution of much of the complex which, according to the above, had shaped and even engendered Mesopotamian mathematics.

Firstly of that sort of state which, since its emergence as a pseudo-redistributional organization, had guarded the pretense to be the upholder of justice and affluence (in spite of often contrary realities). The end of the Old Babylonian epoch was brought about by the Hittites, who sacked Babylon, after which a warrior people (the Kassites) took over power in the Babylonian area. They exploited it, not by taxation however vaguely disguised as redistribution but by direct extorsion, as conquerors would mostly do until the advent of the more sophisticated methods of the Modern era, taking over part of the land, allying themselves with the autochthonous upper class and pressing tribute from a re-communalized peasant class. City life, on that occasion, did not disappear *completely*—but the proportion of town to country dwellers reverted to the level of the pre-literate Middle Uruk period.

The scribe school disappeared. Administration and scribes were still needed, but scribes were from now on trained as apprentices inside their »scribal family«.

The self-asserting individualism of the Old Babylonian period disappeared. The particular scribal expression of Old Babylonian individualism, »humanism«, disappeared, too. Instead, scribal pride was founded on the membership in an age-old *tradition*. That cloak of magic and secrecy which Wittfogel ascribes to the bureaucrats of the managerial and functional state is in fact a product of the intellectual crisis caused by its breakdown.

Even mathematics disappeared—at least from the archaeological horizon. But realities are involved, too. Techniques, of course, survived. But the few texts of later times suggest that the integrity of mathematics as a subject on its own disappeared—while Old Babylonian mathematical texts would contain *nothing but* mathematics, things were now mixed up. »Pure« scribal interest in mathematics disappeared, it seems. The evidence suggests, indeed, that second-degree algebra, even though it turns up again in a few Seleucid texts, survived in a practitioners' (surveyors' and/or architects') rather than in the scribal environment[70]. Moreover, the evolution of metrology suggests that technical mathematical skills declined. As explained above, the routines and procedures associated with the place value system overcame the conflict between mathematical and technical rationality, thus making the use of »natural measures« unnecessary. From the Kassite era onwards, however, the metrological system changes; field measures keyed to the squared length unit, e.g., are replaced by seed measures. Apparently, technical efficiency was no longer compatible with »mathematical efficiency«, i.e., coherence and simplicity.

In the Late Babylonian epoch (from c. 600 B.C. onwards), finally, mathematics reappears above the horizon. Its practitioners are no longer primarily scribes, i.e., accountants and engineering managers; instead, they designate themselves as »exorcists« (*āšipu*) or »priests« (*šangû*). The latter title, oddly enough, coincides with the Sumerian sanga, who was not only a priest but also a manager of temple estates and a teacher in the Fara

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[70] For one thing, the set of Sumerian equivalents for Akkadian technical terms changed—tab, once used as a Sumerogram for *eṣēpum* (»to double« or »repeat concretely«—arithmetically, to multiply by an integer *n* below c. 10) came to designate addition. It thus appears that the scribes translated the language of »algebra« into their favourite Sumerian tongue for a second time without knowing that (or without knowing too precisely *how*) it had been done before.

school; the Late Babylonian *šangû*-mathematician, however, was no practical manager as his proto-literate predecessor but an astrologer.

The astrologer-priests who created Late Babylonian mathematical astronomy performed technical wonders, no doubt. Their skill in developing interpolation schemes, six-place (sexagesimal places!) reciprocal tables etc. is impressing. But if we understand mathematics as »a coherent way of thought, both *in actu* and as impetus and challenge«, then the high point of Mesopotamian mathematics was reached in Hammurapi's Bronze Age and never again.

XII. Supplementary comparative observations

A possible test of the plausibility of the theses advanced above on the connections between the specific process of state formation and development and the emergence and shaping of mathematics would be cross-cultural comparison. Implicitly, of course, much of the analysis *is* already cross-culturally based, through the use of theoretical tools sharpened on non-Mesopotamian whetstones. In the present appendix I shall only point to two possibilities of explicit comparison.

First, of course, Egypt suggests itself, as every time a mirror is to be held up to Mesopotamia[71]. State formation in Egypt was roughly contemporary with that of Uruk, presumably slightly later. Its background, however, was more explicitly in agreement with Carneiro's warfare model. Pharaoh united Egypt through conquest, and courtly art demonstrates that he was proud of that. The early Egyptian state was not built on any redistributive pretext or ideology[72].

[71] Apart from the general literature, the following builds in particular on [Baines 1988]; [Brunner 1957]; and [Høyrup 1990b].

[72] Or, at least, only on redistribution in an utterly distorted form (cf. [Endesfelder 1988]): Pharaoh took hold of the societal surplus and redistributed part of it to his officials while returning perhaps promises of cosmic stability to the general peasant

Writing was also roughly contemporary, and presumably slightly later. But until late in the Old Kingdom, literacy was extremely restricted, and not before the Middle Kingdom, i.e., in the outgoing third millennium, did a scribal school arise.

If no other forces were present which could nourish the process, we should thus expect the development of mathematics as a coherent whole and, especially, of »pure« orientations, to be much slower than in Mesopotamia. As far as it can be judged from the meager evidence from the Old Kingdom, this seems indeed to be the case. Firstly, some generosity is already required to see the development of the Egyptian unit fraction system as evidence of a »pure« orientation; but even if that is granted one will have to observe that the unit fraction system seems only to be created *as a system* in the Middle Kingdom, in the wake of the new scribe school institution. No other branch of Egyptian mathematics can at all be considered non-utilitarian. Secondly, it is questionable how far the unification of single techniques into a coherent whole had developed before it was definitely brought about by the unit fraction system.

While Ancient Egypt is a mirror through which the Mesopotamian development is recurrently observed, Medieval Western Europe is rarely mentioned as an analogue. In one important aspect, however, the Medieval West *is* relevant, *viz* as a dual society. If Gilgameš shares essential features with the Homeric kings, he can also be compared to a Frankish warrior-king. The Church, on the other hand, shares with the Sumerian Temple the status of a purported institutionalization of the common good; to a large extent, its incomes derived from benevolent gifts (often compulsory, it is true, and in the case where the gift was a nobleman's donation of land with appurtenant peasants it could only be made productive through continued compulsion; but these details are irrelevant for my present purpose, and nobody knows whether realities were much different in Sumer). The interesting thing is that *literacy was until the High Middle Ages the exclusive ally of the ecclesiastical »Temple Institution«;* except for a few dreamers of learning like Charlemagne and Otto III one can describe the history of Central Medieval learning without ever presenting the feudal

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population.

power in person. »Who was the feudal lord who donated land for the Cluny monastery? It doesn't matter: feudal lords did that sort of thing!« »Who pacified the French core areas in the late 10th century to the extent that cathedral schools could revive? No single lord or king, it was part of a general trend visible in many places«; etc. And vice versa of course: in the *Poema de Mio Cid*, the tale of this most Christian hero of the Spanish *reconquista*, the role of the Church is as secondary as in *Gilgameš and Agga*. Societal duality is thus a recurrent historical type in state formations not yet fully satisfying Runciman's criteria (quotation F), and literacy and learning belong with the institutionalization of alleged general interest, not with the warrior-robber lordship in its interaction with a pre-state, communal or kinship-based sector.

The Medieval parallel can be pursued further, into the High Middle Ages. Then, as we know, duality was reabsorbed, and royal centralization was well served by literate clerks. But at the same time the environment of learning, rapidly growing and therefore less directly subject to the »Temple« institution, went through a process of intellectual emancipation, first in the »twelfth century renaissance« and then in the universities. But scholars remained *clerks*, firstly because of the general socio-cultural and the particular institutional context, secondly because of the future social position of most university students. As in the Old Babylonian scribal school, though less strictly, the traditional binding to the »Temple« and the actual nexus to scribal (notarial and cameralistic) functions in existing society set limits to the tendencies toward intellectual enfranchisement.

Bibliography

Adams, Robert McC., & Hans J. Nissen, 1972. *The Uruk Countryside. The Natural Setting of Urban Societies.* Chicago & London: University of Chicago Press.

Adams, Robert McC., 1974. "Anthropological Perspectives on Ancient Trade". *Current Anthropology* **15**, 239-249, discussion 249-259.

Adams, Robert McC., 1982. "Die Rolle des Bewässerungsbodenbaus bei der Entwicklung von Institutionen in der mesopotamischen Gesellschaft", *in* J. Herrmann & I. Sellnow (eds) 1982:119-140.

Alster, Bendt, 1975. *Studies in Sumerian Proverbs.* (Mesopotamia: Copenhagen Studies in Assyriology, 3). København: Akademisk Forlag.

Ascher, Marcia, & Robert Ascher, 1986. "Ethnomathematics". *History of Science* **24**, 125-144.

Ascher, Marcia, 1988. "Graphs in Cultures: A Study in Ethnomathematics". *Historia Mathematica* **15**, 201-227.

Baines, John, 1988. "Literacy, Social Organization, and the Archaeological Record: The Case of Early Egypt", *in* Gledhill, Bender & Larsen (eds) 1988:192-214.

Beale, Thomas W., 1978. "Bevelled Rim Bowls and Their Implication for Change and Economic Organization in the Later Fourth Millennium B.C.". *Journal for Near Eastern Studies* **37**, 289-313.

Beale, T. W., & S. M. Carter, 1983. "On the Track of the Yahya Large Kuš: Evidence for Architectural Planning in the Period IVC Complex at Tepe Yahya". *Paléorient* **9**:1, 81-88.

Brunner, Hellmut, 1957. *Altägyptische Erziehung.* Wiesbaden: Otto Harrassowitz.

Carneiro, Robert L., 1970. "A Theory of the Origin of the State". *Science* **169**, 733-738.

Carneiro, Robert L., 1981. "The Chiefdom: Precursor of the State", pp. 37-79 *in* G. D. Jones & R. R. Kautz, *The Transition to Statehood in the New World.* Cambridge: Cambridge University Press.

Childe, V. Gordon, 1950. "The Urban Revolution". *Town Planning Review* **21**, 3-17.

Claessen, Henri J. M., & Peter Skalník (eds), 1978. *The Early State.* (New Babylon. Studies in the Social Sciences, 32). Hague etc.: Mouton.

Damerow, Peter, & Wolfgang Lefèvre (eds), 1981. *Rechenstein, Experiment, Sprache. Historische Fallstudien zur Entstehung der exakten Wissenschaften.* Stuttgart: Klett-Cotta.

Damerow, Peter, & Robert K. Englund, 1987. "Die Zahlzeichensysteme der Archaischen Texte aus Uruk", Chapter 3 (pp. 117-166) *in* M. W. Green & Hans J. Nissen, *Zeichenliste der Archaischen Texte aus Uruk,* Band II (ATU 2). Berlin: Gebr. Mann.

Damerow, Peter, Robert K. Englund & Hans J. Nissen, 1988. "Die Entstehung der Schrift". *Spektrum der Wissenschaften,* Februar 1988, 74-85.

Damerow, Peter, Robert K. Englund & Hans J. Nissen, 1988a. "Die ersten Zahldarstellungen und die Entwicklung des Zahlbegriffs". *Spektrum der Wissenschaften,* März 1988, 46-55.

Damerow, Peter, & Robert K. Englund, 1989. *The Proto-Elamite Texts from Tepe Yahya.* (The American School of Prehistoric Research, Bulletin 39). Cambridge, MA.: Peabody Museum of Archaeology and Ethnology / Harvard University Press.

Dandamajev, Mohammed A., 1971. "Die Rolle des *tamkārum* in Babylonien im 2. und 1. Jahrtausend v.u.Z.", pp. 69-78 *in* Horst Klengel (ed.) 1971:69-78.

Deimel, Anton, 1923. *Die Inscriften von Fara.* II, *Schultexte aus Fara,* in Umschrift herausgegeben und bearbeitet (Wissenschaftliche Veröffentlichungen der deutschen Orientgesellschaft, 43). Leipzig: J.C. Hinrichs'sche Buchhandlung.

Diakonoff, I. M. (ed.), 1969. *Ancient Mesopotamia, Socio-Economic History. A Collection of Studies by Soviet Scholars.* Moscow: »Nauka« Publishing House, Central Department of Oriental Literature.

Diakonoff, I. M., 1969a. "The Rise of the Despotic State in Ancient Mesopotamia", *in* I. M. Diakonoff (ed.) 1969:173-202.

Diakonoff, Igor M., 1971. "On the Structure of Old Babylonian Society", *in* H. Klengel (ed.) 1971:15-31.

Diakonoff, I. M., 1975. "The Rural Community in the Ancient Near East". *Journal of the Economic and Social History of the Orient* **18,** 121-133.

Edzard, Dietz Otto, 1969. "Eine altsumerische Rechentafel (OIP 14, 70), pp. 101-104 *in* W. Röllig (ed.) *Lišān mitḫurti. Festschrift Wolfram Freiherr von Soden zum 19.VI.1968 gewidmet von Schülern und Mitarbeitern.* (Altes Orient und Altes Testament, 1). Kevelaer: Butzon & Bercker / Neukirchen-Vluyn: Neukirchener Verlag des Erziehungsvereins.

Edzard, Dietz Otto, 1974. "»Soziale Reformen« im Zweistromland bis nach 1600 v. Chr.: Realität oder literarischer Topos?" *Acta Antiqua Academiae Scientiarum Hungaricae* **22**, 145-156.

Edzard, Dietz Otto, 1980. "Keilschrift". pp. 544-568 *in Reallexikon der Assyriologie und Vorderasiatischen Archäologie* V. Berlin: Walter de Gruyter.

Endesfelder, Erika, 1988. "Zur Herausbildung von Klassen und Staat im alten Ägypten", *in* in J. Herrmann & J. Köhn (eds) 1988:372-377.

Englund, Robert K., 1988. "Administrative Timekeeping in Ancient Mesopotamia". *Journal of the Economic and Social History of the Orient* **31**, 121-185.

Englund, Robert K., 1990. *Organisation und Verwaltung der Ur III-Fischerei.* (Berliner Beiträge zum Vorderen Orient, 10). Berlin: Dietrich Reimer.

Falkenstein, Adam, 1953. "Die babylonische Schule". *Saeculum* **4**, 125-137.

Finkelstein, J. J., 1969. "The Laws of Ur-Nammu". *Journal of Cuneiform Studies* **22**, 66-82.

Foster, Benjamin, 1981. "A New Look at the Sumerian Temple State". *Journal of the Economic and Social History of the Orient* **24**, 225-281.

Foster, Benjamin R., 1982. "Archives and Record-Keeping in Sargonic Mesopotamia". *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* **72**, 1-27.

Friberg, Jöran, 1978. "The Third Millennium Roots of Babylonian Mathematics. I. A Method for the Decipherment, through Mathematical and Metrological Analysis, of Proto-Sumerian and proto-Elamite Semi-Pictographic Inscriptions". *Department of Mathematics, Chalmers University of Technology and the University of Göteborg* No. 1978-9.

Friberg, Jöran, 1986. "The Early Roots of Babylonian Mathematics. III: Three Remarkable Texts from Ancient Ebla". *Vicino Oriente* **6**, 3-25.

Friberg, Jöran, 1990. "Mathematik". Pp. 531-585 *in Reallexikon der Assyriologie und Vorderasiatischen Archäologie* VII. Berlin & New York: De Gruyter.

Fried, Morton, 1967. *The Evolution of Political Society. An Essay in Political Anthropology.* New York: Random House.

Gadd, C. J., 1956. *Teachers and Students in the Oldest Schools.* An Inaugural Lecture Delivered on 6 March 1956. London: School of Oriental and African Studies, University of London.

Gelb, Ignace J., 1965. "The Ancient Mesopotamian Ration System". *Journal of Near Eastern Studies* **24**, 230-243.

Gledhill, John, Barbara Bender & Mogens Trolle Larsen (eds), 1988. *State and Society. The Emergence and Development of Social Hierarchy and Political Centralization*. (One World Archaeology, 4). London: Unwin Hyman.

Green, M. W., 1981. "The Construction and Implementation of the Cuneiform Writing System". *Visible Language* **15**, 345-372.

Hallo, William W., 1976. "Toward a History of Sumerian Literature", *in Sumerological Studies ...*, 181-203.

Heinrich, Ernst, 1938. "Grabungen im Gebiet des Anu-Antum-Tempels", pp. 19ff *in* A. Nöldeke et al, *Neunter vorläufiger Bericht über die von der deutschen Forschungsgemeinschaft in Uruk-Warka unternommenen Ausgrabungen*. Berlin: Verlag der Akademie der Wissenschaften.

Heinrich, E., 1982. *Die Tempel und Heiligtümer im Alten Mesopotamien*. 2 vols. (Denkmäler Antiker Architektur 14). Berlin: De Gruyter.

Herrmann, Joachim, & Irmgard Sellnow (eds), 1982. *Produktivkräfte und Gesellschaftsformationen in vorkapitalistischer Zeit*. (Veröffentlichungen des Zentralinstituts für Alte Geschichte und Archäologie, 12) Berlin: Akademie-Verlag.

Herrmann, Joachim, & Jens Köhn (eds), 1988. *Familie, Staat und Gesellschaftsformation. Grundprobleme vorkapitalistischer Epochen einhundert Jahre nach Friedrich Engels' Werk "Der Ursprung der Familie, des Privateigentums und des Staats*. Berlin: Akademie-Verlag.

Høyrup, Jens, 1980. "Influences of Institutionalized Mathematics Teaching on the Development and Organization of Mathematical Thought in the Pre-Modern Period. Investigations into an Aspect of the Anthropology of Mathematics". *Materialien und Studien. Institut für Didaktik der Mathematik der Universität Bielefeld* **20**, 7-137.

Høyrup, Jens, 1982. "Investigations of an Early Sumerian Division Problem, c. 2500 B.C.". *Historia Mathematica* **9**, 19-36.

Høyrup, Jens, 1983. "The Geometrical Pattern between Decoration, Art and Structuration of the Plan". I: Discussion Paper. II: Photographs. *Mimeographed manuscript*, Roskilde Universitetscenter, Institut for Uddannelsesforskning, Medieforskning og Videnskabsteori, October 1983.

Høyrup, Jens, 1985. *Babylonian Algebra from the View-Point of Geometrical Heuristics. An Investigation of Terminology, Methods, and Patterns of Thought*. Second, slightly Corrected Printing. Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science.

Høyrup, Jens, 1985a. "Varieties of Mathematical Discourse in Pre-Modern Socio-Cultural Contexts: Mesopotamia, Greece, and the Latin Middle Ages". *Science & Society* **49**, 4-41.

Høyrup, Jens, 1987. "The Formation of »Islamic Mathematics«. Sources and Conditions". *Science in Context* **1**, 281-329.

Høyrup, Jens, 1989. "Zur Frühgeschichte algebraischer Denkweisen". *Mathematische Semesterberichte* **36**, 1-46.

Høyrup, Jens, 1990. "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". To be published in *Altorientalische Forschungen* **17**.

Høyrup, Jens, 1990a. "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* **28**, 63-86.

Høyrup, Jens, 1990b. "On Parts of Parts and Ascending Continued Fractions". *Centaurus* **33**, 293-324.

Høyrup, Jens, 1992. "Sumerian: The Descendant of a Proto-Historical Creole? An Alternative Approach to »the Sumerian Problem«. Revised contribution to the Thirteenth Scandinavian Conference of Linguistics, University of Roskilde, January 9-11, 1992. *Roneo*. Abridged version to be published in the proceedings of the conference.

Hruška, Blahoslav, 1973. "Die innere Struktur der Reformtexte Urukaginas von Lagaš". *Archiv Orientální* **41**, 4-13, 104-132.

Hultsch, Friedrich (ed., transl.), 1876. Pappi Alexandrini *Collectionis* quae supersunt. E libris manu scriptis edidit et commentariis instruxit Fridericus Hultsch. I-III. Berlin: Weidmann, 1876, 1877, 1878.

Humanismus und Menschenbild im Orient und in der Antike. Konferenzvorträge, herausgegeben von der Sektion Orient- und Altertumswissenschaften der Martin-Luther-Universität Halle-Wittenberg. (Martin-Luther-Universität Halle-Wittenberg, Wissenschaftliche Beiträge 1977/28 (I 2)).

Jacobsen, Thorkild, 1943. "Primitive Democracy in Mesopotamia". *Journal of Near Eastern Studies* **2**, 159-182.

Jacobsen, Thorkild, 1957. "Early Political Development in Mesopotamia". *Zeitschrift für Assyriologie*, Neue Folge **18**, 91-140.

Jakobson, Vladimir A., 1971. "Some Problems Connected with the Rise of Landed Property (Old Babylonian Period)", *in* Klengel (ed.) 1971: 33-37.

Jasim, Sabah Abboud, & Joan Oates, 1986. "Early Tokens and Tablets in Mesopotamia: New Information from Tell Abada and Tell Brak" *World Archaeology* **17**, 348-362.

Jestin, Raymond, 1937. *Tablettes sumériennes de Šuruppak au Musée de Stamboul.* (Mémoires de l'Institut Français d'Archéologie de Stamboul, III). Paris: Boccard.

Johnson, Gregory A., 1975. "Locational Analysis and the Investigation of Uruk Local Exchange Systems", *in* Sabloff & Lamberg-Karlovsky 1975:285-339.

Klein, Jacob, 1981. *Three Šulgi Hymns. Sumerian Hymns Glorifying King Šulgi of Ur.* Ramat-Gan, Israel: Bar-Ilan University Press.

Klengel, Horst (ed.), 1971. *Beiträge zur sozialen Struktur des alten Vorderasien.* (Schriften zur Geschichte und Kultur des Alten Orients, 1). Berlin: Akademie-Verlag.

Klengel, Horst, 1974. "Einige Bemerkungen zur sozialökonomischen Entwicklung in der altbabylonischen Zeit". *Acta Antiqua Academiae Scientiarum Hungaricae* **22**, 249-257.

Klengel, Horst, 1977. "Zur Rolle der Persönlichkeit in der altbabylonischen Gesellschaft", *in Humanismus und Menschenbild ...*, 109-117.

Klengel, Horst, 1980. *Hammurapi von Babylon und seine Zeit.* 4., überarbeitete Auflage. Berlin: Deutscher Verlag der Wissenschaften.

Klengel, Horst (ed.), 1982. *Gesellschaft und Kultur im alten Vorderasien.* (Schriften zur Geschichte und Kultur des Alten Orients, 15). Berlin: Akademie-Verlag.

Komoróczy, Géza, 1978. "Landed Property in Ancient Mesopotamia and the Theory of the So-Called Asiatic Mode of Production". *Oikumene* **2**, 9-26.

Komoróczy, Géza, 1979. "Zu den Eigentumsverhältnissen in der altbabylonischen Zeit: Das Problem der Privatwirtschaft", *in* Lipinski (ed.) 1979:411-442.

Kraeling, Carl, & Robert McC. Adams (eds), 1960. *City Invincible.* A Symposium on Urbanization and Cultural Development in the Ancient Near East Held at the Oriental Institute of the University of Chicago. December 4-7, 1958. Chicago: University of Chicago Press.

Kramer, Samuel Noah, 1949. "Schooldays: A Sumerian Composition Relating to the Education of a Scribe". *Journal of American Oriental Studies* **69**, 199-215.

Kramer, Samuel Noah, 1983. "The Ur-Nammu Law Code: Who Was its Author?" *Orientalia*, NS **52**, 453-456.

Kraus, F. R., 1973. *Vom mesopotamischen Menschen der altbabylonischen Zeit und seiner Welt*. (Mededelingen der Koninklijke Nederlandse Akademie van Wetenschappen, Afd. Letterkunde. Nieuwe Reeks - Deel 36 - No. 6). Amsterdam & London: North-Holland.

Krecher, Joachim, 1973. "Neue sumerische Rechtsurkunden des 3. Jahrtausends". *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* **63** (1973-74), 145-271.

Krecher, Joachim, 1974. "Die Aufgliederung des Kaufpreises nach sumerischen Kaufverträgen der Fara- und der Akkade-Zeit". *Acta Antiqua Academiae Scientiarum Hungaricae* **22**, 29-32.

Lamberg-Karlovsky, C. C., 1976. "The Economic World of Sumer", *in* Schmandt-Besserat (ed.) 1976:59-68.

Lambert, Maurice, 1952. "La période présargonique". *Sumer* **8**, 57-77, 198-216.

Lambert, W. G., & A. R. Millard, 1969. *Atra-ḫasīs: The Babylonian Story of the Flood*. Oxford: Oxford University Press.

Landsberger, Benno, 1956. "Babylonian Scribal Craft and Its Terminology" pp. 123-126 *in*D. Sinor (ed.), *Proceedings of the 23rd International Congress of Orientalists, Cambridge, 21-28 August 1954*. London.

Landsberger, Benno, 1960. "Scribal Concepts of Education", *in* Kraeling & Adams 1960:94-102, discussion 104-123.

Larsen, Mogens Trolle, 1988. "Literacy and Social Complexity", *in* Gledhill, Bender & Larsen (eds) 1988:173-191.

Le Brun, A., & François Vallat, 1978. "L'origine de l'écriture à Suse". *Cahiers de la Délégation Archéologique Française en Iran* **8**, 11-59.

Leemans, W. F., 1950. *The Old Babylonian Merchant: His Business and his Social Position*. (Studia et documenta ad iura Orientis antiqui pertinentia III). Leiden: Brill.

Lipinski, Edward (ed.), 1979. *State and Temple Economy in the Ancient Near East*. 2 vols. Proceedings of the International Conference Organized by the Katholieke Universiteit Leuven from the 10th to the 14th of April 1978. (Orientalia Lovaniensa Analecta, 5-6). Leuven: Departement Oriëntalistiek.

Liverani, Mario, 1988. *Antico Oriente. Storia, società, economia*. Roma & Bari: Laterza.

Lucas, Christopher J., 1979. "The Scribal Tablet-House in Ancient Mesopotamia". *History of Education Quarterly* **19**, 305-332.

Luria, Aleksandr R., 1976. *Cognitive Development. Its Cultural and Social Foundations.* Edited by Michael Cole. Cambridge, Mass., & London: Harvard University Press.

Martzloff, Jean-Claude, 1988. *Historie des mathématiques chinoises*. Paris: Masson.

Mellaart, James, 1978. "Early Urban Communities in the Early Near East, *c*. 9000—3400 BC", *in* Moorey 1978:22-33.

MEW: Karl Marx, Friedrich Engels, *Werke*. 39 vols. Berlin: Dietz Verlag, 1965-1972.

MKT: O. Neugebauer, *Mathematische Keilschrift-texte*. I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937.

Moorey, P. R. S., 1979. *The Origins of Civilization*. Wolfson College Lectures 1978. Oxford: Clarendon Pres.

Nemet-Nejàt, Karen Rhea, forthcoming. *Cuneiform Mathematical Texts as a Reflection of Everyday Life in Mesopotamia*. Work in progress (presented at the Fourth Workshop on Concept Development in Babylonian Mathematics, May 1988).

Neugebauer, Otto, 1954. "Ancient Mathematics and Astronomy", pp. 785-803 *in* Ch. Singer, E. J. Holmyard & A. R. Hall (eds), *A History of Technology*. I. *From Early Times to Fall of Ancient Empires*. Oxford: Oxford University Press.

Neumann, Hans, 1987. *Handwerk in Mesopotamien. Untersuchungen zu seiner Organisation in der Zeit der III. Dynastie von Ur*. (Schriften zur Geschichte und Kultur des Alten Orients, 19). Berlin: Akademie-Verlag.

Neumann, Hans, 1988. "Bemerkungen zu den Eigentums- und Wirtschaftsverhältnissen in Mesopotamien gegen Ende des 3. Jahrtausends v.u.Z.", *in* J. Herrmann & J. Köhn 1988:335-343.

Neumann, Hans, 1989. "»Gerechtigkeit liebe ich...«. Zum Strafrecht in den ältesten Gesetzen Mesopotamiens". *Das Altertum* **35**:1, 13-22.

Nissen , Hans J., 1974 . "Zur Frage der Arbeitsorganisation in Babylonien während der Späturuk-Zeit". *Acta Antiqua Academiae Scientiarum Hungaricae* **22**, 5-14.

Nissen, Hans J., 1981. "Bemerkungen zur Listenlitteratur Vorderasiens im 3. Jahrtausend (gesehen von den Archaischen Texten aus Uruk)", pp. 99-108 *in* Luigi Cagni (ed.), *La Lingua di Ebla*. Atti del Convegno Internazionale (Napoli, 21-23 aprile 1980). (Istituto Universitario Orientale, Seminario di Studi asiatici, series minor XIV). Napoli.

Nissen, Hans J., 1982. "Die »Tempelstadt«: Regierungsform der frühdynastischen Zeit in Babylonien?", *in* H. Klengel (ed.) 1982:195-200.

Nissen, Hans J., 1983. *Grundzüge einer Geschichte der Frühzeit des Vorderen Orients*. (Grundzüge, 52). Darmstadt: Wissenschaftliches Buchgesellschaft.

Nissen, Hans J., 1985. "The Emergence of Writing in the Ancient Near East". *Interdisciplinary Science Reviews* **10**, 349-361.

Nissen, Hans J., 1986. *Mesopotamia Before 5000 Years.* (Istituto di Studi del Vicino Oriente. Sussidi didattici). Roma: Istituto di Studi del Vicino Oriente.

Nissen, Hans J., 1986a. "The Archaic Texts from Uruk". *World Archaeology* **17**, 317-334.

Nissen, Hans J., 1986b. "The Development of Writing and of Glyptic Art", pp. 316-331 *in* U. Finkbeiner & W. Röllig (eds), *Ǧamdat Naṣr—Period or Regional Style?* (Beihefte zum Tübinger Atlas des Vorderen Orients, Reihe B Nr. 62). Wiesbaden: Ludwig Reichert.

Oates, Joan, 1960. "Ur and Eridu, the Prehistory". *Iraq* **22**, 32-50.

Oppenheim, A. Leo, 1967. "A New Look at the Social Structure of Mesopotamian Society". *Journal of the Economic and Social History of the Orient* **18**, 1-16.

Powell, Marvin A., 1971. *Sumerian Numeration and Metrology.* Dissertation, University of Minnesota.

Powell, Marvin A., Jr., 1972. "Sumerian Area Measures and the Alleged Decimal Substratum". *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* **62** (1972-73), 165-221.

Powell, Marvin A., 1976. "The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics". *Historia Mathematica* **3**, 417-439.

Powell, Marvin A., 1978. "Götter, Könige und »Kapitalisten« im Mesopotamien des 3. Jahrtausend v. u. Z.". *Oikumene* **2**, 127-144.

Powell, Marvin A., 1986. "Economy of the Extended Family According to Sumerian Sources". *Oikumene* **5**, 9-13.

Pritchard, J. B. (ed.), 1950. *Ancient Near Eastern Texts Relating to the Old Testament.* Princeton, N.J.: Princeton University Press.

Renger, Johannes, 1979. "Interaction of Temple, Palace, and »Private Enterprise« in the Old Babylonian Economy", *in* Lipinski (ed.) 1979:249-256.

Romaine, Suzanne, 1988. *Pidgin and Creole Languages.* London & New York: Longman.

Römer, W. H. Ph., 1980. *Das sumerische Kurzepos "Bilgameš und Akka".* (Alter Orient und Altes Testament, 209/1. Nimwegener sumerologische Studien I. Studien zu sumerischen literarischen Texten 1). Kevelaer: Butzon & Bercker / Neukirchen-Vluyn: Neukirchener Verlag.

Runciman, W. G., 1982. "Origins of States: The Case of Archaic Greece". *Comparative Studies in Society and History* **24**, 351-377.

Tyumenev, A. I., 1969. "The State Economy in Ancient Sumer" [Summary of the Book *State Economy in Ancient Sumer*, Moscow-Leningrad 1956], *in* Diakonoff (ed.) 1969:70-87.

Tyumenev, A. I., 1969a. " The Working Personnel on the Estate of the Temple of Ba-U_2 in Lagaš during the Period of Lugalanda and Urukagina (25th-24th cent. B.C.)" [Slightly abbreviated translation of pp. 136-169 in *State Economy in Ancient Sumer*, Moscow-Leningrad 1956], *in* Diakonoff (ed.) 1969:88-126.

Vaiman, A. A., 1974. "Über die protosumerische Schrift". *Acta Antiqua Academiae Scientiarum Hungaricae* **22**, 15-27.

van Dijk, J. J. A., 1953. *La sagesse suméro-accadienne. Recherches sur les genres littéraires des textes sapientiaux*. Leiden: Brill.

Vanstiphout, H. L. J., 1978. "Lipit-Estar's Praise in the Edubba". *Journal of Cuneiform Studies* **30**, 33-61.

von Soden, Wolfram, 1936. "Leistung und Grenze sumerischer und babylonischer Wissenschaft". *Die Welt als Geschichte* **2**, 411-464, 507-557.

Weiss, Harvey, 1977. "Periodization, Population, and Early State Formation in Khuzistan". *Bibliotheca Mesopotamica* **7**, 347-369.

Whiting, Robert M., 1984. "More Evidence for Sexagesimal Calculations in the Third Millennium B.C." *Zeitschrift für Assyriologie und Vorderasiatische Archäologie* **74**, 59-66.

Wittfogel, Karl, 1957. *Oriental Despotism: A Comparative Study of Total Power*. New Haven, Conn.: Yale University Press.

Wright, Gary A., 1978. "Social Differentiation in the Early Natufian", pp. 201-223 *in* C. A. Redman et al (eds), *Social Archeology. Beyond Subsistence and Dating*. New York: Academic Press.

Wright, Henry T., & Gregory A. Johnson, 1975. "Population, Exchange, and Early State Formation in Southwestern Iran". *American Anthropologist* **77**, 267-289.

Sabloff, Jeremy A., & C. C. Lamberg-Karlovsky (eds), 1975. *Ancient Civilization and Trade*. Albuquerque: University of New Mexico Press.

Schmandt-Besserat, Denise (ed.), 1976. *The Legacy of Sumer*. Invited Lectures on the Middle East at the University of Texas as Austin. (Bibliotheca Mesopotamica 4). Malibu, Ca.: Undena.

Schmandt-Besserat, Denise, 1977. "An Archaic Recording System and the Origin of Writing". *Syro-Mesopotamian Studies* **1**:2.

Schmandt-Besserat, Denise, 1978. "The Earliest Precursor of Writing". *Scientific American* **238**:6 (June 1978), 38-47 (European edition).

Schmandt-Besserat, Denise, 1986. "Tokens: Facts and Interpretation". *Visible Language* **20**, 250-272.

Schmandt-Besserat, Denise, 1988. "Tokens as Funerary Offerings". *Vicino Oriente* **7**, 3-9, Tav. I-V.

Schneider, Nikolaus, 1940. "Die Urkundenbehälter von Ur III und ihre archivalische Systematik". *Orientalia*, NS **9**, 1-16.

Service, Elman R., 1975. *Origins of the State and Civilization. The Process of Cultural Evolution*. New York: W. W. Norton.

Sjöberg, Åke W., 1972. "In Praise of the Scribal Art". *Journal of Cuneiform Studies* **24** (1971-72), 126-129.

Sjöberg, Åke W., 1973 . "Der Vater und sein mißratener Sohn" *Journal of Cuneiform Studies* **25**, 105-169.

Sjöberg, Åke W., 1975 . "Der Examenstext A". *Zeitschrift für Assyriologie und vorderasiatische Archäologie"* **64**, 137-176.

Sjöberg, Åke W., 1976 . "The Old Babylonian Eduba", *in Sumerological Studies ...*, 159-179.

Sollberger, Edmond, & Jean-Robert Kupper, 1971. *Inscriptions royales sumériennes et akkadiennes*. (Littératures anciennes du Proche-Orient). Paris: Éditions du Cerf.

Stone, Elisabeth C., 1982. "The Social Role of the Nadītu Women in Old Babylonian Nippur". *Journal of the Economic and Social History of the Orient* **25**, 50-70.

Sumerological Studies in Honor of Thorkild Jacobsen on his Seventieth Birthday, June 7, 1974. (The Oriental Institute of the University of Chicago, Assyriological Studies, 20). Chicago & London: University of Chicago Press, 1976.

Thureau-Dangin, F., 1907. *Die sumerischen und akkadischen Königsinschriften*. (Vorderasiatische Bibliothek, I. Band Abteilung 1). Leipzig: J. C. Hinrichs'sche Buchhandlung.

H

"Dynamis, the Babylonians, and Theaetetus 147c7 – 148d7". *Historia Mathematica* **17** (1990), 201–222.

ERRATA

For some reason, a large number of my corrections in the proofs of the present paper were not taken into account by the publisher. Fortunately, most errors are merely irritating but do not disturb the argument of the text (»corrections« of quotations against the manuscript, misspelled Greek, or disrespect for the convention by which assyriologists distinguish Sumerian from Akkadian words). A few, however, affect the core of the discussion. The complete list is as follows (superscript $\pm n$ indicates line number, counted from above or below, respectively):

Page	Actual text	Read
203^{+8}	are not	aren't
203^{+18}	another circle of diameter d_2	the diameter d_2 of another circle
204^{+4}	"two feet	"of two feet
205^{-8}	corrected	faithful
206^{-9}	*δῆλος τω γε λογιστικω*	*δῆλον τῳ γε λογιστοκῳ*
207^{+13}	*ζ*	ς
208^{-20}	*íbsi*$_8$	í b - s i$_8$
208^{-18}	*íb-si*$_8$	í b - s i$_8$
208^{-15}	*mithartum*	*mitḫartum*
209^{-22}	*íb-si*$_8$	í b - s i$_8$
209^{-21}	*ib-si*$_8$	í b - s i$_8$
209^{-19}	side (at	side—at
209^{-17}	And in still other instances	In one specific type of texts, finally
211^{+7}	*mithartum*	*mitḫartum*
211^{+16}	Aristotle in	Aristotle refers to in
211^{+17}	1019^{b}33f, and	1019^{b}33f (cf. above), and
212^{+10}	*íb-si*$_8$	í b - s i$_8$
215^{-2}	*a-š à*	a - š à
218^{+13}	mass	masse
218^{-12}	^{c}bq	$^{\supset}$bq
218^{-1}	*íb-si*$_8$	í b - s i$_8$

Remaining errors are my own responsibility, either because I did not notice them in the proof sheets or because I committed them myself in the first place. I have located three:

213^{-9}	*mithartum*	*mitḫartum*
218^{+12}	*ὅκος*	*ὄγκος*

Finally, the ascription of the *Liber podismus* to Nipsus on p. 218^{-21} is mistaken, as shown by Bubnov (Gerberti *Opera mathematica* edd N. Bubnov, Berlin 1899, p. 399).

Jens Høyrup

HISTORIA MATHEMATICA 17 (1990), 201–222

Dýnamis, the Babylonians, and Theaetetus 147c7–148d7

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DEDICATED TO OLAF PEDERSEN ON THE OCCASION OF THE BEGINNING OF HIS EIGHTH DECADE

Traditionally, the Greek mathematical term *dýnamis* is interpreted alternately as "square" and "root/side of square." A survey of the usages of the term and of the related verb *dýnasthai* by Plato, Aristotle, and various mathematical authors including Eudemos/Hippocrates, Euclid, Archimedes, Hero, Diophantos, and Nicomachos shows that all are compatible with a familiar concept of Babylonian mathematics, the square identified by (and hence with) its side. It turns out that a "geometers' *dýnamis*" and a "calculators' *dýnamis*" must be differentiated; that the technical usage for the former became fixed only around the mid-fourth century B.C.; and that it vanished except in specific connections and formulaic expressions by the third century. Along with the conceptual congruity, the Babylonian and Greek terms share a number of everyday connotations. This suggests that the Greek concept may have been inspired or borrowed from the Near East. This hypothesis can be neither proved nor disproved directly by the sources, but it is internally coherent and fruitful with regard to the existing material.

La tradition interprête alternativement le terme mathématique grec *dýnamis* comme "carré" et "racine carrée". Un aperçu sur les modes d'emploi du terme grec chez Platon, Aristote, et chez un nombre de mathématiciens (dont Eudème/Hippocrate, Euclide, Archimède, Héron, Diophante, et Nicomaque) fait pourtant voir que l'on peut comprendre tous ces modes d'emplois à partir d'un concept familier aux mathématiques babyloniennes, à savoir le carré identifié par (et donc avec) son côté. Il s'ensuit aussi qu'il faut distinguer entre la "*dýnamis* des géomètres" et la "*dýnamis* des calculateurs"; que l'usage du premier ne devient fixe qu'au milieu du quatrième siècle avant J.-C; et qu'il disparaît du discours géométrique courant à partir du troisième siècle avant J.-C. et n'est conservé que dans des contextes spécifiques. Le contenu conceptuel commun et l'existence de connotations secondaires partagées suggèrent la possibilité d'un emprunt du concept. Cette hypothèse ne se laisse ni prouver ni réfuter directement par les sources; elle résulte pourtant cohérente et féconde pour l'interprétation des documents existants.

Gewohnheitsgemäß wird der griechische mathematische Terminus *dýnamis* abwechselnd als "Quadrat" und "Quadratwurzel/-Seite" verstanden. Eine Übersicht über die verschiedenen Anwendungen des Ausdrucks und des verwandten Verbum *dýnasthai* bei Platon und Aristoteles und bei mathematischen Autoren von Eudemos/Hippokrates über Euklid, Archimedes und Apollonios bis Heron, Nikomachos und Diophantos zeigt jedoch, daß sie alle mit einem bekannten Begriff aus der babylonischen Mathematik vereinbar sind, nämlich dem Quadrat identifiziert durch (und dadurch auch mit) seiner Seite. Es zeigt sich auch, daß eine "*dýnamis* der Geometer" und eine "*dýnamis* der Rechner" zu unterscheiden sind; daß erstere erst gegen Mitte des vierten Jahrhunderts v.u.Z. als technische Terminologie völlig standardisiert wird; und daß sie zur Zeit Euklids außer in besonderen Verbindungen und formelhaften Ausdrücken wieder verschwindet. Zusammen mit der begrifflichen Übereinstimmung zwischen den babylonischen und griechischen Termini deuten eine Reihe von gemeinsamen alltagssprachlichen Konnotationen darauf hin, daß der griechische Begriff von der nahöstlichen Mathematik angeregt oder übernommen worden ist. Diese Vermutung läßt sich aus den Quellen weder endgültig beweisen noch widerlegen; sie ist aber nicht nur kohärent, sondern auch fruchtbar für die Interpretation des gesamten Quellenmaterials.

Among the most debated single terms of ancient Greek mathematics is the word *dýnamis* [1], the basic everyday meaning of which is "power," "might," "strength," "ability," etc. [GEL 452^{a-b}]. Responsible for this debate are first of all the paradoxical ways in which Plato uses the term in *Theaetetus*, especially because these appear to disagree with Euclid's use of the term in the *Elements*.

The word is absent from Books I through IX of the *Elements*. In Book X, Definition 2, however, we read that straight lines (*εὐθεῖαι*) are "commensurable in respect of *dýnamis* (*δυνάμει σύμμετροι*) whenever the squares on them (*τὰ ἀπ' αὐτῶν τετράγωνα*) are measured by the same area." This indicates that *dýnamis* should be read as "square," while raising the problem of why it is used instead of the current term *tetragon*.

In Plato's *Theaetetus*, a "*dýnamis* of three feet" ([*δύναμις*] *τρίπους*) appears to be a square with an area of 3 square feet (147d 3–4) [2]. A little bit later, however, *dýnamis* is the term chosen for certain *lines* (*γραμμαὶ*)—*viz.*, lines which "square off" (*τετραγωνίζειν*) nonsquare numbers (anachronistically expressed, lines the lengths of which are surds). The latter use of the word has given rise to the other traditional interpretation of the word, as "side of square" or "square root"—at times as "irrational square root."

A third text has often been taken into account in these discussions. In Eudemos' account of Hippocrates of Chios' investigation of the lunes (as quoted by Simplicios [Thomas 1939, 238]) it is stated (in words which are often taken to go back to Hippocrates himself) that similar circular segments have the same ratios "as their bases in respect of *dýnamis*" (*καὶ αἱ βάσεις αὐτῶν δυνάμει*), while circles have the same ratio "as the diameters in respect of *dýnamis*." The Euclidean dative form *dynámei* is thus found (with approximately the same meaning) in a text dating back to the fourth or maybe even the fifth century.

FURTHER OCCURRENCES: THE EARLY EPOCH

In this paper, I intend to show that the apparently equivocal use of the term need not be considered equivocal after all in light of an analogous conceptual structure in Babylonian mathematics. Before presenting this parallel I shall, however, give a more precise survey of the mathematical usages of the Greek term, in order to uncover more fully its context and development.

There are, indeed, a number of less frequently discussed occurrences of the term and of the related verb *dýnasthai* (*δύνασθαι*; non-technical meaning "to be able/strong enough (to do something)," " to be worth," "to be able to produce," etc. [GEL 451^{b}–452^{a}]). As a preliminary (semantically uncommitted) translation integrating connotations of physical power as well as commercial value, I shall use "be worth" when discussing the mathematical uses of the verb. Instead of the expression "in respect of *dýnamis*" I shall mostly use the Greek dative *dynámei*.

The verb is used in close connection with the noun in the central *Theaetetus* passage (148a6–b2):

> THEAETETOS. We defined all the lines that square off equal-sided numbers on plane surfaces as lengths, and all the lines that square off oblong [i.e., nonsquare—JH] numbers as *dynámeis*, since they are not commensurable with the first sort in respect of length but only in respect of the plane figures which they are worth.

This translation reproduces McDowell as quoted by Burnyeat [1978, 493], with these exceptions: "*dynámeis*" is used instead of "powers"; "are worth" instead of "have the power to form"; and "in respect of length" instead of "in length," in order to render the parallel uses of the dative forms δυνάμει and μήκει. It can be seen that the lines which are labelled *dynámeis* "are worth" those squares of which they are the sides (anachronistically: The line of length $\sqrt{3}$ "is worth" the square of area 3).

In the Eudemos/Hippocrates fragment, the diameter d_1 of one circle is said to "be worth" the sextuple of another circle of diameter d_2 when it "is" its sextuple *dynámei*, i.e., when $d_1{}^2 = 6d_2{}^2$ (248^5 and 250^5 combined); the diameter of a circle, being the double of the radius "in length" (μήκει) is its quadruple *dynámei* (250^4). Furthermore, the two short sides in a right-angled triangle "are worth the same" (ἴσον) as the hypotenuse (250^1), while a line a is said to "be worth less" than two others b and c when $a^2 < b^2 + c^2$ (242^4).

In Aristotle's *De incessu animalium* 708^b33–709^a2, on the other hand, the hypotenuse of a right-angled triangle is said to "be worth" (not "worth the same as") the two other sides [3]; according to Heath ([1949, 284] against [GEL 452^a44–45] following the Oxford translation), the same usage is meant in 709^a18–22. An identical formulation of the Pythagorean theorem is found in the pseudo-Aristotelian *De lineis insecabilibus* 970^a12–14.

In connection with a general discussion of "potency" and "potent" (δύναμις and δυνατός, respectively), Aristotle explains in *Metaphysica* 1019^b33–34 that the term *dýnamis* is used in geometry "by metaphor"; in 1046^a6–8 he explains the usage as due to "resemblance" (ὁμοιότης). An explanation of the concept as derived from Aristotelian (or older natural) philosophy should thus be excluded—even though a metaphor along the lines of "the square which a line is able to produce" would perhaps not be far from Aristotle's own understanding of the term [4].

The examples given so far demonstrate beyond a doubt that *dýnamis* and *dýnasthai* belong to accepted fourth- (and maybe fifth-) century geometrical parlance. They might also suggest that the use of *dýnamis* in *Theaetetus* as a designation for a line (be it a specific sort of line) is a Platonic hint of an idiosyncrasy of the young Theaetetos—as suggested by Burnyeat [1978, 496].

The first of these theses is confirmed by another Platonic passage, while the second is falsified (*pace* Burnyeat). *Politicus* 266a–b contains a pun on the word (already discussed by Burnyeat [1978, 496] and by Szabó [1969, 90]): Man, having the ability (*dýnamis*) to walk on two feet (being "of two feet in respect of ability"/ δίπους δυνάμει) is identified with the diagonal [of the unit square], which is also "two feet *dynámei*." Similarly, the swine, being four footed in respect of ability, is the "diagonal of the diagonal" (being of four feet *dynámei* it must be of length 2, and so be the diagonal of a square with side $\sqrt{2}$). We observe that the "human" diagonal is regarded in the second instance as something possessing itself a diagonal, i.e., as a square, in a way which defies both the interpretation of the *dýnamis* as a square pure and simple and the traditional alternative "side"/"square root."

Because Theaetetos and "the young Socrates" participate together in the dialogue as they do in *Theaetetus*, Burnyeat interprets the passage as another reference to Theaetetos' characteristic idiom. The pun is, however, put forward by the "Stranger from Elea," and furthermore with the words "since both of you are devoted to geometry." Had Plato wanted to hint at Theaetetos' own terminological contributions or habits he would hardly have chosen this way to express himself. Instead, the pun must be a play on the familiar and shared terminology of geometers of the period (or, rather, a terminology which a mid-fourth century philosopher would find natural in the mouth of a late fifth-century geometer).

FURTHER OCCURRENCES: THE EPOCH OF MATURITY

As is well known, almost all sources for the history of Greek mathematics date from the third century B.C. or later. Truly, in this age of maturity Greek mathematicians tended to make less use of the *dýnamis*/*dýnasthai* structure than their forerunners appear to have done in the period from Hippocrates to Eudemos. Still, both terms occur a number of times in the great mathematical authors from Euclid onward, and in ways which may serve to elucidate the terminology, revealing continuity with earlier usages of varying character.

In *Data* 64, 65, and 67, Euclid speaks of the amount by which one side of a triangle "is worth" more or less than the other two sides, with the same meaning and in the same connection as Hippocrates/Eudemos. In the ensuing demonstrations, however, he only refers to "the tetragons on" the sides. The same thing occurs in Proposition 86. It appears as if the *dýnamis*/*dýnasthai* usage had been current at a time (fifth and fourth century) when certain theorems and standardized expressions were first formulated (the point in question here being the extended Pythagorean theorem), and that those formulations were handed down faithfully [5]. But the actual proofs of the *Data* were formulated in the current terminology, which spoke of tetragons and not *dynámei*.

Stronger evidence for a changing usage is seen in the *Elements*. Here the *dýnamis* is avoided even in the formulations of the theorems until Book X. So, the Pythagorean theorem, which both Eudemos/Hippocrates and the Aristotelian corpus refer to time and again but in *dýnasthai* dress, deals here with "the tetragons on" the sides (I.47). The same holds for XII.2, in which "circles are to each other as the tetragons on their diameters," whereas Eudemos/Hippocrates had spoken of the ratio "between the diameters *dynámei*."

In Books X and XIII we find the traditional usage—but only in definitions, in theorems, and in proofs referring to definitions or theorems or (in a few cases) summing up a result in formulaic language. During the free discursive argumentation on figures, all references are to "the tetragons on" the lines in question. X, Definition 2, which was quoted above, explains the formula "commensurability *dynámei*" of two straight lines as "commensurability of the tetragons on" the lines, and can thus be taken as a paradigm for the general relation between formulae and free speech.

The formulae which are used belong without exception to types with which we are already familiar from earlier sources. We find the counterposition of "commensurability in respect of length" (μήκει) and *dynámei* (e.g., X, Definition 3); line a "being worth more" than line b (e.g., X.14), "being worth n times" b (e.g. XIII.2), or line a "being worth" lines b and c (e.g., XIII.10). Finally, a line may "be worth" an area (e.g., X.40) or a figure (e.g., XIII.1).

On the faith of Proclos, Archimedes is normally taken to have worked after Euclid. As observed by Schneider [1979, 61f, n. 82] and Knorr [1978, 221], however, his works build on pre-Euclidean mathematics and not on the Euclidean *Elements*; as a witness of early terminology, he can thus be considered on a par with Euclid.

Archimedes' use of the *dýnamis/dýnasthai* terminology varies from work to work—a fact which was used by Knorr as supplementary evidence in his investigation of the relative chronology of the Archimedean corpus [1978, 264, n. 124a]. Most of the occurrences fall under the types also attested to in Euclid: ratio *dynámei* in contrast to ratio *simpliciter* or *mékei*, and a line "being worth" a rectangle or a plane figure. At times, however, a line "is worth the same" as a rectangle (e.g., *De sphaera et cylindro* I.29, 124[1]). Furthermore, there seems to be a tendency (according to Knorr's relative chronology) for earlier works to use occasionally the idiom in free speech and for late works to restrict it to formulaic expressions and quotations of established theorems.

Like Euclid's work, the Archimedean corpus thus suggests that the *dýnamis/dýnasthai* usage was left behind in the free language of third-century geometers but was preserved (and still used) in a frozen state in formulaic expressions. This is further confirmed in Apollonios' *Conica*, with one qualification: Apollonios takes advantage of the possibilities of the terms to compress complicated expressions, creating formulae of his own (e.g., III.54, 440[15], where a ratio is composed from one ratio *dynámei* and another ordinary ratio between areas).

Later geometers would still use the formulae but only by tradition. This is demonstrated by Pappos, in whose *Mathematical Collection* (along with some 20 corrected quotations of the old formulations) the *dýnamis* and *tetragon* formulations of the *Data* are mixed up as "the *dynámeis* of the sides of the triangles" (638[11–13]). Direct and indirect testimony is supplied by an anonymous 2nd century A.D. commentary to *Theaetetus* [Burnyeat 1978, 497]: It tells that "the ancients called tetragons *dynámeis*"; evidently, the readers were supposed not to know—and the commentator for his part seems not to know that the two terms, though somehow semantically connected, had been used differently.

It is then no wonder that even Hero speaks of ratios *dynámei* vs. *mékei* in

Metrica I.19, 54^{18}—nor that a passage of I.34 (82^{28f}) appears to make a rectangle and not a line the subject of the verb *dýnasthai* (appears, since the passage is anyhow illegitimately elliptic and therefore possibly corrupt [6]). At other points, however, striking deviations from familiar expressions turn up. A passage in I.15 (42^{22-25}) runs "and take away from *dynámei* 121 *dynámei* 36, remainder *dynámei* 25, which is *mékei* 5." *Dynámei* 121 is thus simply $\sqrt{121} = 11$, which in a more traditional formulation might appear as "that which *dynámei* is 121," corresponding also to the expression "BΘ *dynámei* 180" found three lines above (freely to be interpreted $B\Theta^2 = 180$ or $B\Theta = \sqrt{180}$). But the phrase in lines 22–24 contains none of those articles and relative pronouns which in normal Greek mathematical texts indicate elided words. *Dynámei N* is simply used for $\sqrt{N}$.

If we go to I.17, 48^{5f}, on the other hand, "the ratio of the ⟨tetragon⟩ on *dýnamis* BΓ to the ⟨tetragon⟩ on BΓ, together with the ⟨tetragon⟩ on AΔ" designates the ratio of $B\Gamma^4$ to $B\Gamma^2 \cdot A\Delta^2$. *Dýnamis N* is thus N^2. So, the Platonic ambiguity between "square" and "square root" turns up again in this rather late and very un-Platonic text (though grammatically distinguished as it should be in an efficient technical terminology).

THE "CALCULATOR'S *DYNAMIS*"

The $B\Gamma^4$ of *Metrica* I.17 is also spoken of as "the *dynamodýnamis* upon BΓ" (48^{21}), Diophantos' term for the fourth power. It might therefore seem that the numerically oriented mathematicians of later antiquity merely embraced a traditional geometrical concept and shaped it for their own purposes. I shall call this concept (that of Hippocrates, Euclid, Archimedes, Apollonios and Pappos) the "geometers' *dýnamis*," in agreement with the passages from *Metaphysica* and *Politicus* quoted above. More likely, however, the similarities between Plato's and Hero's texts should be explained with reference to an old, related but distinct "calculators' *dýnamis*." To this point I shall return; for the moment I shall only argue for the existence of the entity in question.

It turns up rather explicitly in Plato's *Republica* 587d, during the discussion of the distance between the tyrant's phantasmagoric pleasure and real pleasure, which, when regarded as "number of the length" (*τοῦ μήκους ἀριθμός*), is argued by Socrates to be the "plane number" $3 \cdot 3 = 9$. It is then "clear, in truth, how great a distance it is removed according to *dýnamis* and third increase" (*κατὰ δύναμιν καὶ τρίτην αὔξην*)—a statement upon which Glaucon comments: "clear at least to the calculator" (*δῆλος τῷ γε λογιστικῷ*). In this gently ironic portrait of his brother [7] Plato evidently supposes that the mathematically illiterate will have known the word *dýnamis* as belonging to the field of practical calculation (logistics) rather than to that of theoretical geometry. Furthermore, logistics is supposed by Socrates' remark to deal with three different numerical manifestations of one and the same entity, as "number of the length," *dýnamis*, and "third increase." As if to avert misunderstanding, Plato tells us that these are not just the "linear," "square," and "cube numbers" known from Greek theoretical arithmetic (and from *Theaetetus*), the "number of the length" being already a square number; they correspond instead to the first, second, and third power of the entity.

Presumably, the "calculators' *dýnamis*" is also mentioned in *Timaeus* 31c–32a [8]. At most, however, this passage provides us with the extra information that the terminology for the "third power" vacillated. More interesting as an elucidation of the *Republica* passage and of the "calculators' *dýnamis*" are the terms used in Diophantos' *Arithmetica*. As he explains in his foreword, Diophantos speaks of square and cube numbers as "tetragons" (*τετράγωνοι*) and "cubes" (*κύβοι*), respectively (2^{18-22}). *In agreement with general convention,* however, the second and third power of the unknown number (the *ἀριθμός*) are spoken of as *dýnamis* (*δύναμις*, abbreviated Δ^Y) and cube (*κύβος*/K^Y) (4^{15-17}) [9]. Now, it is known that part of Diophantos' algebraic formalism is taken from earlier Greek calculators: the abbreviation ζ for the *ἀριθμός* is used in a ca. 1st century (A.D.) papyrus (see [Robbins 1929] and [Vogel 1930]), and the term *δυναμοδύναμις* for the fourth power was used during the same century by Hero (cf. above). Furthermore, part of Diophantos' material (I.xvi–xix, xxii–xxv) is borrowed from traditions of recreational mathematics ("purchase of a horse," "finding a purse," etc.; see [Tropfke 1980, 606–613]) which already in Plato's time had given rise to theoretical treatments ("Thymarides' flower"; see [Heath 1921, 94ff]). Since the distinction made between *square number* and *dýnamis* coincides with that made in *Republica* 587d, it appears reasonable to assume that even this is due to continuity, and that Diophantos' "general convention" followed the old calculators known to Glaucon in its specific use of *dýnamis* [10].

If this is so, "geometers'" and "calculators' *dýnamis*" are of course related but yet different concepts, and one must be assumed to derive from the other. For the moment, we will have to leave open the question of the direction of influence, and return our attention to the geometers' concept, which is better documented in the sources.

INTERPRETING *DYNAMIS*

The difficulty of explaining *dýnamis* plainly as another name either for *tetragon* or for *side* is as evident as the difficulty of explaining away the evidence in favor of the rival explanation. Instead, two new interpretations (both involving centrally the verb *dýnasthai*) have been proposed by Szabó and Taisbak.

Taisbak [1980; summarized in 1982, 72–76] proposed a reading of *dýnasthai* as "to master," in the sense that a line "masters" that two-dimensional extension which it is able to cover by a square; this *extension* should be understood as an entity different both from the square as a *geometrical figure* and from its area regarded as a *number* resulting from mensuration. In its origin, *dýnamis* should then be a term for the extension. For later times, Taisbak proposed a reduction to an ill-understood rudiment. The use of the term for a line should result from informal speaking among mathematicians.

Szabó's explanation [1969, 46f; reworked 1986] built on the well-documented use of *dýnasthai* as "being worth" in a real commercial sense ("the *shekel* is worth

7 obols"). This is supposed to have inspired a use expressing the notion that *a square* is equal to *some other surface* (a rectangle or a sum of squares); for some reason ("irgendwie" [1986, 359]), the expression involves the side of the square as the subject, and not the square itself. Formally, a *dýnamis* should be a line; in reality, however, it should denote the square constructed upon the line, but only on condition that this square is equal to another surface.

In order to underpin his interpretation, Szabó claimed that the *κατὰ δύναμιν* usage of the passages from *Republica* and *Timaeus* (in fact the earliest certain appearances of the mathematical *dýnamis*) is derivative, while the dative *dynámei* used from the late Platonic dialogues onward reflects the original thinking. Even if this hypothesis is granted, the rather loose language of the remaining pre-Euclidean sources is problematic for a strict reading of Szabó's thesis—a line being sometimes worth other lines, sometimes "the same" as other lines, etc. If the thesis is read more loosely than originally intended, however, as informal speaking, neither the early Platonic occurrences nor the lax formulations are serious challenges; interpreted like this, on the other hand, the explanation comes closer to Taisbak's.

Before considering either of these positions. I shall step outside the circle of Greek language and culture.

A BABYLONIAN PARALLEL

To a historian of Babylonian mathematics, the apparent ambiguity between "square" and "square root" has a familiar ring. Both the basic Old Babylonian term for a geometric square (*mitḫartum*) and the Sumerogram normally translated as "square root" (*íbsi*$_8$) appear (when translated into modern terminology and concepts) to designate alternately the square and its side. The semantic basis of *íb-si*$_8$ is *equality* (*viz.*, equality of the sides of a square), whereas that of *mitḫartum* is the confrontation of equivalents (still as sides of a square). Interestingly, the Babylonian term for "countervalue" or "commercial rate" (*maḫirum*) derives from the same root as *mithartum*, *viz.*, from *maḫārum*, "to stand up against, to encounter, to receive [an antagonist, an equivalent, a peer]." So, the linking of "square," "side of square," "commercial rate," "equivalence," and "confrontation of force," so puzzling in Greek mathematics, is shared with the mathematics of the old eastern neighbor. Could it be that the Greek term translates a borrowed technical concept, using a Greek term possessing the same connotational range as the original Semitic expression [11]? And could a possible borrowing, or simply the conceptual parallel, help us understand the nuances of the Greek term?

Since our earliest sources (be it Plato or the Eudemos/Hippocrates fragment) use the *dýnamis* terminology in developed form, the original idea behind it cannot be established beyond doubt, and conceptual and terminological diffusion (from Babylonia or, indeed, from anywhere) can be neither proved nor ruled out as a possibility. The answer to the first question is an uninteresting "yes—anything *could* be." For the time being, the hypothesis can only be tested for plausibility and fruitfulness, the former depending largely on the latter, i.e., on the answer to our second question. We shall therefore need to take a closer look at the Babylonian concepts.

According to its derivation and to cognate terms, *mithartum* designates an entity arising from the confrontation of equivalents (the confrontation of the line and its *mehrum* or "counterpart"—another derivative from the same root). A number of texts show that the *mithartum*, when a number is ascribed to it, *is* the length of the side and *possesses* an area [12]. No single text can be found where the square is identified with its area, as we would tend to do, and as is inherent in the Euclidean tetragon as a "figure" (σχῆμα), i.e., as something which is "encompassed by some boundary or boundaries" (ὑπό τινος ἤ τινων ὅρων περιεχόμενον) (*Elements* I, Definitions 22 and 14). On the other hand, other evidence shows beyond a doubt that the *mithartum is* a geometrical square and not a mere line adjacent to a square—e.g., BM 15285 [MKT I, 137f], where the squares are *drawn*.

This may seem strange to us. From a culturally neutral standpoint, however, our own ways are equally strange. Why should a complex geometrical configuration—four equal lines at right angles delimiting a plane surface—be considered identical with the measure of the plane surface, rather than with the measure of one of the lines? Once the configuration is given, one parametrization is as good as the other. So, the ambiguity of the *mithartum* concept vanishes: it is not alternately square and square root, but simply the figure identified by—and hence *with*—its side.

The case of *íb-si*$_8$ is similar. Etymologically and in most occurrences the term is a verb. A phrase like "81-e 9 *ib-si*$_8$" must apparently be read as "81 makes 9 equal-[sided]" [13]. In some occurrences, the term is used as a noun related to *mithartum*, i.e., as a square figure parametrized by the length of its side (at times when the side of a square of known area is asked for, but occasionally as a description of the geometrical configuration itself. And in still other instances, the term occurs as a verb denoting the creation from a length of the corresponding quadratic figure (but *not* its area) [14]. Once again, the square is considered under the aspect of a figure made up of equal sides, not as a plane surface surrounded by such sides.

THE "GEOMETERS' *DYNAMIS*"

With this in mind we now return to the Greek material—first to the concepts "commensurable in respect of length" (μήκει σύμμετροί) and "commensurable in respect of *dýnamis*" (δυνάμει σύμμετροί) from *Elements* X, Definitions 2–3. Two straight lines (εὐθεῖαι [γραμμαὶ]) are commensurable "in respect of length" if they have a common measure when each is regarded without sophistication as a length—a μῆκος. They are commensurable "in respect of *dýnamis*" when the tetragons on them have a common measure—that is, when *the two lines themselves* are *commensurable if regarded in the Babylonian way*, as representing squares. The common grammatical form (the dative) of μήκει and δυνάμει suggests that the two terms should stand in the same relation to the straight lines; since the line *can* indubitably be apprehended as a length, it should also be possible to apprehend it as a *dýnamis* (and it *should* be seen so in "commensurability in respect of *dýnamis*"). But the parallel leads still further. Since in the former case the lengths themselves have the common measure, in the latter case the *dynámeis* must be the things measured (remember that the Greek measuring procedure is a process of covering or taking away, cf. the *anthyphairesis*). The *dýnamis* can hence hardly be anything but a *mitḫartum*, a square identified with its side (but still of course possessing an area to which a measuring number can be ascribed). Otherwise expressed, the *dýnamis* is a *line seen under the aspect of square.*

If instead of commensurability we had looked at ratio *dynámei* and *mékei*, as known from Archimedes, the same arguments could have been developed. In both cases it becomes evident why we never find expressions like "commensurability in respect of tetragon" or "ratio in respect of tetragon": tetragons themselves are commensurable (if they are) and in possession of a mutual ratio—they are not aspects of a line. The absence of such expressions also follows from Taisbak's interpretation of the term; it is, however, somewhat enigmatic if "*dýnamis*" is believed to be nothing but another word for "tetragon." Why, in fact, should *Elements* XII.2 when reformulating the Hippocratean theorem that circles have the same ratio "as their diameters *dynámei*" also change the grammatical construction if it had been meaningful to speak of ratios *τετράγονῳ*? Truly, grammatical habits might have changed over the centuries, but this would then affect both terms had they really been synonyms (as, in fact, we see in Pappos' late mix-up).

If we turn to *Theaetetus*, the first use of *dýnamis* as a "square of three [square] feet" is of course in harmony with the interpretation of the term as a *mitḫartum—τρίπους*, "of three feet," is an adjective and hence not necessarily to be regarded as an identity. The later passage, in which the young Theaetetos introduces his definition distinguishing two sorts of lines (*γραμμαὶ*), is more interesting: on the one hand, a line which can be "spoken of" as a length, i.e., a line the length of which can be measured by a rational number which can be used as its name, is called a "length," a *μῆκος*. On the other hand, a line which can only be "spoken of," i.e., be given a numerical name, when regarded under its aspect of *dýnamis*, is called a *dýnamis* (it will be remembered that the Greek term which translates as "rational" is *ῥητός*, meaning "which can be spoken").

According to the *mitḫartum* interpretation, the definitions introduced by Theaetetos are no longer shocking, clumsy, or childish, as they have been regarded by various authors. Theaetetos does not call a square root a square, or anything like that. Truly, any line can be regarded in advance as a *dýnamis*, and Theaetetos restricts the use of the term to such lines which in a certain sense are *only* to be spoken of as *dynámeis*. This is, however, a precise analog of another well-known Greek dichotomy: some numbers are "square numbers"; they can be "engendered as equal times equal" (*ἴσον ἰσάκις γίγνεσθαι*), i.e., produced as the product of two equal factors. In principle, a "square number" is also "oblong"—it can be produced as the product of unequal factors: $4 \cdot 4 = 8 \cdot 2$; $3 \cdot 3 = 9 \cdot 1$. The name "oblong number" (*ἀριθμός προμήκης*) is, however, reserved for such numbers which are *only* oblong, i.e., for nonsquare numbers. This delimitation, introduced by Theaetetos in the same dialogue just before the "shocking" definitions of "length" and "*dýnamis*" (147e9–148a4), has never shocked anyone. Yet, according to the *mitḫartum* interpretation, the logic of the two definitions is strictly the same. No puzzles are left. The passages from *Theaetetus*, as well as the entire material on the "geometers' *dýnamis*," fit the interpretation of the *dýnamis* as a concept of the same structure as the Babylonian conceptualization of the square.

As already stated, the link between *dýnamis* as commercial worth and as confrontation of force is a feature shared with the Babylonian *mithartum*. No Babylonian mathematical term equivalent to *dýnasthai* exists, however. Nor does there appear to be any concept or procedure in Babylonian mathematics which necessitates such a word. So, even if the *dýnamis* may be imported from or inspired by Babylonia, the term *dýnasthai* appears to be a genuine Greek development due to the integration of the *dýnamis* concept into the theoretical structure of Greek geometry. We see in *Theaetetus* 148b2 a possible explanation for such a development, when Plato speaks of "the plane figures" which the lines *dýnantai*, i.e., "have in their power to form when seen *dynámei*" or "are worth" under the same aspect. This could also be the metaphorical sense which Aristotle in *Metaphysica* 1019^b33f, and it suggests that the Greeks may have conceptualized the term in Taisbak's manner in the mid-fourth century (and perhaps earlier), independently of its origin. This, in connection with the verb's connotations of equivalence and being worth, could then easily lead to the general loose usage in which lines or surfaces (Hero!) can be said to *dýnasthai* other lines or surfaces, but where in all cases the equality involved is one of surfaces, not of lengths.

On the other hand, the *dýnamis* might also stand for a *mitḫartum*-like concept without having been borrowed at the conceptual level. Both concepts could have developed independently on the basis of analogous or shared measuring *practices* [15]. In this case, the shared secondary connotations of the two terms must be considered accidental (which, given the connotative richness of both languages, could easily have happened).

THE "CALCULATORS' *DYNAMIS*" REVISITED

Thus, if we restrict our reflections to the "geometers' *dýnamis*," conceptual borrowing and independent development of analogous conceptualizations of the square figure are equally good causal explanations of the apparent *mitḫartum*-structure of the Greek concept. This, however, brings us to the question of the "calculators' *dýnamis*." If, as was argued, Greek calculators may plausibly have been in possession of second-degree algebra showing terminological continuity up to Diophantos, it can hardly have been an indigenous development: it would have been inspired (or, more probably, imported) from some Middle Eastern algebra descending from the Old Babylonian tradition. Now, I have shown elsewhere that Old Babylonian "algebra" cannot have been arithmetical, i.e., conceptualized as dealing with unknown numbers organized by means of numerical operations [16]. Instead it appears to have been organized on the basis of "naive," nondeductive geometry of a sort related to that used by al-Khwārizmī in his *Algebra* to justify the standard algorithms used to solve basic mixed second-degree equations (see

[Rosen 1831, 13–21], or one of the published Medieval Latin translations, e.g., [Hughes 1986, 236–241]), but of course without al-Khwārizmī's Greek-type letter symbolism. Since the Arabic treatise mentioned in note 11 was of a similar sort, a descendant which inspired Greek calculators can hardly have been much different. Even early Greek "calculator-algebra" will consequently have dealt with "real" lines and squares, not with sums and products of pure numbers [17]. Truly, the "real" lines and squares may have been rows and patterns of pebbles on an abacus-board, rather than the continuous lines of a drawing—cf. below.

At the same time, the branch of Old Babylonian mathematics in which *mitḫartum* and *íb-si*$_8$ occur most frequently is the so-called "algebra." So, if a conceptual import into Greece has indeed taken place, the plausible channel is "calculator-algebra" rather than theoretical geometry. This would make the "calculators' *dýnamis*" the primary concept from which the "geometers' *dýnamis*" would be derived.

Hero's curious phraseology ("*dynámei* 25, which is *mékei* 5"—cf. above) might then derive from this calculators' tradition rather than from his Archimedean affiliation. It belongs indeed with a numerical calculation. As in *Republica* 587d, the same concrete entity is represented by several numbers; and as in the second passage from *Theaetetus*, the mathematics of the passage suggests the translation "root." If the segregation of a geometrical *dýnamis* was only taking place during Plato's (and Theaetetos') youth, these specific parallels between Plato and Hero are probably manifestations of the closeness of both to the calculators' usage.

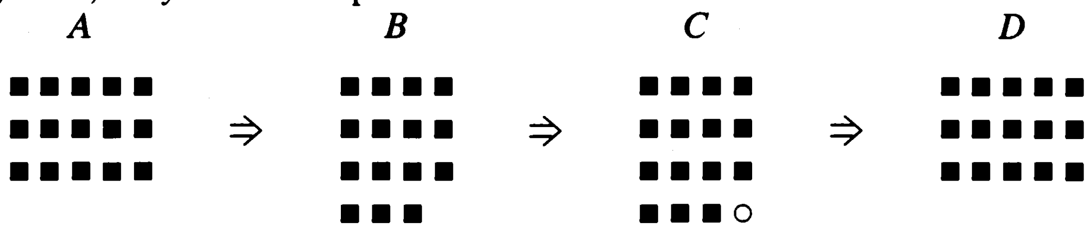
If, on the other hand, the *dýnamis*-concept was indigenously developed, we would rather expect its origin to belong with geometry and mensuration. This would make the "calculators' *dýnamis*" a metaphor, and suggest that, in spite of its dependence on prescientific sources and methods, logistics had already come under the sway of scientific mathematics in respect of metaphorics and conceptualizations around 400 B.C. If one reflects on the balance between references to logistics and to the purer branches of mathematics in the earlier part of the Platonic corpus (including *Republica* and *Timaeus*), this seems highly improbable.

THE *DYNAMIS* OF FIGURATE NUMBERS

Furthermore, seeking the origins of our term in logistics rather than theoretical geometry also better fits its use in the "Pythagorean" theory of figurate numbers. Here, indeed, the word *dýnamis* turns up in a way which could well be related to its use in a "pebble-algebra" but not to its geometrical function.

By "pebble-algebra" I refer to a possible representation of a second-degree "algebra" in Babylonian style by means of pebbles on the abacus board. Indeed, a person who says "calculator" in a Greek context says "pebble" or ψῆφος—the main tool of the calculator being the abacus with appurtenant pebble calculi. It is also a well-established fact that the "doctrine of odd and even," as well as the whole theory of figurate numbers, grew out of the patterns in which pebbles could be arranged (cf. [Lefèvre 1981]). It is therefore natural to assume that *if* some calculator algebra was in use in classical Greece it was performed (exclusively or occasionally) with pebbles on the abacus board [18].

This observation is interesting for several reasons. *First*, the interest in figurate numbers (including the "square" and "oblong" numbers spoken of by Theaetetos) ceases to be the result of some play with abacus pebbles irrelevant to their normal use. Square, gnomonic, and oblong numbers occur naturally as soon as one tries to represent a mixed second-degree problem on the board. So, e.g., the problem $x + y = 8, x \cdot y = 15$ is represented and solved thus:

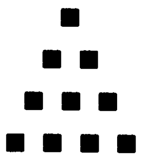


The virtual starting point for the analytical procedure is a pattern of 15 pebbles (A), whose length and width taken separately are unknown, whereas the sum of the length and the width is known to be 8. In the real process of solution we therefore start by laying out a gnomon with $8/2 = 4$ pebbles in each leg, and fill out the inside until all 15 pebbles have been used (B). This shows that a square of $1 \cdot 1 = 1$ pebble is lacking in order to complete the square (C), and that hence 1 row has to be moved from bottom to the right in order to actualize the virtual rectangle (D) [19].

Apart from the occurrence of oblong, gnomonic, and square numbers (all basic entities in the theory of figurate numbers), we see that one of the basic theorems of the theory follows immediately from the procedure—*viz.*, that the sum of the first n odd numbers equals n^2. Even the triangular numbers and the theorem that the sum of two consecutive triangular numbers is a square number are seen from the figure, although these observations play no role in the process. As soon as one starts reflecting on the patterns, these triangular numbers and their properties, as well as those of the gnomonic, square, and oblong numbers, lead to a series of obvious questions [20]; the theory of figurate numbers emerges as a *theory dealing with the general properties of existent tools and practices* instead of being an idle play picked up from nowhere.

Second, an astonishing use of the term *dýnamis* in Pythagorean or Neopythagorean arithmetic becomes meaningful. In configuration C, the *mithartum–dýnamis* is evidently 4. This is the line which "squares off" the complete pattern in Theaetetos' words. Now, the term turns up in Nicomachos' *Introduction to Arithmetic* in a way which could easily be explained as a generalization of this usage but which is otherwise anomalous. If we look at configuration A, we see the number 15 being arranged in thirds—according to Nicomachos in parts which "by name" (*ὀνόματι*) are 3 and *dynámei* (or *κατὰ δύναμιν*—both forms are used) are 5 (see, e.g., I.viii.7, 16[1]). This is no far-fetched transfer of the meaning in C, even though contact with the geometrical meaning is lost.

Other 1st or 2nd century (A.D.) doxographic sources suggest that the usage is not a Nicomachean idiosyncrasy. They concern one of the central Pythagorean concepts, the *tetractys* or decade drawn up as a triangular number:



According to Aëtius (*Placita* I.3.8), the Pythagoreans "declare . . . that the *dýnamis* of ten is in four, and in the tetrad" (*τῶν δέκα . . . φησίν, ἡ δύναμίς ἐστιν ἐν τοῖς τέσσαρσι καὶ τῆι τετράδι*) (Fragment 58 B 15 (Diels 1951 I. 544[1])). Taken in itself this phrase is ambiguous, and could well mean that the power of the magical number 10 resides in its possible triangular arrangement as *tetractys*. Hierocles, however, is more explicit in a commentary to supposedly early Pythagorean writings, stating that "the *dýnamis* of the decad is the tetrad" (*τῆς δε δεκάδος δύναμις ἡ τετράς*) ([Mullach 1875 I, 464B], quoted from [Souilhé 1919, 23]). So, these two doxographers (who will hardly be suspected of innovative mathematical terminology) appear to refer to a generalization of the concept of *dýnamis* different from but very close to that of Nicomachos: once more, the "base" of a nonsquare figurate number is taken as its characteristic parameter and given the name *dýnamis* belonging originally to the same parameter in the case of a square figurate number.

FURTHER OBSERVATIONS

Can we get any nearer to the process, or has the meager material now been exhausted? We can, in fact, squeeze the sources somewhat harder, observing that the two "intermediate" Platonic dialogues contain the expression *κατὰ δύναμιν*; whereas the late dialogues (*Theaetetus, Politicus*) as well as all other authors (except the nongeometrical Nicomachos) invariably use the simple dative *dynámei*. This suggests that the technical use of the term was only crystallizing in Plato's later years, around the mid-fourth century; by then, on the other hand, a fully technical "geometers' *dýnamis*" was crystallizing.

First, this observation makes it seem highly doubtful that Hippocrates' own words are rendered exactly in the Eudemos fragment, which agrees so perfectly with the style of late Platonic, Aristotelian, and Archimedean occurrences [21]. The fragment seems rather to contain Eudemos' reformulations in his own phrase structures of Hippocrates' ideas, concepts and basic terms (including probably some forms of *dýnamis* and *dýnasthai*). *This conclusion is independent of all other hypotheses on the meaning and origin of our terms.*

Second, cautious assumptions on the temporal distance between the introduction of a mathematical terminology and its crystallization in fixed linguistic forms (*viz.*, the assumption that in an interactive environment this distance should be of the order of one or two generations of masters and students) support our earlier conclusion that the segregation of a distinct "geometers' *dýnamis*" from a naive-geometric or pebble-based calculators' concept occurred during Plato's youth or shortly before. A central role could then perhaps be ascribed to Hippocrates and Theodoros.

An observation made by Neuenschwander [1973, 329ff] may indicate in which connection the innovation took place. Time and again, the early books of the *Elements* use a principle which is neither proved nor stated as an axiom, viz.,

$$AB = CD \Leftrightarrow (AB)^2 = (CD)^2$$

Now, it follows from Neuenschwander's analysis that when this principle is applied in Books II and IV, it is most often stated explicitly. When it is used in Books I and III, however, it remains implicit, except in III.35–36; precisely these two propositions deal with areas of parallelograms, and their subject-matter is thus related to that of Book II. We may conclude that only the tradition behind Books II and IV, the "metrical tradition" dealing centrally with areas of plane figures and continuing itself in the theory of irrationals, based itself on a set of concepts making it natural to notice and formulate the application of the principle, which is nothing but the interchangeability of equality *mḗkei* and *dynámei*. This agrees perfectly with the hypothesis of a Near Eastern borrowing, because the branch of geometry which could be inspired by Babylonian "naive-geometric" algebra (or a Greek "calculators' algebra," for that matter) is precisely the so-called "geometric algebra" of *Elements* II (I shall not mix up the discussion of this much-debated term with the present investigation). It also fits well with the branches of geometry which later make use of the *dýnamis* idiom: *Elements* X and XIII, etc.

A final observation concerns the very idea of a "conceptual import." Truly, the translation of *dýnamis* from *mitḫartum* makes good sense of all occurrences of the term prior to Pappos. Still, the "geometers' *dýnamis*" belongs within a conceptual context differing fundamentally from that of the *mitḫartum*: from the principle that the concepts of a connected body of thought are themselves connected we should therefore expect that the idea of a translation can only be approximately true.

This is in fact borne out by closer analysis of some of our Greek texts. In the definition of "commensurability *dynámei*" in *Elements* X, the entities which are explicitly measured by an area (χῶρος) are the tetragons on the lines. Implicitly, however, the expression supposes that the *lines* regarded in their aspect of *dynámeis* are measured (since the lines themselves are commensurable in that aspect). Earlier, in the Eudemos fragment, bases and diameters themselves are said explicitly to have a ratio (*viz.*, the ratio of the areas of their squares) under the same condition. This must mean that the area associated with a line regarded as a parametrization of a square figure is less of an external accessory than the area of a Babylonian *mitḫartum*—the Greeks, apprehending the tetragon–square as well as circles and other plane figures as identical *with their areas*, tended to assimilate the *dýnamis*-square into the same pattern [22]. In the case of the "calculators' *dýnamis*" this becomes even more evident, since in Diophantos's work *dýnamis* has assumed the numerical role in his problems which the area (*a-š à* or *eqlum*) and not the *mitḫartum* assumes in Babylonian texts.

Precisely this conceptual incongruity is probably the reason for the disappearance of the terms *dýnamis* and *dýnasthai* from the active vocabulary of geometers by the early third century, except in specific technical niches (commensurability *dynámei*) and formulaic expressions. The terms did not fit the mental organization of Greek mathematics once its various branches and disciplines blended into the melting-pot of Alexandrian learning.

As to the term *dýnamis* itself, it is clear that the connotational similarity with *mitḫartum* does not reflect a borrowing of the Babylonian understanding of the square as a result of a confrontation of equals or counterparts. If not accidental, the shared connotations (involving physical force and commercial value) have to be explained at the level of the "folk etymology" (the "folk" in question being calculators or possibly geometers)—*viz.*, as an attempt to understand why the Semitic masters called a "line regarded under the aspect of the appurtenant square" by a strange name related to the confrontation of values and force, a usage then reflected in the Greek term chosen to denote the same object.

Such a pseudo-etymology may from the beginning have been connected with explanations proposed on the basis of the Greek language: the square which a line "has the power to form," "is worth," or "masters." Such metaphors may also have been introduced as secondary explanations when memory of a foreign origin had been forgotten (which could have happened quickly). A "Babylonian" and a "Greek" interpretation of the term need not be mutually exclusive; in some way they probably supplement each other.

CONCLUSIONS

As stated by Berggren [1984, 402], there are in the early history of Greek mathematics "sufficient documents to support a variety of reconstructions but an insufficient number to narrow the list of contending theories to one." This pessimism is confirmed by the impossibility of reaching consensus on the merits of such great reconstructions as [Szabó 1969] and [Knorr 1975] [23]. For the time being, no compelling reconstruction can apparently be written; instead, further progress may be made through the construction of scenarios for all or parts of the development which may open our eyes to hitherto unnoticed features in the source material at hand. Such scenarios should be internally coherent and in agreement with available documents, and should be compared with rival interpretations of history on the basis of their merits in these respects; however, they need not claim in advance to be necessary truths.

The above discussion, which includes an abundance of hypothetical formulations, is meant primarily to provide suggestions for such a partial scenario. Still, the knitting is not so tight that all parts of the argument stand or fall together; nor are they equally hypothetical.

Among the positively supported results is the distinction between a "geometers' *dýnamis*" and a "calculators' *dýnamis*." Both groups made use of the term, but they did so for different purposes and within different conceptual frameworks, and hence necessarily in partially different ways–*vide* the quotations from Hero. Direct evidence was also given for the assignment of the crystallization of the geometrical *dýnamis* usage to Plato's late years—and hence also for the doubt concerning the Hippocratean origin of the exact formulations in the Eudemos fragment.

The interpretation of the geometrical *dýnamis* concept as "a square identified by, and hence with, its side" is also supported by the sources regarded as a totality in the sense that the apparent ambiguities in the usage can only be surmounted by an interpretation of this kind. The possibility that such a concept can have been held is established through the *mitḫartum*-parallel.

More hypothetical are the primacy of the "calculators' *dýnamis*" over the "geometers *dýnamis*"; the interpretation of the early "calculators' *dýnamis*" as belonging with a naive-geometric or pebble-based "algebra"; the suggestion that the segregation of a distinct "geometers' *dýnamis*" is connected with the beginnings of the theoretical tradition behind *Elements* II in the later fifth century; and the hypothesis that the *dýnamis* is structurally similar to the *mitḫartum* because it is borrowed. Taken singly, these are nothing but possible hypotheses; together, they appear to form a plausible scenario fitting the complete available evidence, including evidence rarely taken into account (e.g., the finer details of Plato's formulations in their chronology, the hidden presence and absence of the *dynámei*/*mékei* relation in *Elements* I–IV, and the peculiar Neopythagorean usage).

Independent but secondary observations are the disappearance of the *dýnamis* usage and its sole survival in formulaic language (which is not a new idea), and the explanation of this process in terms of the incongruity between the "*dýnamis*-square" and the normal Greek conceptualization of squares and other plane figures as identical with the surfaces covered.

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NOTES

1. Extensive references to the debate prior to the year 1975 will be found in [Burnyeat 1978]. Among later discussions of the term, [Knorr 1975], [Taisbak 1980], and [Taisbak 1982] should be mentioned.

2. Burnyeat [1978, 492f] renders the whole passage 147c7 to 148d7 quoting John McDowell's English translation, rendering δύναμις as "power." In the Loeb edition, Fowler [1921] translates the term as "root."

3. It should be kept in mind that the Greek verb is transitive; "x being worth Y" is thus as different from "x being worth the same as Y" as "x loving Y" is from "x loving the same as Y" (jealousy apart).

4. Formulations like the latter are found in various commentators from late antiquity (see Burnyeat [1978, 500, n. 34]. An explicit derivation from natural philosophy is considered "beyond doubt" by Bärthlein [1965, 45], who, substantiating his claim, mixes up lines and numbers in quite anachronistic ways.

5. Aujac [1984a, b] has investigated such word-by-word preservation of the phrasing of theorems, involving also Euclid and pre-Euclidean spherics.

6. Hero cites Archimedes, *De conoidibus et sphaeroidibus* v, for the statement that "the $^{N/A}$ ⟨rectangle⟩ under the axes [of an ellipse] is worth the circle^A equal^A to the ellipse" [^{N} = nominative case ending, ^{A} = accusative], but afterward uses the correct theorem that the product of the axes equals the *square of the diameter* of the circle in question. In a footnote, Heiberg proposes the correction ". . . is worth ⟨the diameter^A⟩ of the circle^G equal^G . . ." [^{G} = genitive], which still takes the rectangle to be the subject of the sentence; the emendation ". . . is worth ⟨the diameter^N⟩ of the circle^G equal^G . . .", however, would restore normal usage, apart from a legitimate though rather unusual inversion.

7. The reading of the passage as benign irony is supported by the similar portrait of the *jeunesse-doré* attitudes of the other brother Adeimantos in 420a.

8. Souilhé [1919, 124] reads the passage differently, equating δύναμις with "force" and ὄκος with "mass." This is not very plausible in view of the context. This passage exhausts the number of mathematical occurrences of the *dýnamis* in the Platonic corpus, together with another passage in *Timaeus* (54b) where, in the triangle obtained by bisection of the equilateral triangle, one side is said to be the triple of the other "according to *dýnamis*" (κατὰ δύναμιν). (I disregard a possible hint in the notoriously obscure *Republica* 546b, and the occurrences in the pseudo-Platonic *Epinomis*).

9. As pointed out by Rashed [1984, 113], the term *dýnamis* is introduced at an earlier stage than unknown numbers. Only by saying that "it has been approved" (ἐδοκιμάσθη) that in this form the square of numbers becomes one of the "elements of arithmetical theory" (στοιχεῖον τῆς ἀριθμητικῆς θεωρίας), does Diophantos make clear that he is already here aiming at the only actual use of the term later on, *viz.*, as a designation for the square of the unknown ἀριθμός. At the same time, he notes that he is following a general convention from a discipline of "arithmetical theory" which is neither Euclidean nor Neopythagorean (Nicomachos uses the term quite differently, as we shall see). Only Diophantos' own brand of arithmetic seems to be left, i.e., *algebra*.

10. Few instances of ancient second-degree "algebra" below the level of Diophantos have survived in sources from classical antiquity. Some, however, can be found scattered throughout surveyors' and related texts. E.g., in the *Geometrica* ascribed to Hero, xxi 9–10 (380^{15-31}), the dimensions of a circle are found from the sum of diameter, perimeter, and area, while the Roman agrimensor Nipsus (2nd c. A.D?) treats the problem of a right-angled triangle with known hypotenuse and area in his *Podismus* (297f). We can hence be sure that basic second-degree "algebra" was indeed known to the ancient practitioners.

11. Next to nothing is known about the transmission of Babylonian mathematics after the end of the Old Babylonian period (c. 1600 B.C.), but that transmission took place is sure. As I have shown in my [1986, 457–468], a 12th-century Latin translation from the Arabic follows Old Babylonian ways down to the choice of grammatical forms. That the Greek calculators owed part of their technique to the Near East is also apparent from the name of their favorite instrument, the ἄβαξ, the [dust] abacus, which is borrowed from western Semitic ᶜb q, "light dust" (the root is absent in Babylonian). Since finally the term *maḫirum* is testified in Hebrew in the related form *mᵉḥīr*, a Western Semitic (Phoenician?) contact is no less linguistically possible than direct Babylonian influence.

Without taking Proclos' *Commentary* more seriously than it deserves, we may also remember his ascription in 65^5 of "accurate investigation of numbers" (τῶν ἀριθμῶν ἀκριβὴς γνῶσις) to the Phoenicians, which he derives from the needs of logistics.

12. E.g., BM 13901, *passim* [MKT III, 1–5]. The first problem can be translated: "I have added the area and my *mitḫartum*, it is $\frac{3}{4}$." The solution states that the *mitḫartum*, the square identified with its side, is $\frac{1}{2}$.

13. This follows both from the Sumerian ergative suffix -e and from interrogative variants of the phrase showing 9 to be an accusative. Exemplifications can be traced through the glossaries of MKT.

14. A full documentation of the varying uses of *íb-si*$_8$ would lead too far astray. It belongs with a

larger investigation of Babylonian "algebra" (work in progress; preliminary report in [Høyrup 1984], final to appear in [Høyrup 1990]).

15. I am grateful to Professor Tilman Krischer of the Freie Universität Berlin for pointing out the importance of this possibility in his comments on an earlier version of the present paper.

16. Once more, documentation would lead too far astray—cf. note 14 above. The simplest part of the evidence comes from an analysis of the terminological structure of the texts. Two different "additive" operations are kept strictly apart in a way which has no meaning in an arithmetical interpretation, i.e., if the terms are synonyms for the one and only numerical addition. Similarly, two different "subtractions" and four different "multiplications" are distinguished.

17. If we take Plato's testimony at its words, it suggests the same. The third power was spoken of as the "third increase," which fits well with a spatial conceptualization but rather poorly with an arithmetical representation before the introduction of exponential symbolism or spatial representation. Arithmetically, we would have the number itself, the increase (*i.e.*, the second power), and the second increase, *i.e.*, our third power.

18. Since the abacus appears first to have been borrowed in the form of a dust abacus from the Near East (*cf.* above, note 11), and since this device was used for geometric drawings throughout antiquity, occasional use of real drawings on a dustboard is also a possibility and in fact appears to fit Nipsus' problem (see note 10) better than pebble manipulation.

19. If the problem had been $x - y = 2$, $x \cdot y = 15$, we would start step B with the inner gnomon, the one with legs containing 2 pebbles, and add new layers at the outside. Apart from that, the same configurations would have to be used. Odd values of $x \pm y$, on the other hand, require further refinement.

20. In his investigation of the prehistory of incommensurability, Knorr [1975, 142ff] comes to similar pebble-configurations and conclusions from another angle and deals with the matter in much more detail.

21. The same doubt as to the literal precision of Eudemos's quotation was recently formulated by Knorr [1986, 38f] on the basis of other evidence.

22. Conversely, in its exact form the Greek concept could of course have no place with the Babylonians. A Babylonian line (and any other geometrical entity) is identified by, and conceptually not distinguished from its measuring number. A Greek line, however, is conceptually distinct both from the number of unit lengths contained in it when regarded as a length and from the number of unit squares covering it when regarded *dynámei*.

23. *Cf.* also the review of a number of ongoing controversies in [Berggren 1984].

SOURCE EDITIONS USED

Apollonios, *Conica*: [Heiberg 1891].
Archimedes, all works: [Heiberg 1880].
Aristotle, *De incessu animalium*: [Peck & Forster 1937].
Aristotle (ps-), *De lineis insecabilibus*: [Hett 1936].
Aristotle, *Metaphysica*: [Tredennick 1933].
Diophantos, *Arithmetica*: [Tannery 1893].
Eudemos/Hippocrates-fragment: [Thomas 1939 I].
Euclid, *Data*: [Menge 1896].
Euclid, *Elements*: [Heiberg 1883].
Hero, *Geometrica*: [Heiberg 1912].
Hero, *Metrica*: [Schöne 1903].
Nicomachos, *Introductio arithmeticae*: [Hoche 1866].
Nipsus, *Podismus*: [Blume et al. 1848 I].
Pappos, *Collectio*: [Hultsch 1876].
Plato, *Politicus*: [Fowler & Lamb 1925].

Plato, *Republica*: [Shorey 1930].
Plato, *Theaetetus*: [Fowler 1921].
Plato, *Timaeus*: [Bury 1929].
Proclos, *In primum Euclidis Elementorum librum commentarii*: [Friedlein 1873].

REFERENCES

Aujac, G. 1984a. Autolycos de Pitané, prédécesseur d'Euclide. *Cahiers du Séminaire d' Histoire des Mathématiques* **5,** 1–12.

Aujac, G. 1984b. Le langage formulaire dans la géométrie grecque. *Revue d'Histoire des Sciences* **37,** 97–109.

Bärthlein, K. 1965. Über das Verhältnis des Aristoteles zur Dynamislehre der griechischen Mathematiker. *Rheinisches Museum für Philologie* **108,** 35–61.

Berggren, J. L. 1984. History of Greek mathematics: A survey of recent research. *Historia Mathematica* **11,** 394–410.

Blume, F., Lachmann, K., & Rudorff, A. A. F. (Eds.) 1848. *Die Schriften der römischen Feldmesser.* 2 vols. Berlin: Reimer, 1848, 1852.

Burnyeat, M. F. 1978. The philosophical sense of Theaetetus' mathematics. *Isis* **69,** 489–513.

Bury, R. G. (Ed., Transl.) 1929. Plato, *Timaeus. Critias. Cleitophon. Menexenus. Epistles.* Loeb Classical Library, Vol. 234. Cambridge, MA: Harvard Univ. Press/London: Heinemann. [Reprint 1966]

Diels, H. 1951. *Die Fragmente der Vorsokratiker.* Herausgegeben von W. Kranz. 3 vols. 6. Auflage, Berlin: Weidmann. [Reprint 1972]

Fowler, H. N. (Ed., Transl.) 1921. Plato, *Theaetetus. Sophist.* Loeb Classical Library, Vol. 123. Cambridge, MA: Harvard Univ. Press/London: Heinemann. [Reprint 1977]

Fowler, H., & Lamb, W. R. M. (Eds. Transls.) 1925. Plato, *The Statesman. Philebus.* Loeb Classical Library, Vol. 164. Cambridge, MA: Harvard Univ. Press/London: Heinemann. [Reprint 1975]

Friedlein, G. (Ed.). 1873. Procli Diadochi *In primum Euclidis Elementorum librum commentarii.* Leipzig: Teubner.

GEL: *Greek-English Lexicon.* 1968. Compiled by Henry George Liddell and Robert Scott. Revised and Augmented Throughout . . . With a Supplement. Oxford: Oxford Univ. Press. [Reprint 1976]

Heath, Th. L. 1921. *A History of Greek Mathematics.* 2 vols. Oxford: Clarendon.

Heath, Th. L. 1949. *Mathematics in Aristotle.* Oxford: Oxford Univ. Press.

Heiberg, J. L. (Ed., Transl.) 1880. Archimedis *Opera omnia* cum *Commentariis* Eutocii. 3 vols. Leipzig: Teubner, 1880, 1881, 1881.

Heiberg, J. L. (Ed., Transl.) 1883. Euclidis *Elementa.* 5 vols. (Euclidis Opera omnia, Vol. I–V). Leipzig: Teubner, 1883, 1884, 1886, 1885, 1888.

Heiberg, J. L. (Ed., Transl.) 1891. Apollonii Pergaei *quae graece extant cum commentariis antiquis.* 2 vols. Leipzig: Teubner, 1891, 1893.

Heiberg, J. L. (Ed., Transl.) 1912. Heronis *Definitiones* cum variis collectionibus. Heronis quae feruntur *Geometrica.* (Heronis Alexandrini Opera quae supersunt omnia, IV). Leipzig: Teubner.

Hett, W. S. (Ed., Transl.) 1936. Aristotle. *Minor Works.* Loeb Classical Library, Vol. 307. Cambridge, MA: Harvard Univ. Press/London: Heinemann. [Reprint 1980]

Hoche, R. (Ed.). 1866. Nicomachi Geraseni Pythagorei *Introductionis arithmeticae* libri II. Leipzig: Teubner, 1866.

Høyrup, J. 1984. *Babylonian algebra from the view-point of geometrical heuristics. An investigation of terminology, methods, and patterns of thought.* Second, slightly corrected printing. Roskilde:

Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science, 1985.

Høyrup, J. 1986. Al-Khwārizmī, Ibn Turk, and the Liber Mensurationum: On the origins of Islamic algebra. *Erdem* **2** (Ankara), 445–484.

Høyrup, J. 1990. Algebra and naive geometry. An investigation of some basic aspects of Old Babylonian mathematical thought. *Altorientalische Forschungen* **17**, to appear.

Hughes, B. B., O. F. M. 1986. Gerard of Cremona's translation of al-Khwārizmī's *Al-Jabr*: A critical edition. *Mediaeval Studies* **48**, 211–263.

Hultsch, F. (Ed., Transl.) 1876. Pappi Alexandrini *Collectionis* quae supersunt. 3 vols. Berlin: Weidmann, 1876, 1877, 1878.

Knorr, W. R. 1975. *The evolution of the Euclidean elements. A study of the theory of incommensurable magnitudes and its significance for early Greek geometry*. Synthese Historical Library, Vol. 15. Dordrecht & Boston: Reidel.

Knorr, W. R. 1978. Archimedes and the *Elements*: Proposal for a revised chronological ordering of the Archimedean corpus. *Archive for History of Exact Sciences* **19**, 211–290.

Knorr, W. R. 1986. *The Ancient Tradition of Geometric Problems*. Boston, Basel, Stuttgart: Birkhäuser.

Lefèvre, W. 1981. Rechenstein und Sprache. In *Rechenstein, Experiment, Sprache, Historische Fallstudien zur Entstehung der exakten Wissenschaften*, Damerow & Lefèvre, Eds. pp. 115–169. Stuttgart: Klett–Cotta.

Menge, H. (Ed.) 1896. Euclidis *Data* cum *Commentario* Marini et scholiis antiquis. (Euclidis Opera Omnia, Vol. VI). Leipzig: Teubner.

MKT: Neugebauer, O. 1935. *Mathematische Keilschrifttexte*. I–III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster–dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937. [Reprint Berlin etc.: Springer-Verlag 1973]

Mullach, 1875. *Fragmenta philosophorum graecorum*. Paris: Didot. (data from [Souilhé 1919, viii]).

Neuenschwander, E. 1972–1973. Die ersten vier Bücher der Elemente Euklids. Untersuchungen über den mathematischen Aufbau, die Zitierweise und die Entstehungsgeschichte. *Archive for History of Exact Sciences* **9**, 325–380.

Peck, A. L., & Forster, E. S. (Eds., Transls.) 1937. Aristotle, *Parts of Animals, Movement of Animals, Progression of Animals*. Loeb Classical Library, Vol. 323. Cambridge, MA: Harvard Univ. Press/London: Heinemann. [Reprint 1968]

Rashed, R. (Ed., Transl.) 1984. Diophante, *Les Arithmétiques*, tômes III (livre IV), IV (livres V, VI, VII). Texte établi et traduit. Paris: "Les Belles Lettres."

Robbins, F. E. 1929. P. Mich. 620: A series of arithmetical problems. *Classical Philology* **24**, 321–329.

Rosen, F. (Ed., Transl.) 1831. The *Algebra* of Muhammad ben Musa. London: The Oriental Translation Fund.

Schneider, I. 1979. *Archimedes: Ingenieur, Naturwissenschaftler und Mathematiker*. Reihe Erträge der Forschung, Band 102. Darmstadt: Wissenschaftliche Buchgesellschaft.

Schöne, H. (Ed., Transl.) 1903. Herons von Alexandria *Vermessungslehre* und *Dioptra*. (Heronis Alexandrini Opera quae supersunt omnia, Vol. III). Leipzig: Teubner.

Shorey, P. (Ed., Transl.) 1930. Plato, *The Republic*. 2 vols. Loeb Classical Library, Vols. 237, 276. London: Heinemann/Cambridge, MA: Harvard Univ. Press, 1930, 1935. [Reprint 1978, 1963]

Souilhé, J. 1919. *Étude sur le terme δύναμις dans les dialogues de Platon*. Thèse complémentaire présentée à la Faculté des Lettres de l'Université de Poitiers. Paris: Felix Alcan.

Szabó, Á. 1969. *Anfänge der griechischen Mathematik*. München & Wien: R. Oldenbourg/Budapest: Akadémiai Kiadó.

Szabó, Á. 1986. Der Begriff "Dynamis" in der griechischen Mathematik. *Bulletin de la Société Mathématique de Belgique* **38**, 341–365.

Taisbak, C. M. 1980. "Dynamis" og "dynasthai". Et forslag til tolkning af en betydningsfuld geometrisk terminus i den græske lære om usammålelige linjestykker. *Museum Tusculanum* **40–43,** 119–131.

Taisbak, C. M. 1982. *Coloured Quadrangles. A Guide to the Tenth Book of Euclid's Elements*. Opuscula Graecolatina—Supplementa *Musei Tusculani*, Vol. 24. Copenhagen: Museum Tusculanum Press.

Tannery, P. (Ed., Transl.) 1893. Diophanti Alexandrini *Opera omnia* cum graecis commentariis. 2 vols. Leipzig: Teubner, 1893, 1895.

Thomas, I. (Ed., Transl.) 1939. *Selections Illustrating the History of Greek Mathematics*. In two volumes. Loeb Classical Library, Vols. 335, 362. London: Heinemann/Cambridge, MA: Harvard Univ. Press, 1939, 1941. [Reprint 1967]

Tredennick, H. (Ed., Transl.) 1933. Aristotle, *The Metaphysics*. 2 vols. Loeb Classical Library, Vols. 271, 287. Cambridge, MA: Harvard Univ. Press/London: Heinemann, 1933, 1935. [Reprints 1980, 1977]

Tropfke, J. 1980. *Geschichte der Elementarmathematik*. 4. Auflage. Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von K. Vogel, K. Reich, H. Gericke. Berlin & New York: de Gruyter.

Vogel, K. 1930. Die algebraischen Probleme des P. Mich. 620. *Classical Philology* **25,** 373–375.

I

"Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* **28** (1990), 63–86.

Hist. Sci., xxvii (1989)

SUB-SCIENTIFIC MATHEMATICS: OBSERVATIONS ON A PRE-MODERN PHENOMENON

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IN MEMORIAM N. I. BUKHARIN

1. THE CONCEPTS

In the post-Renaissance world, mathematics used for technical or administrative purposes utilizes results and techniques derived from some level of 'scientific' mathematics, though often transformed or simplified in order to be adequate for practitioners. Furthermore, modern practitioners of mathematics have been taught their mathematics by teachers who have been taught by teachers (who ... etc.) who have been taught by mathematicians. In both senses, practitioners' mathematics can then be regarded as 'applied mathematics'.

In the pre-Modern world, the situation was different. A quotation from Aristotle's *Metaphysics* — dealing not with mathematics but with knowledge in general — will help us see how:

> At first he who invented any art whatever that went beyond the common perceptions of man was naturally admired by men, not only because there was something useful in the inventions, but because he was thought wise and superior to the rest. But as more arts were invented, and some were directed to the necessities of life, others to recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences, which do not aim at giving pleasure or at the necessities of life were discovered ....
>
> So ..., the theoretical kinds of knowledge [are thought] to be more the nature of Wisdom than the productive.[1]

First of all, of course, this passage establishes the distinction between 'theoretical' and 'productive' knowledge, between that knowledge which aims at nothing beyond itself (or possibly at moral improvement of the knowing person) and that knowledge which aims at application. We also observe, however, that Aristotle assumes the two kinds of knowledge to be carried by *different persons*.

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This is no new observation if we consider the history of Greek science and technology. In the main, the two kinds of knowledge were indeed carried by *separate social groups and traditions* (the exceptions do not concern us for the moment). Among other things because of the source situation, the history of Ancient science and especially that of Ancient mathematics has normally focused upon that tradition which cared for theoretical knowledge. It has done so to the extent that the specific character (or even the separate existence) of the other traditions has often gone unnoticed. This might be relatively unimportant as long as nothing but Ancient mathematics itself was concerned. When we turn to those mathematical cultures inspiring or inspired by Ancient Greek mathematics, however, neglecting the existence and distinctive character of the 'productive' traditions makes us blind to many facets of history; this is what I shall try to demonstrate in the following by pointing to new observations on mostly well-known material which are made possible by the explicit distinction between 'scientific' and 'sub-scientific' knowledge and traditions.

The distinction concerns the *orientation* of knowledge, the purpose intended in the acquisition, the conservation and the transmission of knowledge. *Scientific knowledge* is knowledge which is pursued *systematically* and *for its own sake* (or at least without any intentions of application) *beyond the level of everyday knowledge*. 'Scientific knowledge' is thus the same as 'theoretical knowledge' in the Greek sense. The term 'scientific' is chosen because this is the kind of knowledge upon which descriptions of the *history of science* will normally concentrate (again, the exceptions are immaterial to the present argument). In order to avoid confusion with the vaguer everyday meaning of the word, the term will be kept in quotes throughout the article.

Sub-scientific knowledge, on the other hand, is *specialists' knowledge* which (at least as a corpus) is acquired and transmitted *in view of its applicability*. Even sub-scientific knowledge is thus knowledge beyond the level of common understanding, and it may well be much more refined than 'scientific' knowledge—as it is evident if one compares the level of the 'scientific' knowledge presented in Nicomachus's *Introduction to arithmetic* with that of Old Babylonian 'algebra', the sub-scientific character of which shall be argued below.[2]

All this might sound as nothing but new words for the familiar division of knowledge into 'pure' and 'applied'. This division, however, is bound up with the specific Modern understanding of the connection between the two levels. Roughly speaking, fundamental knowledge is assumed to be found by 'pure scientists' and then to be worked upon, recast and synthesized in new ways by 'applied scientists' or 'technologists' in consideration of the problems and possibilities of current practice. These implications of the modern terms constitute one reason for avoiding them in descriptions of the non-Modern world, where things were different, where practitioners' mathematics was not a separate *level* but an autonomous *type* of mathematics.

Another reason is the existence of a 'pure' outgrowth of the sub-scientific corpus of knowledge — i.e., that sub-scientific knowledge *itself* consists of (at least) two levels of which one is intentionally *non-applicable*. I shall return to this question in Section 2; first, however, I shall point briefly to the two issues which have led me to formulate the distinction. Both concern *mathematical knowledge*, which is also going to be my sole subject in the following, although the distinction itself is of more general validity.

One issue is the contrast between Greek and Old Babylonian mathematics.[3] The naïve first-order explanation of the difference between the two says that Babylonian mathematics aimed at practical application, while Greek mathematics considered abstract entities and aimed at Aristotle's "Wisdom".[4] This view derives from the formulation of Babylonian problem texts, which formally deal with fields, siege ramps, etc. Many problems, it is true, do train skills in practical computation; other problems, however, and indeed whole branches of Babylonian mathematics, are 'formally applied' but 'substantially pure' — i.e., the entities dealt with are those encountered in practical scribal (surveyor's, military engineer's, accountant's) life, but the problems about these entities are not only unrealistic as far as numerical magnitudes are concerned but also with regard to structure.[5] The second-order explanation is then that the Babylonians had also discovered the joys of pure mathematics.[6] However, if these joys were the same to the Babylonians and, for example, to the Greeks, how shall we explain the fundamental *difference* between Greek and Babylonian 'pure' mathematics — Greek mathematics being roughly speaking determined by *problems*, for the solution of which new methods would have to be developed (*cf.* below), while Babylonian non-applicable mathematics was determined by the *methods already at hand*, for the display of which new problems were continually constructed.[7]

These difficulties are cleared away if we look into the function of Babylonian mathematics for the scribes and in the scribal school. The social basis of Babylonian mathematics is 'sub-scientific' rather than 'scientific' in character, and the peculiar character of Babylonian 'pure' mathematics is close to that of non-applicable sub-scientific mathematics, as I shall discuss below.

The other issue is the question of the sources of Islamic mathematics. When the tenth century Baghdad court librarian al-Nadīm wrote his *Catalogue* (*Fihrist*), he mentioned 21 Greek mathematical authors but not a single pre-Islamic non-Greek mathematician.[8] This corresponds to the current picture, where Greek mathematics is still considered the all-important source. Still, India is also taken into account nowadays, and is in fact unavoidable if we trace, for example, Islamic trigonometry; at times, even Syrian, Pahlavi and Khorezmian learning are mentioned at least as undocumented or indirectly documented possibilities. It is rarely observed, however, that Islamic mathematics cannot have been taken over directly from Indian high-level mathematicians like Āryabhaṭa and Brahmagupta;[9] nor is the question discussed whether (for example) Khorezmian learning will have been comparable to Greek science in structure.

There are, in fact, good and even compelling reasons to take non-Greek sources into account. But why then were they not known to the learned al-Nadīm, who was in a position to know better than anybody else? Once again, the distinction between 'scientific' and 'sub-scientific' may be called in: the sources unknown to al-Nadīm were sub-scientific in character — practitioners' traditions which were transmitted not in books but orally and often 'on the job'.[10]

2. THE CHARACTER AND STRUCTURE OF SUB-SCIENTIFIC MATHEMATICS

Below, I shall address the character of Babylonian mathematics as well as the sub-scientific sources of Islamic mathematics in somewhat more detail. First, however, I shall consider the general structure of sub-scientific mathematics.

Sub-scientific mathematics was carried by specialists' 'professions',[11] and the chief aim for building it up and transmitting it was its practical applicability as a tool for the profession in question. Evidently, then, its main contents were methods and techniques for practical numerical and geometrical computation (and, in certain professional contexts, practical geometrical *construction*). Still, practical computation has mostly been lost from the sources,[12] and often we know only the global results — accounts have been made, heritages have been distributed, taxes have been levied, armies have been supplied with food and pay, goods have been bought and sold. Paradoxically, then, we know the basis and *raison d'être* of sub-scientific mathematics less well than its 'pure' outgrowth: 'Recreational mathematics'.

Recreational mathematics was once described by Hermelink as "problems and riddles which use the language of everyday but do not much care for the circumstances of reality".[13] "Lack of care" is an understatement: a funny, striking or even absurd deviation from the circumstances of reality is *an essential feature* of any recreational problem. It is this deviation from the habitual which causes amazement, and which thus imparts upon the problem its recreational value.

One function of recreational mathematics is that of teaching. Thus the following

> example of reduction of residues: A traveller, engaged in a pilgrimage, gave half his money at *Prayága*; two-ninths of the remainder at *Cáśi*; a quarter of the residue in payment of taxes on the road; six-tenths of what

> was left at *Gáyá*; there remained sixty-three *nishcas*; with which he returned home. Tell me the amount of his original stock of money, if you have learned the method of reduction of fractions of residues,[14]

which obviously serves to train a method of practical use, *viz*, that of "reduction of fractions of residues". The fancy application of a staple method may serve in part as an appetizer, in part also to suggest abstract or general validity: Look, this method may be used for *any* problem of a similar structure, not just for trite commercial calculation.

In this end of the spectrum of recreational mathematics, it passes imperceptibly into general school mathematics, which in the Bronze Ages as now would often be unrealistic in the precision and magnitude of numbers without being fun in any way. Whether fun or not, such problems would be determined *from the methods to be trained.* This organization around an existing stock of methods is also a valid description of the other end of the spectrum, the purpose of which was described in one example by Christoph Rudolph in 1540 as the way "to find out, *not without exceptional amazement of the ignorant*, how many penning, creutzer, groschen or other coin somebody possesses".[15] Recreational problems *stricto sensu* are *riddles*; like an *argot*, they belong to the cultural superstructure of the profession. They set aside the members of the craft as particular, and particularly clever, people (whether in the opinion of others or in their professional self-esteem) – and set aside those who are able to solve the problems as especially clever members of the craft. Since the problems occurring in everyday practice will soon become trivial, these are not fit for kindling anybody's vanity. Instead, more complex problems are constructed, which still look as belonging in the professional domain, and which are still solvable by current professional techniques — but only on the condition that you *are* fairly clever.

As a typical example illustrating the matter we may mention the "purchase of a horse", for example, in Leonardo Fibonacci's first version:

> Two men in possession of money found a horse which they wanted to buy; and the first said to the second that he wanted to buy it. If you give me $\frac{1}{3}$ of your money, I shall have the price of the horse. The second asked the first for $\frac{1}{4}$ of his money, and then he would equally have the price. The price of the horse and the money of each of the two is asked for.[16]

All elements of the problem are familiar to the commercial calculator or merchant; but the situation as a whole is certainly not, both because of the odd coincidence of numbers and because the problem is indeterminate. In which market would the price of a horse be determined only as an arbitrary multiple of 11?

A more extreme example can be found in the Carolingian *Propositiones ad acuendos juvenes*:

> A paterfamilias had a distance from one house of his to another of 30 leagues, and a camel which was to carry from one of the houses to the other 90 measures of grain in three turns. For each league, the camel would always eat 1 measure. Tell me, whoever is worth anything, how many measures were left.[17]

Again, the elements of the problem will have been familiar in the Near or Middle Eastern environment where its origin is to be sought. The solution given, however, is based on an unexpected trick (an intermediate stop after 20 leagues), as is characteristic of riddles (and on the tacit assumption that the camel eats nothing while returning), and not on mathematical reasoning; if a little elementary mathematical reasoning is applied to the trick, furthermore, the solution is seen not to be optimal even on its own premisses.

The latter case is extreme, but still illustrative of an important aspect of recreational mathematics. Actually, the problem-type is still alive in the contemporary Global Village, where one may find it dealing, for example, with jeeps and petrol in the Sahara. Once introduced a specious trick will, in fact, often be adopted into the stock of current techniques (even though it is of no use in professional practice), and new problems will be constructed where it can be used (often but not necessarily amounting to nothing but new fancy dressings of the same mathematical structure).

Over the whole range from school mathematics to mathematical riddles, the *methods* or *techniques* are thus the basic determinants of development, and problems are constructed which permit one to bring the methods at hand into play. This not only contrasts with that foundation of applications out of which the superstructure had grown and to which it referred, but also with the structure of *'scientific' mathematics* as embodied, for example, in Greek mathematics.

That *problems are primary* and the methods used to solve them derivative is a matter of course (and almost of definition) when *practical applications* are concerned: the Eiffel Tower, built in order to demonstrate the possibilities of modern iron constructions, remains an exception. That the same holds *for Greek pure mathematics* follows from historical scrutiny.

For one thing, we know the importance of the three 'classical problems' as foci of interest: doubling the cube, trisecting the angle, squaring the circle. When these were formulated as geometrical problems,[18] no theoretically acceptable methods were known which would allow them to be solved; their whole history throughout Antiquity is the story of recurrent attempts to solve them by means of methods more satisfactory than those found by earlier workers.[19] But we may also look at the theory of irrationals. The first discovery of irrationals led to the problems of *how to construct* according to a general scheme lines which are not commensurate with a given line (or whose squares are not commensurate with a given square); *how to classify* magnitudes with regard to commensurability; and *which are the relations* between different classes of irrationals? The first problem is the one which was addressed by Theodorus according to Plato's *Theaetetus* 147D;[20] further on in the same passage, Theaetetus makes a seemingly first attempt at the second problem; *Elements* X, finally, is a partial answer to all three problems.

This role of *the problem* is no exclusive distinction of Greek mathematics. It is a global characteristic of all those later traditions which can be characterized as 'scientific'.[21] As suggested above, it does distinguish, on the other hand, Greek mathematics from the non-utilitarian level of Babylonian mathematics. To see how, we may look at an Old Babylonian text:[22]

> A trapezoidal field. I cut off a reed and used it as a measuring reed.
> While it was unbroken I went 1 three-score steps along the length.
> Its 6th part broke off for me, I let follow 1,12 steps on the length.
> Again, $\frac{1}{3}$ of the reed and $\frac{1}{3}$ cubit broke off for me; in 3 three-score steps I went through the upper width.
> I extended the reed with that which [in the second instance] broke off for me, and I made the lower width in 36 steps.
> The surface is 1 bur [=30,0 nindan^2]. What is the original length of the reed?
> You, by your making: pose the reed which you do not know as 1.
> Break off its 6th part, then 0;50 remain for you.
> Detach its igi, raise [the resulting] 1;12 to 1 three-score.
> Append [the resulting] 1,12 to 1,12; it gives 2,24, the false length.
> Pose the reed which you do not know as 1.
> Break off its $\frac{1}{3}$, raise [the remaining] 0;40 to 3 three-score, the upper width; it gives 2,0.
> Accumulate 2,0 and 36, the lower width.
> Raise [the resulting] 2,36 to 2,24, the false width; 6,14,24 is the false surface.
> Repeat the [true] surface until twice, that is 1,0,0; raise this to 6,14,24; it gives 6,14,24,0,0,
> and raise $\frac{1}{3}$ cubit, which you broke off, to 3 three-score.
> Raise [the resulting] 5 to 2,24, the false length; it is 12,0.
> Break $\frac{1}{2}$ of 12,0, to two, make it confront itself.
> Append [the resulting] 36,0,0 to 6,14,24,0,0; it gives 6,15,0,0,0.
> 6,15,0,0,0 makes 2,30,0 equilateral.
> Append that 6,0 which you have left back to 2,30,0; it gives 2,36,0.

> The igi of 6,14,24, the false surface, cannot be detached. What shall I pose to 6,14,24 which gives me 2,36,0?
> Pose 0;25.
> Because the 6th part broke off, write 6, let 1 go away; you leave 5.
> <The igi of 5 is 0;12; raise 0;12 to 0;25: it gives 0;5.>
> Append 0;5 to 0;25: it gives you [0;30, i.e.] $\frac{1}{2}$ nindan as original reed.

The problem pretends to deal with surveying practice, that is, with an essential constituent of Babylonian scribal practice. But it is obviously no real-life problem but a puzzle. So far it belongs to the *genre* of recreational mathematics. Mathematically, however, it is above the level of normal recreational problems: firstly, it leads to a mixed, non-normalized second-degree equation;[23] secondly, this equation is itself only established by means of fairly complex first-degree operations. The first-degree operations involve repeated use of the 'single false position', which was a staple method of practical computation. Second-degree problems, on the other hand, would never occur in *real* scribal practice. Their solution built on a special trick (the quadratic completion), which appears to have been designated "the Akkadian method".[24] In one sense, this trick corresponds to the trick by which the paterfamilias avoided having all his grain eaten by the camel. But even though not relevant for daily mensuration practice, the completion is a *mathematical* trick; it is, moreover, the basis of a whole mathematical discipline (second-degree 'algebra') which was explored extensively and systematically in the search of problems permitting the use of the trick.

The problem of the broken reed is thus a nice illustration of the general character of Babylonian 'pure' mathematics. It is similar to the recreational *genre*, but it is much more technical. It is, like recreational mathematics, governed by the stock of available techniques and methods, and its purpose is to display and/or to train these; and like recreational mathematics, it goes beyond the range of practically relevant problems and makes use of techniques of no practical avail. But being taught in a formal and highly organized school-system it becomes systemaic, building up quasi-disciplines according to the possibilities of the fund of methods. In this respect it is very different from medieval recreational mathematics, which was practised by reckoners who (when exposed to the problem of repeated doublings of unity) "strain themselves in memorizing [a procedure] and reproduce it without knowledge or scheme, [and by others who] strain themselves by a scheme in which they hesitate, make mistakes, or fall in doubt", as it was formulated by al-Uqlīdisī in Damascus in A.D. 952.[25] Though basically sub-scientific in character, Babylonian mathematics demonstrates to what extent sub-scientific mathematics could mimic 'scientific' mathematics under appropriate cultural and institutional conditions (and to which extent it could not).[26] Babylonian mathematics could be said to represent *scholasticized* sub-scientific mathematics. Al-Uqlīdisī's reckoners, on the other hand, represent a *lay*, and presumably *orally transmitted*, type of sub-scientific mathematics.

Even Egyptian mathematics belongs to the scholasticized type: it differs from Babylonian mathematics regarding mathematical substance (and differs fundamentally); but it shares the overall sub-scientific orientation coupled to the rigorizing effects of a systematic school system. Quite different, however, is another apparent intermediate form between the sub-scientific and the 'scientific' orientation: Diophantus's *Arithmetic*. This work, too, shares many of the features of sub-scientific mathematics, from single problems to the guiding role of methods. Still, the work is written on the background of Greek '*scientific*' mathematics, and this background had provided Diophantus with his perspective. What he does is to *adopt a corpus of sub-scientific knowledge* into the domain of 'scientific' mathematics (and expand it immensely). Diophantine arithmetic is therefore not to be regarded along with the scholasticized, quasi-'scientific' variants of sub-scientific mathematics but under the heading of sub-scientific *traditions* and their role as *sources* for 'scientific' mathematics.

Before we leave the discussion of *the structure* of sub-scientific mathematics and take up the question of *traditions* we should take note that the subdivision into 'scholasticized' and 'lay' types is not the only relevant sub-division. Firstly, one can differentiate roughly between 'computational' and 'geometrical' orientation, as we shall do in the following discussion. Secondly, the concepts of 'practitioners' is of course vague; the computations of a caravan merchant and those of, say, a Sassanian royal astrologer are distinct not only according to subject-matter but also, and more decisively, when we ask for the level of mathematical sophistication. These distinctions are not only of internal relevance but also important if one wants to investigate the impact of various sub-scientific traditions upon 'scientific' mathematics. They can only be made, however, if we have a reasonably detailed knowledge of the sub-scientific tradition in question, which is often not the case. I shall therefore not pursue them in any detail.

Another highly interesting question regarding the possible relation between 'scientific' and sub-scientific mathematics I shall leave as an open question, this time not because adequate source material does not exist but because I am not sufficiently familiar with it: How are we to describe the relation between the two types in India, in China and in Japan? It is my preliminary impression from the secondary literature that at least Japanese *wasan* exemplifies a process where the 'pure' level of sub-scientific mathematics gives rise to a direct and smooth creation of 'scientific' mathematics. Is this really so? If it is, can a similar process be traced in India and Japan? And can this provide us with new insights into the possibly distinctive nature of Indian, Chinese and Japanese 'scientific' mathematics?

3. TRADITIONS AND INDEX FOSSILS

As stated initially, the term 'sub-scientific' describes an orientation of knowledge. In principle, such an orientation could be individual and idiosyncratic, as are contemporary deviations from the 'scientific' orientation (*cf.* ref. 21). But since it belongs with specialists' 'professions', which themselves are continuous over time, sub-scientific mathematics can be described historically as being carried by *traditions*.

Fundamentally arithmetical and geometrical rules are interculturally true — these are the "$2 \cdot 2 = 4$ – truths" which Karl Mannheim excluded from the concern of the sociology of knowledge;[27] similar global orientations and interest in specific problems may arise not because of diffusion but from similar sociological and technical backgrounds; and even common errors may be explained as parallel simplifications or as random incidents. Much of the subject-matter and many of the techniques of sub-scientific mathematics, however, are so specific that they are indubitable witnesses of the duration and intertwinement of traditions. Often, such shared elements are even the only indications of otherwise purely hypothetical connections; we may regard them as the index fossils of cultural history.

One such index fossil is the usage of 'parts of parts' and its extension into a system of ascending continued fractions.[28] 'Parts of parts' are expressions like "$\frac{2}{5}$ of $\frac{1}{3}$" for $\frac{2}{15}$; ascending continued fractions can be exemplified by the expression "$\frac{1}{3}$ and $\frac{2}{5}$ of $\frac{1}{3}$" ($= \frac{7}{15}$). Both expressions are found in medieval Islamic mathematics, and they are mostly discussed with reference only to this area and to the post-Islamic tradition in medieval and Renaissance Italy. Though rare they can, however, be found in Middle Kingdom Egyptian and Old Babylonian sources, often (in Egypt exclusively) in connections which suggest popular usage or in problems of definite riddle character. In late Greek Antiquity, they turn up in the arithmetical epigrams of the *Anthologia graeca* — but only in problems concerned with trade routes, with the partition of heritages, or with the hours of the day (whether working hours, astrological time-keeping or questions to the gnomon-maker with no further explicit purpose); the only exception among those problems dealing with other subjects concerns a banquet taking place in Hellenistic Syria. In medieval Islam, the usage goes together with the so-called finger-reckoning tradition, which in Arabic was called *ḥisāb al-Rūm wa'l-ᶜArab* ("computation of the Byzantines and the Arabs"). Even in the Carolingian *Propositiones ad acuendos juvenes* can it be seen — but only in a few problems of the same mathematical type as one in which it occurs in the Rhind Mathematical Papyrus.

The evidence does not allow us to determine whether the Babylonian and Egyptian usages were borrowed from some common contact (for example, a commercial intermediary), or whether shared computational tools or techniques or common Hamito-Semitic linguistic structures called forth parallel developments. But there is little doubt that the Greek use of composite fractions was borrowed from the Near East together with the techniques for time-keeping, astrology, and notarial and mercantile computation. The idiom of composite fractions turns out to be an important index fossil demonstrating that these techniques belonged in a common cluster, which was inspired or borrowed from the East as a relatively connected whole. The usage may have been in general use among Semitic speakers in the Near East, as the later Arabic sources suggest; but in the Greco-Roman world it never spread beyond the circle of practitioners of the techniques together with which it had been borrowed. It went inseparably together with a specific sub-scientific tradition.

The usage (and hence probably the living tradition) may even have remained restricted to the Greek orbit. This is actually intimated by the *Propositiones*. Several problems from this collection point to the Eastern trading connections of the Roman world; but none of them contains composite fractions. As observed above, those simple ascending continued fractions which occur suggest an Egyptian connection — which may be indirect but which appears in any case *not* to be mediated by the channel reflected in the *Anthologia graeca*.

The latter conclusion already involved another sort of index fossil: shared problems. Mathematical problems dealing with real applications may of course be shared already because features of social or technological organization are common (exceptions to this rule in the sphere of practical geometry will be discussed in Section 4); recreational problems, on the other hand, will often be so specious (regarding dressing, mathematical structure, and/or numerical constants) that chance identity can safely be ruled out. Shared recreational problems of this category therefore constitute firm evidence of cultural connections.

Much work has been done, especially by German historians of mathematics, on the distribution of specific recreational problems. The basic results on each type are conveniently summarized in the new edition of Tropfke's *Geschichte der Elementarmathematik*,[29] and there is no need to repeat the details. One very important category of problems is found in medieval India and Islam and then again in late medieval Western Europe. It includes the "purchase of a horse", the "hundred fowls", and many others. Some of these problems are also found in China, mostly in a different dressing but recognizable because of a peculiar mathematical structure; the same stock was also drawn upon by Diophantus, who of course stripped the problems of their concrete dressing.[30] Some of the problems, finally, turn up in the *Propositiones*. Disregarding perhaps the latter work, the distribution of this group of problems coincides with the trading network bound together by the Silk Route and its medieval descendants (when the *Propositiones* acquired their final form, Francia was no longer an active part of the network; instead, the very eclectic collection reflects the various influences to which the Roman world had been submitted in Antiquity). At least one problem belonging with the Silk Route group, however, can be traced to much earlier times: the successive doublings of unity treated so confusedly by al-Uqlīdisī's reckoners. In the same passage he says that these problems always deal with 30 or 64 successive doublings. The oldest version of the problem known to date is found in a mathematical tablet from Old Babylonian Mari.[31] This text has direct affinities both to the (Islamo-Indian) chessboard problem and to the version in the *Propositiones*, and probably also to a Chinese version which I have not seen.[32] Other problems may have originated elsewhere, but the Mari problem indicates that one of the centres from which the caravan-route merchants' culture grew was located in the Middle East (from where also the composite fractions appear to have diffused, without meeting with great success outside the Semitic-speaking area).

The "Silk Route group" encompasses many but far from all popular recreational problem types. The Egyptian flavour of certain problems from the *Propositiones* was pointed out above. Similarly, the "non-Near-Eastern" problems of the *Anthologia graeca* may have been borrowed from the same location as the unit fraction system of which they make use — *viz*, from Egypt. True, they are not specifically akin to anything known from Middle Kingdom Egyptian sources (the shared interest in unhomogeneous first degree problems is too unspecific to allow any conclusions); but a large group appears to reflect precisely that kind of elementary mathematics teaching which Plato ascribes to the Egyptians in *The laws* 819A–C.[33]

The riddle-character of recreational mathematics was referred to repeatedly above; like other riddles, recreational mathematics belongs to the domain of *oral literature*.[34] Recreational problems can thus be compared to folktales. The distribution of the "Silk Route group" of problems is also fairly similar to the distribution of the "Eurasian folktale", which extends "from Ireland to India".[35] However, for several reasons (not least because the outer limits of the geographical range do not coincide) we should not make too much of this parallel. Recreational problems belong to a *specific subculture* — the subculture of those people who are able to grasp them. The most mobile members of this group were of course the merchants, who moved relatively freely or had contacts even where communication was otherwise scarce (mathematical problems appear to have diffused into China well before Buddhism).

The parallel between recreational (and most sub-scientific) mathematics and oral literature is more illuminating in another respect. Around A.D. 900, Abū Kāmil described the recreational problem of "the hundred fowls" as

> a particular type of calculation, circulating among high-ranking and lowly people, among scholars and among the uneducated, at which they rejoice, and which they find new and beautiful; one asks the other, and he is then given an approximate and only assumed answer, they know neither principle nor rule in the matter.[36]

This is not far from the scornful attitude of literate writers to folktales up to the Romantic era. Their disdain would not prevent many of them from using folk tale material; but Apuleius and Boccaccio (to name only two of them) would then rework the material and "make it agree with good taste". This is also exactly what Abū Kāmil does with the problem of the hundred fowls: instead of presenting without "principle or rule" a particular haphazard solution to the indeterminate problem, he shows how to find the complete set of solutions. He was not the only 'scientific' mathematician reworking sub-scientific mathematics in this way. Diophantus's treatment of the "purchase of a horse" reflects the same attitude and intention; others, from al-Khwārizmī and Leonardo Fibonacci to Cardano, Stifel and Clavius would follow similar programs. Sub-scientific 'sources' for 'scientific' mathematics were thus accepted and used in a way which differed fundamentally from that in which 'scientific' sources were used. Pointedly and paradigmatically: when medieval mathematical cultures got hold of Euclid he supplied them both with material and with norms for mathematical 'good taste'. With these norms in their luggage, mathematicians would look around in their own world and discover there a vast supply — anonymous and ubiquitous — of mathematical problems and techniques beyond the range of what was already available in acceptable form; they would pilfer it (at times appropriate it wholesale), either without recognizing their debts (as Diophantus, Apuleius and Boccaccio) or while criticizing those poor fools who thought themselves competent (like Abū Kāmil and eighteenth century *literati*). No wonder that sources dealt with like this were not reflected in al-Nadīm's *Catalogue*.[37]

4. PRACTICAL GEOMETRY

Both composite fractions (at least when used outside the Semitic speaking area) and the cluster of "Silk Route Problems" belonged with the calculators' craft. They should not make us forget other traditions of sub-scientific mathematics engaged not in commercial transactions and accounting but in practical geometry. Their practitioners were surveyors, architects, master builders, and the like. In some cultures, all or some of these preoccupations were taken care of, if not by the same persons then at least by the same groups as those engaged in accounting and so forth. So, surveying, accounting, and the allocation of rations were the responsibility of scribes with a common schooling both in Old Babylonia and in Middle Kingdom Egypt[38] (architects and master builders, on the other hand, may have belonged outside the scribal craft). In other cultures, however, the tasks were socially separated (Roman gentlemen might write on surveying but not on accounting). It is therefore reasonable to deal with geometrical practice as a distinctive topic, the sub-scientific character of which need not be argued.

A reason why this topic should at all be discussed and not merely be dismissed with a ditto is the character of its index fossils. Recreational problems reflecting geometrical practice are relatively rare and therefore not very informative (some exceptions will be mentioned below); the methods used *in real life*, however, are more significant than in the case of computation, because they often make use of one of several possible approximations or techniques.

One familiar exemplification of this is the treatment of the circle. The Old Babylonians would find its area as $\frac{1}{12}$ of the square of the circumference (corresponding to the value $\pi = 3$) and the diameter as $\frac{1}{3}$ times the circumference; if needed, they could use a correction factor, corresponding to $\pi = 3\frac{1}{8}$.[39] The Egyptians would find the area as the square on $\frac{8}{9}$ of the diameter.[40] These methods are obviously unconnected. The ratio 3 between the diameter and the circumference is used in the Bible,[41] which fits well with that dependency of Babylonia which follows from borrowed metrologies (but which would of course prove nothing in itself). Ptolemaic Egyptian texts (written well after the conquest of Egypt by Assyrian and Persian armies and shortly after the establishment of Greek rule) determine the circular area as $\frac{3}{4}$ of the square on the diameter,[42] which looks like a grafting of the Babylonian numerical assumption upon the traditional Egyptian habit to determine the area from the diameter (and not from the circumference, as the Babylonians had done). Combined with other evidence this fossil can be taken to imply that the originally separate traditions had amalgamated in the Assyro-Persian melting pot. The adoption of a mathematically poorer way to calculate the circular area in Egypt shows that the amalgamation did not change the sub-scientific character of the domain. Increased mathematical precision ('truth') was apparently no decisive criterion for preferences; the new way may have been accepted because it offered greater calculational ease – or simply because it was used by the engineers and tax-collectors of the conquerors.

Later practical geometries, from the Roman agrimensors to the Hebrew *Mishnat ha Middot*, began using the Archimedean approximation ($\pi = \frac{22}{7}$).[43] This then confronts us with a new phenomenon. $3\frac{1}{7}$ can hardly have had any practical advantages over the ratio $3\frac{1}{8}$—except perhaps in the hodometer, where Vitruvius appears none the less to have considered the latter value satisfactory.[44] For easy computation in current metrologies, $3\frac{1}{8}$ was certainly to be preferred. The only reason to embrace $3\frac{1}{7}$ is that it *derived from 'scientific'* *mathematics*. The adoption is thus a portent of later tendencies to transform the autonomous sub-scientific traditions into *applied science*.[45] For long, however, it remained an isolated portend. The new value occurs as indubitable (or at least undoubted) *truth*, and often in eclectic connection with other techniques with no 'scientific' merit, as is characteristic of the sub-scientific traditions. Not only were societal conditions (in particular teaching systems) not ripe for the autonomous sub-scientific practitioners' traditions to be absorbed; *'scientific' mathematics* itself was hardly developed to a point and in a direction where purposeful simplification and restructuration would make it solve the *specific* problems of geometrical practice. Even in late Antiquity and the early Middle Ages, the sub-scientific category remains important if one wants to understand the specific character of geometrical practice and its relation to the 'scientific' knowledge of the day.

5. ALGEBRA

This also comes true if we consider the distinctive mathematical innovation of the early Middle Ages: algebra.[46] Conventionally, the rise of this discipline is traced to al-Khwārizmī's book on the subject.[47] It is, however, obvious already from al-Khwārizmī's preface that not only the subject but also the name was already established before his time. A few decades after al-Khwārizmī, Thābit ibn Qurrah wrote a treatise "On the rectification of the cases of *al-jabr*",[48] where the discipline was ascribed to a group of "*al-jabr*-people". On close investigation the sources leave no doubt that these were the carriers of a sub-scientific tradition, the doctrines and techniques of which had been systematized, excerpted and exposed by al-Khwārizmī from a 'scientific' perspective and then put on a Euclidean basis by Thābit. The practitioners of the field appear to have been 'reckoners', people engaged in accounting and juridical and commercial computation.[49]

The roots of this sub-scientific *al-jabr*-tradition are not easily extricated, and may be diverse. Indian 'scientific' algebra is out of the game, its organization being clearly different from (and on a higher level than) that of the tradition known to al-Khwārizmī. But some connection to India must be present, since the (far from self-evident) term used metaphorically for the first power of the unknown in a second-degree problem (*jidhr*, 'root', 'stem', 'stub', etc.) is shared with Indian sources from the first century B.C.[50] But the term for the second power (*māl*, 'property', 'possessions', 'fortune', 'assets', etc.; translated *census* in medieval Latin texts) coincides firstly with the term used for the unknown in a whole class of first degree problems, and secondly with the *θησαυρός* used in analogous Greco-Egyptian first-degree problems in the same function.[51] As far as I know no equivalent term is found in the same standardized role in Indian sources.[52]

It is thus not very plausible that the sub-scientific *al-jabr* tradition was borrowed directly from India; since both early writers on the subject are of Turkestanian descent the best guess is perhaps that its home was somewhere in Central Asia (Khorezm? Iran?), and that it had developed in this cultural meeting place *par excellence*. Still, this can be nothing more than a guess. Nor can it be known for sure whether the tradition had roots back to Babylonian 'algebra'; *if* it had, it had been much transformed before reaching the form in which we know it.

This form was that of a *rhetorical algebra*. What is meant by this concept can be shown by an example borrowed from al-Khwārizmī:

> If a person puts such a question to you as: "I have divided ten into two parts, and multiplying one of these by the other, the result was twenty-one"; then you know that one of the two parts is thing, and the other ten minus thing. Multiply, therefore, thing by ten minus thing; then you have ten things minus a square [*māl*], which is equal to twenty-one. Separate the square from the ten things, and add it to the twenty-one. Then you have ten things, which are equal to twenty-one dirhems and a square. Take away the moiety of the [number of] roots, and multiply the remaining five by itself; it is twenty-five. Subtract from this the twenty-one which are connected with the square; the remainder is four. Extract its root, it is two. Subtract this from the moiety of the roots, namely, five; there remain three, which is one of the two parts. Or, if you please, you may add the root of four to the moiety of the roots; the sum is seven, which is likewise one of the parts.[53]

In the first half, everything goes by 'rhetorical', that is, verbal argument. If we replace the term 'thing' with x, and express the arithmetical operations by symbols instead of words, the reduction of the problem follows exactly the same path as we would follow today. The second half, then, solves the reduced problem ("$10x = 21 + x^2$") according to a standard algorithm, without giving any arguments at all. This is indeed characteristic of the tradition.

True, al-Khwārizmī introduces geometrical justifications of these algorithms (justifications of the same character as the procedure used to solve the "broken reed" problem above). These are, however, grafted upon the contribution of the sub-scientific tradition, in agreement with al-Khwārizmī's 'scientific' perspective. This does not mean that he invented them himself,[54] nor that they were borrowed from Greek geometry, from which they differ in style. Instead, they were taken over from *another* sub-scientific tradition.

This tradition is reflected in another book, a treatise *On mensuration* written by some unidentified Abū Bakr and only known in a Latin translation from the twelfth century, the *Liber mensurationum*.[55] Its first half has little to do with practical mensuration; instead, it contains a series of problems which we would today describe as 'algebraic': a square plus its side equals 110; the 4 sides of a square plus its area equals 140; in a rectangle of area 48, the sum of length and width is 14; in a rectangle, where the sum of the area and the four sides is 76, the length exceeds the width by 2; etc. In most cases, two ways to solve the problem are indicated. The first, fundamental method is based on 'naïve' geometric arguments in Babylonian style, and formulated in a language which down to grammatical details repeats the phrasing of Babylonian texts.[56] The second method, which is stated to be "according to *aliabra*" and which is not given in all cases, is identical with al-Khwārizmian *al-jabr*.

Detailed analysis leaves no reasonable doubt that the trunk of the first part of the treatise reflects a sub-scientific tradition going directly back to Old Babylonian 'algebra' (the time-span may inspire doubts, but it coincides precisely with the span from the Mari doublings to the earliest versions of the chessboard-problem and from the Old Babylonian ascending continued fractions to those known from medieval Islam). This will have been the source from which al-Khwārizmī drew his geometrical justifications of the standard algorithms of *al-jabr*.

Who were the carriers of this tradition? The inclusion in a "treatise on mensuration" suggests that they were surveyors and possibly master builders. This guess is bolstered by a source from the late tenth century, Abūl-Wafā"s *Book on what is necessary from geometric construction for the artisan*. One passage of this work[57] refers to discussions with artisans from this category, where these keep to ideals and methods corresponding precisely to those displayed in the *Liber mensurationum*. *Algebra*, as explained to posterity by al-Khwārizmī, is thus a merger of calculators' and practical geometers' sub-scientific traditions.[58]

This is of course a nice point. But does it imply that Old Babylonian scribal mathematics, which in its own times had combined calculation and practical geometry, was only transmitted by practitioners of the latter field?

Not necessarily. Even though the *Liber mensurationum* seems to be a *direct* continuation of Old Babylonian scribal traditions its basis may indeed be different: a practical geometers' tradition which had also once inspired the Old Babylonian scribe school teachers and was systematized by them.

For the moment, this is a hypothesis — but still a hypothesis derived from the sources. The main source is an Old Babylonian text dealing very systematically with problems concerning squares and sides.[59] It starts from the elementary beginning: Given the sum of area and side; then (in symbolic mistranslation) $x^2-x=c$ (problem 2); $ax^2+bx=c$ and variations on that pattern (problems 3, 4, 5, 6, 16); $x^2+y^2=a$, $x\pm y=b$ (problems 8–9); $x^2+y^2=a$, $y=bx$ (problems 10–11, 13); $x^2+y^2=a$, $x \cdot y=b$ (problem 12); $x^2+y^2=a$, $y=bx+c$ (problem 14); $x^2+y^2+z^2+u^2=a$, $x=bu$, $y=cu$, $z=du$ (problem 15); etc. — all in technical and standardized language. Then, suddenly, in problem 23, we find the following (my translation and restitution of damaged passages):

> In a surface, the four fronts and the surface I have accumulated, 0;41,40.
> 4, the four fronts, you inscribe. The igi of 4 is 0;15.
> 0;15 to 0;41,40 you raise: 0;10,25 you inscribe.
> 1, the projection, you append: 1;10,25 makes 1;5 equilateral.
> 1, the projection, which you have appended, you tear out: 0;5 to two you repeat: 0;10 nindan confronts itself.

Abstractly seen, this is once more the simple problem $x^2+ax=b$. But the wording is different from that used in the beginning, and so is the procedure,[60] which locates "the four fronts" as four rectangles of width 1 (the "projection") along the edges ("fronts") of the square.

The same figure is used by al-Khwārizmī in the first of two alternative proofs of the algorithm solving the case $x^2+ax=b$.[61] The problem of a square or rectangle to which is added the sum of all four sides will also be remembered from the *Liber mensurationum*. In contrast to the series of technicalized problems in the beginning of our Babylonian tablet, this looks like a typical *recreational problem* created and transmitted in a surveyors' environment. At the same time, school systematization of the techniques involved in the solution of the recreational problem would automatically lead to something like the initial technical series. It is thus at least a reasonable assumption that the *square/rectangle plus four sides* was originally invented in a surveyors' environment as a recreational problem and then taken over by the scribal school and used to sharpen the wits and kindle the self-esteem even of the future accountants and other calculators.

Another typical recreational problem for surveyors will have been the bisection of a trapezium by a parallel transversal. This was also popular in the Old Babylonian school; but it goes back at least to the 23rd century B.C.[62] Old Babylonian texts presenting its solution share the characteristic vocabulary of second-degree 'algebra'. 'Neo-Sumerian' texts (dating from the 21st century B.C.), on the other hand, seem to ignore these subjects. It is therefore my guess that the surveyors' environment which created the quasi-algebraic recreational problems was Akkadian (in agreement also with the name of the completion trick, see above) and non-scribal (whence illiterate). As the Akkadian tongue (in Babylonian dialect) became a literary language in the Old Babylonian period, adoption and ensuing systematization into the scribal school took place. At the same time, however, the original surveyors' tradition survived, and it is this tradition (I guess) rather than a direct descendant of the Old Babylonian scribal school that surfaces for the last time with its pet recreational problems in the *Liber mensurationum* and in the early ninth century justifications of the algorithms of *al-jabr*.

6. THE END OF SUB-SCIENTIFIC MATHEMATICS

Already Hero attempted to improve the practitioners' tool kit by teaching them Archimedean formulae, and thus to bring the autonomous sub-scientific traditions under the sway of theoretical knowledge — we may regard his work as an early attempt to make *applied science*. He and other Alexandrinians had some success, in so far as they had some practitioners' groups accept the Archimedean value for π and the formula for the triangular area. As we saw, however, the sub-scientific character of practitioners' mathematics was not changed. Nor was it affected when Diophantus and others borrowed sub-scientific material and reworked it from a 'scientific' perspective.

Changes set in, however, in the medieval Islamic world. Islamic scholars were more prone to accept the *specific* problematic of practitioners as legitimate without refusing for that reason the legitimacy of the 'scientific' perspective; they were thus able to combine the 'Heronian' and the 'Diophantine' approaches, transforming material from both sources in a way which was relevant for practitioners. "Applied science' began making its way.[63]. Thābit ibn Qurrah would still know the *al-jabr*-people as a sub-scientific group in the mid-ninth century. Already one or two generations later, however, *al-jabr* would only be known to Abū Kāmil as al-Khwārizmī's discipline, and it would be listed by al-Fārābī exclusively under the heading of "ingenuities" with reference to *Elements* X, and not with sub-scientific, 'practical' arithmetic or geometry (*cf.* above, ref. 37).

The process was never brought to completion in the Islamic world. Nor was it during the European Renaissance, where similar developments would take place from Jean de Murs to Adam Riese. In the mid-sixteenth century, however, from Stifel's and Riese's times onwards, the impetus of the process had become irresistible, and from then on specialists' mathematical practice was no longer semi-autonomous but dependent upon 'scientific mathematics'. In Early Modern Europe, then, the concept of 'sub-scientific knowledge' loses definitively its heuristic value at least as far as mathematics is concerned (in most other fields, the change took place later); instead, we encounter the problem, to which extent (and in which sense!) 'applied science' can be described as *an application* of 'science'. This, however, is a different problem, which I shall leave to historians and philosophers of modern technology.

Recreational mathematics did not die with sub-scientific mathematics. But it stopped being the exclusive property of the minority of specialists who were able to grasp it. Like mathematical literacy, recreational problems became a possession of the majority — eventually the belonging of virtually everybody.

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This article is dedicated to the memory of N. I. Bukharin (1888–1938) who in 1931 taught the International Congress of the History of Science and Technology about the intricacy of the relations between these subjects.

REFERENCES

1. 981^{b}14-982^{a}1 — quoted from W. D. Ross (ed., transl.), *The works of Aristotle, translated into English*, viii: *Metaphysica*, 2nd edn (Oxford, 1928).
2. Very briefly stated (too briefly!), 'scientific' knowledge thus aims at *truth*, while sub-scientific knowledge is directed toward utility. In the mathematical domain this might make us conclude that 'scientific' knowledge is mathematics built on proofs while sub-scientific mathematics builds on recipes and empirical rules. This conclusion is false for several reasons. Firstly, as we know, e.g., from cosmogonies, statements may be considered supremely true because they are old and revered or because they are supposed to stem from sacred revelation. At times, the latter source is encountered even in mathematics: in one of his works, al-Bīrūnī tells that the Indians hold the ratio of circular circumference to diameter to be as "the ratio of 3,927 to 1,250, because it was communicated to them, by divine revelation and angelic disclosure" that such were the proportions of the world — as translated in D. Pingree, "The fragments of the works of al-Fazārī", *Journal of Near Eastern studies*, xxix (1970), 103–23, p. 120. Secondly, Nicomachus's 'science' is argued solely on the basis of empirical rules. Thirdly, complex mathematics, not least complex mathematics aimed at many-sided *application*, can be transmitted successfully only if supported by some level of understanding, whence by some sort of argument or proof. Basically, the need for proofs in mathematics comes from *teaching*; the elevation of the need into a transcendental philosophical principle is historically secondary — *cf.* my "Influences of institutionalized mathematics teaching on the development and organization of mathematical thought in the pre-modern period", *Materialien und Studien: Institut für Didaktik der Mathematik der Universität Bielefeld*, xx (1980), 7–137.
3. The "Old Babylonian" mathematical texts were produced between *c.* 1800 B.C. and 1600 B.C. When speaking in the following of "Babylonian mathematics" I shall refer exclusively to this corpus.
4. This position is asserted in Morris Kline's *Mathematical thought from ancient to modern times* (New York, 1972), 11.
5. When, for instance, will you know the area of a trapezoidal field and the portion which broke off from your measuring reed but not the reed itself (*cf.* below)? The evidence could be multiplied *ad libitum*.
6. So, *grosso modo*, Carl B. Boyer, *A history of mathematics* (New York, 1968), 45.
7. See my "Varieties of mathematical discourse in pre-modern socio-cultural contexts: Mesopo-

tamia, Greece, and the Latin Middle Ages", *Science & society*, xlix (1985), 4–41, pp. 11–17, and below, Section 2.

8. B. Dodge (ed., tr.), *The Fihrist of al-Nadīm: A tenth-century survey of Muslim culture* (2 vols, New York and London, 1970), ii, 634ff. In other parts of the catalogue, al-Nadīm mentions many non-Greek authors; their absence in the chapter on mathematics can hence not be explained as a result of personal prejudice on his part.
9. This follows, e.g., if one compares Indian and early Islamic algebra; *cf.* my "The formation of 'Islamic mathematics': Sources and conditions", *Science in context*, i (1987), 281–329, p. 286.
10. On the whole, this holds even for the mathematical training of astrologers. Part of the Indian mathematical techniques which they took over was incorporated in books — but as technical chapters in astronomical books, only indirectly derived from the great mathematicians, where, e.g., Āryabhaṭa's value for π would occur as something communicated "by divine revelation and angelic discourse" (*cf.* above, ref. 2; a general impression of the mathematics of early Islamic astronomy can be gained from E. S. Kennedy, "The lunar visibility theory of Yaᶜqûb ibn Târiq", *Journal of Near Eastern studies*, xxvii (1968), 126–32; D. Pingree, "The fragments of the works of Yaᶜqûb ibn Ṭâriq", *Journal of Near Eastern studies*, xxvii (1968), 97–125; and *idem*, "The fragments of the works of al-Fazārī" (ref. 2)).
11. This word, like any other modern term, is not completely adequate. We might also speak of 'crafts', if only we keep in mind that no guild institution need be involved, and that the groups in question were composed of 'higher artisans'.
12. Notable exceptions to this rule are made up by Sumerian and Babylonian accounting tablets and the Mycenean Linear B tablets. Others could be mentioned.
13. H. Hermelink, "Arabic recreational mathematics as a mirror of age-old cultural relations between eastern and western civilizations", in A. Y. al-Hassan, Gh. Karmi and N. Namnum (eds), *Proceedings of the First International Symposium for the History of Arabic Science, April 5–12, 1976* (Aleppo, 1978), ii, 44–52, p. 44.
14. Bhascara II, *Lilávati*, 53. Quoted from H. T. Colebrooke (ed., tr.), *Algebra, with arithmetic and mensuration from the Sanscrit of Brahmagupta and Bhascara* (London, 1817; reprinted, Wiesbaden, 1973), 24.
15. Chr. Rudolph, *Künstliche Rechnung mit der Ziffer* ..., 2nd edn (Vienna, 1540), the introduction to the chapter "Schimpfrechnung" (my translation and emphasis).
16. *Liber abaci*, my translation from B. Boncompagni (ed.), *Scritti di Leonardo Pisano matematico del secolo decimoterzo*, i: *Il Liber abaci di Leonardo Pisano* (Rome, 1857), 228.
17. Problem 52, version II. My translation from M. Folkerts, "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*. Überlieferung, Inhalt, Kritische Edition", *Denkschriften der Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse*, cxvi, part 6 (1978), 74.
18. I shall not go into details concerning the distinction between *geometrical* and everyday (practical) problems, but only refer to Aristotle's polemics against various sophists' approaches which simply miss that distinction and thus permit trivial solution (*Analytica posteriora* 75ᵇ40-76ᵃ3; *De sophisticis elenchis* 171ᵇ16-22, 172ᵃ3-7; *Metaphysica* 998ᵃ1-4).
19. See, e.g., Th. L. Heath, *A history of Greek mathematics* (2 vols, Oxford, 1921), i, 218–70, "Special problems", which lists these attempts.
20. Plato, *Theaetetus. Sophist*, ed. and trans. by H. N. Fowler (Loeb Classical Library, London and Cambridge, Mass., 1921), 24.
21. There are of course lots of individual exceptions. When mathematics *per se* has become a

profession, people in need of a dissertation subject or another item in the list of publications will easily end up looking out for problems which are likely to be solved with the methods already at their disposal.

22. VAT 7532, in O. Neugebauer, *Mathematische Keilschrift-texte* (3 vols, *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*. Abteilung A: *Quellen*, iii, erster-dritter Teil (Berlin, 1935-37)), i, 294f. The translation is mine, and builds on my reinterpretation of the Old Babylonian mathematical terminology. Without going into irrelevant details the text should be comprehensible with the following explanations:
(1) Numbers are written in a sexagesimal place-value system (Neugebauer's notation).
(2) 1 cubit = $\frac{1}{12}$ nindan; the nindan is the basic length unit and equals approximate 6m.
(3) To "detach the igi of *n*" means finding its reciprocal ($\frac{1}{n}$).
(4) "To raise" means calculation of a concrete entity through multiplication.
(5) "To repeat until twice" means (concrete) doubling.
(6) "To make *a* confront itself" means constructing a square with side *a*; if we do not care about the real (geometric) method of the Babylonians we may translate it "to square".
(7) "*a* makes *b* equilateral" means "*b* is the side of a square with area *a*"; in numerical interpretation, $b=\sqrt{a}$.
23. $6,14,24 \cdot z^2 - 12,0 \cdot z = 1,0,0$, where z is $\frac{1}{6}$ of the original length of the reed.
24. According to text IX in E. M. Bruins and M. Rutten, *Textes mathématiques de Suse* (Mémoires de la Mission Archéologique en Iran, xxxiv (Paris, 1961)), 63f. See my "Algebra and naive geometry: An investigation of some basic aspects of Old Babylonian mathematical thought", *Filosofi og videnskabsteori på Roskilde Universitetscenter*, 3. Række: *Preprints og reprints*, 1987, nr 2 (to be published in *Altorientalische Forschungen*, xvii), 114f.
25. Quoted from A. S. Saidan (ed., tr.), *The* Arithmetic *of al-Uqlīdisī* (Dordrecht, 1978), 337.
26. This is not the place to go into details about the substance of Babylonian 'pure' mathematics. I shall only refer to my "Algebra and naive geometry" (ref. 24), which gives the reasons why its 'algebra' must have been built on geometrical (though 'naive', not critical) argumentation, and where the overall cognitive orientation of Babylonian mathematics is also discussed (Chapter 9).
27. K. Mannheim, *Ideologie und Utopie*, 4. Auflage (Frankfurt a. M., 1965), 71f., 251f., and *passim*.
28. The detailed arguments for the following discussion of composite fractions are given in my "On parts of parts and ascending continued fractions: An investigation of the origins and spread of a peculiar system", *Filosofi og videnskabsteori på Roskilde Universitetscenter*, 3. Række: *Preprints og reprints*, 1988, nr 2 (to be published in *Centaurus*). Here I also discuss why similar usage in different cultures cannot be explained away as a random phenomenon.
29. J. Tropfke, *Geschichte der Elementarmathematik*, 4. Auflage, Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke (Berlin and New York, 1980), 573–660.
30. *Arithmetica*, I, xxiv-xxv are unmistakable stripped versions of the "purchase of a horse" (P. Tannery (ed., tr.), *Diophanti Alexandrini Opera omnia cum graecis commentariis* (2 vols, Leipzig, 1893–95), i, 56–61).
31. D. Soubeyran, "Textes mathématiques de Mari", *Revue d'Assyriologie*, lxxviii (1984), 19–48, p. 30. The text is discussed and compared to later versions in my "Al-Khwârizmî, Ibn Turk, and the Liber mensurationum: On the origins of Islamic algebra", *Erdem*, ii (Ankara, 1986), 445–84, pp. 477–9.
32. Connected to a tale about a peasant and his servant, whose wages are determined as successively doubled harvests from one grain of rice; reported in S. Thompson, *Motif-index of folk-literature* (6 vols, rev. and enl. edition, London, 1975), v, 542, no. Z 21.1.1.

33. Plato, *Laws*, ed. and trans. by R. G. Bury (2 vols, Loeb Classical Library, London and Cambridge, Mass., 1926), ii, 104f.
34. We observe that this oral character of genuine recreational mathematics sets it apart from Old Babylonian sub-scientific mathematics, which was carried by written texts even though the details of didactical explanation have normally been given orally. Genuine recreational mathematics belongs with 'lay' traditions; the methodical orchestration of scholasticized mathematics negates its recreational value, even though single problems like the above "broken reed" may betray a recreational origin.
35. See S. Thompson, *The folktale* (New York, 1946), 13ff.
36. My English translation from H. Suter, "Das Buch der Seltenheiten der Rechenkunst von Abū Kāmil al-Miṣrī", *Bibliotheca mathematica*, 3. Folge, xi (1910–11), 100–20, p. 100.
37. Another group of encyclopedias does reflect the sub-scientific traditions, those concerned with the *practice of science* and not directly with *books*. A good example is found in Al-Fārābī, *Catálogo de las ciencias*, edición y traducción Castellana por A. G. Palencia, segunda edición (Madrid, 1953), 39–53, *cf.* Arabic, 73. Here, seven branches of mathematics are distinguished, the last of which is *ʿilm al-ḥiyal*, "science of devices/ingenuities", which according to the description appears to refer to *practical applications of 'scientific' mathematics*—i.e., to 'applied mathematics' as defined in chapter 1. At the same time, several of the other branches (so arithmetic and geometry) are subdivided into 'theoretical' and practical'. No doubt, this division is a reflection of Aristotle's conceptions — but Aristotle's categories are apparently understood exactly as done above, as 'scientific' and sub-scientific mathematics, respectively.
38. In Babylonia this is made fully clear, e.g., in line 27 of the "Examination Text A", "Kennst du die Multiplikation, die Bildung von reziproken Werten und Koeffizienten, die Buchführung, die Verwaltungsabrechnung, die verschiedensten Geldtransaktionen, (kannst du) Anteile zuweisen, Feldanteile abgrenzen?" (quoted from Å. Sjöberg, "Der Examenstext A", *Zeitschrift für Assyriologie und vorderasiatische Archäologie*, lxiv (1975), 137–76, p. 145). In the case of Egypt, it follows from the range of subjects dealt with in the Rhind Mathematical Papyrus, as also from the Papyrus Anastasi I, a 'satirical letter' much used in the scribal school to vilify the poor dunce who knows neither how to calculate a ramp, nor to provide rations for troops, nor to find the number of men required to transport an obelisk (etc.); translated in A. H. Gardiner, *The Papyrus Anastasi I* and the *Papyrus Koller*, together with parallel texts, *Egyptian hieratic texts*, Series I: *Literary texts from the New Kingdom*, Part I (Leipzig, 1911).
39. See text III, lines 2, 3 and 30, in Bruins and Rutten, *Textes mathématiques de Suse* (ref. 24), 25f., and commentary pp. 31, 33.
40. See Problems 41ff. in A. B. Chace *et al.*, *The Rhind mathematical papyrus, ii: Photographs, transcription, transliteration, literal translation* (Oberlin, Ohio, 1929).
41. I Kings 7, 23; II Chronicles 4, 2.
42. See, e.g., Problems nos. 32 and 42 in R. A. Parker, *Demotic mathematical papyri* (Providence and London, 1972), 40f. and 54ff.
43. See M. Cantor, *Vorlesungen über Geschichte der Mathematik*, i: *Von den ältesten Zeiten bis zum Jahre 1200 n. Chr*, dritte Auflage (Leipzig, 1907), 551; and S. Gandz (ed., tr.), *The Mishnat ha Middot, the first Hebrew geometry of about 150 C.E., and the Geometry of Muhammad ibn Musa al-Khowarizmi, the first Arabic geometry [c. 820]* (*Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abteilung A: *Quellen*, ii (Berlin, 1932)), 49.
44. Or even the ratio 3! See Vitruvius, *De architectura* X, ix, 1 and 5. The text is doubtful, *cf.* the text with appurtenant notes in Vitruvius, *On architecture*, ed. and trans. by F. Granger (2 vols, Loeb Classical Library, London and Cambridge, Mass., 1931, 1934), ii, 318 and 322.

45. Earlier parallels *could* be pointed out. So, when Middle Kingdom Egyptian scribes began making their accounts in unit fractions instead of metrological sub-units (as had been done in the Old Kingdom), the only reason is that they had learned this system in school — and the reason to introduce it in school will have been that it was easier to argue for (and thus teach unambiguously) the precise calculations in unit fraction notation than to defend the necessarily approximate (and thus ambiguous) solutions of practical problems in metrological units (see my "Influences of institutionalized mathematics teaching ..." (ref. 2), 34).
46. This formulation fits 'Western' civilization (the medieval centre of which was the Islamic world). Indian high-level algebra is older, but it did not influence the Western development (*cf.* above).
47. F. Rosen (ed., tr.), *The Algebra of Muhammad ben Musa* (London, 1831). Another book on the subject and probably also the same title (*Kitāb al-jabr wa'l-muqābalah*) was written by ibn Turk at almost the same time (A. Sayılı, *Abdülhamid ibn Türk'ün katışık denklemlerde mantıkî zaruretler adlı yazısı ve zamanın cebri* (*Logical necessities in mixed equations by ʿAbd al Ḥamîd ibn Turk and the algebra of his time*) (Publications of the Turkish Historical Society, series VII, 41; Ankara, 1962)). The question of priority can not be settled with certainty, but terminological considerations suggest that ibn Turk was at least *independent* of al-Khwārizmī (see my "Al-Khwârizmî, Ibn Turk, and the Liber mensurationum ..." (ref. 31), 474, note 28).
48. P. Luckey, "Ṯābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen", *Berichte der Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-physische Klasse*, xciii (1941), 93–114.
49. The detailed arguments for this (which also involve Abū Kāmil's *Algebra*) are given in my "Al-Khwârizmî, Ibn Turk, and the Liber mensurationum ...". One correction should be given to the exposition in that paper: the distinction between two groups of calculators each with their own specific methods referred on p. 472 to Abū Kāmil is not found in the Arabic text (J. P. Hogendijk (ed.), Abū Kāmil Shujāʿ ibn Aslam, *The book of algebra. Kitāb al-jabr wa l-muqābala* (Publications of the Institute for the History of Arabic-Islamic Science, series C: Facsimile editions, xxiv (Frankfurt a. M., 1986)). It seems to have been inserted into the Hebrew Renaissance translation and thus to be irrelevant to the early history of the subject.
50. See B. Datta and A. N. Singh, *History of Hindu mathematics: A source book* (2 vols, Lahore, 1935–38; reprint edn, Bombay, 1962), i, 169. In India, the term is used in connection with the extraction of a square root, and the interpretation is geometrical and thus meaningful. In the *al-jabr*-tradition, the term is *not* understood geometrically, which deprives it of metaphorical meaning.
51. See problems nos. 13 and 17 in J. Baillet, *Le Papyrus mathématique d'Akhmīm* (Mémoires de la Mission Archéologique Française au Caire, ix, part 1 (Paris, 1892)), 70 and 72. It should be observed that there is nothing strange in the use of the same term for the unknown in problems of type $a \cdot x = b$ and for the second-degree term in problems of type $y^2 + a \cdot y = b$, if only we put $y^2 = x$, thus transforming the latter problem into $x + a \cdot \sqrt{x} = b$. Islamic algebras will normally give the value for both x and for y, thus regarding the *māl* as an unknown in its own right. The transformation is hence justified by the sources.
52. See also Datta and Singh, *History of Hindu mathematics* (ref. 50), ii, 9f.
53. Rosen (ed., tr.), *The Algebra of Muhammad ben Musa* (ref. 47), 41f.
54. For other operations al-Khwārizmī *did* invent geometrical justifications for the rhetorical reductions himself, but then the wording is different ("This is what we intended to elucidate"; "We had, indeed, contrived to construct a figure also for this case, but it was not sufficiently clear", Rosen (ed., tr.), *The Algebra of Muhammad ben Musa* (ref. 47), 32 and 34).

55. Edited critically in H. L. L. Busard, "L'algèbre au moyen âge: Le 'Liber mensurationum' d'Abū Bekr". *Journal des savants*, Avril-Juin 1968, 65–125.
56. Detailed analysis in my "Al-Khwârizmî, Ibn Turk, and the Liber mensurationum ..." (ref. 31), 456–68. It should be noted that the geometrical character of the argument only follows from indirect arguments — the figures belonging with the solutions are lost in the Latin translation.
57. Chapter X, xiii, in "Abu-l-Vafa al-Buzdžani, *Kniga o tom, čto neobxodimo remeslenniku iz geometričeskix postroenij*", ed. and trans. by S. A. Krasnova in A. T. Grigor'jan and A. P. Juškevič (eds), *Fiziko-matematičeskie nauki v stranax vostoka* (Sbornik statej i publikacij, i; Moscow, 1966), 42–140, p. 115.
58. This *might* also be what al-Fārābī tells us when stating in his *Catalogue* that the science of algebra is "common to arithmetic and geometry" (other interpretations are possible) — my translation from Al-Fārābī, *Catálogo de las ciencias*, ed. by Palencia (ref. 37), 52.
59. BM 13901, in Neugebauer, *Mathematische Keilschrift-texte* (ref. 22), iii, 1–5, *cf.* translations and discussions in my "Algebra and naive geometry" (ref. 24), 43–56.
60. Analysis in my "Algebra and naive geometry", 48–51.
61. Rosen (ed., tr.), *The Algebra of Muhammad ben Musa* (ref. 47), 13–15. The procedure fits the algorithm badly; when it is none the less used (and used *first*), the reason must be that it was familiar.
62. Jöran Friberg, "Mathematik", *Reallexikon der Assyriologie*, section 5.4.k (forthcoming).
63. This 'Islamic miracle' (a term coined in imitation of the well-known 'Greek miracle', the creation of the autonomous 'scientific' approach) is dealt with extensively in my "The formation of 'Islamic mathematics' ..." (ref. 9).

J

"Sub-scientific Mathematics: Undercurrents and Missing Links in the Mathematical Technology of the Hellenistic and Roman World". *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1990 nr. 3.

TABLE OF CONTENTS

I. ANCIENT MATHEMATICS: THEORY OR TECHNOLOGY?

Greek mathematics—to anybody who possesses the faintest idea of the history of mathematics, this means something in the style of Euclid's *Elements*, of Archimedes, of Apollonios, of Diophantos and of Pappos. That is, »Greek mathematics« (or, we might generalize, referring to the »mathematics of Classical Antiquity«, or be precise and speak of »Hellenistic mathematics«) is a field of knowledge concerned with theoretical understanding of abstract entities. Those whose ideas are less faint may know about the Heronian corpus, about Ptolemy, and about similar applications of the abstractions to describe material reality—»the more physical of the branches of mathematics«, to speak with Aristotle[1]. Still, *theory* retains the primary role, and the rest remains derivation—»subordinate«, if we stick to Aristotelian parlance[2]. Neo-Pythagorean works like those of Nicomachos, of course, change nothing in this respect, and Hero constitutes a minor exception, the very distinctiveness of which seems to confirm the global rule.

As it has been pointed out by G. J. Toomer[3], this image of Greek mathematics is produced by a somewhat distorting lens: The ideals of the schoolmen of late Antiquity and early Medieval Byzantium. *They* decided which manuscripts were to be copied and be preserved with sufficient care. The effect of this process of spontaneous censorship is revealed by the character of those works which are only known via

[1] *Physica* 194^{a}7; transl. R. P. Hardie and R. K. Gaye 1930.

[2] E.g., *Posterior Analytics* 75^{b}14-16; transl. G. R. G. Mure 1928.

[3] Toomer 1984.

Arabic translations[4]. Among the works which were translated into Arabic around A.D. 800 and thus still available in Greek at that date but often only in one defective manuscript and not in the late Middle Ages are:

- Euclid's treatise of the division of figures.
- A number of presumably Archimedean works, dealing *inter alia* with the construction of the regular heptagon and the construction of water-clocks.
- Menelaos' *Spherics* and his treatise on the mathematics of specific gravities.
- Ptolemy's *Planisphaerium* and *Optics*.
- Books 5-7 of Apollonios' *Conics* and his *On cutting off a ratio*.
- Pappos' *Commentary to Book X of Euclid's Elements*, the passage of his *Collection* dealing with constructions with fixed compass opening and part of Book VIII on mechanics.
- Part of Diophantos' *Arithmetica*[5].

Even though there is no obvious system in this list, it suggests that works which did not agree with the canon of »compass and ruler«, which were too sophisticated, or which belonged to the Aristotelian category of »subalternate sciences« (optics, mechanics, spherics) were more likely to be neglected than others.

It can be easily argued that this bias corresponds to the attitudes expressed by a multitude of later Hellenistic and late Ancient authors from Plutarch to Proclos[6]. To them, mathematics was, in itself, either a way to gain higher insight or, more modestly, a propaedeutic paradigm by which the ability to gain insight was trained—or it was

[4] In several cases, where the Arabic translations have also been lost, their content has been incorporated or paraphrased in surviving texts, or they have been translated into Latin in the twelfth century.

[5] For detailed information, see the relevant articles in DSB and (as far as more recent discoveries are concerned) Toomer 1984.

[6] See, e.g., the oft-quoted passage in *Tabletalk* VIII.2 (ed., transl. Minar et al 1969: 118-131), where Plutarch and his table companions discuss Plato's reasons for claiming (as they suppose he did) that »God is always doing geometry« .

a hermeneutic aid, necessary for the interpretation of Plato and Aristotle. Hermeneutic assistance apart, however, these attitudes are close to those expressed by Plato and Aristotle in the fourth century B.C., in whose vicinity mathematicians like Theaetetos and Eudoxos made their work; they correspond fairly well to the style of the major mathematicians[7], and even to those of the lost works which exist in genuine Arabic translation; they are not contradicted by mathematically competent commentators and compilators from Geminos to Pappos, Theon and Eutocios. Byzantine scholars, furthermore, were not too strict in their criteria, as demonstrated, e.g., by their compilation of the Heronian *Geometrica*. All in all, then, the lens of the late schoolmen can be seen to have been somewhat distorting; but it certainly did not change the total picture, nor *a fortiori* produce an illusion. Greek and Hellenistic mathematics, *in its culturally and quantitatively dominating form, was* theoretical and concerned with abstract entities—»pure«, we would say. What is more: Even works which according to their contents were »applied« (dealing with physical or astronomical reality like, e.g., Archimedean statics or Autolycos' spherics) tended to be *formally pure,* demonstrating thus their dependence on the abstract fundament.

To us, this may seem the natural order of things, heirs to the Hellenistic tradition as we are. Ever since the French École Polytechnique was established in 1794, engineers have been taught their applied mathematics according to the same model. Seen in the context of the Ancient world, however, the pattern of Greek and Hellenistic mathematics is outstanding. The Romans only accepted it halfheartedly if at all. This is clearly stated by Cicero in the *Tusculan Disputations*[8]: »With the Greeks geometry was regarded with the utmost respect, and consequently none was held in greater honour than mathematicians, but we Romans have restricted this art to the practical purposes

[7] Ptolemy, so it seems, did not even regard the high-level logistics of astronomical computations as a *mathematical* activity, as pointed out by Olaf Pedersen (1974: 32-34).

[8] I, ii; transl. King 1971: 7.

of measuring and reckoning«. A demonstration *ad oculos* is provided by Quintilian in the passage of *De institutione oratoria* where the relevance of geometry is explained[9]: Firstly, the term geometry is taken to include plain numerical computation; secondly, the main aim of teaching the subject is to avoid elementary blunders in basic practical numerical and field-surveying calculations. *Roman* mathematics at its best, on this evidence, was not Euclid, nor even Hero's deliberate adaptation of theoretical results for use in practice; it is adequately represented by the agrimensors' secondary adoption of Heronian and similar Alexandrian material.

The Greek and Hellenistic pattern is also radically different from *earlier* mathematical traditions. Babylonian and Egyptian mathematics (the only early traditions which are clearly documented and clearly dated[10]) originated as *technologies*, as techniques for accounting, for field measurement, and for the planning of provisions for workers and soldiers. In the long run, Babylonian mathematics certainly did not stick to this »applied« character: many of its characteristic problems, indeed whole disciplines, are definitely non-utilitarian. But however »pure« the contents, the *form* remained »applied«[11] (on a more modest

[9] I, x, 34ff; ed., transl. H. E. Butler 1969: IV, 176ff.

[10] The Indian geometrical tradition as documented in the *s | ulba sūtras* may antedate the rise of Greek mathematics; but in their present form, the *sūtras* may be roughly contemporary with the early Greek mathematicians (c. 500 to c. 300 B.C.) (Bag 1979: 4-6). Chinese mathematics is only documented from the Han era onwards (Martzloff 1988: 110ff).

[11] One striking example of »pure commercial arithmetic« is the following problem: »I have bought 770 sila [≈liter] oil. From what was bought for 1 šeqel of silver I cut away 4 sila each time. I saw 40 šeqel of silver as profit. How much did I buy and how much did I sell [for each šeqel]?« (my translation from the transliteration in Bruins & Rutten 1961: 82; the meaning is that if r sila are bought per šeqel, r-4 sila are sold at that price). Thanks to the fiction of a merchant who buys and sells at prices he does not know, a second-degree equation is produced—something which would never happen in real practical computation within the Babylonian horizon.

Like the vast majority of Babylonian mathematical texts, this problems dates from the Old Babylonian era (c. 1900 to c. 1600 B.C.).

scale, the same can be stated of Egyptian mathematics). Even when the mathematics of the scribal cultures was non-utilitarian (»pure«), it was never *theoretical*, neither in Greek nor in modern sense[12]. In mathematics (as elsewhere), Greek culture created something radically new—something which was then institutionalized and ripened in the early Hellenistic era, in particular around Alexandria, and conserved and canonized by the Hellenistic schoolmen throughout the Roman period.

Apart from their »pure« outgrowth, Babylonian and Egyptian mathematics had corresponded to obvious social needs of a practical nature. Evidently, these needs were not abolished at the birth of Thales nor through the Macedonian conquest. Nor were they covered by the sparse works applying theoretical results to more practical problems, which, in fact, were either concentrated within select areas (predominantly other sciences like astronomy) or attempts to improve upon the bad methods used by rank-and-file practitioners[13]. Already from first principles we can thus be sure that practical arithmetical and geometrical computation—the two fundamental mathematical technologies—lived on throughout the Classical age; *by name* we also know

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[12] This does not mean, as often claimed, that Babylonian mathematics was based on empirically discovered rules and on rote learning. As I have documented in a large-scale investigation of the techniques and mode of thought of Babylonian »algebra« (Høyrup 1990), it was based on intuitively meaningful manipulation of geometrical figures. But it did not *aim at insight*, and thus was not theoretical in the Greek sense. Nor was it organized in a explicitly formulated coherent conceptual structure, whence it cannot be considered theoretical in a modern sense.

[13] In the introduction to his treatise on the *Dioptra* (ed., transl. Schöne 1903: 188ff), Hero explains his aim to be to correct earlier errors, those committed to writing as well as those committed in actual siege warfare. One of the blunders which Quintilian wants to eliminate through the teaching of geometry is the measure of areas by means of the circumference of figures, accepted according to him by almost everybody (*De institutione oratoria* I, x, 39; ed., transl. Butler 1969: 178f) and apparently also used throughout Classical Antiquity by practitioners. Admittedly, the evidence cited for this by Eva Sachs (1917: 174) is weak when taken in itself—Thukydides (*History of the Peloponnesian War* VI, i, 2; ed., transl. Voilquin 1950: II, 69) does not argue from the time of circumnavigation of Sicily for its area but for the difficulty of the Athenian military adventure—but cf. corroborating evidence cited below,section VII.

about logistics and geodesics from Geminos[14] as well as Aristotle and Plato and a number of commentators, and in the *Corpus iuris civilis*, *calculatores* are mentioned a few times on a par with librarians, nomenclators (slaves telling names of persons met or of fellow slaves to their master), stenographers, stage-players and other performers of practical arts[15]. Unlike its scribal predecessor traditions, however, this practical computation and its carriers had stopped being culturally productive; to a large extent their existence was not even recognized by the culturally productive stratum, and we are thus told virtually nothing about the actual ways and tasks of these lowly people, beyond, e.g., their use of ropes and rulers[16] and of concrete (»sensible«), not abstract numbers.

Some supplementary evidence comes from administrative Greco-Egyptian papyri, from descriptions of and materials for elementary teaching[17], from pictorial representations of calculators manipulating calculi on an abacus, and from surviving specimens of this device. On the whole, however, material of Classical provenience tells us fairly little about the basic mathematical technologies of the Hellenistic and Roman world. In particular, it does not inform us whether (or to which extent) they were ultimately derived from the theoretical mathematics of the age, indigenously but autonomously developed, or borrowed from older neighbouring cultures.

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[14] Fragment on the mathematical sciences; ed., transl. Aujac 1975: 115f.

[15] *Digest* XXVII, i,15,§5; XXXVIII,i,7,§5; L,xiii,1,§6. Cf. Marrou 1956: 431 and Kinsey 1979.

[16] Geminos, fragment on mathematics, ed., transl. Aujac 1975: 115f.

[17] The most important specimen being the compendium in Guéraud & Jouguet (eds, transls) 1938. Such material for primary education, of course, carries little information on the more specific ways of professional calculators.

II. SUB-SCIENTIFIC MATHEMATICS

Left with Greek and Roman sources alone, we would thus have to content ourselves with the observation that practical arithmetic and geometry existed and were distinguished sharply from theoretical mathematics. Happily, however, we are not left with Greek and Roman sources. Earlier and later mathematical cultures have given us their own documents, which happen to make new sense of scattered and otherwise unapparent evidence in the Classical sources. Before discussing this directly we shall, however, introduce some general observations on the different varieties of mathematical activity in the pre-Modern world.

A passage from Aristotle's *Metaphysics*—dealing not with mathematics but with productive arts and theoretical knowledge in general—may introduce the problem:

> At first he who invented any art whatever that went beyond the common perceptions of man was naturally admired by men, not only because there was something useful in the inventions, but because he was thought wise and superior to the rest. But as more arts were invented, and some were directed to the necessities of life, others to recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered, [...]
>
> So [...], the theoretical kinds of knowledge [are thought] to be more the nature of Wisdom than the productive.[18]

[18] 981^b14-982^a1—ed., transl. Ross 1928.

First of all, this introduces the distinction between productive and theoretical knowledge and establishes precisely that mutual ranking which was noted above for the case of mathematical disciplines. Secondly it presupposes that the two kinds of knowledge are carried by different (groups of) persons: logistics and geodesics are not supposed to be performed by arithmeticians and geometers, the theoretical mathematicians. Thirdly, even productive knowledge is pointed to as going »beyond the common perceptions of man«, i.e., to be *specialists' knowledge.*

We might also speak of *craft knowledge*. Specialists in the practical arts, indeed, belong to different crafts, whose members were until the onset of the Modern era (in most cases, until the present century) trained within the profession, either as apprentices or, in exceptional cases (Babylonian and Egyptian scribe schools, the Abacus school of Late Medieval and Renaissance Italy), in specific schools. The diffusion of knowledge from the theoretical sciences (after the emergence of these during Classical Antiquity) was slow and random, not systematized as in Modern engineering schools, where teachers who have themselves been trained at a university teach future engineers their physics and mathematics, thus ensuring the diffusion of relevant results within a single generation. Minor exceptions disregarded, the knowledge of practical specialists was thus autonomous, and not to be understood as »applied science«[19]. At the same time, the knowledge of a craft constitutes, in the likeness of a scientific discipline, an *organized body of knowledge* and not a mere heap of random and disconnected rules; but cognitive coherence is no primary aim in itself but only a by-product of the practical coherence of the activity of the craft, whose members (e.g., geometrical practitioners) will often attend to a number of different practical tasks united by the fact that they can be dealt

[19] Even today, it is true, the idea of technology as »applied science« should be handled with uttermost care, as it has been established by historians of technology in recent decades. Still, the diffusion of scientific knowledge through the network of teachers and the deliberate search for relevant knowledge makes the »application of science« *one* important aspect of modern technologies.

with by means of the tools and specific methods of the craft[20]. In order to emphasize both the organized character of this kind of specialists' knowledge going »beyond the common perceptions of man« and the distinct character of this organization, I have suggested the term »sub-scientific knowledge«[21].

In the following, we shall concentrate on sub-scientific *mathematics*, even though the concept has wider currency. Like Babylonian scribal mathematics, sub-scientific mathematics in general possesses a »pure«, i.e., non-utilitarian level, which can be regarded as its »cultural superstructure«. None the less, the *raison-d'être* of a body of practitioners' knowledge remains its adequacy with respect to the practical tasks of the professional group in question. The utilitarian basis of a body of sub-scientific mathematical knowledge is thus determined by problems, and its characteristic methods and conceptual tools have been developed with the aim of coping with these problems. To this extent, the basic structure of sub-scientific mathematics is similar to the central principle of theoretical mathematics of Greek type. The key to the development of Greek mathematics, too, was the *problem*, notwithstanding its usual textual presentation in the form of axioms, theorems, etc.

Firstly, the importance of the three »classical problems« is well-known: *viz.* doubling the cube, trisecting the angle, squaring the circle. When these were first approached as specific geometrical problems, presumably in the late fifth century B.C., no theoretically acceptable methods were at hand allowing solution; from Hippocrates of Chios and Archytas onwards, incessant attempts were made to solve them

[20] In the case of professions carried by a scribal or similar school, a supplementary source for cognitive coherence is the systematizing dynamics of the school—a fact to which we shall return below. The aggregate outcome of these two drives for secondary cognitive coherence—the practical coherence of professional tasks as reflected in schoolmasters' ideals—is that casuistic organization which characterizes not only Babylonian (and Egyptian) mathematical texts but also the »Codex Hammurapi« and the Babylonian omen literature.

[21] The most thoroughly discussion of the concept will be found in Høyrup 1990a, on which the present article draws on a number of points.

by means of methods more satisfactory than those found by earlier workers[22]. But a case of even greater consequence is provided by the theory of irrationals. The first discovery of irrationals—itself a result of theoretical investigation—highlighted problems which could not have been formulated at the level of common sense (as could the »classical problems): *how to construct* according to a general scheme lines which are not commensurate with a given line (or whose squares are not commensurate with a given square); *how to classify* magnitudes with regard to commensurability; and *which are the relations* between different classes of irrationals? The first problem was addressed by Theodoros according to Plato's *Theatetus* 147D[23]; according to the same passage, the young Theaetetos made a (seemingly first) attempt at the second problem; *Elements* X, finally, is a partial answer to all three problems. Later on, all of them were taken up by Apollonios in work which is now lost but described by Pappos[24].

Paradoxically, the »pure« level of sub-scientific mathematics is different. It is determined not by problems but by its stock of methods and selects its problems according to their tractability by this stock at hand. To understand why, we may look at the expressions and functions of this cultural superstructure. There are two such functions, though interconnected and not always to be clearly distinguished: teaching, and the formation of professional identity and pride.

Teaching of future practitioners, evidently, aims at transmitting acquaintance with existing methods and skill in using them. This is a question of training, not of understanding or familiarizing with abstract theorems, and since the Bronze Ages the main medium for this has

[22] See, e.g., Heath 1921: I, 218-270, "Special Problems". The wider perspectives of the issue are dealt with extensively in Knorr 1986, which it would lead too far to discuss in detail.

[23] Ed., transl. Fowler 1977.

[24] *Commentary on Book X of Euclid's Elements,* ed., transl. Thomson & Junge 1930: 63f. According to the same passage in the commentary, the mature Theaetetos was responsible for the substance of *Elements* X. (Some doubts as to the identity between the conserved Arabic text and Pappos' original commentary have been raised, see Bulmer-Thomas, "Pappus of Alexandria", DSB X, 299f).

always been the *exercise problem*—in so far as it has not been supervised participation in genuine practice. Participation has left few detectable textual traces, while collections of exercise problems constitute our main sources for several mathematical cultures (not least Babylonian and Egyptian mathematics). The problems in question, however, are not in themselves fundamental, in the sense that they are posed because somebody needs or wants their solution—they are nothing but pretexts for the application of existing methods, and constructed so as to allow the practice of these. Problems, in other words, are *a means*, geared to the core of the subject-matter to be taught, the existing stock of methods.

The *formation of professional identity and pride* is served in particular by so-called »recreational problems«, one specimen of which we may look at:

> A paterfamilias had a distance from one house of his to another of 30 leagues, and a camel which was to carry from one of the houses to the other 90 measures of grain in three turns. For each league, the camel would always eat 1 measure. Tell me, whoever is worth anything, how many measures were left.[25]

The problem is found in a Carolingian anthology of which more shall be said below. For the moment, we shall concentrate on the characteristics of the problem text itself.

Firstly, it is strikingly unrealistic in spite of its apparently daily-life subject-matter. Unless an astute trick (an intermediate stop after 20 leagues) is introduced, exactly *nothing* will be left. »Recreational problems« owe their entertainment value precisely to such grotesquerie and unexpected coincidences.

Secondly, the format is that of a riddle. No wonder that the anthology in question went together with a collection of riddles in many Medieval manuscripts, nor that Book XIV of the *Anthologia graeca*

[25] *Propositiones ad acuendos iuvenes*, problem 52, version II, ed. Folkerts 1978: 74; my translation.

(to which we shall also return) combines »recreational problems«, riddles and oracles.

Thirdly, however, the riddle is for specialists only. As far as the present problem is concerned, this is perhaps not obvious to readers in a world where basic (and even not quite basic) numeracy is widespread. In the pre-Modern world, however, only the professional specialists would be able to follow the solution—not to speak of finding it. In the Roman world, even the majority of the generally educated would be at a loss, as was the (apparently not uncommon) orator told about by Quintilian, who »contradicts the calculation which he states in words by making an uncertain or inappropriate gesture with his fingers«[26]. Thus, by being able to solve the riddle you demonstrate (to yourself as well as to others) that you belong to the select members of the calculators' craft—that you are »worth something«.

This point may stand out more clearly in another »recreational problem«, belonging to the widespread class »purchase of a horse«:

> Two men in possession of money found a horse which they wanted to buy; and the first said to the second that he wanted to buy it. If you give me $^1/_3$ of your money, I shall have the price of the horse. The second asked the first for $^1/_4$ of his money, and then he would equally have the price. The price of the horse and the money of each of the two is asked for.[27]

That this problem is intended for specialists will be obvious. Even in our times, few but those who remember their school algebra will know how to approach it, and even the majority of these might give up at the versions involving three or more buyers. All elements of the

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[26] *De institutione oratoria* I, x, 35; transl. Butler 1969: 177. According to the context, the inappropriate finger-reckoning gestures imply that the orator has learned by heart a result found by others, not being able to find it himself.

[27] Leonardo Fibonacci, *Liber abaci*, ed. Boncompagni 1857: 228; my translation. The type was most popular in the Middle Ages (Leonardo gives a number of variants with three, four or five participants); but as we shall see below, it was already known to Diophantos.

problem would of course be familiar to merchants of Antiquity and of the Middle Ages. The total situation, however, is as unrealistic as anything could be—already for the reason that the price of the horse can be any multiple of 11. Similarly, to combine the staple methods of commercial arithmetic in a way which solves the problem requires skill and dexterity in a world not yet in possession of symbolic algebra—and even more than ordinary skill. If you find the solution without hesitation you are really »worth something« within the community of reckoners.

This (and not plain and vaguely defined fun, as the misleading name of the genre suggests) is precisely the function of the mathematical puzzles. To a large extent, professional identity and pride consists in awareness of one's professional skill. In principle, of course, this skill is displayed in actual professional practice. But the mathematical problems presenting themselves in the everyday practice of an accountant or merchant will soon become trivial, and hence not fit for kindling anybody's vanity. Here problems like the »purchase of a horse« come to serve: more complex than everyday problems yet still looking as if they belong within the professional domain, and still solvable by current professional techniques—but only on the condition that you *are* fairly clever. Problems of this category set aside the members of the craft as particular, and particularly clever, people (whether in the opinion of others or in their professional self-esteem)—and set aside those who are able to solve the problems as craft members *par excellence*. In order to do this they have to make use of the characteristic techniques of the craft. Like the problems made for teaching, they are thus constructed around the stock of existing methods—at times enlarged by specific tricks like the intermediate stop, which, once found, become a sanctioned part of that stock and of professional sub-culture in general without possessing any utilitarian function; as it shall be explained below (in note 79), a process of this kind appears to be the origin of second-degree algebra.

So far, all sub-scientific mathematical activity was treated as a uniform phenomenon. This is certainly a rough approximation, and distinctions may be introduced along many dimensions—level, degree of specialization, reckoning versus geometry. One dichotomy of importance for the understanding of the difference between the Roman world and the Bronze Age cultures is that between scribal versus non-scribal organization, between school and apprenticeship transmission of the professional tradition.

This dichotomy reveals itself (*inter alia*) in the attitude to non-utilitarian problems. The typical attitude of non-scribal reckoners is described by the mid-tenth century Damascene textbook author al-Uqlīdisī. He tells about reckoners who (when exposed to the problem of repeated doublings of unity, of type »chess-board problem«)

> strain themselves in memorizing [a procedure] and reproduce it without knowledge or scheme, [and by others who] strain themselves by a scheme in which they hesitate, make mistakes, or fall in doubt.[28]

Scribal reckoners, on the other hand (be they Babylonian scribes or Medieval clerks), will have been trained in agreement with the typical spirit of the school, according to a fixed curriculum constructed with some degree of systematic progress and involving some sort of explanation or description of principles[29].

Certain Greco-Egyptian papyri demonstrate the survival of some kind of »scribal schooling« moulded upon the traditional Egyptian pattern albeit presumably in weakened form—in particular the slightly postclassical Papyrus Akhmîm[30]. Elsewhere, where no such antecedents could make their influence felt, whatever scanty evidence we have suggests the »apprenticeship model«[31], with what that implies for the character of sub-scientific mathematical knowledge (no drive toward systematization and order, etc.).

[28] *Book on the Chapters of Hindu Reckoning*, ed., transl. Saidan 1978: 337.

[29] This is precisely what is often denied concerning Babylonian and Egyptian scribal mathematics; but cf. above, note 12.

[30] Ed., transl. Baillet 1892; see, e.g., pp. 34f and 59ff of the commentary.

[31] See, e.g., Kinsey 1979. But the evidence *is* scanty.

III. TRADITIONS

As long as they have existed, crafts have transmitted their cunning from one generation of practitioners to the next, and they have borrowed (as a rule selectively) from neighbouring cultures. The same can be supposed regarding the mathematical techniques of computation, geometrical calculation, and practical-geometrical construction. But independent invention of similar techniques is a no less recurrent phenomenon, and no less to be assumed in the case of practical mathematics.

In the case of simple applied arithmetic, it is impossible to decide whether shared knowledge and similar techniques indicates diffusion from one culture to another or common response to similar practical problems. Even as complex a procedure as the division on the Medieval abacus[32] is demonstrably devised anew time and again. Shared basic arithmetical techniques do not prove the existence of connections between mathematical practitioners of different cultures. The same holds for elementary geometrical constructions and simple area calculation, including a number of »wrong«, approximate formulae which are near at hand.

This is the reason that »recreational« problems are important, not only for the understanding of the cultural sociology of the craft of reckoners but as »index fossils«. One thing is to observe that the

[32] For instance: 3390 divided by 188. Instead of counting off 188 repeatedly, you see how often you can count off 200. Answer: 16, leaving 190. But each time you have removed 200 you have taken away 12 too much, all in all $12 \cdot 16 = 192$; the real remainder is thus $190+192=382$, from which you may take away 200 once, leaving a true remainder of $182+12=194$, i.e., an extra 188 and a remainder of 6. 3390 divided by 188 thus gives a result of 18 and a remainder of 6.

problem of repeated doublings of unity is found in Bronze Age Babylonia, Roman Egypt, Carolingian France, and Medieval Damascus and India. This could still be a non-utilitarian play occurring naturally to anybody trading in numerical computation. But when al-Uqlīdisī observes that »this is a question many people ask. Some ask about doubling one 30 times, and others ask about doubling it 64 times«[33]; when we know that the still famous »chess-board« version consists in 64 doublings, while all the other versions cited have precisely 30 doublings, there can hardly be any doubt that the motif was borrowed: nothing in the nature of numbers or geometrical series suggest the choice of 30 members, only few of the problems speak of days, and only one (late) explicitly to the days of a month[34].

Apart from odd problems, certain peculiar expressions and weird geometrical approximations can serve as index fossils. Taken together, the evidence demonstrates the existence of a number of enduring diffusion patterns, the identity of which shall be briefly mentioned in the present section of the article—more detailed information follows below[35].

One can be defined as the »Silk Road community«, the community of traders interacting in Antiquity along this combined caravan and sea route and its extensions, reaching from China to Cadiz, and encompassing in the Middle Ages at least the Mediterraneo-Islamo-Indian trade network with its offshoots. Within this whole area, recreational mathematical puzzles appear to have migrated as »camp fire riddles« for professional traders.

Evidently, we have no direct testimony of this oral mathematical culture. But from all over the area we know either problem collections or, more often, arithmetical textbooks including favourite problems. Everywhere, indeed, mathematicians behaved towards *their* oral tradition as did Apuleius, Boccaccio and others with regard to the

[33] Ed., transl. Saidan 1978: 337.

[34] Bhaskara II, *Līlāvatī*, ed., transl. Colebrooke 1819: 55.

[35] See also Høyrup 1990a for details. A general overview of widespread problem types with references to single occurrences is given in Tropfke/Vogel 1980: 513-660.

treasure of anonymous folk tales known to them: Borrowing, pilfering, and putting in »better taste«—which last thing means in mathematics giving explicit rules and proofs, and ordering according to mathematical principles (cf. the first quotation from al-Uqlīdisī, which mentions the oral tradition and criticizes its lack of principles).

Another network of diffusion, revealing itself in a particular way to speak of fractions, seems to be restricted to the Semitic-speaking area and its immediate Mediterranean contacts in Antiquity and the Middle Ages. A third network, finally, is connected to surveyors or other practical geometers; it has links backward to Old Babylonian mathematics, and certain of its characteristic ways turn up in Hellenistic and Roman sources.

Since the evidence for the existence of these networks is always indirect, it is not possible to determine to which extent they were carried by distinct professional groups within, e.g., Hellenistic and Roman society. Evidence for partial overlap of carriers will be mentioned below.

IV. »SILK ROAD« INFLUENCE IN THE CLASSICAL WORLD

Above, the simplest version of the »purchase of a horse« was quoted. More often, the problem involves three or more potential buyers, of which the first needs (e.g.) one third of what the others have together, the second (e.g.) one fourth of the total possessions of the others, etc. This is one of the problem types which turns up everywhere along the »Silk Road« trading network. Most of the evidence, it is true, comes from the Medieval era[36], and there is no reason to cite it in detail. But three examples can be found in Classical sources: Firstly, Diophantos' *Arithmetica* I,xxiv[37]. As always in Diophantos, it treats of pure numbers and not of money. The structure, however, is unmistakable: to find three numbers (say, A, B and C), so that $A+{}^1/_3(B+C) = B+{}^1/_4(A+C) = C+{}^1/_5(A+B)$. Apart from the abstract formulation, this problem coincides precisely (coefficients included) with another purchase found in Leonardo Fibonacci's *Liber abaci*[38]. Secondly, *Arithmetica* I,xxv, which involves four unknown numbers and the successive fractions ${}^1/_3$, ${}^1/_4$, ${}^1/_5$ and ${}^1/_6$. Thirdly and finally, a hint in passing at the characteristic clothing (»to buy in common or sell a horse«) as to something familiar occurs in Book I of Plato's *Republic*[39].

Another widespread type is the »give and take«: A says »if you give me P, I shall have m times as much as you«; B answers »but if you give me Q, I shall have n times as much as you«[40]. Even this problem is treated (in pure numbers) by Diophantos, *viz.* in *Arithmetica* I,xv. It would of course be possible, if these Diophantine problems had been quite isolated in their epoch, to claim that Diophantos was the original source and the later »recreational« versions nothing but derivations in disguise[41]. But they are not, as shown by earlier Greek as well as Chinese evidence. One Greek source is the Greco-Egyptian mathematical papyrus Michigan 620, which dates from no later than the early second century C.E.[42] Like quite a few problems from

[36] All examples mentioned by Tropfke/Vogel (1980: 609f) except the Chinese ones are Medieval or early Modern.

[37] Ed., transl. Tannery 1893: I, 56-59.

[38] Ed. Boncompagni 1857: 245.

[39] 333b-c, ed., transl. Shorey 1978: I, 332f. I am grateful to Benno Artmann for directing my attention to this passage.

[40] See Tropfke/Vogel 1980: 610f.

[41] This was in fact intimated by Kurt Vogel (1954:219), before his translation of the Chinese *Nine chapters*. Woepcke's pure-arithmetical translation (1853: 77 n° 26) of a problem in al-Karajī which in the Arabic original deals with horse-trade (Jacques Sesiano, private communication) seems to have called forth the misunderstanding.

[42] Published in Robbins 1929; cf. Vogel 1930.

Diophantos' *Arithmetica* I, it deals with linear problems with several unknowns, and solves them by means of the ἀριθμός (abbreviated ς) representing the unknown number in a way reminding much of Diophantos (and even more, perhaps, of the more straightforward procedure presumably added by a scholiast in I,xviii, I,xix and I,xix, where Diophantos becomes too elegant and sophisticated); it seems to excel in the same reference to »ratio with excess« which abounds in Diophantos, and which is one of the key concepts of Euclid's *Data* (cf. below, section VI); and depending on an ambiguous restoration, one of the problems may even coincide in mathematical detail with *Arithmetica* I,xx. But in contradistinction to Diophantos and in the likeness of Medieval material of sub-scientific origin, its problems seem to deal with quantities of *drachmas,* not with pure numbers, i.e., numbers of *monads*[43].

Another interesting piece of evidence is Iamblichos' discussion of »Thymaridas flower« in his commentary to Nicomachos' *Introduction to Arithmetic*[44]. In order to show the general applicability of the rule he illustrates its strength by means of two examples which are fully in the spirit of *Arithmetica* I,xvi-xxi though actually not to be found in Diophantos. The argument presupposes that such problems were somehow considered of importance, i.e., that some group in Iamblichos' third century (and, we may assume with some confidence, in Thymaridas' fourth century B.C.) took interest in linear algebraic problems with several unknowns.

The Chinese evidence is found in Chapter VIII of the *Nine Chapters on the Mathematical Art,* the *Jiuzhang suanshu*[45], which dates no later than the early Christian era[46]. N° 10 is a precise mathematical analogue

[43] Vogel 1930: 373f.

[44] Cited from Heath 1921: 94ff.

[45] Ed., transl. Vogel 1968.

[46] Vogel (1968: 5) cites the statement of the mathematician and commentator Liu Hui from A. D. 263, according to whom the final version of the work dates from the first century B.C. Lǐ and Dù (1987: 35) point out that the work is absent from a catalogue of books from the late first century B.C. where it should probably have

to the two-person »purchase of a horse« quoted above from Leonardo Fibonacci. N^{os} 12 and 13 are analogues of the 3- and 5-person purchases of a horse (though in variant dress[47]) in the version where each potential buyer asks the following and not everybody (a type dealt with extensively by Leonardo and related to *Arithmetica* I,xxii-xxiii).

The technique used in the *Nine Chapters* to solve these problems differs from the one used by Diophantos; it consists in a sophisticated manipulation of numerical arrays ordered in a matrix, which can hardly be imagined to be the way used by those who carried the characteristic riddles along the trade routes. In any case, the difference in method excludes both that the Chinese should have borrowed everything from an early Greek precursor of Diophantos and, vice versa, that Diophantos should have had direct access to the Chinese textbook for future mandarins.

Diophantos' method, as we saw, was close to that of earlier Greco-Egyptian algebra, employing the same symbol for the unknown number and using it in the same way. This method, in contradistinction to that of the *Nine Chapters*, thus seems to have been in use among practitioners in at least part of the Classical world. It may also

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been mentioned had it been in existence, while a text from c. A.D. 50 refers correctly to the contents of the single chapters. According to Martzloff (1988: 118ff), finally, certain parts of the work go back to early Han or even further, but precisely chapter VIII (and IV, which does not concern us here) contains no socio-cultural or metrological chronological cues suggesting an early date. Nor does its contents seem to be represented on the arithmetical bamboo strips found in a tomb from the second century B.C. (according to the preliminary information given in Li & Dù 1987: 57). Since precisely chapter VIII is, as formulated by Martzloff (1988: 124), »de loin, le plus original de tous«, it is probably safe to date it to the first century C.E.

[47] The different guises should not astonish us. In general, the editors of the *Nine Chapters* seem to have used their phantasy most creatively in order to vary the clothing of problems. But the Chinese collection shares many problem types apart from those mentioned here with the recreational mathematics characteristic of the other parts of the »Silk Road area«. A striking coincidence in *mathematical structure* alone should therefore be sufficient evidence that a particular Chinese problem is connected to one known from the remaining area.

have accompanied the cluster of »Silk Road problems« further down through time. Leonardo, indeed, during his discussion of »give and take« problems, presents what turns out to be exactly the procedure used in *Arithmetica* I,xv (Diophantos' version of the »give and take«) under the name *regula recta,* telling it to be most commendable and in use among the Arabs[48]. Further on in the *Liber abaci,* the name and the method turns up repeatedly, showing the term to cover rhetorical algebra of the first degree and in one variable based on the name *res* for the unknown, corresponding to the Arabic *šay'* and later Italian/German *cosa/coss,* and calculating »de principio ad finem questionis«[49].

The presence of sub-scientific »Silk Road« material in the Classical world is confirmed by two slightly post-Classical sources. One of them is the above-mentioned collection of arithmetical epigrams in Book XIV of the *Anthologia graeca*[50]. The collection was presumably put together by Metrodoros around A.D. 500, but the single epigrams are of earlier and varied origin. More will have to be said about the collection below, but in the present connection it should be observed that two of the epigrams (N^os^ 145 and 146) are of the »give and take« type, while N^os^ 7, 130, 131, 132, 133 and 135 deal with the »filling of a container« from a number of sprouts with different capacity—a type which is also testified as N° VI,26 of the *Nine Chapters*[51].

[48] *Liber abaci,* ed. Boncompagni 1857: 191.

[49] *Liber abaci,* ed. Boncompagni 1857: 203. Here, the *regula recta* is opposed to the *regula versa,* the backward computation from result to unknown values. The »direct« computation, which begins at the beginning where essential nubers are still unknown, evidently presupposes that these be represented by some all-purpose name or symbol.

[50] Ed., transl. Paton 1979.

[51] Ed., transl. Vogel 1968: 68. A mathematically analogous type treats of combined work performance. This is found as *Anthologia graeca* XIV,136 and in the *Nine Chapters* as N^os^ VI,20-25. It may be of interest that the solution in the case of two workers doing a job together (or two sources filling the vessel) can be conveniently expressed by ways of the harmonic mean.

The other post-Classical source of interest is the Carolingian collection *Propositiones ad acuendos iuvenes*[52], maybe put together by Alcuin (in any case connected to the Carolingian educational effort) but similarly composed from older (rather disparate) material circulating in the north-western provinces of the Roman Empire and adopted into Monastic recreational lore in late Antiquity. In any case, an Ancient origin can safely be ascribed to those problem types whose geographical distribution connects them to the transcontinental trading network, the characteristic of the early Frankish Middle Ages being precisely the extreme attenuation of international commercial relations—with ups and downs, it is true, but on an extremely modest level compared with the situation which had prevailed during the Principate.

One of these types is represented by another »give and take« problem (N° 16). Two others are the »hundred fowls« (N^{os} 5, 32, 33, 34, 38, 39, 47) and the »pursuit«. In the first type[53], a number (typically one hundred) of animals or objects (typically fowls) are bought at different prices per piece for different categories but totalling the same number of monetary units (N° 39, dealing with »animals bought in the Orient«, has camels at a price of 5 *solidi*, asses at one per *solidus*, and 20 sheep per *solidus*). In the second type[54], a pursuer and a pursued person or animal move at different paces, and the moment of catching up is asked for. At the simplest level both speeds are constant[55]; this is the case in the *Propositiones* (N° 26, a hound pursuing a hare), but elsewhere arithmetically increasing and decreasing speeds occur. Both types are also testified in early Chinese mathematics, the »hundred fowls« in a treatise by the late fifth century author Zhang Qiujian[56], and the pursuit in several versions in the *Nine Chapters* (III,12 is the

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[52] Ed. Folkerts 1978.

[53] See Tropfke/Vogel 1980: 613ff.

[54] Tropfke/Vogel 1980: 588ff.

[55] In this simple case, the reverted problem (»meeting«) comes mathematically close to the filling of a vessel from two sources.

[56] See Tropfke/Vogel 1980: 613f.

simple version corresponding to *Propositiones* N° 26, while III,14 is a more complex problem dealing with hound and hare).

A curiosity of some consequence in the *Propositiones* is N° 8. It borrows the dress of the »filling of a vessel« (actually, the vessel is not filled but emptied through three outlets of unequal capacity), showing thus familiarity with that tradition. But the mathematics of the usual filling problem being apparently too difficult, the mathematical substance is changed into something simpler. More significant, however, is N° 13. This is one of the many trigesimal doubling problems spoken of in the beginning of section III, and which should now be presented in more detail.

The oldest appearance of the problem is in a cuneiform tablet from Old Babylonian Mari[57] and runs as follows:

> To one grain, one grain has been added:
> Two grains on the first day;
> Four grains on the second day;
> .
> .
> .

going on until 30 days, but expressing the larger amounts not in numbers but in metrological units (when used as a weight unit, a »grain« is $^{1}/_{180}$ of a *šeqel*, itself some 8 g).

The following occurrence is a tabulation found in a Greco-Egyptian papyrus[58] (probably to be dated to the Principate but perhaps as late as the fourth century). It starts at 5 drachmas, contains again 30 steps (nothing is said about days) and makes use of the copper talent (=6000 drachmas) and a still larger unit of 13200 talents when reaching sufficiently large numbers.

Next in time follows *Propositiones* N° 13[59]:

[57] Ed. Soubeyran 1984: 30; my translation.

[58] P. Ifao 88, ed. Boyaval 1971.

[59] Ed. Folkerts 1978: 51. My translation.

A certain king ordered his minister to gather an army from 30 domains in such a way that from each he should levy as many men as he brought to there. But he came alone to the first domain, and to the second with another man; now three came [with him] to the third. Let the one who is able to say how many men were gathered from the 30 domains.

Then, finally, we have al-Uqlīdisī's observation, made in tenth century Damascus, that »many people« ask for 30 (or 64) doublings of unity, and the indisputable presence of the problem everywhere in the »Silk Road« area. As already stated above, the possibility of independence can be safely disregarded. At least one of the problems belonging to the Medieval »Silk Road« cluster can thus be demonstrated to have a Babylonian (or even older) origin; and to have been widespread within the Roman Empire (from Egypt to the northwestern corner). Others, well known from Medieval and Ancient Chinese sources, have left their traces in Diophantos *Arithmetica,* the *Anthologia graeca,* and elsewhere in the *Propositiones.*

We may conclude that a whole fund of sub-scientific mathematics, connected to the transcontinental trade routes and including a superstratum of »recreational«, non-utilitarian problems, was diffused throughout Greco-Roman society though at the »culturally subliminal« level. It might be worth asking whether it is reflected in other ways. One possibility was already hinted at: The relation between the concept of the harmonic mean and the problems of combined performances or the »filling of a vessel«. Another example might be Zeno's paradox of Achilles and the tortoise; the point of this might be even sharper than usually assumed if it does not refer to common sense understanding only but also to the ways of vulgar computation. More close at hand than both possibilities is, however, the possibility (equally touched on above) that the algebraic ἀριθμός-technique used Diophantos (and by that tradition which he hints at in the introduction, including Papyrus

Michigan 620) was borrowed from the same sub-scientific tradition, which also transmitted it as Leonardo's *regula recta*[60].

V. COMPOSITE FRACTIONS

Other diffusion networks span smaller regions. One is connected to a particular idiom for fractions, best known from Medieval Arabic sources and from Leonardo Fibonacci[61]. Its basis is the »composite fraction«, »$^1/_p$ of $^1/_q$« instead of »$^1/_{(p\cdot q)}$«, and its most highly developed form the »ascending continued fraction«[62], »$^p/_q$, and $^r/_s$ of $^1/_q$, and $^t/_u$

[60] It should be emphasized that this is a suggestion and no firmly proved fact. The *regula recta* is, after all, only brought in as a secondary method, not as the technique going »naturally« with the problem, and at the moment where Leonardo made himself a disciple of the Arabs al-Khwārizmī's *Algebra* (which uses the technique amply) had already circulated for almost 400 years.

On the other hand, the relation between the »thing« terminology (used, e.g., exclusively in the treatment of inheritance problems) and the »treasure and root« terminology used when second degree problems are solved suggest that these terminologies and techniques are of different origin. Of particular interest are problems of the type »I have divided ten into two parts, and multiplying one of these by the other, the result was twenty-one«. In the first step, such division problems (which have a definitely Diophantine ring, cf. *Arithmetica* I,xxvii) are first expressed in »thing« terminology (one number is the »thing«, the other is »ten minus a thing«) and next translated into »treasures and roots«.

Detailed documentation of these suggestions would lead too far; but as far as the present issues are concerned, Rosen's translation (1831) can be safely used, even though Rozenfeld's Russian (1983) or Gerard of Cremona's Latin (in Hughes 1986) are to be preferred (Gerard, however, omits the legacy part). Cf. also Høyrup 1990b.

[61] For more complete documentation and discussion I refer to Høyrup 1988.

[62] It should be emphasized that this concept has *nothing* to do with ordinary continued fractions except for the graphic similarity between the two when both

of $^1/_s$ of $^1/_q$« (or going on to even more members)—in numerical examples »$^1/_3$ of $^1/_5$« and »$^2/_5$, and $^4/_5$ of $^1/_5$«. Though rare, both varieties also do turn up in a number of Old Babylonian sources—either as a final recourse when other notations fail or in problems of riddle character; which fits an existence as a popular or sub-scientific usage known to the scribes but not accepted as fully legitimate by them. In Medieval Arabic, composite fractions are evident as normal language—»one third of one fifth« *was* simply the current name for $^1/_{15}$. Ascending continued fractions went together with the »finger reckoning tradition« preferred by merchants. Once more we encounter a popular usage and connections to a sub-scientific practice.

Even in Egypt, composite fractions turn up (though only rudimentary ascending continued fractions). Again, it happens when popular usage is portrayed (a herdsman speaking to an official defines his due as »$^2/_3$ of $^1/_3$« of the cattle entrusted to his care) or in the riddle »go down I [*viz.*, a jug of unknown capacity] times 3 into the *ḥeḳat*-measure, $^1/_3$ of me is added to me, $^1/_3$ of $^1/_3$ of me is added to me, $^1/_9$ of me is added to me; return I, filled am I. Then what says it?«[63] (i.e., $3+^1/_3+^1/_3\ ^1/_3+^1/_9$ times an unknown quantity equals 1 *ḥeḳat*).

Once again, we seem to be confronted with a popular usage, normally avoided by the scribes when they had developed that sophisticated unit fraction system which was eventually borrowed by the Greeks. Evidently, the composite fractions and the additive unit fraction system differ fundamentally; but some evidence exists that the latter system developed from the same set of simple unit fractions ($^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$ and $^1/_6$—and, notwithstanding *our* conceptions of system, $^2/_3$) which was extended in more popular usage through multiplicative composition.

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are written with fraction lines. Ordinary continued fractions are a way to write down the outcome of an *anthyphairesis*-procedure (use of the »Euclidean algorithm«); »ascending continued fractions« are a generalization of the principle of measurement by a system of decreasing units.

[63] Rhind Mathematical Papyrus, N° 37, in Chace's literal translation (1929, Plate 59). The herdsman is put on the stage in N° 67.

I have not come across the system in texts from the Classical epoch, but it turns up in *Anthologia graeca* XIV as well as in the *Propositiones*—strangely enough in two different versions.

Strictly speaking, it is not the ordinary system which is found in the *Anthologia* but a curious travesty: »Twice two-third« (N° 6); »One-eighth and the twelfth part of one-tenth« (N° 121); »The fifth part of seven-elevenths« (N° 128); »Twice two-fifths« (N° 129); »A fifth of the fifth part« (N° 137); »Four times three-fifths« (N° 139); »Twice two-sixths and twice one-seventh« (N° 140); »Six times two-sevenths« (N° 141); »A fifth part of three-eighths« (N° 142); and »Twice two-thirds« (N° 143). Everywhere else, fractions are expressed in the usual Greek (and Egyptian) manner.

The choice of one or the other usage has nothing to do with the mathematical substance of the problems (most of which are anyway of the same type, reducible in symbolic form to an equation $x \cdot (1\text{-}p)=A$, where p is a sum of fractional expressions). Nor is it, however, random: it is geared to the clothing of the problem. Composite fractional expressions turn up in all problems dealing with the Mediterranean extensions of the Silk Road (N^{os} 121 and 129), with the legal partition of heritages (N^{os} 128 and 143), and with the hours of the day (N^{os} 6, 139, 140, 141, and 142; N° 141 is connected to astrology). A final instance is found in N° 137, dealing with a catastrophic banquet apparently meant to be held in Hellenistic Syria. Problems which refer to Greek mythology or history make use of Greek/Egyptian fractions. The same applies to problems dealing with apples or walnuts stolen by girl friends, with the filling of jars or cisterns from several sources, with spinners', brickmakers' or gold- or silversmiths' production, with wills, and with the ages of life.

The most plausible explanation of this striking distribution is that a number of recreational problems belonging to (at least) two different professional contexts (providing the guises of the problems) have been brought together in the anthology, each conserving its own distinctive idiom for fractions: on one hand the traditional Greek idiom based on unit fractions (and occasional rudimentary general fractions); on the other the usage of the trading community and of juridical calculators, and of astrologers and makers of celestial dials, which is different. The association between astrology and »Chaldaeans« as well as the Syrian banquet and the use of composite fractional expressions in juridical calculations from Seleucid Uruk suggest that the origin this usage (and hence the source for the corresponding technologies of time-measurement etc.) should be looked for in the Semitic-speaking Near East. Here, as we have observed, the system of composite fractions had indeed been in use at least since the early second millennium B.C.

Of course, the composite expressions found in the *Anthologia graeca* can not be expected to have been those in practical use among traders etc. It is not conceivable that »two-thirds« should be expressed as »twice two-sixths« for any everyday purpose. But even the »Greek« group of epigrams contains similar deviations from computational »real life«. The »double« and »triple« seventh of N^os 116 and 119, of course, are fairly regular, as are the »two fifths« of N° 132; but the »double sixth« and the »two quarters« of N^os 117 and 119 are certainly not (N^os 116, 117 and 119 deal with division of apples, and N° 132 with the filling of a cistern). Most likely, the irregular expressions are to be explained from the recreational character of the problems: by being queer, they make the riddles more funny and more obscure.

The composite fractional expressions seem to have remained strangers in the Classical world, and to have been unable to spread from those specific groups of practitioners who brought them or adopted them along with other techniques. Admittedly, composite fractions are also found in the *Propositiones*,—but in a way which suggests an ultimate root in Egypt (the way from Egypt to Charlemagne's Aachen may of course have been highly tortuous). They are found in N^os 2, 3, 4 and 40, which all belong to the same type. We may quote N° 40 as an example[64]:

> From a mountain, a man saw sheep grazing, and said: If only I had as many and as many once more, and half of the half, and further the half

[64] Ed. Folkerts 1978: 68; my translation.

of that half, and then I would enter my house together with them as one of hundred. Let the one who is able to find out how many sheep he saw grazing there.

Thus, the unknown number taken twice, with its $^1/_2 \cdot {}^1/_2$ and its $^1/_2 \cdot {}^1/_2 \cdot {}^1/_2$, is 99. The fraction is the same sort of rudimentary ascending continued fraction as found in the Egyptian *ḥeḳat*-problem—and the mathematical structure of the problem is also strikingly similar[65]. The composite fractions found in N^{os} 2, 3 and 4 are $^1/_2 \cdot {}^1/_2 + {}^1/_2 \cdot {}^1/_2 \cdot {}^1/_2$, $^1/_3 + {}^1/_2 \cdot {}^1/_3$ and $^1/_2 \cdot {}^1/_2$, respectively, i.e., two of the same type and one reduced.

The quintuple occurrence of the same problem type (N^{os} 45 is similar but only contains the fraction $^1/_3$) shows that it must have been quite popular. It remained so in later Medieval problems collections[66], retaining even the *numerical value* of the characteristic series of fractions; but instead of speaking of »$^1/_2$ and $^1/_2$ of $^1/_2$«, the »Columbia Algorism« speaks simply of »$^1/_2$ and $^1/_4$«[67]. Once again, the peculiar technique of composite fractions proved able to survive for a while when attached to a specific and isolated tradition—which must have been the situation in the Classical world; but when the tradition in question left the Hamito-Semitic language area and was absorbed into a broader current, the composite fractions were replaced by more familiar expressions. The occurrence of composite fractions in the West

[65] On the abstract mathematical level, of course, both belong the class of so-called *ʿḥʿ*-problems, $a \cdot x=b$ (*ʿḥʿ* is Egyptian for »heap« or, more abstractly, »quantity«; the term refers to problems in the Rhind Mathematical Papyrus dealing with such indefinite »heaps«). This class, however, is too wide-spread and too unspecific to be evidence of anything. Conclusions only become possible if the precise structure of a is noticed. The regular Egyptian *ʿḥʿ*-problems have $a=1+{}^1/_p+{}^1/_q...$; the apple problems of the *Anthologia graeca* have $a=1-{}^1/_p-{}^1/_q...$; problems with $a=n+r$, where r stands for a »rudimentary« ascending continued fraction are rare, in fact known to me only from the Rhind Papyrus and from the *Propositiones*.

[66] Tropfke/Vogel 1980: 574f lists its occurrence in the tenth-century Iranian al-Ṭabarī, the twelfth-century Spanish-Hebrew ibn Ezra, and in a 14th-century Italian algorism.

[67] Ed. Vogel 1977: 109.

is thus a reliable index fossil, demonstrating the survival of an autonomous sub-scientific tradition (*different* traditions, indeed, if we look at the *Anthologia* and the *Propositiones*[68].

VI. »SURVEYORS' ALGEBRA« AND »CALCULATORS' ALGEBRA«

After the sequence of propositions apparently inspired by »recreational« first degree problems in several variables in Diophantos' *Arithmetica* I comes a sequence dealing with problems of the second degree: To find two numbers with given sum and product (xxvii); to find two numbers, when their sum and the sum of their squares are given (xviii); to find two numbers, when their sum and the difference between their squares are given (xxix); to find two numbers with given difference and product (xxx); and to find two numbers with a given ratio, when the sum of or difference between their squares has a given ratio to their sum or their difference (xxxi-xxxiv), or when the square of the smaller number has a given ratio to the smaller or greater number or to their sum or difference (xxxv-xxxviii).

[68] A third instance could be pointed out. As mentioned above, Leonardo Fibonacci introduced the ascending continued fractions in his *Liber abaci,* together with an ingenious notation borrowed from the Maghreb school of mathematics. From the *Liber abaci* they went into the Italian abacus school, in itself a sub-scientific institution; there they survived until the sixteenth century (Clavius still discusses them), ultimately to disappear when this sub-scientific tradition dissolved in the late Renaissance (see Vogel 1982).

In his introduction, Diophantos promises to teach the solution of mixed second-degree equation with one unknown[69] (a promise which he does not keep in the conserved parts of his text); at various places in Book VI, furthermore, he refers to the solvability conditions for non-normalized second-degree equations in one variable, and at others he states actual solutions of such equations without explanation[70]. Apart from that, however, non-trivial (i.e., mixed) numerical second-degree problems only turn up in utterly few Greco-Roman sources. One place is the quasi-Heronian compilation *Geometrica*, where the same problem turns up twice, in **21,9** and again in **24,46**[71]: To find the diameter of a circle when the sum of the diameter, the perimeter and the area is 212. The solution follows from a numerical algorithm given without comments, but corresponding to the way we would treat the problem $(11d)^2+2\cdot29\cdot(11d)=32648$ (which agrees with the original statement if $\pi={}^{22}/_{7}$).

Two other sources both deal with right triangles. One is the anonymous *Liber podismi*[72]. This opuscule is part of the *Corpus agrimensorum,* which was collected in the mid-fifth century C.E. from older material. One of the problems dealt with refers to a right triangle, whose hypotenuse and area are given. Algebraically, the problem can be expressed as $x+y=A$, $x^2+y^2=B$; but the solution seems to build on a simple piece of geometrical insight, which follows from this diagram: [see next page]
If H is the square on the hypotenuse, and A is the area, then $H+4A=(x+y)^2$, and $H-4A=(x-y)^2$.

The other text dealing with right triangles is the Greco-Egyptian Papyrus Genève 259, which contains three problems and should

[69] Ed., transl. Tannery 1893: I, 14f. Possibly, however, Diophantos had something more complex in mind, as suggested by Sesiano (1982: 78).

[70] Ed., transl. Tannery 1893: I, 392-449. References to solvability conditions are found, e.g., in VI,vi and VI,xxii, actual solutions, e.g., in VI,vi and VI,vii.

[71] Ed., transl. Heiberg 1912: 380f and 444-447.

[72] Ed. Bubnov 1899: 510-516, cf. p. 399 and Folkerts 1970: 95-98 on manuscripts and authorship. The problem mentioned is in Bubnov 1899: 511f.

probably be dated to the second century C.E.[73]. We may denote the hypotenuse c and the other sides a and b. The first problem (a=3, c=5) is trivial once the Pythagorean theorem (which also follows from the above diagram) is known, and the third ($a+b$=14, c=10) is too damaged to allow any certain reconstruction[74]. But the solution of the second ($a+c$=8, b=4) appears to make use of the rule that $b^2= c^2-a^2=(c+a)\cdot(c-a)$, which can be claimed to be algebraic in nature, but which (given the Pythagorean theorem) can be easily ascertained on a diagram similar to the above.

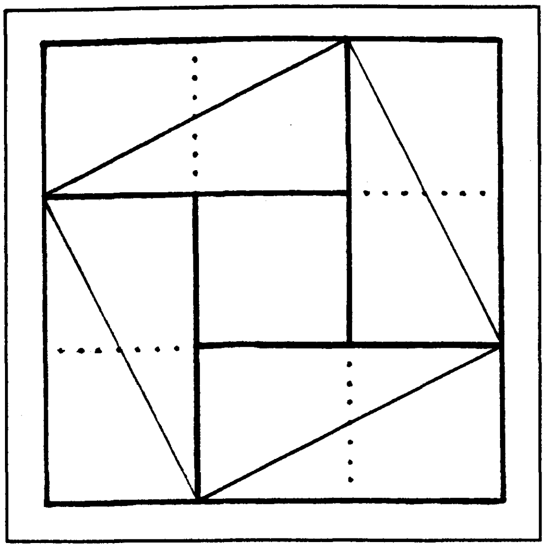

FIGURE 1

It is not conceivable that these isolated Latin and Greek geometrical computations should have popped up from nowhere—their way of obtaining the solutions from unexplained sequences of numbers demonstrates that well-known procedures were used. Together the two sources thus establish the existence of yet another concealed mathematical undercurrent, somehow connected, it appears, with practical geometrical computation. In this context, they seem to have belonged to the non-utilitarian superstratum—a practical geometer will hardly ever know the sum of the hypotenuse and another side of a right triangle before he knows them separately, nor need to construct one from such data. Isolated and laconic as the texts in question are they tell us little more—in particular not whether the methods were really founded on insights of a geometrical nature or on an ἀριϑμός-algebra *à la* Diophantos.

[73] Published, translated and discussed by Rudhardt (1978). Further discussion in Sesiano 1986.

[74] Rudhardt suggests the first step to be a squaring of the hypotenuse. But all that is sure is a ρ̄, which might just as well (and no less reasonably, cf. below, note 83) be the first »digit« of 196, the square on $a+b$. Sesiano's complete reconstruction is pure conjecture.

Once more, sources from earlier and later epochs will be of help, showing us the river before it goes underground and after it reappears. At the same occasion, they will inform us about some of Diophantos' sources and, so it appears, about other aspects of the history of Greek mathematics.

The central elements in the argument will be the Old Babylonian second degree »algebra« and an Arabic text written by an unidentified Abū Bakr and known from a Latin translation *Liber mensurationum* due to Gerard of Cremona.

Since Neugebauer's and Thureau-Dangin deciphered and interpreted the Babylonian mathematical texts in the 1930es, it has been a prevailing belief that the whole class of texts dealing with squares and their sides and with rectangular lengths, widths and areas was nothing but algebra in geometrical disguise, and it has been taken for implicitly granted that »algebra« would treat of *numbers*, and would, if it did not possess the modern (Cartesian) symbolism, do so by means of »rhetorical« techniques in the vein of Diophantos' ἀριθμός-algebra and the Arabic *šay*ʾ-/thing-representation.

A detailed comparative investigation of the »algebraic« texts shows this conclusion to be precipitate and even erroneous[75]. Their rectangles and squares are not metaphors for products and second powers of

[75] The argument for this is complex, involving a structural investigation of the total terminology and a close comparative reading of many texts. Part of the outcome of this investigation is that the Old Babylonian scribal mathematicians distinguished two different »additive operations« (i.e., operations which when read as operations with abstract numbers are both additions), two different subtractive operations, and no less than four different »multiplications«. Nothing of this makes sense in a numerical interpretation, where there *is* only one addition, one subtraction, and one multiplication. But if one »multiplicative« operation consists in constructing a rectangle, another one in repeating a geometrical figure concretely (e.g., by joining it to a mirror image), a third in calculating a concrete magnitude through an argument of proportionality, and a fourth in making repeated additions of a number, then the operations *are* really different, and it makes sense to label them differently.

The details of the investigation are presented in Høyrup 1990; a summary exposition will be found in Høyrup 1989.

numbers but real geometrical figures (abstract »fields«, in fact). The procedures which are described, furthermore, are not numerical algorithms but reports of geometrical cut-and-paste procedures. As an example we may look at the simplest text of all[76]:

[1] The surface and my confrontation I have accumulated: $^3/_4$.
[2] 1 the projection you pose.
[3] The moiety of 1 you break, $^1/_2$ and $^1/_2$ you make span,
[4] $^1/_4$ to $^3/_4$ you append: 1 makes 1 equilateral.
[5] $^1/_2$ which you have made span, from the body of 1 you tear out:
[6] $^1/_2$ is the confrontation.

This cries for explanation. The »confrontation« (*mitḫartum*) is a configuration characterized by the confrontation of equal sides, i.e., a geometrical square. But since the Babylonians understood the magnitude of a square as characterized by its side as distinctive parameter, the »confrontation« is also identified with the numerical value of the side. This is less strange than we may find at first. To us (and mostly to the Greeks), a square (which is after all a complex geometrical configuration with sides, angles, diagonals, area, circumscribable circle etc.) *has* a side of two feet and *is* four square feet; to the Babylonians, on the other hand, it *had* a surface of 4 square feet and *was* two feet. We shall return below to a specific geometrical Greek term (the δύναμις) which reflects the same understanding.

In [1] we are thus told that a square has a sum of the numbers measuring the area and the side equal to $^3/_4$—»accumulation« (*kamārum*) is the real addition among the two »additions«, and it allows the addition of numbers without regard for their significance. The rest of the text is best explained on a diagram:

[76] The first problem from the tablet BM 13 901 (ed. Neugebauer 1935: III, 1-5). The translation is my own, and is extremely literal, except for the numbers (the particularities of the Babylonian numerical notation are irrelevant in the present connection). The tablet contains a long sequence of problems dealing with one or more squares, and we shall have to return to it repeatedly.

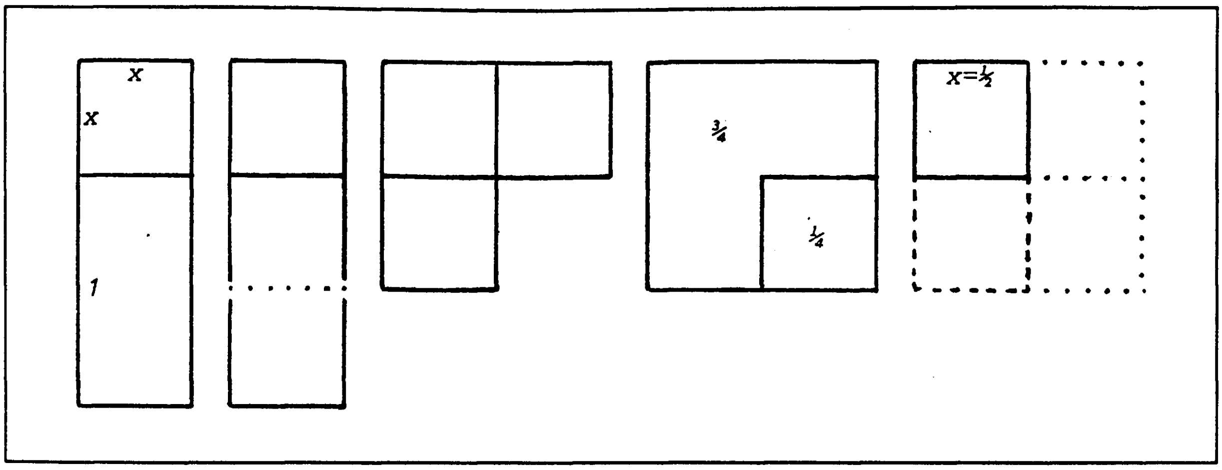

FIGURE 2

In [2], the square (whose side we may for brevity designate with a Cartesian x) is provided with a »projection« (*waṣītum*) of 1. As we see, this corresponds to appending a rectangle of length 1 and width x, i.e., of area $x \cdot 1 = x$. The area of the total figure will then be x^2+x, which is known to be $^3/_4$.

Next [3] this »projection« is »broken« (*ḫepûm*) into »moieties« (*bāmtum*). »Moieties« (literally rather »rib-sides«) are »natural« or »customary« halves, as the radius of a circle is the natural half of a diameter. »Breaking« is bisection into »moieties« (the two terms thus go together). The two moieties (with appurtenant sections of the rectangle) are »made span« (*šutākulum*), i.e., they are used to form a rectangle (actually a square), whose area is seen [4] to be $^1/_2 \cdot {}^1/_2 = {}^1/_4$. When this is »appended« (*waṣābum*) to the area ($^3/_4$) of the transformed figure, the outcome is a larger square with area $^3/_4 + {}^1/_4 = 1$. This area »makes 1 equilateral« (íb-si$_8$), i.e., if it is formed as a square it causes 1 to be the side of this square. Finally [5], that part of the broken rectangle which was moved and »made span« is »torn out« (*nasāḫum*) from the »body« (*libbum*, literally »heart« or »bowels«) of the side of this larger square (meaning from the concrete, bodily entity, not from a measuring number), [6] leaving the original unknown »confrontation«, which thus equals $^1/_2$.

The correctness of the procedure is intuitively obvious, even though it is »naive«, as opposed to the »critical« approach which characterizes

Greek mathematics (Euclid, in the very similar proof of *Elements* II,6, does not loosely move a rectangle, but constructs another one, proving it to have the same area, etc.[77]). The method is *analytic*, i.e., that which is unknown is taken to be known and moved around until something really known eventually drops out—as it happens when we represent an unknown number by x and write down what we know about its relations. It is, moreover, homomorphic with the analytical procedure which *we* would apply: $x^2+x={}^3/_4 \Rightarrow x^2+2\cdot({}^1/_2\cdot x)+({}^1/_2)^2={}^3/_4+{}^1/_4=1 \Rightarrow (x+{}^1/_2)^2=1 \Rightarrow x+{}^1/_2=\sqrt{1}=1 \Rightarrow x=1-{}^1/_2={}^1/_2$.

In the scribal school, a highly systematic teaching was built up around these techniques. The aim was not to create *theory*, but it was still non-utilitarian; just as the mastery of written and spoken Sumerian, proficiency in second-degree »algebra«, so it seems, was one of the ways in which a scribe could display professional virtuosity. But certain indications exist[78] that the techniques did not originate inside the Babylonian school but were taken over from a non-scholarly sub-scientific tradition (carried, we may surmise, by surveyors and other practical geometers), where it served in more genuinely recreational problems[79].

As it was argued concerning accountants and merchants, the mathematical problems used in everyday practice by a surveyor will soon become trivial. Everybody within the craft will be familiar with the determination of a rectangular area from length and width, and will be able to add up partial areas. In order to demonstrate pro-

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[77] Ed., transl. Heiberg 1883: I, 132ff; cf. Heath (ed., transl.) 1926: I, 385f.

[78] See Høyrup 1990a: 79f.

[79] In this context, the quadratic complement (the essential trick, in fact, in the solution of mixed second-degree problems) will have played a role similar to that of the intermediate stop in the camel problem from the *Propositiones*. The trick seems to have carried the name »the Akkadian method«, suggesting that it originated among Akkadian practitioners, not among the Sumerian scribes of the third millennium B.C. At the emergence of Akkadian scribe-hood in the Old Babylonian era, it will have followed the language into the school curriculum, making thus second-degree »field« problems the distinctive characteristic of Old Babylonian (as opposed to Sumerian) scribal mathematics.

fessional dexterity beyond the ordinary level you should be able to answer more specious questions, which, in agreement with the familiar psychology of recreational mathematics, should at the same time contain something striking. A first question of this type would be precisely to ask for the side when you known the sum of the area and the side of a square. But while the next question occurring naturally to a school teacher is then the sum of the area and another multiple of the side, and next the difference, and a multiple of the area together with a multiple of a side, the obvious next *funny* question concerns the sum of the area and *all four sides*.

The tablet containing as its first problem the »area plus side« exhibits both features. It proceeds systematically, exactly as a school text could be expected to do. Towards the end comes, however, precisely the question of area and four sides; the formulation, however, is unorthodox, and the procedure makes use of a special trick which only works in this case. The function is clearly that of entertainment during the »last lesson before Christmas«, and the language suggests the square field in question to be imagined as less abstract than the others. Everything fits a problem borrowed from a living, non-scholastic tradition.

This tradition proved astonishingly hard to kill. The *Liber mensurationum*[80] mentioned above, a work whose Arabic original was probably written around or shortly after A.D. 800, still appears to remember it. The evidence for this is multifarious.

Firstly, there is what might be called the »rhetorical structure« of the problem texts. The Old Babylonian text quoted above exhibited some characteristic features, which when more (and longer) texts are included amount to a system:

The text begins with, or presupposes, the phrase »If somebody/the teacher has said to you«. Then follows the statement of the problem, which is held in the first person singular of the preterit tense, *with one*

[80] Ed. Busard 1968. The text is analyzed in Høyrup 1986, and again in Høyrup 1990b, to which publications I refer for the sake of documentation.

exception: if the length of a rectangle exceeds the width by a certain amount, this is stated in the third person singular of the present tense as a neutral fact, not as something which the speaker has caused it to do. Then comes a phrase (implicit above) »you, by your method«, and then a description of the procedure, formulated in the present tense, second person singular, or in the imperative.

Occasionally, a certain step in the procedure is justified by a quotation from the statement. Such quotations are literal (grammatical forms included), and indicated by the phrase »because he has said«. At other points, an intermediate result is to be remembered, not taken down. This number is then followed by the phrase »which your head shall retain«.

At first sight, the corresponding structure of the *Liber mensurationum* is more complex. For one thing, the second part of this treatise deals with *real* mensuration of Heronian character, and thus does not concern us here. But apart from that, the first part combines two traditions. After the statement and the description of the procedure to be used for the solution of each problem comes in most cases the observation that »there is another method according to *aliabra*«, which is then described. The solution »according to *aliabra*« turns out to make use of *al-jabr* (»treasure-root-algebra«) as presented by al-Khwārizmī (but not exactly in his formulation). If we disregard this alternative method, however, the rhetorics of Abū Bakr's text follows the Old Babylonian scheme in every particular, with the sole exceptions that »your method« has become »the method«, and that »your head« has changed into »memory« in the Latin version.

As the basic fund of Old Babylonian second-degree algebra, Abū Bakr's problems deal with squares and rectangles (rhombs are treated too, but in fact trivially reduced to the rectangles in which they are inscribed). Apart from the determination of a diagonal from the side(s) or vice versa, and a few similar issues, the questions have no relevance for practical mensuration—they belong to the same family as the Old Babylonian »square area plus side« problem quoted above. That a number of simple problem types are shared (e.g., in symbolic interpretation, $x^2 \pm x=A$; and $x+y=A$, $x \cdot y=B$) is then in itself not astonishing: after all, the number of simple problem types concerning squares and rectangles is quite restricted. But more striking coincidences are found, involving certain very idiosyncratic Old Babylonian problems together with their no less idiosyncratic methods (amounting in modern language to a »change of variable«). Even the distinctions between two different additive and two different subtractive operations is found, together with traces of the distinction between different multiplications.

The text used by Gerard for his very literal translation must have been corrupt in several respects, as demonstrated by the presence of repeated and permuted problems. The most serious flaw is the absence of a number of diagrams to which the text refers. None the less, the text as it stands may make us confident that the basic method, the one that is used in the first solution of each problem, was precisely that »naive« cut-and-paste geometry which the Babylonians handled with such skill.

All in all, there can be no reasonable doubt that Abū Bakr had access to a tradition going back to the Old Babylonian era and used it as his fundament for the first part of his treatise (while demonstrating that the same solutions could be found by means of *al-jabr*). On the other hand, important Old Babylonian problem types are absent from his collection, most notably all mixed problems necessitating the use of operations of proportionality—in particular problems of the types $a \cdot x^2+b \cdot x=c$ (BM 13 901 N° 3) and $x^2+y^2=A$, $y=p \cdot x+q$ (BM 13 901 N^{os} 9, 10, 11, 13, and 14); in other words, problems which cannot be solved by cut-and-paste geometry alone but involve changes of scale or complex coefficient accounting. At the same time, problems involving the sides of squares or rectangles will mostly involve *one* side, *one* length and *one* width, or *all four* sides; this is quite different from the style of the Babylonian school tablets, but agrees (as observed above in connection with a rare Babylonian specimen) with that predilection of genuine »recreational« traditions for striking formulations which was referred to above. We may hence conclude, *either* that the tradition which Abū Bakr used as his fundament did not derive directly from the Old Babylonian scribal tradition but from an even earlier sub-scientific source tradition from which even the Old Babylonian school had borrowed; *or* that the scribal mathematical tradition was fitted to the non-scholastic needs of that sub-scientific surveyors' environment which appears to have carried the tradition onwards after the collapse of the Old Babylonian school system.

Scribal-scholarly second-degree »algebra« turns up again in a few Seleucid texts, in a way which makes manifest a passage through a non-scholarly environment[81]. These Seleucid texts appear to represent in themselves a dead alley, but they derive from a stage of the tradition between what we know from the Old Babylonian era and Abū Bakr. One of them, in particular[82], exhibits a strong interest in the diagonal of the rectangle and in the right triangle, embracing in fact all the three problem types of the Genève papyrus though formulated as questions concerned with rectangles with diagonal and not with right triangles. Some very particular problems from this Seleucid tablet turn up once more in the *Liber mensurationum*—and the problem corresponding to No 2 of the Genève papyrus, which is solved there in a way which differs from that used in the cuneiform text (N^{os} 3, 4 and 11), is solved by Abū Bakr (N° 30) precisely as in the papyrus[83].

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[81] A characteristic of the Babylonian scholarly environment is the use of Sumerian terms for spoken Babylonian. But some of the of Sumerograms in the Seleucid texts turn out to be results of a recent retranslation: e.g., a term which in Old Babylonian texts had meant »repetition« (one of the four »multiplications« has suddenly come to mean »addition«, which is, in fact, a possible extension of its general semantics but not of its established meaning as a mathematical *terminus technicus*.

[82] BM 34 4568, ed., transl. Neugebauer 1935: III, 14-22.

[83] Abū Bakr's No 28, whose statement coincides with the damaged N° 3 from the Genève papyrus, begins (like N° 10 of the Seleucid tablet) by squaring the sum of the sides, and not by squaring the diagonal, as Rudhardt and Sesiano conjecture for the papyrus problem. But as observed in note 74, the conserved papyrus text fits one beginning just as well as the other.

It can thus be taken for granted that both the Genève papyrus and the *Liber podismi* reflect the presence in the Greco-Roman world of that surveyor's tradition which connects Abū Bakr with Old Babylonia. Similar roots can be claimed for the quasi-Heronian circle problem. This very specious problem is, in fact, found in the Old Babylonian catalogue text BM 80 209[84]. It is important to observe, however, that this problem is not normalized, and thus calls for an operation of proportionality. It can only have survived in a less reduced descendant of the Old Babylonian tradition than the one Abū Bakr had access to.

Because of their closeness to the *Liber mensurationum* it can also be safely assumed that both the Genève papyrus and the *Liber podismi* base their method on »naive« geometrical understanding. Since »Hero« relates differently to the tradition, we are on less firm ground in his case.

Hero's way to deal with the normalization reminds of the Old Babylonian technique, which consists in multiplying with the coefficient to the second-degree term instead of eliminating it[85]. The same technique is presupposed by Diophantos when he states solvability conditions for second-degree equations in *Arithmetica* VI. In itself this proves little; yet Diophantos' way to state the solution to these equations without further ado suggests that he refers to well-known procedures, and the indubitable Babylonian inspiration behind »Hero« together with the terminology for powers shared by Hero and Diophantos[86] indicates that these procedures are of Babylonian origin. What then about the second-degree problems with two unknowns in *Arithmetica* I?

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[84] Ed. Friberg 1981: 61.

[85] Thus transforming (in the symbolic interpretation) $ax^2+bx=c$ into $X^2+bX=ca$, with $X=ax$.

[86] See the use of the term δυναμο-δύναμις in Hero, *Metrica* I,xvii (ed., transl. Schöne 1903: 48f) and in Diophantos' introduction, ed., transl. Tannery 1893: I, 4-7. (Liddell and Scott's *Greek-English Lexicon* mentions no other authors using the term). That he had his terminology for the powers of the unknown from established custom is actually what Diophantos himself tells (cf. Høyrup 1990c, note 9).

In Old Babylonian »algebra«, rectangles with known area and known sum of or difference between the sides abound. Translated into numbers and their product, this corresponds to *Arithmetica* I,xxvii and xxx. In both cases, Diophantos proceeds via the semi-sum and the semi-difference between the two unknown numbers (of which, in both cases, the one is known and the other taken to be the ἀριθμός). This agrees with what the Babylonians had done. In *Arithmetica* I,xxviii, Diophantos asks for two numbers, of which the sum and the sum of their squares are known ($x+y=20$, $x^2+y^2=208$ is the example given). The same problem occurs as N° 8 on that Old Babylonian tablet (BM 13 901) whose N° 1 was quoted above, although solved in a slightly different way.

In all three cases, a diorism (solvability condition) is stated, followed by the remark that this is πλασματικός which might mean that it can be verified on a diagram[87]. All three diorisms, indeed, follow easily from the πλάσμα (standardized diagram) which is shown above in Figure 1, and which is also familiar from Old Babylonian texts.

Arithmetica I,xxxix, where Diophantos asks for two numbers, of which the sum and the difference between their squares are known, has no known Old Babylonian parallel. Nor does it state anything about being πλασματικός; but since no diorism is needed there is no pretext to state it. From internal evidence alone it is, all in all, difficult to claim that the cluster I.xxvii-xxx must by necessity be inspired from a tradition going back to the Babylonians. If we look at the totality of Book I, however, where the initial first-degree problems seem inspired by current recreational and similar mathematics, and where the trivially casuistic sequence I,xxxi-xxxviii could have been represented by one or two specimens without theoretical loss, it is a reasonable assumption

[87] This is Ver Eecke's interpretation (1926: 38 n. 3). Because the distribution of the term in the Arabic books of Diophantos' *Arithmetica* agrees badly with an interpretation through Euclidean geometry, both editors of that text have rejected Ver Eecke's proposal (see Rashed 1984: III, 133-138 and Sesiano 1982: 192f); but if »naive« geometry in the style of Figure 1 is meant, their objections are not compelling—see Høyrup 1990, chapter X.3.

that the whole book is inspired by existing sub-scientific traditions. Some problems may have been taken over directly without any other change than the removal of the concrete interpretation of numbers (this is obviously the case in xv, the »give and take« problem, and in xxiv, the »purchase of a horse«). Others may perhaps have been developed as analogues and generalisations by Diophantos himself and included for completeness' sake.

The triangular problems in *Liber podismi* and the Genève papyrus appeared to belong to a practical geometers' tradition, while the »purchase of a horse« (etc.) would rather go with traders, calculators and accountants. Diophantos, however, would use the same ἀριϑμός in problems of all degrees; his δύναμις, the second power of the ἀριϑμός, is told by Plato to have been used by calculators in this function already around 400 B.C.[88] Whatever their origins, the various traditions drawn upon for *Arithmetica* I will thus have been merged by practical calculators already in the early Classical period into *one* field. This will have been the source for Hero's second-degree equation and for both categories of second-degree problems in Diophantos; the simple surveyor-»algebra« will apparently have followed a separate way.

It has usually been assumed that Diophantos took his term δύναμις from geometry. Here, too, the term was used from early times, and it has been much discussed whether it meant »square« or »square root«/»side of square«. The puzzle is solved if one observes that all occurrences of the »geometers' δύναμις« fit the use of the Babylonian *mitḫartum*—a square identified *by* and hence *with* its side. A thorough discussion of this and of the relation between »calculators'« and »geometers' δύναμις« would be extensive[89]; the main outcome is that the calculators' concept seems to be primary, and to have served in a naive-geometrical »algebra« of Babylonian type and descent. It will then have been borrowed by geometers in the late fifth century B.C. and used

[88] *Republica* 587d, ed., transl. Shorey 1978: II, 396f. The implications of this passage are discussed in Høyrup 1990c, text around note 7.

[89] I deal with these issues in Høyrup 1990c.

naive-geometrical »algebra« of Babylonian type and descent. It will then have been borrowed by geometers in the late fifth century B.C. and used when they launched the enterprise which eventually gave rise to *Elements* II (etc.). This so-called »geometrical algebra«, as it has been called, will not have been a »dressing in geometrical garment« of a Babylonian numerical algebra, as it has repeatedly been maintained since the discovery of Babylonian second-degree »algebra« (and has in recent decades been vehemently denied). It will rather have been a critical investigation of the foundations of the naive-geometrical procedures of the calculators, which was then worked up as a discipline of its own with its own systematics and its own inherent problems (among which the theory of irrationals, cf. above)[90]. Parts of Euclid's *Data* may represent a stage in this process which comes closer to the starting point: Are the sides of a rectangle really given when the area and the difference between/sum of its sides are given (Prop. 84-85[91])? When the area is given and the sides have a given ratio (Prop. 78[92])? When the area is given and the squares upon sides have a given ratio with excess (Prop. 86[93])? Or when it arises through the application of a given area to a given line directly or with excess or deficiency (Prop. 57-59[94])? A number of the ways in which magnitudes

[90] Already before the geometrical reinterpretation of Babylonian »algebra«, Wilbur Knorr (1975) proposed a connection between the »metric« geometry of *Element* II (etc.) and the techniques of calculators, more precisely the patterns of calculi (ψῆφοι) in figurate numbers etc. These patterns may, indeed, have much to do with what I have here called the »calculators' algebra«, see Høyrup 1990c.

Several other interpretations of the origin of the techniques and propositions of *Elements* II have been proposed in recent years (Fowler 1987; Herz-Fischler 1987). I shall refrain from discussing whether these conjectures contradict the one suggested here or might serve as compatible complements.

[91] Ed., transl. Menge 1896: 164-167.

[92] Ed., transl. Menge 1896: 150-153. In arithmetical translation, we observe, this proposition comes close to Diophantos' trivial *Arithmetica* I,xxxi-xxxviii.

[93] Ed., transl. Menge 1896: 168-173.

[94] Ed., transl. Menge 1896: 102-109.

can be given according to the definitions[95] remind of Diophantos' *Arithmetica* I and of the Babylonian tablet BM 13 901 mentioned above: To be given in ratio, with excess or deficiency, or in ratio with excess or deficiency.

Part of Euclid's *Book on the Division of Figures*, on its part, seems to reflect a similar generalizing reflection of the cut-and-paste geometry of the surveyors' tradition. Prop. 33[96], in particular, which requires the partition of a trapezium by means of a parallel transversal in a given ratio, corresponds precisely to a clay tablet from the 23d century B.C.[97], with the only difference that this early partition is in ratio 1:1. This tablet constitutes the earliest positive trace of that Akkadian surveyors' tradition which seems to be behind the scribal »algebra« of the Old Babylonian era. As late as the 10th century, on the other hand, the Arabic mathematician Abūl-Wafāʾ, tells about partitions and about the cut-and-paste predilections of practical geometers[98].

It thus appears that *all* Classical second-degree purported »algebra«, including the hotly disputed »geometrical algebra« (and even other branches of scientific mathematics), grew out of or were inspired by the same sub-scientific soil[99]. No wonder that Hero, whose familiarity with calculators' second-degree algebra is demonstrated in various places, was able to give an »analytical«, i.e., quasi-algebraic, interpretation of *Elements* II[100].

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[95] Ed., transl. Menge 1896: 2-5.

[96] Counted as in Archibald's reconstruction (1915: 72f).

[97] Friberg, forthcoming, section 5.4.k.

[98] *Book about that which is Necessary for Artisans in Geometrical Construction*, ed., transl. Krasnova 1966. See especially p. 115.

[99] The one exception to this rule is the classification of irrationals and the study of the relations between classes in *Elements* X—actually the only piece of Ancient mathematics which relates in spirit to certain aspects of modern, »post-Noether« algebra. But for some obscure reason precisely this subject is normally left out from the search for Greek »algebra«.

[100] Se al-Nayrīzī's report of Hero's commentary to *Elements* II,1-10 in Besthorn & Heiberg (eds, transls) 1893: II, 4-61. Mueller's discussion of the relation between Hero's single-line analysis and Euclid's two-dimensional proofs (1981: 46-50) is

VII. OTHER NETWORKS

The »Silk Road« family of arithmetical problems, the composite fractions, and the surveyors' and calculators' »algebras« were certainly not the only sub-scientific networks connecting the Hellenistic and Roman world with earlier, surrounding and later cultures. Much of Greco-Roman metrology was borrowed, as we know; the channel will have been contact with the original practitioners of the metrologies in question, i.e., mainly traders and surveyors. The adoption of the Egyptian unit fraction system and the amalgamation of this system with the Greek alphabetic numerals is a well-known phenomenon, which certainly took place first at the sub-scientific level. Good reasons could be given that another family of arithmetical problems found its way into Hellenistic culture that way[101]. Certain Archimedean results, most notably those connected to his determination of the circular circumference and area, were adopted by practitioners and became

perspicacious; but if a »naive-geometric« interpretation is applied to Hero's »algebra«, it is no longer significantly different from Mueller's alternative interpretation, »geometric assertions about the equality of certain areas useful for the transformation of one area or areas into another«.

[101] Many of the problems from *Anthologia graeca* XIV making use of »Greek«, i.e., Egyptian fractions, deal with precisely such matters which Plato (*Laws* VII, 819B-C, ed., transl. Bury 1967: II, 104f) tells to be part of Egyptian elementary teaching (dividing up heaps of apples, etc.). In Anania of Širak's collection of arithmetical problems, N° 22 (ed., transl. Kokian 1919: 116) deals with the distribution of wine to Pharaoh's officials at his birthday according to a scheme which is already familiar from Rhind Mathematical Papyrus. Since this collection owes much to Anania's stay in Byzantium, this familiarity with Egyptian mathematics has probably passed via Greece.

part of the sub-scientific traditions[102]. Even certain definitely sub-standard (specifically Greco-Roman) practices appear to have been taken over in the bargain during the wholesale Western European appropriation of diluted Classical culture. One instance of this is the use of triangular numbers as measures of the area of the equilateral triangle, which was diffused together with the agrimensor writings and troubled the mathematically interested Adelbold around A.D. 998[103]. Another instance is the measurement of areas of figures by means of their circumference. In the Ancient sources it was not quite clear, as observed in note 13, what was meant by this, and how seriously it was meant. But in the *Propositiones* there is no doubt. In order to find out how many square *perticas* are contained within a circular field of circumference 400 *perticas*, the circumference is distributed as 4 times 100 perticas and the area then found as $100 \cdot 100$ (N° 25)[104],—we may presume that the field is transformed implicitly into a square to fit the square *perticas*. This is confirmed in N° 29, where rectangular houses of 30 feet times 20 feet are to be fitted into a circular city with circumference 8000 feet. This time,the circumference is explicitly divided as 4800 feet and 3200 feet along the length and width of the houses, respectively, and these then bisected and multiplied[105].

This questionable method was not reserved for recreational puzzles. In 1050, Franco of Liège tells (dissociating himself from the technique)

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[102] Thus in the Hebrew *Mišnat ha-Middot*, which not only gives the value of the circular circumference as $3^1/_7$ of the diameter but also cites Rabbi Neḥemia (c. A.D. 150, and according to Gandz the plausible author of the treatise) for the statement that this is what »the people of the world« (or, in another reading, »the landmeasurers«) say (ed., transl. Gandz 1932: 49).

[103] In a letter to Gerbert, whose explanation was edited by Bubnov 1899: 43. A more complete text is found in translation in Lattin 1961: 299-301.

[104] Ed. Folkerts 1978: 59. 1 *pertica* equals 10 feet. Strictly speaking, what is asked for is the contents in square *aripenni*. Since 1 *aripennus* equals 120 feet, the resulting 10 000 are divided twice by 12.

[105] Ed. Folkerts 1978: 61. Certain manuscripts present a different solution to both problems, which happens to be numerically better but looks as a combination of disparate elements from Greek and Babylonian mensuration—cf. Folkerts 1978: 28.

that »there are also some who split the circular circumference in 4 parts, from which they span a square, claiming it to be equal to the circle[106].

Summing up we may conclude that the indifference of the Classical sources toward basic mathematical technologies does not mean that these did not exist. Highly organized as it was on the administrative and commercial level, the Greco-Roman world could not do without them; and knowledge of the corresponding technologies used in geographically and temporally adjacent cultures allow us to extricate more information from the Classical sources than these would yield without supporting evidence. At the culturally subliminal level, the Classical world was traversed by a multitude of sub-scientific networks, more or less merged with each other.

We may also conclude that some of these technologies and networks were important for what went on at the culturally conscious level. Just as in the case of literature, the hidden undercurrents of non-literate and often oral culture provided an important part of the water and the nutrients which made literate scientific culture flourish.

[106] *De quadratura circuli*, ed. Folkerts & Smeur 1976: 65. My translation.

BIBLIOGRAPHY

Archibald, Raymond Clare, 1915. Euclid's *Book on the Division of Figures* with a Restoration Based on Woepcke's Text and on the *Practica Geometriae* of Leonardo Pisano. Cambridge: Cambridge University Press.

Aristotle. *Works*. Translated into English under the Editorship of W.D. Ross. 12 vols. Oxford: The Clarendon Press, 1908-1952.

Aujac, Germaine (ed., transl.), 1975. Géminos, *Introduction aux phénomènes*. Paris: »Les Belles Lettres«.

Bag, Amulya Kumar, 1979. *Mathematics in Ancient and Medieval India*. Varanasi: Chaukhambha Orientalia.

Baillet, J. (ed., transl.), 1892. *Le Papyrus mathématique d'Akhmîm*. (Mission Archéologique Française au Caire, Mémoires 9, 1). Paris: Leroux.

Besthorn, R. O., & J. L. Heiberg (eds, transls), 1893. *Codex Leidensis 399, 1. Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii*. 3 vols. København: Gyldendalske Boghandel, 1893-1932.

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Boyaval, B., 1971. "Le P. Ifao 88: Problèmes de conversion monétaire". *Zeitschrift für Papyrologie und Epigraphik* 7, 165-168, Tafel VI.

Bruins, E. M., & M. Rutten, 1961. *Textes mathématiques de Suse*. (Mémoires de la Mission Archéologique en Iran, XXXIV). Paris: Paul Geuthner.

Bubnov, Nicolaus (ed.), 1899. Gerberti postea Silvestri II papae *Opera mathematica* (972—1003) Berlin: Friedländer, 1899. Reprint Hildesheim: Georg Olm.

Bury, R. G. (ed., transl.), 1967. Plato, *Laws*. 2 vols. (Loeb Classical Library). London: Heinemann / Cambridge, Mass.: Harvard University Press, 1967, 1961.

Busard, H. L. L., 1968. "L'algèbre au moyen âge: Le »Liber mensurationum« d'Abû Bekr". *Journal des Savants*, Avril-Juin 1968, 65-125.

Butler, H. E. (ed., transl.), 1969. Quintilian, *Institutio Oratoria*. 4 vols. (Loeb Classsical Library 124-127). Cambridge, Mass.: Harvard University Press / London: Heinemann, 1969, 1977, 1976, 1961.

Chace, Arnold Buffum, Ludlow Bull & Henry Parker Manning, 1929. *The Rhind Mathematical Papyrus*. II. Photographs, Transcription, Transliteration, Literal Translation. Oberlin, Ohio: Mathematical Association of America.

Colebrooke, H. T. (ed., transl.), 1817. *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*, Translated. London: John Murray. Reprint Wiesbaden: Martin Sändig, 1973.

Corpus iuris civilis. Editio nova prioribus correctior. 2 vols. Amsterdam: P. & I. Blaev, 1700.

DSB: Dictionary of Scientific Biography. Charles Coulston Gillispie, Editor in Chief. 16 vols. New York: Scribner, 1970-1980.

Folkerts, Menso, 1970. *"Boethius" Geometrie II. Ein mathematisches Lehrbuch des Mittelalters*. Wiesbaden: Franz Steiner.

Folkerts, Menso, & A. J. E. M. Smeur (eds), 1976. "A Treatise on Squaring the Circle by Franco of Liège, about 1050". *Archives Internationales d'Histoire des Sciences* **26**, 59-105, 225-253.

Folkerts, Menso, 1978. "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*. Überlieferung, Inhalt, Kritische Edition". *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band, 6. Abhandlung.

Fowler, David H., 1987. *The Mathematics of Plato's Academy. A New Reconstruction*. Oxford: Oxford University Press.

Fowler, Harold North (ed., transl.), 1977. Plato, *Theaetetus, Sophist*. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Friberg, Jöran, 1981. "Methods and Traditions of Babylonian Mathematics, II: An Old Babylonian Catalogue Text with Equations for Squares and Circles". *Journal of Cuneiform Studies* **33**, 57-64.

Friberg, Jöran, Forthcoming. "Mathematik". To appear in *Reallexikon der Assyriologie und Vorderasiatischen Archäologie*.

Gandz, Solomon (ed., transl.), 1932. The *Mishnat ha Middot*, the First Hebrew Geometry of about 150 C.E., and the *Geometry* of Muhammad ibn Musa al-Khowarizmi, the First Arabic Geometry <c. 820>, Representing the Arabic Version of the *Mishnat ha Middot*. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen, 2. Band). Berlin: Julius Springer.

Guéraud, O., & P. Jouguet (eds), 1938. *Un livre d'écolier du* III^{e} *siècle avant J.-C.* (Publications de la Société Royale Égyptienne de Papyrologie. Textes et documents, 2). Cairo: Imprimerie de l'Institut Français d'Archéologie Orientale.

Hardie, R. P., & R. K. Gaye (eds, transls), 1930. Aristotle, *Physica*, *in* Aristotle, *Works* (ed. W. D. Ross), vol. II.

Heath, Thomas L., 1921. *A History of Greek Mathematics*. 2 vols. Oxford: The Clarendon Press.

Heath, Thomas L. (ed., transl.), 1926. *The Thirteen Books of Euclid's Elements*, Translated with Introduction and Commentary. 2nd revised edition. 3 vols. Cambridge: Cambridge University Press / New York: Macmillan.

Heiberg, J. L. (ed., transl.), 1883. Euclidis *Elementa*. 5 vols. (Euclidis Opera omnia, vol. I-V). Leipzig: Teubner, 1883-1888.

Heiberg, J. L. (ed., transl.), 1912. Heronis *Definitiones* cum variis collectionibus. Heronis quae feruntur *Geometrica*. (Heronis Alexandrini Opera quae supersunt omnia, IV). Leipzig: Teubner.

Herz-Fischler, Roger, 1987. *A Mathematical History of Division in Extreme and Mean Ratio*. Waterloo, Ontario: Wilfrid Laurier University Press.

Hughes, Barnabas, O.F.M., 1986. "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr:* A Critical Edition". *Mediaeval Studies* **48**, 211-263.

Høyrup, Jens, 1986. "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra". *Erdem* **2** (Ankara), 445-484.

Høyrup, Jens, 1988. "On Parts of Parts and Ascending Continued Fractions. An Investigation of the Origins and Spread of a Peculiar System". *Filosofi og Videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1988 Nr. 2. To be published in *Centaurus*.

Høyrup, Jens, 1989. "Zur Frühgeschichte algebraischer Denkweisen". *Mathematische Semesterberichte* **36**, 1-46.

Høyrup, Jens, 1990. "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". To appear in *Altorientalische Forschungen* **17** (1990).

Høyrup, Jens, 1990a. "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* **28** (1990), 63-86.

Høyrup, Jens, 1990b. "»Algèbre d'*al-ǧabr*« et »algèbre d'arpentage« au neuvième siècle islamique et la question de l'influence babylonienne«. To appear in a volume edited by Jean Michel Delire and Francine Mawet (Bruxelles, 1990).

Høyrup, Jens, 1990c. "*Dýnamis*, the Babylonians, and Theaetetus 147c7—148d7". To appear in *Historia Mathematica* **17**:3.

King, J. E. (ed., transl.), 1971. Cicero, *Tusculan Disputations*. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Kinsey, T. E., 1979. "Melior the Calculator". *Hermes. Zeitschrift für klassische Philologie* **107**, 501.

Knorr, Wilbur R., 1975. *The Evolution of the Euclidean Elements. A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*. (Synthese Historical Library, vol. 15). Dordrecht & Boston: D. Reidel.

Knorr, Wilbur R., 1986. *The Ancient Tradition of Geometric Problems*. Boston, Basel, Stuttgart: Birkhäuser, 1986.

Kokian, P. Sahak (ed., transl.), 1919. "Des Anania von Schirak arithmetische Aufgaben". *Zeitschrift für die deutsch-österreichischen Gymnasien* **69** (1919-20), 112-117.

Krasnova, S. A. (ed., transl.), 1966. "Abu-l-Vafa al-Buzdžani, *Kniga o tom, čto neobxodimo remeslenniku iz geometričeskix postroenij*", pp. 42-140 *in* A. T. Grigor'jan & A. P. Juškevič (eds), *Fiziko-matematičeskie nauki v stranax vostoka*. Sbornik statej i publikacij. Vypusk I (IV). Moskva: Izdatel'stvo »Nauka«..

Lattin, Harriet Pratt (ed., transl.), 1961. *The Letters of Gerbert with his Papal Privileges as Sylvester II. (Records of Civilization, LX). New York: Columbia University Press.*

Lǐ Yǎn & Dù Shírán, 1987. Chinese Mathematics: A Concise History. Oxford: Clarendon Press.

Liddell, Henry George, & Robert Scott. *Greek-English Lexicon*. Revised and Augmented Throughout ... With a Supplement. Oxford: Oxford University Press, 1968.

Marrou, Henri-Irénée, 1956. *A History of Education in Antiquity*. London: Sheed and Ward.

Martzloff, Jean Claude, 1982. *Histoire des mathématiques chinoises*. Paris: Masson.

Menge, Heinrich (ed.), 1896. Euclidis *Data* cum *Commentario* Marini et scholiis antiquis. (Euclidis Opera Omnia vol. VI). Leipzig: Teubner.

Minar, E. L., F. H. Sandbach & W. C. Helmbold (eds, transls), 1969. Plutarch, *Moralia*, vol. IX (697 C —771 E). (Loeb Classical Library). London: Heinemann / Cambridge, Mass.: Harvard University Press.

Mueller, Ian, 1981. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. Cambridge, Mass., & London: MIT Press.

Mure, G. R. G. (ed., transl.), 1928. Aristotle, *Analytica posteriora, in* Aristotle, *Works* (ed. W. D. Ross), vol. I.

Neugebauer, O., 1935. *Mathematische Keilschrift-texte*. I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937.

Paton, W. R. (ed., transl.), 1979. *The Greek Anthology*. 5 vols. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann.

Pedersen, Olaf, 1974. "Logistics and the Theory of Functions. An Essay in the History of Greek Mathematics". *Archives Internationales d'Histoire des Sciences* **24**, 29-50.

Rashed, Roshdi (ed., transl.), 1984. Diophante, *Les Arithmétiques*, tômes III *(livre IV)*, IV *(livres V, VI, VII)*. Paris: »Les Belles Lettres«.

Robbins, Frank Egleston, 1929. "P. Mich. 620: A Series of Arithmetical Problems". *Classical Philology* **24**, 321-329.

Rosen, Frederic (ed., transl.), 1831. The *Algebra* of Muhammad ben Musa. London: The Oriental Translation Fund.

Rozenfeld, Boris (transl.), 1983. Muxammad ibn Musa al-Xorezmi, *Kratkaja kniga ob isčislenii algebry i almukabaly, in* Muxammad ibn Musa al-Xorezmi *Matematičeskie traktaty* (ed. C. X. Siraždinov). Taškent: Izdatel'stvo »FAN« Uzbekskoj CCP.

Ross, W. D. (ed., transl.), 1928. Aristotle, *Metaphysica, in* Aristotle, *Works* (ed. W. D. Ross), vol. VIII (second edition).

Rudhardt, Jean, 1978. "Trois problèmes de géométrie, conservés par un papyrus genevois". *Museum Helveticum* **35**, 233-240.

Sachs, Eva, 1917. *Die fünf platonischen Körper. Zur Geschichte der Mathematik und der Elementenlehre Platons und der Pythagoreer.* (Philologische Untersuchungen, 24. Heft). Berlin: Weidmann.

Saidan, Ahmad S. (ed., transl.), 1978. The *Arithmetic* of al-Uqlīdisī. The Story of Hindu-Arabic Arithmetic as Told in *Kitāb al-Fuṣūl fī al-Ḥisāb al-Hindī* by Abū al-Ḥasan Aḥmad ibn Ibrāhīm al-Uqlīdisī written in Damascus in the Year 341 (A.D. 952/53). Dordrecht: Reidel.

Schöne, Hermann (ed., transl.), 1903. Herons von Alexandria *Vermessungslehre* und *Dioptra*. (Heronis Alexandrini Opera quae supersunt omnia, vol. III). Leipzig: Teubner.

Sesiano, Jacques (ed., transl.), 1982. Books IV to VII of Diophantus' *Arithmetica* in the Arabic Translation Attributed to Qusṭā ibn Lūqā. (Sources in the History of Mathematics and Physical Sciences 3). New York etc.: Springer.

Sesiano, Jacques, 1986. "Sur un papyrus mathématique grec conservé à la Bibliothèque de Genève". *Museum Helveticum* 43, 74-79.

Shorey, Paul (ed., transl.), 1978. Plato, *The Republic*. 2 vols. (Loeb Classical Library 237, 276). London: Heinemann / Cambridge, Mass.: Harvard University Press, 1978, 1963.

Soubeyran, Denis, 1984. "Textes mathématiques de Mari". *Revue d'Assyriologie* 78, 19-48.

Tannery, Paul (ed., transl.), 1893. Diophanti Alexandrini *Opera omnia* cum graecis commentariis. 2 vols. Leipzig: Teubner, 1893-1895.

Thomson, William, & Gustav Junge (eds, transls), 1930. The *Commentary* of Pappus on Book X of Euclid's *Elements*. (Harvard Semitic Series, 8). Cambridge, Mass.: Harvard University Press.

Toomer, G. J., 1984. "Lost Greek Mathematical Works in Arabic Translation". *The Mathematical Intelligencer* 6:2, 32-38.

Tropfke, J./Vogel, Kurt, et al, 1980. *Geschichte der Elementarmathematik*. 4. Auflage. Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.

Ver Eecke, Paul (ed., transl.), 1926. Diophante d'Alexandrie, *Les six livres arithmétiques* et *le livre des nombres polygones*. Bruges: Desclée de Brouwer.

Vogel, Kurt, 1930. "Die algebraischen Probleme des P. Mich. 620". *Classical Philology* 25, 373-375.

Vogel, Kurt (ed.)., 1954. Die *Practica* des *Algorismus Ratisbonensis*. Ein Rechenbuch des Benediktinerklosters St. Emmeram aus der Mitte des 15. Jahrhunderts. (Schriftenreihe zur bayerischen Landesgeschichte, 50). München: C. H. Bech.

Vogel, Kurt (ed., transl.), 1968. *Chiu chang suan shu. Neun Bücher arithmetischer Technik. Ein chinesisches Rechenbuch für den praktischen Gebrauch aus der frühen Hanzeit (202 v. Chr. bis 9 n. Chr.).* (Ostwalds Klassiker der Exakten Wissenschaften. Neue Folge, Band 4). Braunschweig: Friedrich Vieweg & Sohn.

Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13).* (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München: Deutsches Museum.

Vogel, Kurt, 1982. "Zur Geschichte der Stammbrüche und der aufsteigenden Kettenbrüche". *Sudhoffs Archiv* **66**, 1-19.

Voilquin, Jean (ed., transl.), 1950. Thucydide, *Histoire de la guerre du Péloponnèse*. 2 vols. Paris: Garnier.

Woepcke, Franz, 1853. *Extrait du Fakhrî, traité d'algèbre* par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; *précédé d'un mémoire sur l'algèbre indéterminé chez les Arabes*. Paris: L'Imprimerie Impériale.

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"On Parts of Parts and Ascending Continued Fractions". *Centaurus* **33** (1990), 293–324.

On Parts of Parts and Ascending Continued Fractions
An Investigation of the Origins and Spread of a Peculiar System

by

JENS HØYRUP*

O. Neugebauer in Memoriam

I. Introduction

The following article deals with two particular ways to denote fractional numbers, one of them multiplicative ('parts of parts') and the other multiplicative-additive ('ascending continued fractions'). They turn up in sources from several cultures and epochs, but as a standard idiom only in Arabic mathematics, where their occurrence has been amply described. In certain other contexts (Babylonia, High and Late Medieval Europe) their occasional presence has been taken note of though rarely investigated systematically. Finally, a few scattered occurrences in Ancient Greek and Egyptian sources have not been commented upon until this day.

Widespread occurrence of similar practices raises the question of interdependence versus independent development by accident or in response to analogous situations. Thus also in this case. Posing the question, however, turns out to be more easy than answering it, not least because some of the cultures to be dealt with only present us with utterly few examples of the usage, and only the combination of evidence and arguments of many kinds will allow us to construct

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Centaurus 1990: vol. 33: pp. 293–324

a scenario which is at least well-founded if not definitively verified on all points.

As a by-product, the inquiry will cast new light on the origins of the Egyptian unit fraction system.

II. Islamic and Post-Islamic Evidence

In chapter V of Leonardo Fibonacci's *Liber abaci* (second version, 1228) a number of complex writings for fractional numbers are introduced. One of them – the others are irrelevant for the present purpose – is what later has come to be called the 'ascending continued fraction' ('Aufsteigende Kettenbrüche' in German), which Leonardo exemplifies by the number

$$\frac{1\ \ 5\ \ 7}{2\ \ 6\ \ 10}$$

meaning 7 10ths plus 5 6ths of a 10th plus $\frac{1}{2}$ of a 6th of a 10th[1] – in more compact writing $\{\frac{1}{10}\cdot[7+(\frac{1}{6})\cdot(5+\frac{1}{2})]\}$. In general

$$\frac{a_3\,a_2\,a_1}{b_3\,b_2\,b_1}$$

stands for

$$\frac{a_1}{b_1}+\frac{a_1}{b_1\cdot b_2}+\frac{a_3}{b_1\cdot b_2\cdot b_3}=\frac{a_1+\dfrac{a_2+\dfrac{a_3}{b_3}}{b_2}}{b_1}$$

The generalization to two or four or more levels is obvious. Incidentally, the latter expression demonstrates that 'ascending continued fractions' have nothing but an inverted visual image in common with genuine continued fractions.

The notation for ascending continued fractions was not invented by Leonardo but apparently in the Maghreb mathematical school, probably during the 12th century. They are discussed in ibn al-Bannā''s 13th century *Talkhīṣ a'māl al-ḥisāb*[2] though without indication of the way they were to be written. Various commentaries show, however, that standardized notations were in use. In one late commentary, al-Qalaṣādī's *Arithmetic*[3] (1448), it is furthermore required that the denominators in an ascending continued fraction stand in descending order from the right ($b_1 > b_2 > b_3$), as it is actually the case in Leonardo's examples. Even though some of the examples given by other commentators[4] do not observe this rule, which I shall denote *al-Qalaṣādī's canon* in the following, it was probably not of al-Qalaṣādī's own making. The purpose of the canon may have to do with the value of the first member as an approximation. The error committed by throwing away all members but the first will necessarily be less than $1/b_1$ (a's are supposed to be less than corresponding b's). Choosing b_1 as large as possible will ensure that a_1/b_1 is a good approximation (though not necessarily the optimal approximation, cf. note 28). Thus, in Leonardo's example, dividing first by 10 ensures that the first member will at most be 0.1 off the true value. If the reverse canon ($b_1 < b_2 < b_3$) had been used, the result had been $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6}$, and the error committed by taking the first member alone would have been $\frac{7}{24} \approx 0.29$.

The invention of *notations* was part of the general drive of Maghreb mathematics, but verbally expressed ascending continued fractions and other composite fractional expressions belonged to the common lore of Arabic mathematics. They had been amply used and discussed in the later 10th century by Abūl-Wafāʾ in his *Book on What Scribes, Officials and the Like Need from the Science of Arithmetic*[5] They are also present in al-Khwārizmī's early ninth century *Algebra*[6] and as well as in the *Liber mensurationum* by one Abū Bakr, translated by Gherardo of Cremona into Latin in the 12th century and presumably written in the first place around 800 A.D.[7]. Among the occurrences in al-Khwārizmī's work are the following (page references to Rosen's translation):

- P. 24: $\frac{25}{36}$ is transformed into 'two-thirds and one-sixth of a sixth' $[\frac{2}{3} + \frac{1}{6} \cdot \frac{1}{6}]$.
- P. 45: 1 *māl* is found as 'a fifth and one-fifth of a fifth' of $4\frac{1}{6}$ *māl* $[\frac{1}{5} + \frac{1}{5} \cdot \frac{1}{5}]$.

- P. 53: 'three and three-fourths of twenty parts' [$\frac{3}{20}+\frac{3}{4}\cdot\frac{1}{20}$] is transformed into 'fifteen eightieths'.
- P. 54: a twelfth is expressed as 'the moiety of one moiety of one-third' [$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}$].
- P. 72: as one of several rules for finding the circular area we find the square of the diameter minus 'one seventh and half one-seventh of the same'.
- P. 88: the third of 'nine dirhems and four-fifth of thing' is found to be 'three dirhems, and one-fifth and one-third <of> one-fifth of thing' [$3+\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5}$].
- P. 99: 'two-sevenths and two-thirds of a seventh of the share of a son' [$\frac{2}{7}+\frac{2}{3}\cdot\frac{1}{7}$].

The *Liber mensurationum* (which contains mostly integer numbers) presents us with the following relevant passages:

- N° 19 (p. 90), 7 *et dimidium septime.*
- N° 89 (p. 107), 43 *et due quinte et quattuor quinte quinte*, resulting from the computation of $169-(11\frac{1}{5})^2$. Similarly but in greater computational detail in N° 128 (p. 115).
- N° 113 (p. 112), the root of $\frac{3}{16}$ census is expressed as *radix octave census et medietatis octave census.*
- N° 144 (p. 118), the area of the circle is expressed as the square on the diameter minus *septimam et septime eius medietatem.* Similarly in N^{os} 146, 156 and 158 (pp. 119 and 124).

The elementary building stones of the ascending continued fractions are the 'parts of parts', the *partes de partibus* as they came to be called in the Medieval Latin tradition, i.e., expressions of the form '$\frac{p}{q}$ of $\frac{1}{r}$'. The extent to which these were natural to Arabic speakers of early Islam is demonstrated in the first treatise of the 10th century *Epistles of the Brethern of Purity*, the *Rasā'il ikhwān al-ṣafā'*. In this exposition of the fundaments of arithmetic great care is taken to explain that the first of a collection of two is called a half, while the first of three is a third, that of four a fourth, and that of eleven one part of eleven; the first of twelve, however, is labeled a half of a sixth, without a single word commenting upon the reasons for or meaning of this composition. Similarly, the first of fourteen is expressed without explanation as a half of a seventh, and that of fifteen as a third of a fifth.[8]

The origin of both the parts of parts and of the ascending continued fractions has been ascribed to a variety of causes, in particular to the peculiarities of the Arabic vocabulary. Unit fractions from $\frac{1}{2}$ to $\frac{1}{10}$ possess a particular name of their own while those with larger denominators require a full phrase, $\frac{1}{n}$ being expressed as 'one part of n' or 'one part of n parts' *unless* it can be composed from unit fractions with smaller denominators. This might indeed explain that the Arabic authors transformed the $\frac{1}{14}$ of Hero's (or rather pseudo-Hero's) rule for finding the circular area[9] into 'half one-seventh', and that they expressed $\frac{1}{25}$ as 'one-fifth of a fifth'.

On the other hand, 'the moiety of one moiety of one-third' is somewhat at odds with the hypothesis: Why not 'one-third of a fourth', when in the actual case the number 12 arises as $3 \cdot 4$? Or at least 'one-half of a sixth', which according to Abū'l-Wafā' is to be preferred to 'one-third of a fourth',[10] and which still circumvents the difficulties created by the Arabic language while using only two factors? Al-Khwārizmī, moreover, had no particular difficulty with general fractions, at times with denominators exceeding 10, which abound even in those very calculations where the 'parts of parts' turn up. The reason that the reciprocal of $\frac{25}{6}$ is expressed in the form of an ascending continued fraction on p. 45 of the *Algebra* while another ascending continued fraction is, reversely, reexpressed as $\frac{15}{80}$ on p. 53 seems simply to be that both reformulations fit the further calculations better. The conventional explanation of the use of composite expressions based *solely* on Arabic linguistic particularities is apparently insufficient, even if these particularities have evidently tainted the way the system was used.

III. Classical Antiquity and its Legacy

The need for an explanation which goes beyond the peculiarities of the Arabic language is confirmed by certain older sources. One of them is the collection of arithmetical riddles in *Anthologia Graeca* XIV.[11] A study of these gives the fascinating result that the types of fractional expressions used vary with the subject of the problem. Problems which refer to Greek mythology or history make use of unit or general fractions. So do all problems dealing with apples or walnuts stolen by girl friends, with the filling of jars or cisterns from several sources, with spinners', brickmakers' or gold- or silversmiths' production, with wills, and with the epochs of life – none of them make use of 'parts of parts'.

'Parts of parts' and related composite expressions, on the other hand, turn up in all problems dealing with the Mediterranean extensions of the Silk Road (N^os^ 121 and 129), with the legal partition of heritages (N^os^ 128 and 143), and with the hours of the day (N^os^ 6, 139, 140, 141, and 142; N° 141 is connected to astrology). A final 'fifth of a fifth' is found in N° 137, dealing with a catastrophic banquet probably meant to be held in Hellenistic Syria. It appears that a number of recreational problems belonging to (at least) two different contexts (providing the dress of the problems) have been brought together in the anthology, each conserving its own distinctive idiom for fractions: on one hand the traditional Greek idiom, which makes use of general and unit fractions; on the other, the usage of the trading community and of juridical calculators (and perhaps of astrologers and makers of celestial dials), which is different.

We may list the various composite fractional expressions:[12]

- N° 6 (the hour of the day): 'Twice two-third'.
- N° 121 (travelling from Cadiz to Rome): 'One-eighth and the twelfth part of one-tenth'.
- N° 128 (a textually and juridically corrupt heritage): 'The fifth part of seven-elevenths'.
- N° 129 (travelling from Crete to Sicily): 'Twice two-fifths'.
- N° 137 (the Syrian banquet): 'A fifth of the fifth part'.
- N° 139 (a dial-maker asked for the hour of the day): 'Four times three-fifths'.
- N° 140 (the hour of a lunar eclipse): 'Twice two-sixths and twice one-seventh'.
- N° 141 (the hour of a birth, to be used for a horoscope): 'Six times two-sevenths'.

- N° 142 (The hour for spinning-women to wake up): 'A fifth part of three-eighths'.
- N° 143 (The heritage after a shipwrecked traveller): 'Twice two-thirds'.

We observe that the character of these composite expressions is similar to but does not coincide with what we know from the Arabic texts. Firstly, of course, these do not contain integer multiples of fractions like those of N^os 6, 129, 139, 140, 141 and 143, and they would speak of 'three fifths of an eighth', not of 'a fifth part of three-eighths'. Secondly, the Arabic sources mostly follow the canon made explicit by al-Qalaṣādī, while for instance N° 121 of the *Anthologia* does not – and $\frac{1}{12}$ they split further, *viz* into $\frac{1}{2}$ of $\frac{1}{6}$, into $\frac{1}{3}$ of $\frac{1}{4}$ or even, as we have seen, into $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{3}$.

Most likely, the integer multiples of the *Anthologia* are to be explained from the recreational character of the arithmetical riddles: being unusual, the multiples make the riddles more funny or more obscure at first sight – it is hardly imaginable that 'two-thirds' would be expressed as 'twice two-sixths' for any everyday purpose. The demands of versification may have played a supplementary rôle – but since problems with a traditional 'Greek' subject make no use of the stratagem hardly more than a supplementary rôle.

The deviation from 'al-Qalaṣādī's canon', however, gives no impression of grotesquerie and can therefore not be an effect of the recreational purpose of the epigrams. It is thus probable that it reflects the daily usage of the practitioners trading in 'parts of parts', which will not have respected the later Arabic canon and customs in full.

Another, Latin, source of interest for our purpose is the Carolingian collection *Propositiones ad acuendos juvenes* conventionally ascribed to Alcuin and dating from c. A.D. 800.[13] Chronologically, it is roughly contemporary with al-Khwārizmī and probably with the *Liber mensurationum*. The material, however, appears to be inherited from late Antiquity, and the Carolingian scholar (be it Alcuin or somebody else connected to the Carolingian educational effort) has only acted as an editor.

A brief exposition of the global character of the collection will serve the double purpose of locating its composite fractional expressions with respect to their background and of introducing some notions concerning the function of recreational problems from which the further discussion will benefit. In general, the collection is highly eclectic, bringing together material and methods from a variety of traditions, combining at times mutually incompatible approximations within the same problem solution.[14] Of particular interest in the present context is the very diverse network of connections behind the arithmetical problems. N° 13, dealing with 30 successive doublings of 1, points back to a very similar problem from Old Babylonian Mari[15] and eastward to the Arabo-Indian chess-board problem and even to China. N^{os} 5, 32–34, 38–39 and 47 all belong to the type of 'A hundred fowls' known from earlier Chinese and contemporary or earlier Indian sources[16] and presented by Abū Kāmil as a type of question

> circulating among high-ranking and lowly people, among scholars and among the uneducated, at which they rejoice, and which they find new and beautiful; one asks the other, and he is then given an approximate and only assumed answer, they know neither principle nor rule in the matter.[17]

Other problems too point to the 'oral technical literature', the treasure of recreational problems shared and carried by the community of traders and merchants interacting along the Silk Road, the combined caravan and sea route reaching from China to Spain.[18]

Connections to the *Anthologia graeca* and thus to the Greco-Roman orbit are also present. Most significant is probably N° 35, which is a puzzle on heritages – one of the types, we remember, which made use of multiples of parts. It can be traced back to Roman jurisprudential digests, even though the editor of the *Propositiones* has got the solution wrong.[19]

A final type represented by N^{os} 2, 3, 4, 40 and 45 seems to bypass what we know from the *Anthologia graeca* and points directly to Egyptian traditions (even though matters may in reality be more complex, cf. below, p. 314). Admittedly, when expressed in algebraic symbolism, the problems in question are of a type identical with the one dominating the *Anthologia graeca*, both being represented by first degree equations. The equations of the *Anthologia*, however, are variations on the pattern

$$x \cdot (1 - \tfrac{1}{p} - \tfrac{1}{q} - \tfrac{1}{r}) = R$$

(p, q, and r being integers), while N^os^ 2, 3, 4, 40 and 45 of the *Propositiones* build on the scheme

$$x \cdot (n + \alpha + \beta) = T$$

(n being an integer larger than 1 and α and β being unit fractions or 'parts of parts'). Both types possess analogues in the Ancient Egyptian Rhind Mathematical Papyrus[20]. The former type corresponds approximately to N^os^ 24–27 and 31–34; these are problems which consider an unspecified quantity or 'heap' (*ꜥḥꜥ*), and which only differ from those of the *Anthologia* by adding the unit fractions instead of subtracting them. The first-degree problems of the *Propositiones* just spoken of, on the other hand, belong to the same type as Rhind Mathematical Papyrus N^os^ 35–38, problems dealing with the *ḥeḳat-measure.*[21]

The reason for this lengthy presentation of the *Propositiones* and of a particular group of first-degree problems is that four of the five problems in this group employ 'parts of parts':

- N° 2: *medietas medietatis, et rursus de medietate medietas* (meaning $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$).
- N° 3: *ter et medietas tertii* ($\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}$)
- N° 4: *medietas medietatis* ($\frac{1}{2}\cdot\frac{1}{2}$).
- N° 40: *medietatem de medietate et de hac medietate aliam medietatem* ($\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$).

Composite fractions thus seem to go naturally with this problem type. On the other hand, they occur nowhere else, neither in the problems which point to the 'Silk Road corpus', nor those which remind of one or the other group from the *Anthologia graeca*, nor in the inheritance problem. One observes that al-Khwārizmī's predilection for taking successive halves instead of a simple fourth is equally present here, and is even extended to the use of $\frac{1}{2}$ of $\frac{1}{3}$ instead of $\frac{1}{6}$. This is all the more remarkable since the simple terms *quadrans* and *sextans* were at hand,[22] and the composite *quarta pars* and *sexta pars* are actually used in other parts of the text (e.g., N[os] 8 and 47). It will also be noticed that three of the four cases are rudimentary ascending continued fractions.

IV. Babylonia

Some scattered instances of 'parts of parts' and of simple ascending continued fractions can thus be dug out from sources belonging to or pointing back to classical Antiquity though not to the core of Greek mathematical culture.[23] Antecedents for the fuller use of ascending continued fractions, on the other hand, must be looked for further back in time – much further, indeed.

They can be found in the Babylonian tablet MLC 1731, which was analyzed by Abraham Sachs[24] and which dates from the Old Babylonian period (c. 2000 to c. 1600 B.C.; the mathematical texts belong to the second half of the period). It presents us with the following examples of composite fractions:[25]

- N° 1: 'One-sixth of one-fourth of [the unit] a barley-corn'.
- N° 3: 'One-fourth of a barley-corn and one-fourth of a fourth of a barley-corn'.
- N° 4: 'One-third of a barley-corn and one-eighth of a third of 20'.[26]
- N° 5: 'Two-thirds of 20 and one-eighth of two-thirds'.
- N° 6: 'A barley-corn and one-sixth of a fourth of 20'.
- N° 7: 'A barley-corn, two-thirds of 20 and one-eighth of two-thirds of 20'.
- N° 9: '17 bar<ley-corns>, one-third of 20, and one-fourth of a third of a barley-corn'.

All these composite expressions result from the conversion of numbers belonging to the 'abstract' sexagesimal system into metrological units. Sachs has convincingly pointed out that the notation in question is used because no unit below the barley-corn existed[27] – fractions could not be expressed in terms of a smaller unit, as done in other conversions to metrological notation. Still, the tablet shows that the parlance of 'parts of parts' *was* at hand, and even that there was an outspoken tendency to make use of ascending continued fractions rather than of sums of unit fractions with denominators below 10.[28] We observe that two-thirds is the only general fraction to turn up, while everything else consists of unit fractions and their combinations,[29] and that 'al-Qalaṣādī's canon' is inverted – be it accidentally or by principle.

This tablet presents us with the most systematic Old Babylonian use of composite fractions. It is not quite isolated, however, and scattered occurrences can be found here and there in other Old Babylonian tablets.

One instance was pointed out by Sachs: YBC 7164 N° 7 (line 18), where the time required for a piece of work is found to be '$\frac{2}{3}$ of a day, and the 5th part of $\frac{2}{3}$ of a day'.[30]

In another text from the Yale collection, 'parts of parts' (though no ascending continued fractions) occur in all five times: YBC 4652 N^{os} 19–22,[31] problems of riddle-character dealing with the unknown weight of a stone. Here, 'the 3d part of the 7th part', 'the 3d part of the 13th part', 'the 3d part of the 8th part' (twice) and '$\frac{2}{3}$ of the 6th part' turn up. We observe that the ordering of factors agrees with 'al-Qalaṣādī's canon', and that even a '13th part' is present (Babylonian, in contrast to Arabic, had a name for this fraction).

In the series text YBC 4714, N° 28, line 10 (and probably also in the damaged text of N° 27), 'a half of the 3d part' turns up in the statement.[32] This is evidently meant as a step toward greater complexity from the previous problems having 'the n'th part' (n = 7, 4, and 5) in the same place.

A text of special interest is the Susa tablet TMS V.[33] All the way through the tablet, sequences of numbers are used as abbreviations for complex numerical expressions involving parts of parts. Recurrent from section to section (albeit with some variation), 13 times in total, is the following series (the right column gives the interpretation)

a:	'2'	2
b:	'3'	3
c:	'4'	4 (cf. the different meaning in *g*)
d:	'$\frac{2}{3}$'	$\frac{2}{3}$
e:	'$\frac{1}{2}$'	$\frac{1}{2}$

f:	‘$\frac{1}{3}$’	$\frac{1}{3}$
g:	‘4’	$\frac{1}{4}$
h:	‘$\frac{1}{3}$ 4’	$\frac{1}{3}$ of $\frac{1}{4}$
i:	‘7’	$\frac{1}{7}$
j:	‘2 7’	2 times $\frac{1}{7}$
k:	‘7 7’	$\frac{1}{7}$ of $\frac{1}{7}$
l:	‘2 7 7’	2 times $\frac{1}{7}$ of $\frac{1}{7}$
m:	‘11’	$\frac{1}{11}$
n:	‘2 11’	2 times $\frac{1}{11}$
o:	‘11 11’	$\frac{1}{11}$ of $\frac{1}{11}$
p:	‘2 11 11’	2 times $\frac{1}{11}$ of $\frac{1}{11}$
q:	‘11 7’	$\frac{1}{11}$ of $\frac{1}{7}$
r:	‘2 11 7’	2 times $\frac{1}{11}$ of $\frac{1}{7}$
s:	‘$\frac{2}{3}$ $\frac{1}{2}$ $\frac{1}{3}$ 11 7’	$\frac{2}{3}$ of $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{11}$ of $\frac{1}{7}$
t:	‘2 $\frac{2}{3}$ $\frac{1}{2}$ $\frac{1}{3}$ 11 7’	2 times $\frac{2}{3}$ of $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{11}$ of $\frac{1}{7}$

In section 10 we also find

A:	‘1 $\frac{2}{3}$’	1 plus $\frac{2}{3}$
B:	‘1 $\frac{1}{2}$’	1 plus $\frac{1}{2}$
C:	‘1 $\frac{1}{3}$’	1 plus $\frac{1}{3}$
D:	‘1 4’	1 plus $\frac{1}{4}$
E:	‘1 $\frac{1}{3}$ 4’	1 plus $\frac{1}{3}$ of $\frac{1}{4}$
F:	‘1 7’	1 plus $\frac{1}{7}$
G:	‘1 2 7’	1 plus 2 times $\frac{1}{7}$
H:	‘1 7 7’	1 plus $\frac{1}{7}$ of $\frac{1}{7}$
I:	‘1 2 7 7’	1 plus 2 times $\frac{1}{7}$ of $\frac{1}{7}$
J:	‘2 $\frac{1}{2}$’	2 plus $\frac{1}{2}$
K:	‘3 $\frac{1}{3}$’	3 plus $\frac{1}{3}$
L:	‘4 4’	4 plus $\frac{1}{4}$ (not $\frac{1}{4}$ of $\frac{1}{4}$)
M:	‘7 igi-7’	7 plus $\frac{1}{7}$
N:	‘7 2 igi-7’	7 plus 2 times $\frac{1}{7}$

In all cases, the expressions multiply the side of a square (literally: count the number of times the side is to be taken).

In order to make *his text* as unambiguous as possible, the scribe has followed a fairly strict format, most clearly to be seen in *t* and *N*: starting from the right, he lists (with increasing *denominator*) those fractions which in full writing would be written igi-*n*-gál, and which he abbreviates as the integer numeral *n*; next come, in increasing *magnitude*, fractions possessing their own ideogram ($\frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$). This entire section of the sequence is to be understood as 'parts of parts'. Then follows an (optional) integer numerator (>1), and finally an (equally optional) integer addend. As long as the numerator is kept at 2 and the addend at 1, the system is unambiguous. If we violate these restrictions (as in *c* and *L*), however, it stops being so. Inside the text, the systematic progress eliminates the ambiguities: if used as a general notation, on the other hand, the system would lead to total confusion – a fact which is obviously recognized by the scribe, since he introduces *ad hoc* the sign igi in *M* and *N*.

These observations entail the conclusion that we are confronted with a specific, context-dependent shorthand, not with a standardized notation for general fractions, as claimed by Evert Bruins.[34] Behind the shorthand, moreover, stick not just general fractions but the *system of 'parts of parts'*; the summation required by the ascending continued fractions, on the other hand, is not visible through the notation.

In the end of the above-mentioned article, Sachs[35] reviews a number of Seleucid notarial documents making use of composite expressions often involving 'parts of parts' (all examples apart from N° 15 deal with the sale of temple prebends corresponding to parts of the day):

(1) 'A fifth of a day and a third from a 15th of a day'.
(2) 'A sixth, an 18th, and a 60th'.
(3) 'A 30th, and a third from a 60th'.
(4) 'A half from three quarters'.
(5) 'A fifth from two thirds'.
(6) 'Two thirds of a day and an 18th of a day'.
(7) 'A sixth and a ninth of a day'.

(8) 'A 20th from one day, of which a sixth from a 60th of a day is lacking'.
(9) 'A 16th and a 30th of a day', added to 'a 16th of a day', giving 'an eighth and a 30th of a day'.
(10) 'an eighth from a seventh'.
(11) 'A half from an eighteenth'.
(12) 'A third from a twelfth'.
(13) 'An 18th from a seventh'.
(14) 'A twelfth from a seventh'.
(15) 'A half from a twelfth' (as a share of real estate).

Sachs rightly observes that the system seems less strict than the old one. In cases where the number is expressed as a sum, no particular effort is made to assure that the first member is an optimal approximation, nor to follow the strict pattern of an ascending continued fraction.[36] From the present perspective, it may be of interest that all 'parts of parts' *except those involving the irregular* $\frac{1}{7}$ respect 'al-Qalaṣādī's canon'.[37] The Arabic avoidance of denominators larger than 10, of course, is not observed.

V. Egypt

Its building stones being unit fractions with small denominators, the 'parts of parts' scheme has often been connected to the Egyptian unit fraction system. In its mature form, as we know it from Middle Kingdom through Demotic sources, however, the Egyptian system had no predilection for those small denominators which it is the purpose of the 'parts of parts' scheme to achieve. The Egyptians, furthermore, were not interested in such splittings where the first member can serve as a good first approximation, whereas a fair first approximation is a key point in the extension of the 'parts of parts' into ascending continued fractions (as we met it already in the Old Babylonian tablet, cf. note 28). Attempts to explain the schemes of 'parts of parts' and ascending continued fractions by reference to the Egyptian unit fractions system thus appear to be misguided.

'Parts of parts' as discussed above are not common in Egypt. In fact, I only know of three places where the usage is employed *to* *indicate a number*[38] (cf. below on other applications). The first of these is Rhind Mathematical Papyrus (RMP). Problem 37, one of the *ḥeḳat*-problems which were mentioned above in connection with the *Propositiones ad acuendos juvenes*:

> 'Go down I [i.e., a jug of unknown capacity] times 3 into the *ḥeḳat*-measure, $\frac{1}{3}$ of me is added to me, $\frac{1}{3}$ of $\frac{1}{3}$ of me is added to me, $\frac{1}{9}$ of me is added to me; return I, filled am I. Then what says it?'.[39]

The second is Problem 67 of the same papyrus,

> 'Now a herdsman came to the cattle-numbering, bringing with him 70 heads of cattle. The accountant of cattle said to the herdsman, Small indeed is the cattle-amount that thou hast brought. Where is then thy great amount of cattle? The herdsman said to him, What I have brought to thee is: $\frac{2}{3}$ of $\frac{1}{3}$ of the cattle which thou hast committed to me...'.[40]

The third example of 'parts of parts' used to indicate a number, finally, belongs in the Moscow Mathematical Papyrus (MMP), Problem 20, where $2\frac{2}{3}$ is told to be $\frac{1}{5}$ *of* $\frac{2}{3}$ of 20.[41]

The latter example is put into perspective in RMP, 'Problem' 61B, which explains the method to find $\frac{2}{3}$ of any unit fraction with odd denominator, and uses $\frac{2}{3}$ *of* $\frac{1}{5}$ as a paradigm.[42] The $\frac{1}{5}$ *of* $\frac{2}{3}$ which appears as a regular number in the MMP is thus (reversion of factors apart, which was trivial to the Egyptians) not recognized as such in the RMP, N° 61B: a composite expression like $\frac{1}{5}$ *of* $\frac{2}{3}$ was to be considered a *problem* and no number *per se* (a problem whose answer is $\frac{1}{10}+\frac{1}{30}$). The same observation can be made on RMP, 'Problem' 61, which is in fact a tabulation of a series of solutions to such problems.[43]

A final use of what appears at first like composite fractional expressions α *of* β turns up in the description of reversed metrological computations and conversions (RMP 44, 45, 46 and 49). As an example we may take RMP 45,[44] which connects the two. A granary is known to contain 1500 *khar* and is supposed to have a square base of 10 cubits by 10 cubits (1 *khar* is $\frac{2}{3}$ of a cube cubit), and the height is looked for. The calculation then proceeds in the following steps:

1	1500;
$\frac{1}{10}$	150;
$\frac{1}{10}$ of $\frac{1}{10}$ of it	15;
$\frac{2}{3}$ of $\frac{1}{10}$ of $\frac{1}{10}$ of it:	10.

The key to the calculation is provided by Problem 44, which supplies the corresponding direct computation of the content of a cubic container of 10 cubit by 10 cubit by 10 cubit: the volume is first computed as $10 \cdot 10 \cdot 10$ [cube cubits] and then transformed into $1000 + \frac{1}{2} \cdot 1000 = 1500$ *khar*. A solution of the reverse Problem 45 by geometric reasoning would have to go through these steps in reverted order, transforming first the volume of 1500 *khar* into 1000 cubic cubits, and then dividing by the area of the base or, alternatively, by length and width separately. The text, as we see, proceeds differently, reversing the multiplications of Problem 44 one by one without changing their order. The reversal is thus taking place at the level of computational steps, where the order of divisions does not matter, and not on that of analytical reasoning. The composite expression '$\frac{2}{3}$ *of* $\frac{1}{10}$ *of* $\frac{1}{10}$' is not meant as another way to express the number $\frac{1}{150}$ but rather as a way to recapitulate the sequence of computational steps (in other words: To display the algorithm to be used).[45] Its single constituents ($\frac{2}{3}$, $\frac{1}{10}$ and $\frac{1}{10}$) are numbers but the composition is neither an authentic number nor a numerical expression to be transformed into a number (a 'problem' in the sense which makes '$\frac{2}{3}$ *of* $\frac{1}{5}$' a problem and '$\frac{1}{10} + \frac{1}{30}$' the answer in RMP 61B).[46]

Though exceptional, the few occurrences of composite fractional expressions *used as legitimate numbers* are sufficient proof that the schemes of 'parts of parts' and ascending continued fractions are indeed connected to Egypt though not to be explained with reference to the preferred unit fraction notation of the Egyptian scribes. The Egyptians *were* able to understand 'parts of parts' not only as problems or as sequential prescriptions but also as numbers in their own right. When would they do so?

It is difficult to deduce a rule from only three isolated instances. At least two of the present cases, however, are not isolated but embedded in a specific context, on which I shall make some observations in order to answer the question.

Firstly, the *ḥeḳat*-problems are formulated as riddles. When searching the Rhind Papyrus for other riddles I only found one – *viz.* the cattle problem in N° 67 (this is actually how I first discovered my second instance). Stylistically, these five problems are intruders into a problem collection which is otherwise written in a didactically neutral style.

Secondly, we note that the '$\frac{2}{3}$ of $\frac{1}{3}$' of the cattle-problem is put into the mouth of the herdsman and not into that of the accountant-scribe (similarly, the '$\frac{1}{3}$ of $\frac{1}{3}$' is put into the mouth of a jug).

Thirdly, the similarity was already noted between the *ḥeḳat*-problems and those problems of the *Propositiones* which make use of 'parts of parts'. The *ḥeḳat*-problems are thus connected to the whole fund of recreational mathematics.

All this matches a comprehension of recreational mathematics as a 'pure' outgrowth of practitioners' mathematics.[47] 'Parts of parts' appear to have belonged to non-technical, 'folk' parlance, i.e., to the very substrate from which the riddles of recreational problems were drawn. Scribal mathematics, on the other hand, made use of the highly sophisticated scheme of unit fractions; this was a technical language, and the tool which the scribe would use to solve the recreational riddles even when these were formulated in a different idiom.[48]

A parallel to the Old Babylonian situation is obvious. Even here, the ascending continued fractions appeared when the result of calculations in the 'technical system' of sexagesimals had to be transformed into 'practical' units, while the 'parts of parts' turned up in the statement of the riddles on stones of unknown weight, and when supplementary complication had to be added to purely mathematical problems.

'Parts of parts' *could* have arisen as a non-technical simplification and consecutive extension of the unit fraction system, inspired by the sequential prescriptions of reversed computational schemes. Alternatively, it could be the basis from which the unit fraction system had developed. It is as yet not possible to decide the question with full certainty. Strong chronological arguments can be given, however, for the priority of the folk parlance and the secondary character of the unit fraction system. In order to see that we will have to determine the epoch during which the latter system was developed – a question which has never been seriously approached before.

The unit fraction system is used in fully developed form in the RMP. The original from which this papyrus has been copied is dated to the Middle Kingdom, i.e. to the early 2nd millennium. Other papyri computing by means of the unit fraction system, some of them genuine accounts and not materials for teaching or tables for reference, belong to the same period. By this time, general unit fractions had thus become a standard tool for scribal calculators.[49]

Older sources, however, are almost devoid of unit fractions. Old Kingdom scribes made use of metrological sub-units and of those fractions which are *not* written in the standardized way (i.e., $\frac{1}{n}$ written as the numeral *n* below the sign *ro*), *viz.* $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{1}{3}$.[50] Only the Fifth Dynasty Abū Sir Papyri (24th century B.C.) present us with the unit fractions $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{6}$.[51] At the same time, however, they present us with striking evidence that the later *system* was not developed. The sign for $\frac{1}{5}$, indeed, appears in the connection '$\frac{1}{5}$ $\frac{1}{5}$', meaning $\frac{2}{5}$. Later, $2\cdot(\frac{1}{5})$ (or, as it is expressed in the RMP, '2 *called out of* 5') would be no number but a problem, the solution of which was $\frac{1}{3}+\frac{1}{15}$ – about one-third of the text of the Rhind Mathematical Papyrus is in fact occupied by the solution of $\frac{2}{n}$, *n* going from 3 to 101.[52] There are thus good reasons to believe that a notation for simple aliquot parts was gradually being extended toward the end of the Old Kingdom, but was not yet developed into its mature form. True, Reineke[53] thinks that it will have been needed in the complex administration of the Old Kingdom, and thus dates the development to the first three dynasties. As far as I can see, however, real practical tasks are better solved by means of metrological sub-units (which are standardized and can thus be marked out on measuring instruments). The advantage of the unit fraction system is *theoretical*; it will only become manifest in the context of a school system.

This conclusion is supported by analysis of the pyramid problems of the RMP (N^os^ 56, 57, 58, 59A, 59B, 60). Those of them which appear to deal with 'real', traditional pyramids, i.e., which have a slope close to that of Old Kingdom pyramids (N^os^ 56–59B) measure the slope in adequate metrological units (*viz.* palms [of horizontal retreat per cubit's ascent].[54] The result of N° 60, which deals with some other, unidentified structure, is given as a dimensionless, abstract number. At the same time, the dimensions of the first five, 'real' pyramids are given without the unit, as it would be adequate for master-builders who knew what they were speaking about; N° 60 states the data as numbers of cubits, as suitable for a teacher instructing students who do not yet know the concrete practices and entities spoken about. It is thus likely that the author of the papyrus took over the first 5 problems with their metrological units from an older source but created or edited the final, abstract problem himself.[55]

The time when teaching changed from apprenticeship to organized school teaching is fairly well-established.[56] Schools were unknown in the Old Kingdom (if we do not count the education of sons of high officials together with the royal princes), which instead relied upon an apprentice-system. Only after the collapse of the Old Kingdom do we find the first reference to a school (and the absence of a God for the school shows that schools only arose when the Pantheon had reached its definitive structure). By the time of the early Middle Kingdom, on the other hand, scribal education is school education. There is thus a perfect coordination between the changing educational patterns, the move from metrological toward pure number, and the development of the full unit fraction system as far as it is reflected in the sources.

It is therefore fairly certain that the systematic use of unit fractions was a quite recent development when the original of the Rhind Papyrus was written – and implausible, as a consequence, that a non-technical usage built on 'parts of parts' should already have been derived from it. On the other hand, the traces of an incipient use of the unit fraction notation in the Abū Sir Papyri fits a development starting from a set of elementary aliquot parts in popular use but extending and systematizing this idiom in agreement with the requirements of school teaching.

VI. A Scenario

The single occurrences of 'parts of parts' and ascending continued fractions are easily established. When it comes to questions of precedence and to possible connections, however, conclusions will have to be built on indirect evidence and on plausibility. Instead of proposing candidly a theory and claiming it to be necessary truth I shall therefore propose a *scenario* and, in cases where this is needed, try to evaluate the merits of alternative interpretations. Instead of treating the matter in chronological order I shall begin with the most obvious, leaving the more intricate matters to the end.

Most obvious of all are the connections within Western Asia. The Old Babylonian 'parts of parts' and ascending continued fractions are so close to the usage later testified in Arabic sources that the existence of unbroken habits in the Babylonian-Aramaic-Arabic-speaking region is beyond reasonable doubt. The minor differences between canons and materializations of shared principles can easily be explained as effects of the peculiarities of the single languages and from the use of different computational tools or techniques.

In the early Islamic period, the composite fractions belonged to the 'finger-reckoning' tradition and thus to the non-scholarly discourse of merchants and other practical reckoners.[57] One may assume this to have been the case already in earlier times – not least because most of the Old Babylonian occurrences suggest so. The intense interaction of merchants along the Silk Road, which was able to carry a shared culture of recreational problems, will also have been able to spread a Semitic merchants' usage to traders and calculators of neighbouring civilizations. The early rôle of the Phoenicians and the persistent participation of Syrian and other Near Eastern merchants in Mediterranean trade, in particular, will have been an excellent channel for the spread of the system to the West (as it was probably the channel through which a shared system of finger-reckoning spread from the Near East to the whole Mediterranean region and as far as Bede's Northumbria).[58] The striking coincidence that problems from the *Anthologia graeca* concerning parts of the day refer to the very usage which also turns up in Seleucid calculations dealing with that subject, as well as the references to astrology and to dial-makers in the *Anthologia*, suggests that not only traders but 'Chaldean' astrologers and instrument-makers were involved in the spread of the usage from the Near Eastern to the Greek orbit.

To the Greek orbit, but not general spread *within* the orbit of Greek culture. The reason that we can speak of striking coincidences is, in fact, that no such spread took place. 'Parts of parts' and derived expressions are restricted to those very domains where their original practitioners employed them, using probably an idiom borrowed together with other professional instruments from the Near East. Other domains were not affected.

The above argument pressupposes that diffusion took place, and that a channel for that diffusion has to be found. Caution requires, however, that this presupposition be itself examined critically. After all, 'parts of parts' seems to be an idea close at hand. Everybody who understands the fractions will also understand their composition, we should think. Ascending continued fractions, furthermore, is a generalization of the metrological principle of descending sub-units; any culture possessing a linearly ordered and multi-layered metrology should be able to invent them.

So it seems. But the actual evidence contradicts the apparent truisms. Greek Antiquity, though having demonstrably the schemes before its eyes, did not grasp at a notation which was so near at hand. It accepted the notation in a few select places – precisely the ones to where it can be assumed to have been brought. But the Greeks did not like it. For everyday use, they stuck to the Egyptian system; for mathematical purposes, they developed something like general fractions; and in astronomy, they adopted the Babylonian sexagesimal fractions.

The same holds for Latin Europe. The *Propositiones* became quite popular and influenced European recreational mathematics for centuries. But a 14th century problem coming very close to those dealing with *medietas et medietas medietatis* transforms this number into '$\frac{1}{2}$ *and* $\frac{1}{4}$'.[59] The usage 'at hand' did not spread – on the contrary, it was resorbed.

The ascending continued fractions had a similar fate. As told above, they were taken over from Arabic arithmetic as an obligatory subject in Italian arithmetic from Leonardo onwards without acquiring ever any importance. Outside Italy, only Jordanus de Nemore tried to naturalize them as part of theoretical mathematics. He did so in his treatises on 'algorism', computation with Hindu numerals. For this purpose he invented a special concept 'dissimilar fractions'. To explain what the concept was about he connected it precisely to systems of metrological subunits.[60] Not even his closest followers, however, appear to have found anything attractive in the idea, and no echo whatsoever can be discovered. Ascending continued fractions, no more than 'parts of parts', came naturally to the minds of Medieval European reckoners and mathematicians.

If a concept cannot spread inside a given culture but remains restricted to a very particular use (ultimately to be resorbed) it is not likely to have been invented by this culture – at least not if there is no specific need for it in the context where it establishes itself. On this premise the 'parts of parts' occurring in the *Anthologia graeca* and the *Propositiones* can safely be assumed to be there as the result of a borrowing.

In the case of the *Anthologia*, as we have seen, the only conceivable source is Western Asia; as far as the *Propositiones* are concerned, the question of the direct channel is less easily decided. As we have observed, composite fractions are absent even from the problems inspired by the Eastern trade. Only one specific type of riddle employs them – a type which ultimately points toward ancient Egypt and not to the trading network. During the Achaemenid and Hellenistic eras, however, Egyptian and Western Asiatic methods and traditions had largely been mixed up. Even if the composite fractions of the *Propositiones* can ultimately be traced to Egypt, the way from Aachen to Egypt may therefore have passed through anywhere between Kabul and Seville.

Tracing the composite fractions of the *Anthologia* to the Semitic-speaking world of Western Asia and those of the *Propositiones* to Egyptian sources brings us back to the most intricate question: How did these (or, more precisely: the Babylonian and Egyptian usages) relate to each other?

We have found the traces of an Old Kingdom Egyptian as well as an Old Babylonian 'folk' usage of elementary aliquot parts (including $\frac{2}{3}$). We have seen, moreover, that these were combined in both cultures into 'parts of parts'; that they were expanded at least in Babylonia into a system of ascending continued fractions, and that they presumably provided Middle Kingdom Egypt with the foundation on which the full unit fraction system was built.

In principle, the Babylonian and Egyptian composite fractions may have developed in complete independence; two arguments, however, contradict this assumption. For one thing, 'parts of parts' seem not to come naturally to an 'average' culture, if we trust the Greek, Latin and Italian evidence. The Ancient Mesopotamian compositions appear, moreover, to be strictly bound to the *Babylonian* language. Third millennium Sumerian texts employ elementary unit fractions freely; but they never combine them as 'parts of parts'; these, and the ascending continued fractions, only appear when mathematical traditions carried by the Babylonian language took possession of the scribal school in the Old Babylonian epoch. *Shared origins* or at least *shared roots* are thus more credible than full independence.

Shared origins are by no means excluded. Both the Semitic (including the Babylonian) and the Ancient Egyptian languages belong to the Hamito-Semitic language family. Furthermore, a socio-cultural need for simple fractions can reasonably be ascribed to the (presumably pastoral) carriers of the language before the Semitic and the Egyptian branch broke away from each other.[61] Already at this early epoch, the habit of combining them as 'parts of parts' *may* also have existed, even though the (scarce) comparative evidence suggests no need for such arithmetical subtleties in a non-monetary economy. Alternatively, diffusion of the habit via trade routes from one culture to the other at a later moment can be imagined: during the fourth as well as the third millennium B.C., connections existed, in all probability via Syrian territory.[62]

Yet, whether such commercial links were able to influence the development of arithmetical idioms is an open question. They may have involved a whole chain of intermediaries. An argument in favour of diffusion through trading contacts one way or the other (or from an intermediary) could be the common 'institution' of recreational mathematics, which is not likely to have existed when the Semitic and Egyptian branches of the family separated (probably no later than the fifth millennium); but since Babylonian and Egyptian scribes have only the institution but no members (i.e., problem-types) in common, independent development of the recreational *genre* as a response to the similar social environments of professional reckoners – i.e., *shared* (sociological) *roots* of the *genre* – is an alternative explanation at least as near at hand as *shared origins* through common descent or through diffusion.

Similarly, *shared roots* (though linguistic or computational and not sociological) may be the better explanation that composite fractions are found in both Egypt and Babylonia. As one will remember, the objection against fully independent development of systems of composite fractions was founded on the observation that the creation of a scheme of 'parts of parts' is *not* near at hand, in spite of what might look like reasonable *a priori* expectancies. Strictly speaking, however, this observation was only made *on a Greek, Latin, or Italian* linguistic background and on the background of the computational techniques and tools in common use in classical Antiquity and Medieval Europe. But developments in Egypt and Babylonia will not have been fully independent: they will have taken place on structurally similar linguistic backgrounds, and maybe on the background of shared techniques and tools. A common heritage of Babylonians and Egyptians could be a set of elementary fractions and a pattern of linguistic or computational habits being naturally open to specific developments – in particular the development of a scheme of 'parts of parts'.[63] This would be *parallel developments from shared roots.*

Summing up we may conclude with a high degree of certainty that later occurrences of 'parts of parts' and ascending continued fractions outside the Egypto-Semitic area are due to borrowings from developed usages (in some cases distorting or rudimentary borrowings). We may also assume that the parallel Semitic and Egyptian idioms can be ascribed to a shared heritage. But we cannot know whether the shared heritage was *an actual way to speak* about fractional entities or *only a potential scheme* inherent in language structures or computational practices. Personally, I confess to be inclined toward belief in the potential scheme.

BIBLIOGRAPHY AND ABBREVIATIONS

Boncompagni, Baldassare (ed.).
1857: *Scritti di Leonardo Pisano matematico del secolo decimoterzo. I. Il Liber abaci di Leonardo Pisano.* Roma: Tipografia delle Scienze Matematiche e Fisiche.

Brentjes, Sonja.
1984: "Die erste Risâla der Rasâ'il Iḫwân as-Ṣafâ' über elementare Zahlentheorie: Ihr mathematischer Gehalt und ihre Beziehungen zu spätantiken arithmetischen Schriften". *Janus* 71, 181–274.

Brunner, Hellmut.
1957: *Altägyptische Erziehung*. Wiesbaden: Otto Harrassowitz.

Busard, H. L. L.
1968: "L'algèbre au moyen âge: Le "Liber mensurationum" d'Abû Bekr". *Journal des Savants*, Avril-Juin 1968, 65–125.

Chace, Arnold Buffum, Ludlow Bull & Henry Parker Manning.
1929: *The Rhind Mathematical Papyrus*. II. Photographs, Transcription, Transliteration, Literal Translation. *Bibliography of Egyptian and Babylonian Mathematics* (Supplement), by R. C. Archibald. *The Mathematical Leather Roll in the British Museum*, by S. R. K. Glanville. Oberlin, Ohio: Mathematical Association of America.

Cocquerillat, Denise.
1965: "Les calculs pratiques sur les fractions à l'époque seleucide". *Biblioteca Orientalis* 22, 239–242.

Diakonoff, Igo, M.
1965: *Semito-Hamitic Languages*. Moscow: Nauka Publishing House.

Djebbar, A.
1981: *Enseignement et recherche mathématiques dans le Maghreb des XIII^e-XIV^e siècles (étude partielle)*. (Publications mathématiques d'Orsay, 81–02). Orsay: Université de Paris-Sud.

Eneström, Georg.
1913: "Das Bruchrechnen des Nemorarius". *Bibliotheca Mathematica*, 3. Folge 14 (1913–14), 41–54.

Folkerts, Menso.
1978: "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*. Überlieferung, Inhalt, kritische Edition". *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band 6. Abhandlung.

Friedlein, Gottfried.
1871: "Der Calculus des Victorius". *Zeitschrift für Mathematik und Physik* 16, 42–79, 253–254, Tabelle I-IX.

Graefe, Erhart.
1979: "Ein unerkannter Gebrauch des indirekten Genitivs in Zahlkonstruktionen", pp. 174–184 *in* Manfred Görg & Edgar Pusch (eds), *Festschrift Elmar Edel*. (Ägypten und Altes Testament. Studien zur Geschichte, Kultur und Religion Ägyptens und des Alten Testament). Bamberg.

Heiberg, J. L. (ed., trans.).
1912: *Heronis Definitiones cum variis collectionibus. Heronis quae feruntur Geometrica.* (Heronis Alexandrini Opera quae supersunt omnia, IV). Leipzig: Teubner.

Høyrup, Jens.
1986: "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra". *Erdem* 2:5 (Ankara), 445–484.

1987: "The Formation of 'Islamic Mathematics'. Sources and Conditions". *Science in Context* 1, 281–329.

1988: "Jordanus de Nemore, 13th Century Mathematical Innovator: An Essay on Intellectual Context, Achievement, and Failure". *Archive for History of Exact Sciences* 38, 307–363.

1990: "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* 28, 63–86.

Klengel, Horst.

1979: *Handel und Händler im alten Orient*. Leipzig: Koehler und Amelang.

Kokian, P. Sahak (ed., trans.).

1919: "Des Anania von Schirak arithmetische Aufgaben". *Zeitschrift für die deutsch-österreichischen Gymnasien* 69 (1919–20), 112–117.

Kraeling, Carl & Robert McC. Adams (eds).

1960: *City Invincible*. A Symposium on Urbanization and Cultural Development in the Ancient Near East Held at the Oriental Institute of the University of Chicago, December 4–7, 1958. Chicago: University of Chicago Press.

MCT: O. Neugebauer & A. Sachs.

Mathematical Cuneiform Texts. (American Oriental Series, vol. 29). New Haven, Connecticut: American Oriental Society, 1945.

Medovoj, M. I.

1960: "Ob arifmetičestkom traktate Abu-l-Vafy". *Istoriko-Mathematičeskie Issledovanija* 13, 253–324.

MKT: O. Neugebauer.

Mathematische Keilschrift-Texte. I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937.

Moorey, P. R. S.

1987: "On Tracking Cultural Transfers in Prehistory: The Case of Egypt and Lower Mesopotamia in the Fourth Millennium BC". Pp. 36–46 *in* M. Rowlands, M. Larsen & K. Kristiansen (eds.) *Centre and Periphery in the Ancient World*. (New Directions in Archaeology). Cambridge: Cambridge University Press.

Paton, W. R. (ed., trans.).

1979: *The Greek Anthology*. Volume V (Loeb Classical Library). Cambridge, Mass.: Harvard University Press/London: Heinemann. 1st ed. 1918.

Peet, T. Eric.

1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058*. Introduction, Transcription, Translation and Commentary. London: University Press of Liverpool.

Posener-Kriéger, Paule & Jean Louis de Cenival.

1968: *Hieratic Papyri in the British Museum*. Fifth Series, *The Abu Sir Papyri*. Edited, together with Complementary Texts in Other Collections. London: The Trustees of the British Museum.

Posener-Kriéger, Paule.

1976: *Les Archives du temple funéraire de Néferirkare-kakaï (les papyrus d'Abousir)*. Traduction et commentaire. 2 vols. Paris: Institut Français d'Archéologie Orientale du Caire.

Reineke, Walther Friedrich.
1978: "Gedanken zum vermutlichen Alter der mathematischen Kenntnisse im alten Ägypten". *Zeitschrift für ägyptische Sprache und Altertumskunde* 105, 67–76.
Rosen, Frederic (ed., trans.).
1831: *The Algebra of Muhammad ben Musa, Edited and Translated.* London: The Oriental Translation Fund.
Sachs, Abraham J.
1946: "Notes on Fractional Expressions in Old Babylonian Mathematical Texts". *Journal of Near Eastern Studies* 5, 203–214.
Saidan, Ahmad S.
1971: *ʿIlm al-ḥisāb al-ʿarabī.* Amman. Not seen.
1974: "The Arithmetic of Abū'l-Wafā'". *Isis* 65, 367–375.
Sethe, Kurt.
1916: *Von Zahlen und Zahlworten bei den alten Ägyptern, und was für andere Völker und Sprachen daraus zu lernen ist.* (Schriften der Wissenschaftlichen Gesellschaft in Straßburg, 25. Heft). Straßburg: Karl J. Trübner.
Silberman, David P.
1975: "Fractions in the Abu Sir Papyri". *Journal of Egyptian Archaeology* 61, 248–249.
Soubeyran, Denis.
1984: "Textes mathématiques de Mari". *Revue d'Assyriologie* 78, 19–48.
Souissi, Mohamed (ed., trans.).
1969: Ibn al-Bannā', *Talkhīs aʿmāl al-ḥisāb.* Texte établi, annoté et traduit. Tunis: L'Université de Tunis.
1988: *Qalaṣādī, Kašf al-asrār ʿan ʿilm ḥurūf al-ġubār.* Carthage: Maison Arabe du Livre.
Struve, W. W.
1930: *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau.* Herausgegeben und kommentiert. (Quellen und Studien zur Geschichte der Mathematik. Abteilung A: Quellen, 1. Band). Berlin: Julius Springer.
Suter, Heinrich.
1910: "Das Buch der Seltenheiten der Rechenkunst von Abū Kāmil al-Miṣrī". *Bibliotheca Mathematica,* 3. Folge 11 (1910–1911), 100–120.
Thibaut, G. (ed., trans.).
1875: "The Śulvasútra of Baudháyana". I: *The Pandit* 9 (1874–75), 292–298. II: *The Pandit* 10 (1875–76), 17–22, 44–50, 72–74, 139–146, 166–170, 186–194, 209–218. III: *The Pandit,* New Series 1 (1876–77), 316–324, 556–578, 626–642, 692–706, 761–770.
Thompson, Stith.
1975: *Motif-Index of Folk-Literature. A Classification of Narrative Elements in Folktales, Ballads, Myths, Fables, Mediaeval Romances, Exempla, Fabliaux, Jest Books and Local Legends.* I-VI. Rev. and enl. edition. London: Indiana University Press.
TMS: E. M. Bruins & M. Rutten.
Textes mathématiques de Suse. (Mémoires de la Mission Archéologique en Iran, XXXIV). Paris: Paul Geuthner, 1961.
Tropfke, J., Vogel, Kurt, et al.
1980: *Geschichte der Elementarmathematik.* 4. Auflage. Band 1: *Arithmetik und Algebra.*

Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.

Vogel, Kurt.

1977: *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13).* (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C. Quellentexte und Übersetzungen, Nr. 33). München: Deutsches Museum.

1982: "Zur Geschichte der Stammbrüche und der aufsteigenden Kettenbrüche". *Sudhoffs Archiv* 66, 1–19.

Youschkevitch, Adolf P.

1970: "Abū'l-Wafā'". *Dictionary of Scientific Biography* I, 39–43. New York: Scribner.

1976: *Les mathématiques arabes (VIII^e-XV^e siècles).* (Collection d'Histoire des Sciences, 2). Paris: J. Vrin.

NOTES

1. Ed. Boncompagni 1857: 24. A number of later Italian occurrences until Clavius are discussed by Vogel (1982).
2. Ed., transl. Souissi 1969: 70f.
3. Ed. Souissi 1988: Arabic 59f, transl. (with left-right inversion of the scheme) pp. 41f.
4. Quoted in Djebbar 1981: 46f.
5. See Youschkevitch, "Abū'l-Wafā'"; *idem* 1976: 25ff; or Saidan 1974. My poor Russian has not permitted me to make much use of Medovoj's fuller description (1960) of Abū'l-Wafā''s treatise. Nor have I been able to use Saidan's Arabic edition (1971: 64–368) of the work.
6. Ed., transl. Rosen 1831.
7. Ed. Busard 1968. As to the dating (built on terminological considerations), see Høyrup 1986.
8. Transl. Brentjes 1984: 212f.
9. The square on the diameter minus $\frac{1}{7}$ and $\frac{1}{14}$ of the square. *Geometrica* 24.40, ed., transl. Heiberg 1912: 442f. Cf. *Geometrica* 17.4, *ibid.* 332[h], 333[h].
10. Saidan 1974: 368.
11. Ed., transl. Paton 1979. The editor of this part of the *Anthologia* was probably Metrodoros (fl. c. A.D. 500), but the single epigrams are older.
12. I follow Paton's translation, even though a somewhat more literal translation of some fractional expressions *could* be made. Paton's concessions to English rhythm are immaterial for the present purpose.
13. Ed. Folkerts 1978.
14. See Høyrup 1987: 291 n. 38 ('an/42.9' in line 9 from bottom should read 'and').
15. Published in Soubeyran 1984: 30. The connection and similarities between the Carolingian doublings and those from other epochs and places (except China) are discussed in detail in Høyrup 1986: 477–479. On China, see Thompson 1975: V, 542 (Z 21.1), or Høyrup 1987: 288f.
16. See the survey in Tropfke/Vogel 1980: 613–616.

17. My English translation from Suter 1910: 100.
18. The classification of recreational mathematics as a parallel to folk-tales and riddles, and thus as a special *genre* of oral literature, is discussed in Høyrup 1987: 288f and 1990: 74f.
    The influence of eastern trading routes on the stock from which the *Propositiones* were drawn is also made clear by problems N^{os} 39 and 52, dealing, respectively, with the purchase of animals (including camels) *in oriente* and with transport on camel back.
19. Folkerts 1978: 33.
20. Ed., transl. Chace et al. 1929.
21. Another group from the *Propositiones*, consisting of N^{os} 36, 44 and 48, deviates from both models but comes closest to the *ḥekat*-type.
22. In the sense that the use of these subdivisions of the *as* as names for abstract fractions is described explicitly in the preface to the fifth-century *Calculus* of Victorius of Aquitania (ed. Friedlein 1871: 58f).
23. This peripheral status of the Greek 'parts of parts' is borne out by Ananias of Shirak's 7th century arithmetical collection (ed., transl. Kokian 1919), a work strongly dependent on contemporary Byzantine teaching. 'Parts of parts' are as absent from this work as from the 'Greek' problems of the *Anthologia graeca*.
24. Sachs 1946. Besides the fractional expressions of that tablet, the article presents and discusses similar usages in other Babylonian tablets.
25. In my translation of Babylonian texts, I follow the following conventions:
    - 'The *n*'th part' renders the expression 'igi-*n*-gál'.
    - Fractions and numbers written with numerals ($\frac{2}{3}$, $\frac{1}{2}$, etc.; 1, 2, etc.) render special cuneiform signs for these fractions and numbers.
    - Fractions and numbers written as words render corresponding expressions in syllabic writing.
26. In all metrological systems, the barley-corn is 0;0,0,20 times the fundamental unit. '20' is thus a shorter way to write 'a barley-corn'.
27. Except in the system of weights, where $\frac{1}{2}$ barley-corn existed as a separate unit – cf. Sachs 1946: 208f and note 16. Most likely, however, the text is concerned with area units (among other things because the numbers to be converted are obtained as products of two factors, both of which vary from problem to problem).
28. In N° 4, the result could have been given as '$\frac{1}{4}+\frac{1}{8}$' (or as '$\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}$'). In N^{os} 5 and 7, '$\frac{1}{2}+\frac{1}{4}$' (or '$\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}$'), and in N° 9, '$\frac{1}{4}+\frac{1}{6}$' would have been possible.
    The actual choices of the texts secure that the first member alone approximates the true value as closely as possible. They demonstrate that 'al-Qalaṣādī's canon', even though ensuring that the first member of the expansion is a *fair* approximation, of course does not guarantee it to be *optimal*.
29. Naturally enough, this reminded Sachs of the Egyptian unit fraction system (as also borrowed by the Greeks): Even there, $\frac{2}{3}$ is treated on a par with the sub-multiples $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. He did not make much of the fact that '$\frac{1}{3}$ *of* $\frac{1}{5}$' would be no number to an Egyptian scribe but a problem with the solution '$\frac{1}{15}$'. Nor was he apparently aware that much closer parallels to his notation could be found in the Arabic orbit.
30. MCT, 82. Discussed in Sachs 1946: 212.
31. MCT, 101.
32. MKT I, 490.

33. TMS, 35–49. The tablet has probably been prepared toward the end of the Old Babylonian period.
34. TMS, 36.
35. 1946: 213f. In the present case I take over Sachs's translation, except that I translate *ina* as 'from' instead of 'in'.
36. Thus, N° 2 could have been rearranged as $\frac{1}{5}+\frac{1}{45}+\frac{1}{60}$ or, alternatively, as $\frac{1}{5}+\frac{1}{6}\cdot\frac{1}{5}+\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{5}$; N° 7 either as $\frac{1}{4}+\frac{1}{9}\cdot\frac{1}{4}$ or as $2\cdot\frac{1}{9}+\frac{1}{2}\cdot\frac{1}{9}$. N^{os} 3 and 6 only need reformulation and no rearrangement in order to agree with the pattern of ascending continued fractions.
37. 'May be of interest' but need not, at least as far as the history of *mathematical* ideas and notations is concerned. Indeed, in an article discussing some of the same examples and a number of others Denise Cocquerillat (1965) points out that the expressions are chosen in a way which will make the merchandise look as impressing as possible to a mathematically naive customer. The governing principle may thus have been sales psychology rather than any general idiomatic preference.
38. True enough, as pointed out by a referee, these numbers are no pure numbers: they represent the value of one quantity measured by another – a *ḥekat*-measure gauged by a jug, the toll on a herd of cattle as part of the original herd, the number $2\overline{\frac{2}{3}}$ measured by the number 20. But this is precisely what numbers are mostly used for in daily practice, in Ancient Egypt as elsewhere, and also the way numbers most often occur *within* calculations in Egyptian mathematical texts.
39. Chace et al. 1929, Plate 59. The grammatical construction used is $\frac{1}{3}$ *n* $\frac{1}{3}$, the indirect genitive, which is also used in expressions like $\frac{1}{10}$ *of this* 10 (RMP 28), $\frac{1}{2}+\frac{1}{4}$ *of cubit* (RMP 58), $\frac{1}{3}+\frac{1}{5}$ *of this* 30 (MMP, 3), etc. (here as in all transcriptions of Egyptian unit fraction sums I modernize the writing; the original text merely juxtaposes the denominators with the superscript dot meaning *ro*, 'part'.). This construction should be distinguished from the reverse construction *z n* 5, 'persons until [a total of] 5' discussed by Graefe (1979). We observe that the sequence $\frac{1}{3}$ *and* $\frac{1}{3}$ *of* $\frac{1}{3}$ suggests the idea of ascending continued fractions (as do the successive *medietates* in the related *Propositiones*-problems).
40. *Ibid.*, Plate 67. I have straightened somewhat the opaque language of the extremely literal translation.
41. Ed., transl. Struve 1930: 95.
42. Chace et al. 1929, Plate 83.
43. $\frac{2}{3}$ of $\frac{2}{3}$, $\frac{1}{3}$ of $\frac{2}{3}$, $\frac{2}{3}$ of $\frac{1}{3}$, $\frac{2}{3}$ of $\frac{1}{6}$, $\frac{2}{3}$ of $\frac{1}{2}$, etc. (*loc. cit.*). Peet (1923: 103f) makes a point out of a terminological distinction inside the table, which uses the construction α *of* β in cases where α is $\frac{2}{3}$ or can be obtained from $\frac{2}{3}$ by halving or successive halvings, but a construction β, *its* α ($\beta\ \alpha\ \cdot f$) in other cases. Since some of the formulations have been corrected by the scribe it seems indeed that the distinction is determined by a specific canon (which, as we observe, is broken by the $\frac{1}{5}$ *of* $\frac{2}{3}$ of MMP 20).
44. Chace et al. 1929, Plate 67.
45. What looks like 'parts of parts' and ascending continued fractions in the Indian *śulva sūtras*, e.g. in the passage customarily interpreted as an approximation $\sqrt{2}\approx 1+\frac{1}{3}+\frac{1}{34}-{}^{1}/_{3\cdot 4\cdot 34}$, has the same character, i.e., it is a prescription of a (geometric) procedure and no arithmetical number in itself (see *Baudhāyana śulva sūtra*, ed. Thibaut 1875: II,21). Genuine 'parts of parts' are absent from Indian mathematics (as confirmed to me by Guy Mazars in a private communication).

46. The non-numerical function of the composite expressions is confirmed by the non-observance of the canon deduced by Peet from RMP 61 in RMP 44, 45, 46 and 49, which all speak of $\frac{1}{10}$ *of* $\frac{1}{10}$ (44–46 also have $\frac{2}{3}$ *of* $\frac{1}{10}$ *of* $\frac{1}{10}$).
47. See Høyrup 1990: 66–71.
48. The $\frac{1}{3}$ *of* $\frac{2}{3}$ of MMP 20, it is true, turns up inside the calculation. It looks like a slip, like the reformulation of a description of computational steps (which in the present case would rather give $\frac{2}{3}$ *of* $\frac{1}{3}$) inspired by non-scholarly but familiar idiom.
49. The scribal corrections in RMP 61 would suggest, however, that the canon deduced by Peet may only have emerged after the writing on the original, but before the copy was made.
50. My main basis for this description of Old Kingdom sub-unity arithmetic is the material presented in Sethe 1916.
51. I am indebted to Professor Wolfgang Helck for referring me to the publications on the Abū Sir Papyri. The fractional signs in question are found in Posener-Kriéger & de Cenival 1968: Plates 23–25, cf. translation in Posener-Kriéger 1976 and the discussion in Silberman 1975.
52. Silberman (1975: 249) suggests that the writing be explained as a product of scribal ignorance. In view of the central position occupied in Egyptian arithmetic by doubling and ensuing conversion of fractions this is about as plausible as finding a modern accountant ignorant of the place value system.
53. 1978: 73f.
54. See the comparison of real and 'Rhind' slopes in Reineke 1978: 75 n. 28.
55. This is also plausible from 'a serious [conceptual] confusion [which] has taken place' in the text of N° 60, and which is pointed out and discussed by Peet (1923: 101f).
56. See Brunner 1957: 11–15, and Wilson in Kraeling & Adams 1960: 103.
57. After the mid-eleventh century, the originally separate 'finger-reckoning' and 'Hindu reckoning' traditions merged (cf. Høyrup 1987: 309–11). Al-Qalaṣādī, like Leonardo Fibonacci, would hence combine the two.
58. References in Høyrup 1987: 291.
59. Ms. Columbia X 511 A13, ed. Vogel 1977: 109.
60. See the preface to *Demonstratio de minutiis*, ed. Eneström 1913. Cf. Høyrup 1988: 337f.
61. See the table of shared vocabulary in Diakonoff 1965: 42–49, and other shared vocables mentioned elsewhere in the book. Common property is, *inter alia*, the term *ḥsb*, 'to count', 'to reckon', 'to calculate'.
62. See Moorey 1987 on the 4th millennium, and Klengel 1979: 61–72 on the third.
63. In his book (1965) on the Hamito-Semitic language family, Diakonoff mentions many instances where different languages of the family have developed similar features independently; thus as complex a phenomenon as the *pluralis fractus* (p. 68). We might speak of 'structural causation', the effect of shared linguistic structures determining that specific developments are near at hand and compatible with general linguistic habits.

    'Structural causation', however, need not be linguistic. Non-linguistic instruments for accounting and computation (be they mental or material) may in the same way open the way for specific inventions and block others which are not compatible with existing habits, tools or conceptualizations.

Knowledge of the way fractions are spoken about in other Hamito-Semitic languages might seem to offer a way to distinguish linguistic from non-linguistic causation. However, native and ethnically conscious Berber speakers studying mathematics whom I interviewed in Algeria confessed to speak about fractions in Arabic and to be ignorant of any Berber idiom for fractions.

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"«Algèbre d'*al-ǧabr*» et «algèbre d'arpentage» au neuvième siècle islamique et la question de l'influence babylonienne". Pp. 83–110 *in* Fr. Mawet & Ph. Talon (eds), *D'Imhotep à Copernic. Astronomie et mathématiques des origines orientales au moyen âge.* Actes du Colloque international, Université Libre de Bruxelles, 3–4 novembre 1989. Leuven: Peeters, 1992

«Algèbre d'Al-ǧabr» et «algèbre d'arpentage» au neuvième siècle islamique et la question de l'influence babylonienne

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Dédié à I.M. Diakonoff

I. Al-ǧabr

À la cour du Calife al-Maʾmūn (813-833 P.C.) à Baghdad, toute une foule de mathématiciens étaient à l'oeuvre. Parmi eux figurait al-Khwārizmī (fl. 800-847), qui est connu entre autres choses pour son *Algèbre* (*Kitāb al-ǧabr wa'l-muqābalah*, «Livre sur *al-ǧabr* et *al-muqābalah* »), le premier traité complet sur ce sujet qui nous soit parvenu[1]. Naturellement, son contenu et ses méthodes sont différents de ce que l'on trouve dans les livres modernes, aussi bien les livres enseignant l'algèbre littérale que ceux qui présentent la théorie des groupes et les autres domaines de l'algèbre abstraite. D'autre part, l'algèbre d'al-Khwārizmī est étonnamment proche de ce qui a porté ce nom dans l'Occident latin jusqu'à Pierre de la Ramée.

Pour comprendre ce qu'est l'algèbre pour al-Khwārizmī nous pouvons considérer quelques passages extraits de son livre

> Si une personne te demande ceci: «J'ai divisé *dix* en deux parties, et quand j'ai multiplié l'une par l'autre, *vingt et un* advint»; alors tu sais que l'une des deux parties est *chose* et l'autre *dix moins chose*. Multiplie donc *chose* par *dix moins chose*; tu auras alors *dix choses moins un trésor*, ce qui égale *vingt et un*. Sépare le *trésor* des *dix choses* et ajoute-le au *vingt et un*. Alors tu auras *dix choses*, qui égalent *vingt et un dirhems et un trésor*. Enlève la *moitié des racines* et multiplie le *cinq* qui reste par lui-même; c'est *vingt-cinq*. Enlèves-en le *vingt et un* associé avec le trésor; le reste est *quatre*. Extrais sa racine, c'est *deux*. Enlève-le de la *moitié des racines*, à savoir *cinq*; reste *trois*, qui est une des deux parties. Ou, si tu préfères, tu peux ajouter la *racine de quatre* à la *moitié des racines*. La somme est *sept*, ce qui est aussi une des parties.[2]

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[1] Le sens originel des deux termes n'est pas tout à fait clair, surtout parce que les textes arabes les emploient sans cohérence (voir Saliba 1972). Normalement, *al-ǧabr* signifie «restauration» dans une équation, soit en multipliant par 2 l'équation $1/2\,x^2 + 5\,x = 28$ (al-Khwārizmī, dans Rosen 1831: 10), soit en éliminant les membres soustractifs par addition des deux côtés d'une équation; *al-muqābalah* (réduction) signifie le plus souvent l'élimination d'un membre par soustraction des deux côtés. Au total pourtant, l'usage terminologique varie tellement qu'on peut supposer que ses origines étaient déjà perdues lorsque la tradition écrite arabe a commencé.

[2] Traduction anglaise de Rosen (1831: 41f), collationnée avec et corrigée d'après les traductions latine de Gérard de Crémone (éd. Hughes 1986) et russe de Boris Rozenfeld (en Siraždinov 1983), plus littérales toutes les deux; mon arabe plus que rudimentaire ne m'a permis, ici et dans les citations qui suivront plus bas, que des contrôles spécifiques (faits sur le texte arabe donné par Rosen et utilisé aussi par Rozenfeld) de certains passages où les traductions divergeaient.

Comme toutes les versions françaises qui suivent, celle-ci est faite par l'auteur de l'article; l'idéal poursuivi est une littéralité compréhensible plutôt que le style poli.

Les caractères italiques sont ajoutés pour faciliter la compréhension et la comparaison avec la notation symbolique.

Pour suivre l'argument, il faut savoir que la *chose* (*šay'*) occupe le même rôle qu'un *x* moderne; que le *trésor* (*māl*) est le carré de l'inconnue et que la *racine* (*jidhr*) est la racine de ce carré (ou plutôt: le *trésor* est *un nombre carré inconnu*, tandis que la *racine* est la racine de ce nombre); ici, la *chose* égale la *racine*. Le *dirhem* est une unité monétaire, qui sert comme unité des nombres purs (ce qui évidemment correspond bien à l'identification du nombre inconnu avec »un trésor«). La »moitié des racines«, enfin, est à comprendre comme la moitié du *coefficient* ou du *nombre* des racines (on rencontre souvent cette même manière de s'exprimer dans les sources arabes). Avec ces explications, la première section de la procédure se laisse facilement traduire en symboles modernes:

$$x\cdot(10-x) = 21 \quad => \quad 10x-x^2 = 21 \quad => \quad 10x = 21+x^2$$

ou bien, avec $Y = x^2$

$$\sqrt{Y}\cdot(10-\sqrt{Y}) = 21 \quad => \quad 10\sqrt{Y} - Y = 21 \quad => \quad 10\sqrt{Y} = 21+Y$$

Ce qui se passe après est d'un style différent et correspond plutôt à une solution suivant une formule fixe:

$$ax = b+x^2 \quad => \quad x = \frac{a}{2} \pm \sqrt{[(\frac{a}{2})^2-b]}$$

En d'autres mots, la dernière partie de la procédure fait usage d'un *algorithme standardisé*.

De fait, cet algorithme a déjà été exposé dans un chapitre précédent, où l'on trouve ceci:

> *trésor et nombre égalent racines*: c'est comme si tu dis, «un trésor et vingt et un en nombres égalent dix racines du même trésor». C'est-à-dire, quel sera le montant du trésor qui, quand on y ajoute vingt et un dirhems, égale l'équivalent de dix racines du même trésor? Solution: Divise en deux les racines; la moitié est cinq. Multiplie-le par lui-même; il en advient vingt-cinq. Enlèves-en le vingt et un associé avec le trésor; le reste est quatre. Extrais sa racine, c'est deux. Enlève-le de la moitié des racines, qui est cinq; reste trois. Ceci est la racine du trésor que tu demandais et le trésor est neuf. Ou tu peux ajouter la racine à la moitié des racines; ce sera sept; c'est la racine du trésor que tu demandais et le trésor lui-même est quarante-neuf.
> Quand tu rencontres un exemple qui te conduit à ce cas-ci, essaie la solution par addition, et si cela n'aide pas, la soustraction servira certainement. Parce que dans ce cas addition aussi bien que soustraction peut être employée, ce qui ne vaut aucun autre des trois cas où il faut diviser en deux les racines.[3]

Les trois «cas» en question sont les équations du deuxième degré à trois membres, nommées le «quatrième», «cinquième» et «sixième cas». «Trésor et nombre égalent racines» en est le cinquième. Le quatrième est décrit de cette manière:

> *Racines et trésor égalent nombre;* c'est comme si tu dis, «un trésor et dix racines du même, égalent trente-neuf dirhems»; c'est-à-dire, quel sera le trésor qui, quand on l'augmente de dix de ses propres racines, se monte à trente-neuf?

[3] Traduction anglaise Rosen 1831: 11f, collationnée avec et corrigée d'après les traductions latine et russe.

> La solution est celle-ci: Tu divises en deux les racines, ce qui dans la question présente donne cinq. Tu multiplies ceci par lui-même; ce sera vingt-cinq. Ajoute ceci à trente-neuf; la somme est soixante-quatre. Prends-en maintenant la racine, qui est huit, et enlèves-en la moitié des racines, qui est cinq; reste trois. C'est la racine du trésor que tu cherchais; le trésor lui-même est neuf.[4]

Le dernier cas «où il faut diviser en deux les racines» (le sixième cas) est évidemment «racines et nombre égalent trésor», qui possède son propre algorithme[5]. Ces trois cas complexes sont précédés de trois cas plus simples, à savoir «trésor égale des racines», «trésor égale des nombres» et «racines égalent nombre», où l'on ne trouve pas d'algorithme proprement dit, mais où les solutions données sont regardées comme (et sont en effet) intuitivement évidentes.

Après avoir formulé et illustré les divers algorithmes pour résoudre les trois équations complexes, al-Khwārizmī donne enfin des preuves géométriques que ses algorithmes sont corrects. Le cas le plus simple est celui où «un trésor et dix racines égalent trente-neuf dirhems»:

> La figure pour expliquer ceci est un carré[6], dont les côtes sont inconnus. Il représente le trésor, lequel, et la racine duquel, tu demandes à connaître. Ceci est la surface AB, dont chaque côté peut être considéré comme une de ses racines; et si tu multiplies un de ces côtés par un nombre quelconque, alors le montant de ce nombre peut être regardé comme le nombre de racines qui est ajouté au trésor. Chaque côté du carré représente la racine du trésor; et, comme dans ce cas, dix racines étaient associées avec le trésor, nous pouvons prendre un quart des dix, à savoir deux et demi, et faire de chaque quart ensemble avec un des côtés de la surface une surface. Donc, avec la surface originale AB, quatre nouvelles surfaces égales sont combinées, chacune ayant une racine comme longueur, et deux et demi comme largeur; ce sont les surfaces C, G, T et K. Nous avons maintenant une surface à côtés égaux et également inconnus, mais à laquelle il manque dans chacun des quatre coins une pièce de deux et demi multiplié par deux et demi. Pour compenser ce défaut et compléter la surface carrée il faut ajouter quatre fois deux et demi multiplié par lui-même, c'est-à-dire, vingt-cinq. Nous savons que la première surface, à savoir, la surface représentant le trésor, en même temps que les quatre surfaces qui l'entourent et qui représentent les dix racines, égalent trente-neuf en nombres. Si à cela nous ajoutons vingt-cinq, ce qui est l'équivalent des quatre carrés aux coins de la surface AB par lesquels la grande surface DH est complétée, alors nous savons que cela fait ensemble soixante-quatre, et qu'un de ses côtés est sa racine, c'est-à-dire huit. Si nous enlevons deux fois un quart de dix, c'est-à-dire cinq, de huit, comme des deux extrémités du côté de la grande surface, c'est-à-dire la surface DH, alors le reste d'un tel côté sera trois; ceci est le côté de la surface originale AB ou la racine du trésor. Il faut observer que nous n'avons divisé les dix racines et ajouté à trente-neuf le produit de la moitié multipliée par elle-même que pour compléter la grande figure dans ses quatre coins; parce qu'un quart d'un nombre quelconque multiplié par lui-même et après par quatre, égale le produit de la moitié de ce nombre multipliée par elle-même. Pour cette raison nous avons seulement multiplié la

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[4] Traduction anglaise Rosen 1831: 8, collationnée avec et corrigée d'après les traductions latine et russe.

[5] Puisqu'al-Khwārizmī ne considère que les nombres positifs, les seules équations normalisées à trois membres possédant des solutions sont $x^2 + ax = b$, $x^2 + b = ax$ et $x^2 = ax + b$.

[6] Le mot arabe est *murabba'*, littéralement «[figure] à quatre [côtés]»; quand les Arabes ont assimilé la géométrie grecque, ils l'ont employé en particulier au sens de «quadrilatère régulier» ou «carré».

moitié des racines par elle-même, au lieu de multiplier le quart par lui-même et après par quatre. Ceci est la figure:

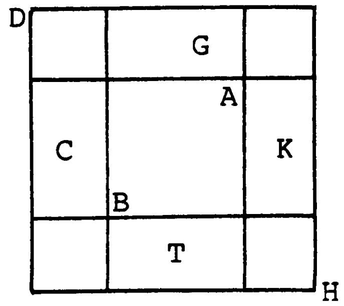

Figure 1

Il y a une autre figure qui conduit à la même chose. C'est la surface AB, qui représente le trésor. Nous voulons donc lui ajouter ses dix racines. Pour ce faire nous divisons les dix en deux, ce qui devient cinq, et nous construisons deux surfaces sur deux côtés d'AB, à savoir les surfaces G et D, dont les longueurs égalent cinq, ce qui est la moitié des dix racines, tandis que la largeur de chacun d'eux égale le côté du carré AB. Alors cinq sur cinq nous manque opposé au coin de AB: ce cinq étant cette moitié des dix racines que nous avons ajoutées à deux des côtés de la première surface. Nous savons donc que la première surface, qui est le trésor, et les deux surfaces sur ses côtés, qui sont les dix racines, font ensemble trente-neuf. Pour compléter la grande surface en carré, seul cinq sur cinq fait défaut, ou vingt-cinq. Nous ajoutons ceci à trente-neuf, pour compléter la grande surface SH. La somme est soixante-quatre. Nous extrayons sa racine, huit, qui est un des côtés de la grande surface. En lui enlevant la même quantité que nous lui avons ajoutée antérieurement, à savoir cinq, nous obtenons trois comme reste. Ceci est le côté de la surface AB, qui représente le trésor; c'est la racine de ce trésor et le trésor lui-même est neuf. Ceci est la figure: [7]

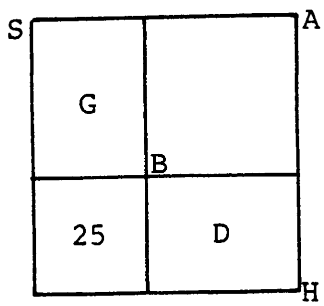

Figure 2

[7] Traduction anglaise Rosen 1831: 13-16, collationnée avec et corrigée d'après les traductions latine et russe.

Nous avons vu ici les trois composantes de l'algèbre al-Khwārizmien: les problèmes complexes sont réduits aux «cas» fondamentaux moyennant des techniques dites «rhétoriciennes», c'est-à-dire des transformations analogues à nos transformations symboliques, mais utilisant des noms, des mots et des phrases complètes. Comme c'est le cas pour les transformations symboliques les procédures rhétoriciennes sont (intuitivement) leurs propres preuves. Les équations fondamentales sont résolues au moyen d'algorithmes fixes, eux-mêmes proposés sans aucun argument en dépit de leur impénétrabilité pour l'intuition immédiate. Enfin, des démonstrations géométriques de la justesse des algorithmes sont données.

Plusieurs observations peuvent êtres faites sur les textes cités. D'abord, les démonstrations géométriques ne concernent en vérité que des exemples spécifiques, bien que la généralisation aux cas parallèles est évidente. De plus, elles sont «naïves»: Si on les compare aux démonstrations analogues dans le livre II des *Éléments* d'Euclide, on remarque qu'il n'y a pas de doute pour al-Khwārizmī que, par exemple, les figures qui font défaut dans les coins des grands carrés sont, elles aussi, des carrés; tout ce qui «se voit» immédiatement est accepté. Deux signes seulement d'inspiration grecque sont présents: l'utilisation de lettres pour désigner les figures et l'emploi de la première personne pluriel au lieu des formes verbales prescriptives.

Deuxièmement, il faut remarquer que seules les démonstrations sont géométriques. Le «trésor» est *un nombre* et sa racine de même[8]. Apparemment, al-Khwārizmī prend grand soin d'expliquer encore et encore que le carré des démonstrations *représente* le trésor; évidemment les deux appartiennent à des catégories tout à fait différentes. De même, les «choses» et «racines» désignent des nombres et les transformations rhétoriciennes sont d'ordre purement arithmétique. Après une introduction dédicatoire au Calife, en fait, l'oeuvre commence avec l'explication suivante:

> Quand je considérais ce dont les gens ont normalement besoin dans le domaine du calcul, je trouvai que c'était toujours un nombre.
> [...]
> J'observai que les nombres qui sont nécessaires pour calculer par *al-ğabr wa'l-muqābalah* sont de trois sortes, à savoir, racines, trésors et nombres simples sans regard ni à racine ni à trésor.

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[8] Ceci n'est pas tout à fait clair dans la plupart des traductions courantes. Rosen, par exemple, traduit *māl* par «square» et *murabba'* par «quadrate», ce qui peut aisément conduire à des interprétations géométriques mal fondées du premier terme.

Le sens numérique et quasi-monétaire du terme *māl* correspond parfaitement à son emploi dans les problèmes du premier degré, du type «Si tu ajoutes à un trésor sa moitié; si tu ajoutes un quart du résultat; si enfin tu enlèves un dixième du total: alors il te reste 20 dirhems. Quel est le montant du trésor?» (al-Karajī, *Kāfī fī'l-ḥisāb* lxx, traduction allemande Hochheim 1878: III, 14). Chez certains historiens des mathématiques cela a conduit à des confusions amusantes. Ainsi, connaissant mieux l'algèbre médiévale que les problèmes sur les transactions commerciales, Libri (1838: I, 304ff) a trouvé le terme *census* (la traduction établie du *māl*) dans un *Liber augmenti et diminutionis* ne contenant que des problèmes du premier degré résolus au moyen de la méthode des «fausses positions»: il considère la terminologie «confondue», mais traduit systématiquement le montant monétaire inconnu comme x^2, bien que le plus souvent cela conduira à un x irrationnel (et que, bien sûr, le x supposé n'est ni demandé ni jamais trouvé).

On peut remarquer que le terme arabe *māl* correspond au terme θησαυρός employé dans certains problèmes gréco-égyptiens du premier degré (Papyrus Akhmîm, éd. Baillet 1892: 86ff).

> Une racine est une quantité quelconque qui sera multipliée par elle-même, consistant en unités ou nombres ascendants ou fractions descendantes.
> Un trésor est le montant total d'une racine multipliée par elle-même.
> Un nombre simple est un nombre quelconque qui peut être prononcé sans référence à racine ou trésor.[9]

Le traité d'al-Khwārizmī est le premier à nous être parvenu dans toute son étendue. Mais al-Khwārizmī n'a pas été le seul à son époque à écrire sur le sujet. En effet, un certain ibn Turk, plus ou moins son contemporain, a écrit un traité du même titre, dont un seul chapitre a survécu[10], contenant des démonstrations très proches de celles d'al-Khwārizmī, mais apparemment indépendantes des siennes (d'après des critères terminologiques). Les formulations sont un peu plus raffinées, mais les démonstrations sont au fond du même caractère «naïf» que celles d'al-Khwārizmī. Les différences les plus grandes sont peut-être, premièrement, qu'ibn Turk donne une démonstration géométrique pour le cas simple «trésors égalent des racines»; deuxièmement qu'il ne donne pas la première des démonstrations du cas «trésors et racines égalent nombre» (celui utilisant quatre rectangles supplémentaires); troisièmement que ses cas mixtes parlent toujours d'un seul trésor, tandis qu'al-Khwārizmī en parle au pluriel (mais présuppose, avant de donner l'algorithme, que l'équation aura été normalisée).

Un demi-siècle peut-être après al-Khwārizmī, Thâbit ibn Qurra (c. 834 à 900) a écrit un petit traité avec des démonstrations géométriques sur la justesse des méthodes des «gens d'algèbre»[11]. Une fois encore, il s'agit des algorithmes utilisés pour résoudre les équations mixtes normalisées. Thābit parle des cas fondamentaux comme le fait ibn Turk, avec un seul trésor. Ses preuves sont euclidiennes et consistent, en fait, en réductions aux théorèmes II.5 et II.6 des *Éléments*. Il ne dit mot de l'existence de démonstrations géométriques d'un autre style – évidemment, *al-ǧabr* pour Thābit (le nom qu'il utilise pour toute la discipline) ne s'identifie pas avec le livre d'al-Khwārizmī; c'est une technique appartenant à tout un groupe de praticiens des mathématiques. Ce qui les caractérise, c'est l'application des *algorithmes fixes*, et c'est à ces algorithmes qu'il trouve des justifications euclidiennes.

Al-Khwārizmī lui-même, en fait, nous informe qu'il n'a pas inventé la discipline. Dans l'introduction dédicatoire il raconte que le Calife l'a encouragé à

> «composer un bref traité sur le calcul par *al-ǧabr* et *al-muqābalah*, réduit à ce qui est brillant et d'importance dans les arithmétiques utilisées constamment dans les affaires d'héritages et les legs, dans les partages et les procès, dans le commerce et dans toutes leurs affaires d'arpentage des terres, de creusement de canaux, de calculs géométriques et d'autres choses variées de pareille sorte»[12].

Ce qui est «d'importance dans les arithmétiques utilisées [...]» est traité dans les deux derniers tiers du traité: La règle de trois, l'arpentage et le calcul des héritages (fait par algèbre rhétoricienne du premier degré). Le premier tiers, l'algèbre de deuxième degré, est inutilisable et doit donc appartenir à la catégorie du «brillant» ou «agréable» (*laṭīf*). En tout cas, les deux catégories existent déjà, la «brillante» non moins que l'utilitaire, et al-Khwārizmī n'a fait «que» produire une oeuvre de syn-

[9] Traduction anglaise Rosen 1831: 5, collationnée avec et corrigée d'après les traductions latine et russe.

[10] Éd., trad. Sayili 1962.

[11] Éd., trad. Luckey 1941.

[12] Traduction anglaise Rosen 1831: 3, collationnée avec la traduction russe (l'introduction est omise par Gérard de Crémone) et avec corrections inspirées par Ruska 1917: 5.

thèse des disciplines et techniques des calculateurs pratiques, y compris la superstructure «pure».

Cette dernière expression nécessite une explication. La sagesse conventionnelle distingue les mathématiques «appliquées» ou «pratiques», aujourd'hui le domaine des ingénieurs et des comptables et avant l'ère moderne de divers «praticiens mathématiques», et les mathématiques «pures» ou «scientifiques» inventées par les Grecs et le domaine des vrais mathématiciens. Parler d'une superstructure «pure» appartenant aux praticiens peut donc surprendre. En fait, pourtant, tout groupe professionnel de calculateurs tend à produire une telle superstructure: des problèmes qui ressemblent formellement aux problèmes rencontrés dans la vie professionnelle, mais qui demandent plus d'ingéniosité que les tâches de tous les jours. Cette superstructure remplit toute une gamme de fonctions: l'entraînement des apprentis; l'amusement et la récréation; enfin, l'affermissement de l'identité et la solidarité professionnelle.

L'importance de l'amusement dans l'enseignement mathématique est bien connus de tous les pédagogues qui ne croient pas, comme les anciens maîtres d'école égyptiens, que «l'écolier écoute le mieux avec son dos». Le rôle de la récréation pour l'identité professionnelle peut paraître plus surprenante. Néanmoins, on en parle parfois dans les sources; les formules peuvent être «dis-moi, si tu es un calculateur accompli, combien [...]», indiquant que le problème est une énigme qui ne peut être résolue que par les membres (les vrais membres!) de la profession; ou bien «comment trouver, à l'ébahissement des non-initiés, [...]». Normalement, bien sûr, on ne trouve pas de telles explications dans les anciennes collections de problèmes de récréation. Les milieux de praticiens, en effet, ne nous ont pas laissé des écrits; leur enseignement et leurs amusements professionnels étaient, sinon oraux, du moins supportés par des écrits éphémères. Les recueils ont été faits par des mathématiciens, des savants ou des antiquistes et ont à peu près le même rapport avec leur origine professionnelle que les nouvelles de Boccaccio, les contes de Perrault et les fables de la Fontaine avec les traditions orales populaires d'où ils prennent une partie appréciable de leur matière.

Dans quelques cas rares, la situation a été différente. Dans la Babylonie de l'époque de Hammurapi et dans l'Égypte du deuxième millénaire avant J.C., l'éducation des *scribes* (pour qui le calcul était non moins important que l'écriture) était du ressort d'une école bien organisée et institutionnalisée. Là, l'écriture était employée pour fixer et systématiser les méthodes de calcul mathématique. Dans ce cas, pourtant, comme dans celui des «énigmes pour calculateurs» des traditions orales, les problèmes sont sélectionnés ou construits afin de faire employer les méthodes existantes; c'est que le but des problèmes est ou de *démontrer ce qu'on peut faire avec le stock de méthodes possédées par la profession* ou d'exercer ces méthodes chez les apprentis ou écoliers. Donc, les méthodes ont la primauté et les problèmes en sont dérivés.

Dans les mathématiques «scientifiques», c'est-à-dire de tradition grecque, ces rôles respectifs des méthodes et des problèmes sont inversés. Ici (comme naturellement dans la *vraie pratique* des calculateurs professionnels) les problèmes sont primaires et, si les méthodes permettant leur solution n'existent pas déjà, il faut les inventer. Pour cette raison, j'ai essayé[13] d'introduire le terme «sous-scientifique» pour le savoir des professions pratiques pré-modernes, comme par exemple des calculateurs: un terme qui le distingue du savoir comme but en soi-même de la philosophie et des mathématiques grecques, mais qui en même temps indique qu'il n'est pas restreint à une collection de recettes transmises sans compréhension ni ambition intellectuelle.

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[13] Par exemple dans mes [1987] et [1990a].

L'algèbre du deuxième degré est donc le niveau pur et inutilisable du calcul pratique de l'entourage d'al-Khwārizmī, nécessitant un haut niveau d'ingéniosité en calcul pratique; en bref le niveau «brillant». Apparemment, c'est ce niveau seulement qui est désigné *al-ǧabr*, à en croire l'introduction, selon laquelle *al-ǧabr wa'l-muqābalah* s'occupe de «trésors», «racines» et «nombres simples» (une autre source citée plus bas pourrait pourtant indiquer que l'algèbre rhétoricienne de la «chose» y appartient aussi). En combinant les témoignages d'al-Khwārizmī, d'ibn Turk et de Thābit ibn Qurra nous pouvons conclure que cette discipline reposait sur une tradition; qu'elle comprenait les réductions rhétoriciennes aux cas fondamentaux et la solution de ces cas au moyen d'algorithmes fixes; finalement, que les démonstrations géométriques de la justesse de ces algorithmes n'y appartenaient pas.

D'autre part pourtant, les démonstrations naïves furent traditionnelles elles aussi. Comment expliquer sans cela que les mêmes démonstrations furent utilisées indépendamment par al-Khwārizmī et ibn Turk, en dépit de leur déviation des normes euclidiennes bien connues d'eux – l'usage de lettres pour désigner les parties des figures géométriques est une importation grecque et des collègues d'al-Khwārizmī à la cour du Calife ont traduit et commenté les *Éléments* et d'autres oeuvres grecques. De plus, al-Khwārizmī nous raconte indirectement qu'il n'a pas inventé ces preuves géométriques lui-même. Après la présentation des cas fondamentaux et des algorithmes correspondants et après les démonstrations viennent des explications d'un nombre de règles arithmétiques, parmi elles les réductions $(20-\sqrt{200})+(\sqrt{200}-10)=10$, $(20-\sqrt{200})-(\sqrt{200}-10)=30-2\sqrt{200}$ et $(50+10r-2t)+(100+t-20r)=150-t-10r$ (r=racine, t=trésor). Pour les deux premiers, des démonstrations géométriques sont données, d'un style assez différent des démonstrations montrées plus haut (les nombres sont représentés par des segments coupés sur des droites); pour le troisième cas, al-Khwārizmī explique,

> l'on ne peut construire aucune figure, parce qu'il y a trois différentes espèces, à voir trésors, racines et nombres, et rien qui y correspond et par quoi ils pourraient être représentés. En effet nous avions imaginé une figure pour ce cas aussi, mais elle n'était pas suffisamment claire.
> L'explication par des mots est très simple [...][14]

et donnée par les techniques rhétoriciennes coutumières. En ce qui concerne ces démonstrations géométriques-ci, al-Khwārizmī les a donc construites lui-même, tandis que les précédentes sont présentées comme existant déjà.

II. Le *Liber mensurationum*

Bien sûr, des arguments qui reposent sur la tournure précise des phrases conduisent aisément à l'erreur. Sans autre témoignage, la préexistence des techniques de géométrie naïve resterait une hypothèse. Heureusement, il existe encore un traité de mathématique qui éclaircit la question de façon inattendue.

Malheureusement, pourtant, le traité en question n'est connu qu'en traduction latine[15] – une traduction faite par Gérard de Crémone dans la deuxième

[14] Traduction anglaise Rosen 1831: 34, collationnée avec et corrigée d'après les traductions latine et russe.

[15] Édition critique par Busard (1968). Il se peut qu'il existe des manuscrits arabes non publiés. Roshdi Rashed (communication personnelle faite loin des sources) pense en avoir vus

moitié du douzième siècle et, semble-t-il, très littérale, comme le sont d'habitude ses traductions.

L'auteur du traité est un certain Abū Bakr; mais, du fait qu'il en existe partout dans le monde islamique et à toutes les époques, cela ne nous aide guère pour situer le traité dans le temps ou dans l'espace. D'autre part, les traducteurs latins n'ont trouvé que très peu de travaux récents en Espagne au douzième siècle; pour cette raison, il semble raisonnable de supposer une origine au neuvième ou dixième siècle. Des considérations de vocabulaire[16] suggèrent une date pas trop différente de celles d'al-Khwārizmī et d'ibn Turk; la même chose vaut pour le contenu mathématique.

Le titre latin du traité est *Liber mensurationum*, «Traité d'arpentage». Il est composé de deux parties, dont la deuxième (propositions 65-158) est précisément ce que promet le titre, un traité dans la tradition Héronienne sur la mesure des champs et des corps – plus précisément, sur la détermination de l'aire et du volume à partir des dimensions linéaires. La première partie, pourtant, celle qui nous occupera ici, est tout à fait différente. Un premier chapitre (propositions 1-19) est consacré aux «quadrilatères équilatéraux et à angles droits». Des 19 propositions, les deux premières seulement concernent réellement l'arpentage (la détermination de l'aire et de la diagonale à partir du côté), tandis que les numéros 10 et 11 peuvent à la rigueur être acceptés. Toutes les autres sont artificielles et appartiennent au domaine du «brillant». En abréviation symbolique (où *c* est le côté, *A* l'aire et *d* la diagonale) les données sont les suivantes:

3.	$c + A = 110$
4.	$4c + A = 140$
5.	$A - 4c = 60$
7.	$4c = \frac{2}{5}A$
8.	$4c = A$
9.	$4c - A = 3$
10.	$d = \sqrt{200};\ c = ?$
11.	$d = \sqrt{200};\ A = ?$
12.	$4c + A = 60$
13.	$A - 3c = 18$
14.	$4c = \frac{3}{8}A$ (texte corrompu)
15.	$A/d = 7\frac{1}{2}$
16.	$d - c = 4$
17.	$d - c = 5$

et pense se souvenir de figures analogues à celles (manquantes dans la version latine) dont il sera question plus bas.

Une étude assez approfondie du traité se trouve dans mon [1986], qui est le fondement de bien des éléments de la présente étude.

[16] Bien sûr, «Wer terminologiegeschichtliche Studien an Hand einer *Übersetzung* machen will, dem ist doch nicht zu helfen» («qui veut étudier l'histoire des terminologies par des traductions ne peut de toute façon être sauvé»), comme dit Neugebauer (MKT III, 5). Mais la traduction de Gérard est assez précise pour nous assurer qu'Abū Bakr emploie encore, même en géométrie, le mot *murabbaʿ* dans le sens général de «quadrilatère» seulement et qu'il parle d'un carré comme d'un *murabbaʿ équilatéral et à angles droits* (Gérard a *quadratum equilaterum et orthogonium*), précisément comme le fait ibn Turk.

18. $d = c + 4$

19. $A/d = 7\frac{1}{14}$

D'un point de vue moderne, les problèmes artificiels sont des problèmes algébriques. Pour voir comment Abū Bakr les considérait, nous pouvons regarder son traitement du numéro 3, analogue au premier cas mixte d'al-Khwārizmī:

> Si quelqu'un t'aura dit: J'ai additionné le côté et l'aire et ce qui en advint était 110[17]. Combien est donc son côté?
> La méthode dans ceci sera que tu prends la moitié du côté comme moitié et que tu la multiplies avec elle-même, dont advient $\frac{1}{4}$, ce que tu ajoutes à 110. Ce sera $110\frac{1}{4}$, dont tu prends la racine, qui est $10\frac{1}{2}$, de laquelle tu enlèves la moitié et 10 resteront qui est le côté. Voici!
> Il y a pour cela aussi un autre mode selon *al-ǧabr* qui est que tu poses le côté comme chose et multiplies celle-ci avec elle-même et ce qui en advient sera le trésor qui sera l'aire. Ajoute-le donc au côté selon ce que tu as posé et ce qui en advient sera un trésor et une chose qui égalent 110. Agis donc selon ce qu'on t'a appris en *al-ǧabr*[18], c'est-à-dire que tu divises en deux la chose et tu la multiplies avec elle-même et ce qui en advient tu l'ajoutes à 110; tu prends la racine de ce que tu as additionné et tu l'enlèves de la moitié des racines. Ce qu'il en reste alors sera le côté.[19]

Le problème peut donc être résolu selon deux méthodes proclamées différentes, dont la dernière est identique à *al-ǧabr* d'al-Khwārizmī, parlant de trésor et choses et utilisant l'algorithme familier du «quatrième cas»[20]. Comme dans le passage cité d'al-Khwārizmī, aussi, Abū Bakr explique soigneusement que le trésor représente l'aire du carré. Paradoxalement, pourtant, les pas numériques sont les mêmes selon les deux méthodes. Ce qui manque à la première n'est apparemment que les mots-clefs; d'autre part, elle a encore le mystérieux «voici».

Le mot latin est «intellige». La traduction «voici» peut paraître mal fondée, mais s'explique par une autre traduction Gérardienne. Dans un fragment sur la construction de l'heptagone, la dernière partie critique une méthode approximative utilisée par les Indiens; en conclusion, le texte nous raconte que les Indiens ne possèdent comme démonstration de la validité de leur méthode que «l'invention *intellige ergo*»[21]. Cela, pourtant, correspond précisément à la manière répandue chez les mathématiciens (ou les commentateurs) indiens de conclure la présentation d'une règle ou d'un algorithme par un exemple d'application, souvent en guise de figure, introduit par le mot

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[17] Abū Bakr a probablement écrit ses nombres en toutes lettres. Mais puisque sur ce point précis Gérard n'est pas littéral (ou, plutôt, utilise un manuscrit qui est corrompu à cet égard), mélangeant des nombres verbaux, les chiffres «hindous» et les chiffres romains sans aucun système, je traduirai tout en chiffres hindous.

[18] D'autres propositions emploient même des formules du style «fais donc selon ce qui précède sur le quatrième cas d'*al-ǧabr*». Le présent traité est donc écrit comme pièce-compagne à une introduction à *al-ǧabr* similaire à celle d'al-Khwārizmī (mais probablement pas identique à celle-là, puisque le mot *muqābalah* semble être employé de manière différente, cf. note 24).

[19] Éd. Busard 1968: 87.

[20] Gérard a dû remarquer qu'Abū Bakr utilise le terme de manière très spécifique; ceci doit être la raison pour laquelle il le traduit *aliabra* et non, comme d'habitude (et comme il le fait lui-même en d'autres occasions) *algebra* – cf. Hughes (éd.) 1986: 233.

[21] Éd. Clagett 1984: 599.

nyāsa, littéralement «on pose», «on écrit», «on trace» – en effet, opérations et figures étaient tracées sur le sol[22]; cette application ou figure fonctionne alors comme démonstration heuristique ou explication. Du moins dans ce fragment Gérardien, la traduction «voici» semble dont adéquate, puisque le mot arabe employé par Abū Bakr a dû couvrir l'utilisation dans les textes mathématiques du mot *nyāsa*, mais en même temps a pu suggérer le mot latin *intellige*, ce qu'une traduction plus littérale du sanscrit à l'arabe n'aurait pas fait.

Naturellement, dans le *Liber mensurationum*, on pourrait encore lire la remarque simplement comme «comprends!». On se demanderait pourtant alors ce qu'il y aurait à comprendre: comme il se présente à nous, le texte ne décrit apparemment qu'une séquence de pas de calcul ou une prescription d'un algorithme aussi peu compréhensible que les algorithmes présentés par al-Khwārizmī. Finalement, «intellige» revient souvent dans les autres problèmes, mais toujours après la description de la première méthode et jamais dans la partie «selon *al-ğabr*»[23]. Il paraît ainsi naturel de comprendre l'*intellige* comme dans le texte parallèle et de conclure que le traité original a comporté des figures là où la traduction (et, il faut croire, le manuscrit utilisé par Gérard) n'en conserve que les références dans le vide (des exemples numériques comme on en trouve dans les textes indiens sont exclus, puisque le texte en consiste déjà; de toute façon, ce qui est suggéré est la présence du même mot *arabe* dans le texte d'Abū Bakr – il n'y a aucune raison de croire que ce mot n'apparaîtrait que dans des textes dérivés du sanscrit ou discutant les méthodes indiennes). Ces figures appartiendraient alors à la première méthode; le support géométrique serait ce qui distinguerait cette méthode de la méthode numérique d'*al-ğabr*.

La troisième proposition n'est donc pas la seule à comporter deux différentes solutions, qui parfois sont numériquement identiques (comme dans la cas cité) et parfois diffèrent. Pour illustrer cette dernière possibilité nous pouvons considérer deux exemples pris dans le chapitre sur les «quadrilatères plus longs d'un côté». D'abord la proposition 29:

> Et s'il t'aura dit: J'ai additionné les deux côtés et ce qui en advint fut 14 et un côté excède l'autre par 2; combien est donc chaque côté?
> Le mode pour trouver ceci sera que tu ajoutes 2 au 14 et ce qui en advient sera 16, dont tu prends la moitié, qui est 8; c'est le côté long; tandis que si tu veux le plus court, tu le trouveras par cette méthode: Enlève 2 de 14 et prend la moitié [...] et ce sera le côté court.
> D'autre part, le mode selon *al-ğabr* est que tu poses le côté court comme chose et le long comme chose et 2 parce que sa parole fut «excède l'autre par 2», addi-

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[22] Guy Mazars, communication personnelle. Des exemples concernant précisément la construction de polygones réguliers se trouvent dans la *Līlāvatī* de Bhāskara (édition sanscrite Banerji 1893: appendice, 87-91; dans sa traduction anglaise (1817: 94f), Colebrooke traduit simplement comme «see»). D'autres exemples concernant précisément l'interprétation géométrique des identités algébriques se trouvent dans le *Bīğagaṇita* (trad. Colebrooke 1817: 222-225).

[23] A la fin de la proposition 50 on pourrait soupçonner une exception à cette règle. En fait, elle finit par la phrase «intellige ergo et invenies», «comprends donc et tu trouveras». Mais si on regarde ce qui se passe dans le calcul, on découvre que tout dans cette proposition est pure parodie. La proposition 49 bâtit déjà sur le fait que tout triangle rectangle où les côtés forment une série arithmétique est proportionnel au triangle 3-4-5. La proposition 50 nous raconte que dans un tel triangle l'hypoténuse égale 10 et tout devrait être clair. Au lieu de cela, pourtant, viennent des calculs absolument fous et délibérément opaques; vraiment, «comprenne qui peut!»

tionne-les donc et oppose-les à 14[24]. Ce qui en revient [après la solution de l'équation] sera le côté court. Enlève-le donc de 14 et il restera le long.[25]

Avant de continuer on peut observer que ce problème est du premier degré; puisqu'il possède une solution «selon *al-ǧabr*», il semble que cette discipline ne soit pas restreinte au deuxième degré: *al-ǧabr* est tout ce qui se traite par *choses, trésors et racines.*

Comme deuxième exemple, nous pouvons citer la proposition 43:

> Mais s'il t'aura dit: : J'ai additionné ses 4 côtés et son aire et ce qui en advint fut 76 et l'un des côtés excède l'autre par 2; combien est donc chaque côté?
> Le mode pour trouver cela sera que tu multiplies ce dont l'un côté augmente l'autre, toujours [c'est-à-dire indépendamment de la valeur de l'augmentation] par 2 et ce qui en advient sera 4. Enlève ceci de 76 et 72 resteront. Additionne le nombre des côtés du quadrangle, qui est 4, et ajoutes-y l'augmentation d'un côté sur l'autre et ce qui en advient sera 6. Prends-en la moitié qui est 3 et multiplie-le avec lui-même, dont advient 9. Ajoute-le à 72, dont advient 81. Prends la racine de cela, qui est 9, et enlèves-en la moitié des 6 qui est 3 et il restera le côté court qui est 6. Ajoute-lui 2 et le côté long sera 8. Voici!
> Pourtant, le mode pour trouver la même chose selon *al-ǧabr* est que tu poses le côté court comme chose. Alors le côté long sera chose et 2: multiplie donc la chose avec la chose et 2 et l'aire sera un trésor et 2 choses. Après, additionne les côtés du quadrilatère, qui sont 4 et 4 choses; ajoute-les au trésor et 2 choses et ce qui en advient sera un trésor et 6 choses et 4, qui égaleront 76; enlève donc 4 de 76 et 72 resteront qui égaleront un trésor et 6 choses. Fais donc selon ce qui précède sur le quatrième cas d'*al-ǧabr*.[26]

La solution selon *al-ǧabr* est facile à suivre et illumine bien les raisons du succès de cette technique. D'autre part, les pas numériques de la première méthode sont assez incompréhensibles (en dépit du fameux *intellige*); ce qui s'y passe demande une clarification, par exemple par algèbre symbolique (le côté long est désigné x, le côté court y):

$$xy+2x+2y = 76\ ,\ x-y = d\ ,\ d = 2$$

Remplaçant donc x par $y+2$ nous trouvons

$$xy+4y+2d = xy+4y+4 = 76$$

ou

$$xy+4y = 76-4 = 72$$

ou encore

$$(x+4)y = 72$$

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[24] Évidemment, le verbe arabe doit être *qabila*, dont est dérivé *muqābalah*. Le texte présent est donc un des exemples d'usages «aberrants» de ce mot – en fait, probablement, de son usage primitif (une raison de plus de ne pas attribuer au traité une date tardive): «Opposer» la somme des côtés à 14 veut dire former l'équation «longueur+largeur=14».

[25] Éd. Busard 1968: 92.

[26] Éd. Busard 1968: 95f.

En introduisant $X = x+4$ nous avons par conséquent réduit le problème au problème

$$Xy = 72,\ X-y = 4+2 = 6,$$

qui est résolu selon une méthode bien connue dans les mathématiques paléobabyloniennes:

$$(X-y)/2 = 6/2 = 3;\ [(X+y)/2]^2 = Xy+[(X-y)/2]^2 = 72+3^2 = 81;$$

$$(X+y)/2 = \sqrt{81} = 9,\ y = (x+y)/2-(x-y)/2 = 9-3 = 6.$$

III. «L'algèbre» babylonienne

L'emploi de cette méthode caractéristique de «l'algèbre» paléobabylonienne[27] n'est pas la seule trace d'un rapport étonnant entre cette très vieille discipline et la méthode de base d'Abū Bakr – que nous pouvons appeler «algèbre d'arpentage» pour souligner à la fois son caractère algébrique[28] et le fait qu'elle est distincte de «l'algèbre d'*al-ğabr*». Avant d'approcher cette question en profondeur, il serait pourtant utile de présenter «l'algèbre» paléobabylonienne elle-même d'une manière plus fidèle à son propre caractère que ne le font la plupart des introductions générales à l'histoire des mathématiques. Regardons d'abord un texte bref:

> La surface et ma rencontre j'ai additionné: 3/4
> 1 le forjet tu poses.
> La moitié de 1 tu brises, 1/2 et 1/2 tu fais se tenir,
> 1/4 à 3/4 tu ajoutes: 1 fait 1 équilatéral.
> 1/2 que tu as fait tenir, du corps de 1 tu arraches:
> 1/2 est la rencontre.[29]

Cette traduction est assez différente des traductions courantes[30]. Elle est fondée sur une étude comparée et approfondie du vocabulaire, des méthodes et de la structure conceptuelle des textes paléobabyloniens dites «algébriques», dont je n'expliquerai ici ni les méthodes ni les résultats généraux en détail[31]. Ce qui importe ici est de savoir que le texte décrit une procédure géométrique, ainsi que le sens et les rapports mutuels des termes employés.

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[27] L'époque dite «paléobabylonienne» recouvre la période de 2000 à 1600 avant J.-C. (chronologie «moyenne»); le règne de Hammurapi se situe au XVIII^e siècle. La plupart des textes mathématiques semblent appartenir à la dernière moitié de l'époque.

[28] Je ne discuterai pas ici dans quelle mesure ce «caractère algébrique» justifie qu'on parle vraiment d'une *algèbre*. Sur ce sujet, je renvoie aux considérations correspondantes dans mon [1989], où je discute plus à fond si «l'algèbre paléobabylonienne» est une vraie *algèbre* ou seulement une technique faisant usage de *modes de pensée algébriques*.

[29] BM 13901 #1, éd. MKT III, 1; cf. traduction et discussion dans mon [1990, section V.2]. Il faut remarquer que les nombres du texte (qui sont écrits dans le système sexagésimal de position) ont été impitoyablement modernisés; la manière par laquelle les Babyloniens écrivaient leurs nombres n'a aucune importance pour la présente discussion.

[30] Neugebauer dans MKT III, 5f; Thureau-Dangin dans TMB, 1.

[31] L'étude complète est décrite dans mon [1990]; un exposé moins spécialisé se trouve dans mon [1989]

Une «rencontre» (*mitḫartum*), d'abord, est la rencontre de quatre lignes égales comme les côtés d'un carré. Le mot «rencontre» signifie à la fois la longueur de chacune de ces lignes et la configuration géométrique; en d'autres termes, le carré géométrique est identifié numériquement à ce paramètre caractéristique qu'est la longueur du côté[32]. La «surface» (*eqlum*, littéralement «champ») est évidemment l'aire contenue par la figure.

Ensuite il y a les opérations. Notre petit texte nous présente deux différentes opérations «additives». L'une, «additionner» (*kamārum*) doit être une vraie addition arithmétique, puisqu'elle peut s'appliquer à des nombres sans considération de leur signification (ici, les nombres mesurant une surface et une longueur sont additionnés). Elle est symétrique (les deux nombres à additionner sont toujours liés par un «et»), et elle fait disparaître les deux composantes dans la somme. L'autre, «ajouter» (*waṣābum*), est concrète; elle n'additionne pas des nombres abstraits, mais des grandeurs (des grandeurs mesurables, bien sûr, ce qui la fait tout de même *correspondre* à une addition arithmétique). Les deux grandeurs ne sont pas traitées de manière symétrique, puisque l'opération conserve «l'identité» d'une d'elle, tandis que l'autre lui est ajoutée. Celui pour qui cela n'a aucun sens n'a qu'à penser à son compte bancaire. À l'expiration de l'année, les intérêts y sont ajoutés, ce qui ne change pas l'identité du compte, qui restera *son* capital; seulement, son montant est devenu un peu plus grand (en fait, *intérêts* en babylonien se dit *ṣiptum*, qui est dérivé de *waṣābum*).

«Briser» (*ḫēpum*) veut dire diviser en deux, mais seulement quand les deux parties sont des «moitiés naturelles» ou coutumières (*bāmtum*) – par exemple une des deux moitiés de la base d'un triangle qui est *toujours* multipliée par la hauteur quand on veut trouver l'aire. «Briser» est une opération strictement distincte de la multiplication par 1/2.

«Faire se tenir» (*šutākulum* – la traduction est assez incertaine; il se peut qu'on doive traduire «faire se dévorer») est d'habitude compris comme «multiplier». En fait, «faire que *a* et *b* se tiennent» signifie «construire un rectangle à côtés *a* et *b* ». Mais puisque les Babyloniens ne s'occupent que des droites mesurables, construire un rectangle implique toujours que son aire sera calculée; bien qu'elle-même une opération *de construction*, l'opération *implique* donc une multiplication dont le résultat est normalement donné sans explication supplémentaire, ce qui est aussi le cas ici (plus bas nous verrons un exemple de mention séparée de l'opération arithmétique).

Que «*A* fait *b* équilatéral» (*A*-e íb-si$_8$ *b*) veut dire, arithmétiquement, que $b=\sqrt{A}$, et pour un Babylonien que l'aire *A* formée en carré aura le côté *b*. «Arracher» (*nasāḫum*), est finalement une soustraction concrète – il s'agit en fait de l'inversion de l'opération d'«ajouter». Comme celle-là, elle conserve donc «l'identité» de l'entité à laquelle une partie est arrachée, tout en la changeant quantitativement.

Enfin, il y a trois termes qui ne correspondent à aucune opération arithmétique et qui, dans l'interprétation consacrée, ont toujours été regardés comme pro-

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[32] Cette identification n'est pas plus étonnante que notre propre identification du carré avec cet autre paramètre caractéristique qu'est son aire. Pour nous, le carré *est* son aire et *possède* un côté; pour un Babylonien, le carré *est* son côté et *possède* une aire.

En conséquence de l'interprétation traditionnelle et purement numérique de ces textes, on l'a considéré comme énigmatique et l'on a beaucoup discuté sur le fait présumé que les Babyloniens ne distinguaient pas le carré et la racine carrée. Comme on le voit, il n'y a plus d'énigme quand le *mitḫartum* est compris comme carré *géométrique*, seulement une conceptualisation différente de la nôtre. La même chose vaut, d'ailleurs, pour l'énigme analogue offerte par l'emploi du mot grec δύναμις comme terme mathématique (voir mon [1990b]).

blématiques. D'abord il y a le «forjet». Le mot babylonien est *waṣûm*, littéralement quelque chose qui sort ou saille. Ici le mot est employé pour dénoter la largeur *1* qui, quand on la pose de façon à saillir orthogonalement d'un segment de longueur L, produit un rectangle de l'aire $1 \cdot L = L$ – ce qui correspond bien à l'utilisation du mot dans la terminologie architecturale au sens de «forjet». L'emploi du terme «poser» (*šakānum*) dans le texte présent est alors évidente; en général, il faut le dire, le sens du mot, même dans les textes mathématiques, est assez diffus et il semble pouvoir désigner tout processus où quelque chose est marqué ou noté matériellement (la mémorisation d'un résultat intermédiaire est, d'autre part, imposée par la phrase «que ta tête retienne»).

Reste le «corps» (*libbum*, littéralement «coeur» ou «entrailles»). L'examen d'un grand nombre de textes montre que l'on peut «ajouter au corps» ou «arracher du corps» d'une surface aussi bien qu'y «ajouter» ou en «arracher» simplement. Mais, tandis qu'on peut aussi «élever» la surface à 3 coudées (ce qui veut dire calculer le volume d'un parallélépipède avec la surface en question comme base et une hauteur de 3 coudées), jamais on n'«élève au corps de» quelque chose. Le «corps» n'intervient donc que quand la métaphore a un sens dans l'interprétation géométrique.

Le texte décrit donc d'abord une transformation de la donnée originale concernant la somme arithmétique des deux mesures en problème géométrique: Poser un forjet de *1* au côté du carré en fait un rectangle avec une aire mesurée par la même somme. 3/4 représente donc un rectangle avec une largeur égale à la «rencontre» (disons x) et une longueur égale à $x+1$. Ce rectangle est transformé en gnomon et le reste ressemble exactement à la deuxième des deux démonstrations citées d'al-Khwārizmī:

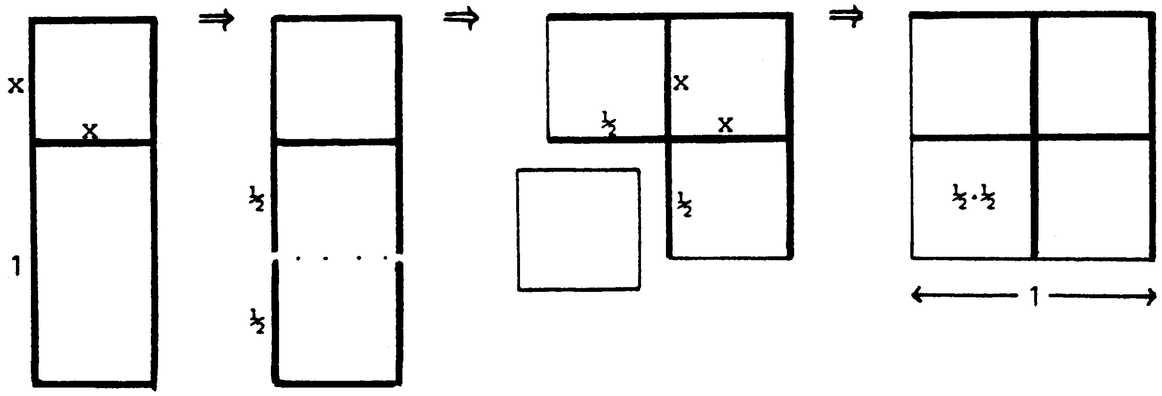

Figure 3

Ce problème simple est le premier d'une série de 24 problèmes sur les carrés contenus dans la même tablette, présentés en progression assez systématique du simple vers le compliqué. Le deuxième problème est de type $x^2-x=S$, le troisième de type $ax^2+bx=S$. Ensuite, le niveau augmente graduellement. Enfin, après bien des problèmes complexes à plusieurs inconnues, vient comme numéro 23 un problème très simple:

Une surface. La surface et les quatre fronts j'ai additionnés: 25/36.
4, les quatre fronts, tu inscris. L'inverse de 4 est 1/4.
1/4 tu élèves à 25/36. 25/144 tu inscris.

1, le forjet, tu ajoutes: $1\frac{25}{144}$ fait $1\frac{1}{12}$ équilatéral.
1, le forjet que tu as ajouté, tu arraches: 1/12 tu répètes jusqu'à deux fois.
1/6 se rencontre.[33]

La plupart des opérations nous sont déjà familières; deux, pourtant, sont nouvelles: «Élever» et «répéter». Également nouveau est le mot «front». Comme mentionné ci-dessus, «élever» (*našûm*) dénote la multiplication par une hauteur par laquelle une aire est transformée en volume; il est même probable que cela explique l'étymologie du mot comme terme mathématique. Sa sphère d'application, toutefois, est beaucoup plus vaste. L'aire d'un triangle et d'un trapèze, par exemple, sont calculées par «élévation» du mi-«front» ou du «front»-moyen à la longueur; de plus, toute opération de proportionnalité implique une élévation; la «division» au moyen d'une multiplication avec l'inverse du nombre diviseur est finalement désignée de cette manière. Tout compte fait, le mot peut être expliqué comme *calcul d'une valeur concrète par multiplication*. L'opération d'«élévation» est donc différente aussi bien de la quasi-multiplication de «faire se tenir» que du terme a-rà, dérivé du verbe «aller» et traduisible comme «pas de», employé dans les tables de multiplication et donc pour des vraies multiplications arithmétiques d'un nombre par un nombre.

«Répéter» (*eṣēpum*) est une autre (quasi-)multiplication et, en fait, la dernière opération multiplicative. Elle désigne une répétition (visuellement saisissable) d'une entité concrète, ce qui implique naturellement que le nombre mesurant l'entité sera multiplié; l'explication géométrique du texte nous montrera un exemple typique de l'emploi du terme.

«Front» (*pūtum*) désigne (dans l'arpentage) le côté court d'un champ, celui qui fait front au canal d'irrigation. Dans les textes mathématiques, son équivalent sumérien (sag) est employé pour la largeur d'un rectangle plus abstrait, mais jamais le mot babylonien lui-même. Son apparition ici démontre donc que le mot «surface» devrait en réalité être lu dans son sens littéral, comme «champ».

Le problème concerne donc un champ carré[34], dont la somme de l'aire et des nombres mesurant les quatre côtés est donnée comme 25/36 (l'unité, le nindan, égale approximativement 6 m). La succession des opérations peut être suivie sur cette séquence de figures:

[33] BM 13901 #23, éd. MKT III, 4f; cf. traduction et discussion dans mon [1990, section V.4].

[34] Que le champ est un carré n'est pas dit explicitement au début. Qu'il le soit réellement résulte d'abord du fait que le champ possède quatre fronts – s'il était rectangulaire, il posséderait deux fronts et deux longueurs et, s'il était irrégulier, il posséderait un front supérieur et un front inférieur et une longueur supérieure et une longueur inférieure. La dernière ligne, de plus, le dit aussi, en utilisant l'expression «se confronter».

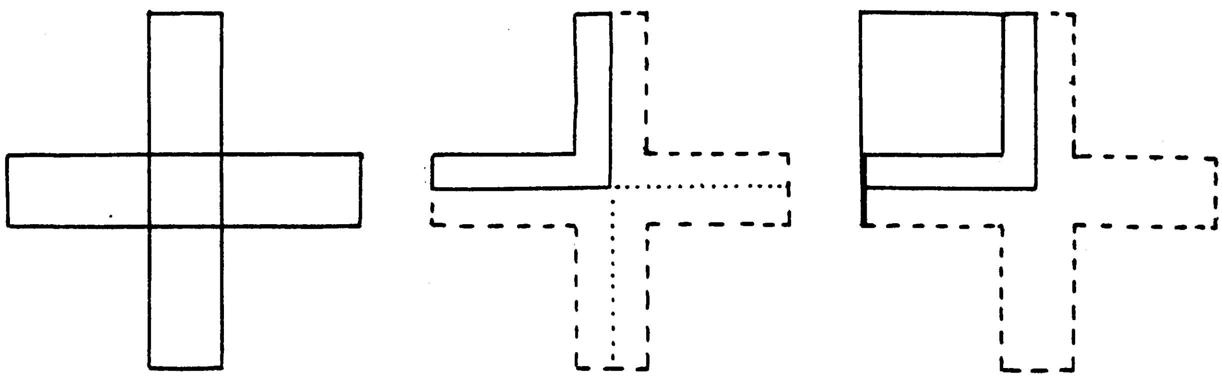

Figure 4

D'abord, la somme est représentée par le champ carré cerné par quatre rectangles à largeur égale au front et longueur *1*. Deuxièmement, l'aire de cette figure est multipliée par 1/4, ce qui correspond à la prise d'un quart de la figure, c'est-à-dire d'un gnomon. Ce gnomon est complété en carré d'une manière remarquable. Un mathématicien moderne ajouterait 1^2, ce qui correspond bien à ce qui se passe dans d'autres problèmes de la même tablette. Ici, pourtant, c'est le forjet lui-même qui est ajouté. Nous avons déjà vu que la «rencontre», numériquement identique au côté du carré, désigne la figure complète. De même, le forjet représente le carré manquant, évidemment *comme figure* (comme nombre, il représente son côté). Ce qui est ajouté est donc *la figure* et non pas le nombre mesurant son aire. Le carré complété a donc une aire de $1\frac{25}{144} = \frac{169}{144}$ et, en conséquence, un côté de $\frac{13}{12} = 1\frac{1}{12}$. Pour retrouver le demi-front, le «forjet que tu as ajouté» est encore arraché, ce qui souligne une fois de plus que la figure carrée ajoutée comme complément est identique à son côté, qui est de fait la grandeur arrachée. Finalement, il en résulte le front lui-même quand le demi-front est «répété», c'est-à-dire combiné avec son image de miroir.

D'après sa structure mathématique, le problème est du même type que le numéro trois de la tablette (et même plus simple). La solution, pourtant, est obtenue de manière différente. Différente aussi est la formule employée, qui selon la manière du numéro deux aurait dû être «J'ai additionné la surface et mes quatre confrontations». L'emplacement entre les problèmes complexes et la formule aberrante aussi bien que la méthode insolite ont toujours été regardés comme mystérieux – Neugebauer a même proposé que le texte serait corrompu et ne donnerait de sens mathématique que par hasard[35]. Pris ensemble, pourtant, les trois mystères semblent se résoudre mutuellement. La formule («Un champ. [...] les quatre fronts [...]») semble indiquer que le problème n'est pas compris comme une équation type. Il appartient plutôt au stock des énigmes mathématiques professionnelles des arpenteurs. Qu'une énigme soit résolue d'une manière surprenante et élégante n'est naturellement que bienséant; il est tout aussi naturel, d'autre part, qu'elle soit alors placée entre les problèmes complexes auxquels les techniques de base peuvent être appliqués.

La ressemblance entre le problème présent et la première démonstration d'al-Khwārizmī n'a pas besoin d'être expliquée. Ce qu'il faut peut-être signaler c'est l'intérêt partagé avec Abū Bakr pour *les quatre côtés* d'un quadrilatère – rien que dans

[35] MKT III, 14.

le chapitre du *Liber mensurationum* sur les carrés, sept problèmes les concernent (les numéros 4, 6, 7, 8, 9,12 et 14), et dans le chapitre sur les rectangles plusieurs (dont le numéro 43 cité plus haut) le font aussi. Cette observation s'accorde bien avec le rôle des problèmes de ce recueil comme énigmes d'arpenteurs. Évidemment, si un arpenteur (ou quelqu'un d'autre dont la motivation n'est pas cet esprit de système qui caractérise l'instruction mathématique des institutions scolaires) doit formuler une énigme combinant l'aire et les côtés d'un carré, les premières possibilités qui lui viendront à l'esprit seront *l'aire et le côté* et *l'aire et les côtés*.

Notre troisième exemple vient d'une autre tablette et concerne un rectangle:

> Longueur, largeur. La longueur et la largeur j'ai fait se tenir: Une surface j'ai bâtie.
> J'en ai fait le tour. Ce par quoi la longueur excède la largeur j'ai ajouté au corps de la surface: 183.
> Je suis retourné. La longueur et la largeur j'ai additionnées: 27.
> Longueur, largeur et surface sont quoi?
>
> | | | |
> |---|---|---|
> | 27 | 183 | les additionnés |
> | 15 | longueur | 180 surface |
> | 12 | largeur | |
>
> Toi, pour ta méthode, 27, les additionnés de longueur et largeur, ajoute au corps de 183: 210.
> 2 à 27 ajoute: 29.
> Sa moitié, celle de 29, tu brises: $14\frac{1}{2}$.
> <$14\frac{1}{2}$ et $14\frac{1}{2}$ tu fais se tenir>[36].
> $14\frac{1}{2}$ pas de $14\frac{1}{2}$, $210\frac{1}{4}$.
> Du corps de $210\frac{1}{4}$, 210 tu arraches: 1/4 est le reste.
> 1/4 fait 1/2 équilatéral.
> 1/2 au premier $14\frac{1}{2}$ ajoute: 15 est la longueur.
> 1/2 du second $14\frac{1}{2}$ coupe: 14 est la largeur..
> 2 que tu as ajouté à 27, de 14, la largeur, tu arraches:
> 12 est la vraie largeur.
> 15, la longueur, 12 la largeur j'ai fait se tenir:
> 15 pas de 12, 180 est la surface.
> 15, la longueur, excède 12, la largeur, par quoi?
> Il l'excède par 3. 3 au corps de 180, la surface, ajoute.
> 183 est la surface.[37]

Du point de vue des opérations, ce texte ne nous apporte pas beaucoup de neuf. On peut observer que l'expression «ce par quoi *A* excède *B* » revient dans beaucoup de textes – elle désigne une «soustraction par comparaison», donc une soustraction dont le résultat ne partage pas l'identité du diminuendum; on remarque aussi qu'elle correspond à l'expression utilisée par Abū Bakr, «ce dont *A* augmente *B*». «Couper *B* de *A*» est, d'autre part, une soustraction qui conserve l'identité, précisément comme fait «arracher»; les deux mots peuvent être regardés comme de simples synonymes, entre

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[36] Ce complètement d'une ellipse suit des passages parallèles de la même tablette, dont l'une se trouve huit lignes plus bas dans la présente traduction.

 [37] AO 8862 #1, éd. MKT I, 108f.

lesquels les Babyloniens ont choisi selon leurs connotations et valeurs métaphoriques (on ne «coupe» que des entités linéaires).

Les mots «longueur, largeur» au commencement nous informent que le problème concerne une surface déterminée par *une* longueur et *une* largeur – c'est-à-dire un rectangle. La spécification selon laquelle le procès de «faire que se tiennent» la longueur et la largeur entraîne qu'une surface est «bâtie» démontre que l'opération en question est vraiment une *construction* et non pas une multiplication arithmétique (une multiplication *présupposant* une opération de bâtissement serait mentionnée *après* celle-là). Plus remarquable encore est la remarque selon laquelle celui qui a jalonné la surface en fait la ronde; il s'agit vraiment d'une surface dans le terrain – à savoir d'un «champ». Une fois encore, le problème prétend traiter de la pratique de l'arpentage, tout en appartenant en réalité dans la catégorie «brillante» des énigmes.

La méthode peut paraître quelque peu opaque, mais s'explique symboliquement comme suit: Des données

$$xy+(x-y) = 183 \ , \ x+y = 27$$

suivent par addition et l'introduction de $Y = y+2$

$$xy+(x-y)+(x+y) = xy+2x = x(y+2) = xY = 210,$$

et d'autre part

$$x+Y = (x+y)+2 = 27+2 = 29, \text{ donc } (x+Y)/2 = 14\tfrac{1}{2},$$

et donc

$$[(x-Y)/2]^2 = [(x+Y)/2]^2 - xY = 210\ \tfrac{1}{2} - 210 = 1/4,\ (x-Y)/2 = 1/2.$$

Ceci nous donne

$$x = [(x+Y)/2]+[(x-Y)/2] = 14\ \tfrac{1}{2}+1/2 = 15$$

et

$$Y = [(x+Y)/2]-[(x-Y)/2] = 14\ \tfrac{1}{2}-1/2 = 14,$$

dont il résulte finalement

$$y = Y-2 = 12.$$

Mis à part le remplacement de $x+y$ par $x-y$ (et vice versa) dans les données, le problème est strictement analogue au dernier problème cité ci-dessus d'Abū Bakr. Plus haut déjà il a été dit qu'Abū Bakr emploie une méthode paléobabylonienne caractéristique pour résoudre ce problème, soit le calcul par demi-somme et demi-différence. En fait, la ressemblance va plus loin, incluant l'introduction d'une «variable auxiliaire» X ou Y, dont le présent texte parle explicitement comme une «largeur», différente bien sûr de la «vraie largeur». Qu'Y soit réellement une largeur se verra sur la figure suivante, qui nous montrera aussi la situation d'un point de vue d'arpenteur:

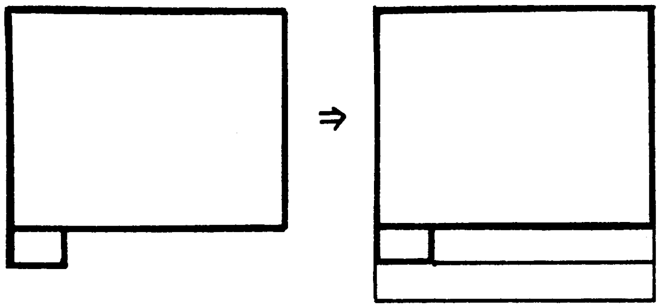

Figure 5

D'abord, le rectangle est bâti et «ce par quoi la longueur excède la largeur» est «ajouté» – ce qui présuppose qu'il est pourvu d'une largeur *1* et ce qui donne une surface totale égale à *183*. Au pas suivant, «les additionnés de longueur et largeur» (le pluriel est indubitable dans le texte Babylonien) y sont ajoutés. Comme on le voit, le résultat est un nouveau rectangle d'aire *210*, dont la nouvelle largeur (Y) excède la largeur initiale par *2* et dont la somme des deux côtés est $27+2 = 29$.

Le rectangle à aire connue (ici *210*) avec la somme des deux côtés connue (ici *29*) est un problème type de l'«algèbre» paléobabylonienne. Sa solution peut être suivie dans la figure suivante, tirée en fait de la démonstration d'al-Khwārizmī du cas «trésor et nombres égalent racines»[38]:

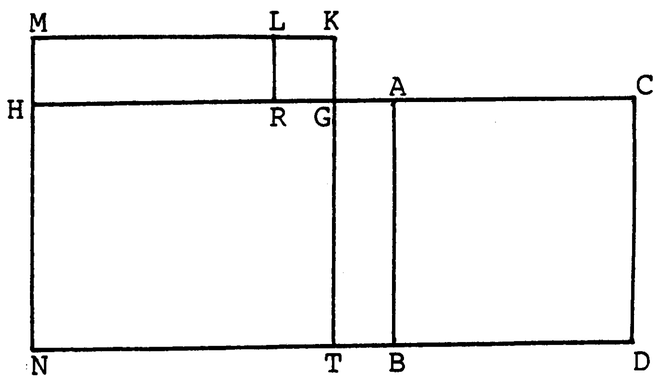

Figure 6

Le rectangle inconnu est représenté par AN, dont l'aire est *210*. Si BD est rendu égal à AB (Y), il s'ensuit que ND égalera *29* ($=x+Y$). Nous divisons ce segment en deux parties égales (NT et TD) et les faisons «se tenir», ce qui produit le carré NK, dont l'aire sera $(14\frac{1}{2})^2 = 210\frac{1}{4}$ ($= [(x+y)/2]^2$). Si HR est rendu égal à AB ($=Y$), on verra à l'aide d'un peu de comptabilité élémentaire (HM, comme GA, égale $(x-Y)/2$) que MR égale GB et donc que l'aire du gnomon MLRGTN égale *210*, qui est arraché des $210\frac{1}{4}$ représentant NK. Le reste (dont l'aire sera *1/4*) sera le carré RK, dont le côté

[38] Ed., trad. Rosen 1831: 18.

égale $(x-Y)/2 = 1/2$. Cette demi-différence est ajoutée «au premier $14\frac{1}{2}$» (NT), ce qui nous donne la longueur ($x = 15$), et ensuite coupé «du second $14\frac{1}{2}$» (MN), ce qui donne la largeur ($Y = 14$).

La même figure aurait pu être reprise au traité d'ibn Turk[39]. Il semble bien, finalement, que la figure à laquelle se réfère Abū Bakr par son «voici!» soit similaire[40]. Somme toute, il y a de bonnes raisons de penser que les savants musulmans bâtissent sur une tradition née durant l'âge du bronze babylonien.

D'autres observations d'ordre stylistique et linguistique étaient cette hypothèse. D'abord, l'organisation «rhétoricienne» des problèmes. Comme nous l'avons vu, les problèmes d'Abū Bakr commencent «Si quelqu'un t'aura dit». Ce que ce «quelqu'un» a dit est dit à la première personne singulier parfait – avec une seule exception: Si la longueur excède la largeur, ceci est expliqué à la troisième personne singulier présent, comme un fait neutre et non pas comme quelque chose «qu'il» a fait.

Alors vient une référence explicite à la méthode, qui est enfin expliquée comme quelque chose que «tu» dois faire, à l'impératif alternant avec la deuxième personne singulier du présent. Parfois, une étape est justifiée par une référence à l'énoncé employant la formule «parce que sa parole fut» (*quoniam sermo eius fuit*). Parfois aussi, un résultat intermédiaire qui ne correspond pas directement à une entité géométrique doit être «retenu en mémoire».

Une partie de ces caractéristiques se retrouvent dans les problèmes paléobabyloniens cités ici. Ceux qui manquent se trouvent tous dans d'autres textes, en particulier la référence au «quelqu'un» introduisant l'énoncé. À elle seule, l'alternance entre la première et la deuxième personne pourrait être expliquée comme un reflet de la situation didactique; la même chose pourrait peut-être se dire de l'alternance entre présent et parfait. Si les deux alternances sont considérées ensemble et combinées avec les autres caractéristiques stylistiques beaucoup plus singulières, toute explication par la situation didactique et par le hasard devient extrêmement invraisemblable.

Une particularité des textes paléobabyloniens est la multiplicité des opérations distinctes. Dans leurs grandes lignes, les mêmes distinctions sont respectées par Abū Bakr concernant les opérations additives et soustractives. Il est plus difficile de voir ce qu'il en est des opérations multiplicatives, parce que les opérations correspondant à «élever» et «pas de» sont absentes de son traité. Il y a des indices, pourtant, qu'il distingue quelque chose de similaire à la «répétition» concrète paléobabylonienne; d'abord, Abū Bakr emploie «doubler» là où un texte babylonien aurait «répété jusqu'à deux» – mais plus remarquable est une construction double au numéro 57, où «quadrupler» est suivi par une explication numérique («multiplication» par 4), précisément comme dans le problème babylonien où le résultat numérique d'un procès de «faire *a* et *b* se tenir» est trouvé comme «*a* pas de *b*».

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[39] Trad. Sayili 1962: 164.

[40] Bien entendu, elle doit être différente dans le détail pour la simple raison qu'il part d'un rectangle où l'aire et la *différence* entre les côtés sont connues; selon toute vraisemblance, elle correspond précisément aux figures 2 et 3. Le numéro 58 du *Liber mensurationum*, pourtant, part de l'aire (plus précisément, de l'aire du losange inscrit dans le rectangle) et de la somme des côtés (égale dans ce cas à la somme des diagonales du losange) et semble suivre exactement la même voie géométrique que notre problème babylonien. Cf. aussi plus bas le numéro 9.

IV. Une tradition

À en croire le mystérieux «voici!» (*intellige*), le traité d'Abū Bakr semble présupposer une compréhension heuristique basée sur des figures géométriques. Beaucoup de problèmes sont du même type que les problèmes paléobabyloniens, y compris les types assez singuliers concernant l'aire et les quatre côtés d'une figure et la somme de l'aire et d'une combinaison linéaire des côtés d'un rectangle. La méthode de base coïncide avec les méthodes particulières et caractéristiques de l'«algèbre» paléobabylonienne, par exemple l'emploi d'une demi-somme et d'une demi-différence et le «changement de variable». La distinction entre opérations qui diffèrent dans une interprétation géométrique, mais pas dans une interprétation arithmétique est respectée. La structure «rhétoricienne» coïncide, finalement, avec celle des textes mathématiques babyloniens dans les détails de grammaire et des expressions fixes. Somme toute, il est donc plus que difficile de rejeter l'idée que le traité d'Abū Bakr (ou du moins une partie de sa substance, puisqu'il contient d'autres matières) appartient à «une tradition née durant l'âge du bronze babylonien», comme il a été suggéré plus haut. Puisqu'il s'agit d'un traité sur l'arpentage, il faut croire que la tradition a été transmise par le milieu des praticiens-géomètres – arpenteurs, architectes et maîtres maçons – le même milieu qui semble avoir inspiré plusieurs des problèmes babyloniens cités plus haut[41]. Comme presque tous les milieux de praticiens antiques et médiévaux, outre celui des scribes, celui-ci ne nous a laissé aucune source écrite sur ses méthodes, bien que l'existence d'un fonds de méthodes ne peut pas être mise en doute; le fait qu'une tradition a pu survivre depuis l'ère paléobabylonienne jusqu'au neuvième siècle sans avoir laissé de traces est donc moins surprenant qu'on pourrait le croire à première vue[42].

Cette conclusion peut être étayée par deux observations d'ordre différent. On peut, d'abord, signaler un autre cas frappant de continuité mathématique réunissant l'ère paléobabylonienne et le moyen-âge islamique (aussi bien que le moyen-âge latin et indien). En 952/53, un certain Abū'l-Ḥasan al-Uqlīdisī, travaillant à Damas, a écrit un traité de grande envergure sur le calcul avec les chiffres hindous. Le dernier chapitre, qui porte le titre «Doubler l'unité, soixante-quatre fois», nous dit que

> Cela est une question posée par beaucoup de gens. Il y en a qui doublent l'unité 30 fois et d'autres qui la doublent 64 fois[43].

Environ un siècle et demi plus tôt, un recueil de problèmes de récréation (les *Propositiones ad acuendos iuvenes*) fut composé en pays franc, peut-être à l'école

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[41] Il est même vraisemblable que le point de départ de «l'algèbre» méthodique des scribes babyloniens aura été les devinettes mathématiques apparaissant dans un milieu d'arpenteurs durant le troisième millénaire. Pendant l'époque paléobabylonienne ces devinettes auront alors été adoptées et systématisées dans l'école de scribes; en même temps, la tradition originale aura continué son existence extra-scolaire (voir mon [1990a, section V]), survivant jusqu'à l'époque d'Abū Bakr.

[42] Il faut savoir que les textes mathématiques de l'époque basse babylonienne (l'époque seleucide) n'utilisent pas les méthodes caractéristiques partagées par Abū Bakr et les textes paléobabyloniens et ne distinguent pas différentes opérations additives, soustractives et multiplicatives (voir mon [1990, section X.2]); la tradition qui réunit Abū Bakr aux sources paléobabyloniennes ne passe donc pas par les prêtres-astrologues seleucides auteurs des textes mathématiques cunéiformes tardifs

[43] Trad. Saidan 1978: 337

palatine de Charlemagne à Aix-la-Chapelle et peut-être par Alcuin. Le 13ème problème explique que

> un roi a commandé à son ministre de lever une armée de 30 villes de cette manière, qu'on conscrive de chaque ville tant d'hommes qu'on y aurait conduits. Le ministre, pourtant, est venu seul à la première ville et à la seconde avec un autre; à la troisième, maintenant, trois sont venus [avec lui]; que celui qui peut dise combien d'hommes ont été levés de ces 30 villes.[44]

Les 64 doublements se réfèrent évidemment au «problème de l'échiquier» raconté par plusieurs auteurs arabes[45] et peut-être aussi discuté par al-Khwārizmī dans une oeuvre perdue[46]. Les 30 doublements se retrouvent plus tard chez Bhāskara, mais aussi dans des sources d'une époque beaucoup plus ancienne. Tout d'abord, un papyrus grec, provenant probablement du Haut-Empire romain, mais peut-être des troisième ou quatrième siècles, calcule 30 doublements d'un montant initial de 5 drachmes[47]. Ce qui est plus intéressant encore, une tablette datant du XVIII^{e} siècle avant J.-C. et venant de Mari en Iraq contient le problème suivant:

> À un seul grain, 1 grain a été ajouté:
> 2 grains le premier jour,
> 4 grains le second jour,
> [...][48]

et ainsi de suite jusqu'au trentième jour. Comme chez «Alcuin», le doublement est formulé de manière additive; comme dans le problème de l'échiquier, il s'agit de grain (et on trouve en effet dans la Babylonie des «échiquiers» à 30 cases!); et comme chez al-Uqlīdisī, Alcuin et Bhāskara et dans le papyrus grec, on double précisément 30 fois. La continuité semble être hors de doute.

L'autre observation concerne un «Traité sur ce qui est nécessaire concernant les constructions géométriques pour les artisans-spécialistes» écrit par Abū'l-Wafā' vers la fin du dixième siècle. À propos du problème qui consiste à additionner trois carrés égaux, Abū'l-Wafā' raconte avoir présenté ce problème à l'occasion d'une rencontre entre géomètres et «artisans-spécialistes» (*ṣunna'*, traduisible aussi comme «praticiens»). Les géomètres, bien sûr, ont vite trouvé une solution, qui

> pourtant n'était pas du tout satisfaisante pour les artisans, parce qu'ils n'étaient pas en état de diviser ces trois carrés en pièces qui pourraient être composées de telle manière qu'il en résulte un seul carré, comme nous l'avons fait pour deux et cinq carrés. En ce qui concerne les artisans, ils ont proposé plusieurs méthodes pour le faire. Pour quelques-unes de ces méthodes des démonstrations ont été faites, tandis que d'autres se montraient fausses. Celles pour lesquelles des démonstrations n'étaient pas données étaient très près d'être correctes et celui qui voyait ces constructions les croirait vraies.[49]

Les «artisans-spécialistes» dont parle Abū'l-Wafā' ont donc pratiqué une géométrie de découpage et d'assemblage des figures, correspondant précisément à la géométrie «naïve» que nous trouvons dans l'«algèbre» paléobabylonienne, dans les

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[44] Éd. Folkerts 1978: 51.
[45] Voir Wiedemann 1970: I, 442-453.
[46] Vois Sesiano 1987: 492 + note 23.
[47] Voir Boayaval 1971.
[48] Éd. Soubeyran 1984: 30ff.
[49] Trad. russe Krasnova 1966: 115.

démonstrations d'al-Khwārizmī et d'ibn Turk et, selon toute vraisemblance, chez Abū Bakr.

La première observation montre la possibilité d'une survie silencieuse durant les deux millénaires et demi séparant l'époque paléobabylonienne d'al-Khwārizmī[50]; la deuxième confirme la préférence des géomètres pratiques pour les techniques de découpage et d'assemblage des figures.

«Découpage et assemblage» est une description assez précise des techniques employées dans «l'algèbre» et la «géométrie naïve» des Babyloniens et donc, semble-t-il, dans «l'algèbre d'arpentage» d'Abū Bakr. Peut-être est-ce aussi le nom par lequel Abū Bakr désignait lui-même cette méthode. C'est du moins une interprétation plausible d'une remarque faite dans le problème numéro 9:

> Et s'il t'aura dit: J'ai enlevé l'aire [d'un carré] de ses côtés et trois sont restés, combien est alors chacun de ses côtés?
> La méthode de ceci sera que tu divises en deux le nombre des côtés, ce qui sera 2, lequel tu multiplies alors par lui-même et ce qui en advient sera 4. De ceci enlève donc 3 et 1 restera; prends sa racine, qui est 1, lequel, si tu l'ajoutes à 2, donnera les trois côtés; et si tu l'enlèves de 2, 1 restera qui sera un quelconque de ses côtés, et ceci, en effet, est selon augmentation et enlèvement.
> D'autre part, la méthode selon *al-ǧabr* est que tu poses toujours un des côtés comme la chose, laquelle tu multiplies par elle-même [...][51]

La structure mathématique du problème coïncide (en interprétation géométrique, où il n'importe pas de savoir s'il y a une ou deux inconnues) avec celle du numéro 43: Rectangle à aire donnée et avec somme donnée des côtés[52]. La procédure peut donc être suivie sur la figure 6, avec HC égal à 4 et le rectangle HD égal donc aux quatre côtés. Le carré inconnu est représenté par AD et le rectangle qui reste quand l'aire du carré est enlevée des quatre côtés est donc représenté par HB. L'adjonction de 1 à 2 est représenté par l'adjonction de TB à NT; elle donne NB (disons $4-x$), et *dans le cas actuel* ($x = 1$) donc 3 ou $3x$, comme on l'affirme dans le texte (l'identification s'explique facilement si l'on suppose que l'argument a été fait à partir d'une figure à proportions correctes). L'enlèvement est représenté par l'enlèvement de HM de NM et le résultat HN identique au côté.

En principe, l'«augmentation et l'enlèvement» (*augmentatio et diminutio*) pourraient être une référence au cas correspondant d'*al-ǧabr*, qui est précisément celui où l'addition aussi bien que la soustraction peut donner l'inconnue, comme nous l'a déjà expliqué al-Khwārizmī. Mais si Abū Bakr avait eu l'intention de nous notifier cela, il aurait probablement dit que l'addition nous donne *aussi* une solution possible ($x = 3$), ou du moins il aurait rejeté cette solution comme différente de celle qui est

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[50] Non pas tout à fait silencieuse, il est vrai: Il y a le papyrus grec. En outre, on trouve dans le corpus Héronien et chez les agrimenseurs romains des problèmes du deuxième degré tout à fait isolés, mais très proches de certains problèmes paléobabyloniens ou présents dans le *Liber mensurationum*. Par exemple, le *Geometrica* Héronien (éd. Heiberg 1912: 380 et encore 444) contient un problème où la somme du diamètre, de la périphérie et de l'aire d'un cercle est donnée; exactement le même problème se trouve dans une tablette paléobabylonienne (éd. Friberg 1981: 61).

[51] Éd. Busard 1968: 88.

[52] En interprétation d'*al-ǧabr*, évidemment, il appartient au cas «trésor et nombre égalent côtés». Il est remarquable, pourtant, que l'énoncé du problème correspond à la formule utilisée dans les textes paléobabyloniens, «aire enlevée de côtés: nombre». Non moins remarquable est le fait que la solution explicite aussi bien la longueur («les trois côtés») que la largeur (le côté) du rectangle inconnu.

recherchée[53]. Qui plus est, beaucoup d'autres problèmes parmi ceux qui traitent des rectangles ou des losanges (dont ceux correspondant au «quatrième cas» d'*al-ğabr* «trésor et côtés égalent nombre») peuvent aussi être résolus par «addition [donnant le côté long ou la diagonale longue] et soustraction [donnant le côté court ou la diagonale courte]» et quelques-uns le disent directement (par exemple les numéros 57 et 58). Dans le contexte du *Liber mensurationum*, il n'y a donc rien qui lie le présent problème *spécifiquement* à «*augmentatio et diminutio*» et il paraît plus probable que l'expression doit être considérée comme opposée à l'*al-ğabr* de la phrase suivante. La méthode de découpage et d'assemblage géométrique naïve semble être désignée par un nom traduisible comme «assemblage et découpage».

Si cela est bien le cas, il faut se demander quelle a été l'expression correspondante arabe. Un candidat possible est *al-ğam' wa'l-tafrīq*[54]. Plusieurs traités sur ce sujet non identifié ont été écrits (dont l'un par al-Khwārizmī) jusqu'au commencement du dixième siècle, bien qu'aucun d'eux n'ait survécu. *Ğam'* vient du verbe *ğama'a* dont le sens originel semble être «assembler, joindre» (concrètement); *tafrīq* vient du verbe *faraqa*, «séparer, diviser, découper». L'expression, un peu énigmatique[55], correspond donc bien à «assemblage et découpage» et pourrait bien avoir été traduite par Gérard comme «*augmentatio et diminutio*».

Bien sûr, cette identification reste hypothétique. Ce qui semble être bien établi, c'est qu'une tradition «d'algèbre d'arpentage» de descendance paléobabylonienne était encore vivante et connue à l'époque d'al-Khwārizmī et d'ibn Turk et que cette tradition faisait usage de méthodes correspondant précisément (l'utilisation de l'alphabet pour désigner des entités géométriques mise à part) aux démonstrations géométriques données par ces auteurs des algorithmes appartenant à *al-ğabr*. C'est donc avec bonne raison que nous avons proposé plus haut que ces démonstrations étaient tout aussi traditionnelles qu'*al-ğabr* lui-même. Ce qu'al-Khwārizmī et ibn Turk ont fait n'est pas de réunir l'*al-ğabr* traditionnel et les mathématiques rigoureuses grecques – Thābit ibn Qurra l'a bien vu –, mais de voir comment les différentes traditions «sous-scientifiques» pourrait être synthétisées avec un résultat beaucoup plus apte à servir de base aux futurs développements que ne l'étaient l'automatisme algorithmique sans preuves ou une technique fondée seulement sur la manipulation des figures, s'ils étaient pris chacun pour soi.

Ceci laisse encore ouverte la question de l'origine d'*al-ğabr*. Il est d'usage de lui attribuer des racines babyloniennes et de proposer un passage par l'Inde. Tant qu'on n'a pas prouvé (ou seulement trouvé) une origine différente, il est évidemment difficile de réfuter l'hypothèse babylonienne; il faut savoir, pourtant, qu'elle se fonde premièrement sur une interprétation purement numérique (et donc fausse!) de

[53] Si l'on s'en tient à la méthode géométrique, le problème de la solution double ne se pose pas; pour représenter géométriquement l'équation $x \cdot (2a-x) = b$, il faut décider d'avance si $x < a$ ou $x > a$, puisque les représentations géométriques des deux cas sont différentes.

[54] Le *Liber augmenti et diminutionis* mentionné plus haut (note 8), en tout cas, n'a rien a voir avec la présente méthode.

[55] La seule trace positive du contenu des traités a été trouveé par A. S. Saidan (1987: 440): Abū Manṣūr al-Baghdādī a dit au commencement de l'onzième siècle qu'un certain problème d'arithmétique pratique est mentionné dans le traité d'al-Khwārizmī sur *al-ğam' wa'l-tafrīq*; Saidan en conclut que les traités sur le sujet en question semblent avoir consisté en, ou du moins comporté des «opérations arithmétiques populaires appliquées à l'arithmétique de tous les jours, utilisant probablement l'expression des nombres par les doigts et l'échelle de 60» («folk arithmetical problems as applied to everyday arithmetic, probably using finger-reckoning and the scale of sixty»). Étant donné le caractère éclectique du traité d'al-Khwārizmī sur *al-ğabr*, il semble pourtant bien risqué de reconstituer le contenu total d'un traité perdu à partir d'une seule citation.

«l'algèbre babylonienne» et deuxièmement sur le fait que les Babyloniens, comme les «gens d'*al-ǧabr*», s'intéressaient aux équations de deuxième degré. D'autre part, à tous les niveaux des détails, les deux disciplines diffèrent, soit en ce qui concerne le choix des problèmes-type préférés, soit dans les formules employées, soit dans les méthodes, soit dans la conceptualisation des entités concernées. S'il y a eu une route menant de l'ancienne Babylonie jusqu'à l'*al-ǧabr* des praticiens pré-al-Khwārizmiens, elle a dû passer par bien d'autres contrées.

L'association indienne est mieux étayée. La métaphore «racine» (*jidhr*) pour le nombre qui multiplié par lui-même donne «le trésor» peut nous paraître naturelle, parce que nous l'avons empruntée nous-mêmes et que nous nous y sommes accoutumés; mais dans l'interprétation numérique que lui ont donnée les Arabes, elle est en fait inexplicable. L'explication présuppose une interprétation géométrique, comme «ce côté sur lequel un carré reste comme sur un pied» – et cette interprétation est bien attestée dans les mathématiques indiennes au moins depuis Āryabhaṭa. En plus, en sanscrit le mot *mūla* signifie «base» et «fondement» aussi bien que «racine d'un arbre»[56].

D'autre part, comme l'a déjà observé Léon Rodet il y a plus d'un siècle, il y a une différence frappante, aussi bien de niveau que d'approche, entre l'algèbre de mathématiciens indiens comme Āryabhaṭa et Brahmagupta et *al-ǧabr* comme nous la rencontrons chez al-Khwārizmī. De toute évidence, les praticiens-calculateurs (et même les astronomes) arabes n'ont *pas* appris leur *al-ǧabr* chez ces maîtres indiens. L'association indienne s'expliquera plutôt comme dépendance par rapport à une tradition sous-scientifique qui a *aussi* inspiré les mathématiciens indiens dans leur développement scientifique du sujet. Il est pour le moment impossible de dire s'il s'agit là d'une tradition d'origine indienne ou venant, peut-être, de Khwārezm (le pays d'origine de la famille d'al-Khwārizmī) ou d'une autre région de l'Asie centrale. Des réponses pourraient peut-être être trouvées au moyen d'analyses philologiques précises des sources indiennes.

Moins d'un siècle après la mort d'al-Khwārizmī, l'existence d'une tradition d'*al-ǧabr* plus vieille que lui semble avoir été oubliée. En ce qui concerne l'existence d'une «algèbre d'arpentage» indépendante d'*al-ǧabr*, la situation n'a pas été bien différente. Après le milieu du dixième siècle, les traités sur *al-ǧamʿ wa'l-tafrīq* ne s'écrivaient plus. Dans l'Espagne lointaine, une copie du traité d'Abū Bakr a encore pu être trouvée au douzième siècle[57]; Savasorda, écrivant au début du même siècle, conserve aussi quelques traces d'une «algèbre» aberrante dans son *Recueil sur l'arpentage*[58], comme le fait encore Léonard de Pise dans sa *Geometria practica*. Dans les deux cas, pourtant, la synthèse avec Euclide et avec la tradition post-al-Khwārizmienne est déjà si mûre que seule la connaissance du traité d'Abū Bakr nous permet de déceler ces traces. Grâce à Gérard de Crémone, «l'algèbre d'arpentage» fut transmise au Moyen-Age latin, mais sans cette technique géométrique qui était son plus fécond aspect. Celle-là n'a été transmise qu'avec l'algèbre d'al-Khwārizmī, en forme rudimentaire, bien sûr, mais déjà accommodée au goût grécisant. Si, en fin de compte, elle va jouer un rôle durant la renaissance (ce qui est bien probable en ce qui concerne Cardan, mais qui vaut peut-être aussi pour Viète), c'est grâce au travail de synthèse d'al-Khwārizmī.

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[56] Voir Datta & Singh 1962: I, 169f.

[57] Une copie assez corrompue, il faut le dire. Il n'y a pas que la présence des «voici» sans figures à voir et le mélange des différents systèmes numériques mentionnés en note 17. Parmi les autres signes de corruption textuelle on trouve, par exemple, des références «au précédent» qui en fait renvoient à des problèmes venant plus tard.

[58] Édition de la traduction latine faite par Platon de Tivoli avec traduction allemande dans Curtze 1902.

BIBLIOGRAPHIE ET ABRÉVIATIONS

J. BAILLET, "Le Papyrus mathématique d'Akhmîm", *Mission Archéologique Française au Caire*, Mémoires 9, 1, Paris, Leroux, 1892.

H. C. BANERJI (ed.), *Colebrooke's Translation of the Lílávatí*. With Notes [and a Sanscrit Text], Calcutta, Thacker, Spink and Co, 1893.

B. BOYAVAL, "Le P. Ifao 88: Problèmes de conversion monétaire", *Zeitschrift für Papyrologie und Epigraphik*, 7, 1971, pp. 165-168, Tafel VI.

H. L. L. BUSARD, "L'algèbre au moyen âge: Le «Liber mensurationum» d'Abū Bakr", *Journal des Savants*, Avril-Juin 1968, pp. 65-125.

M. CLAGETT, 1984. "Archimedes in the Middle Ages. Volume V. Quasi-Archimedean Geometry in the Thirteenth Century", *Memoirs of the American Philosophical Society*, 157 A+B, Philadelphia:, The American Philosophical Society, 1984.

H. T. COLEBROOKE (ed., tr.), *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*, 1817, Translated. London: J. MURRAY. Reprint Wiesbaden: Martin Sändig, 1973.

M. CURTZE (ed.), "Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance", *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, vol. 12-13, Leipzig, Teubner, 1902.

B. DATTA & A. N. SINGH, *History of Hindu Mathematics. A Source Book*, Parts I and II, Bombay, Asia Publishing House, 1st ed, Lahore: Motilal Banarsidass, 1935-38; 2nd ed. 1962.

M. FOLKERTS, "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*. Überlieferung, Inhalt, Kritische Edition", *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band, 6. Abhandlung, 1978.

J. FRIBERG, "Methods and Traditions of Babylonian Mathematics, II: An Old Babylonian Catalogue Text with Equations for Squares and Circles", *Journal of Cuneiform Studies*, 33, 1981, pp. 57-64.

J. L. HEIBERG (éd., tr.), *Heronis Definitiones cum variis collectionibus. Heronis quae feruntur Geometrica. (Heronis Alexandrini Opera quae supersunt omnia*, IV, Leipzig, Teubner, 1912.

A. HOCHHEIM (ed., tr.), *Kafî fîl Hisâb (Genügendes über Arithmetik) des Abu Bekr Muhammed ben Alhusein Alkarkhi*, I-III, Halle, Louis Nebert, 1878.

J. HØYRUP, "Al-Khwārizmī, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra", *Erdem*, 2, 1986, Ankara, pp. 445-484.

J. HØYRUP, "The Formation of «Islamic Mathematics». Sources and Conditions", *Science in Context*, 1, 1987, pp. 281-329.

J. HØYRUP, "Zur Frühgeschichte algebraischer Denkweisen", *Mathematische Semesterberichte*, 36, 1989, pp. 1-46.

J. HØYRUP, "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought", *Altorientalische Forschungen*, 17, 1990 (à paraître).

J. HØYRUP, "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon", *History of Science*, 28, 1990 a, pp. 63-86.

J. HØYRUP, "*Dynamis* and *mitḫartum* : On Analogous Concepts in Greek and Old Babylonian Mathematics", *Historia Mathematica*, 17, 1990 b, pp. 201-222.

B. O. F. M. HUGHES, "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr* : A Critical Edition", *Mediaeval Studies*, 48, 1986, pp. 211-263.

S. A. KRASNOVA (éd., tr.), 1966. "Abu-l-Vafa al-Buzdžani, Kniga o tom, čto neobxodimo remeslenniku iz geometričeskix postroenij", in A. T. GRIGOR'JAN & A. P. JUŠKEVIČ (eds), *Fiziko-matematičeskie nauki v stranax vostoka*, Sbornik statej i publikacij, Vypusk I (IV), Moskva, Izdatel'stvo «Nauka», 1966, pp. 42-140.

G. LIBRI, *Histoire des mathématiques en Italie,* 4 vols, Paris, 1838-1841. Reprint Hildesheim, Georg Olms, 1967.

P. LUCKEY, 1941. "Ṯābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen", *Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-physische Klasse. Berichte,* 93, 1941, pp. 93-114.

MKT: O. NEUGEBAUER, *Mathematische Keilschrift-texte.* I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil), Berlin, Julius Springer, 1935, 1935, 1937. Reprint Berlin etc., Springer, 1973.

F. ROSEN (éd., tr.), *The Algebra of Muhammad ben Musa,* Edited and Translated, London, The Oriental Translation Fund, 1831.

J. RUSKA, "Zur ältesten arabischen Algebra und Rechenkunst", *Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Philosophisch-historische Klasse,* Jahrgang 1917, 2. Abhandlung, 1917.

A. S. SAIDAN (éd., tr.), *The Arithmetic of al-Uqlīdisī. The Story of Hindu-Arabic Arithmetic as Told in Kitāb al-Fuṣūl fī al-Ḥisāb al-Hindī by Abū al-Ḥasan Aḥmad ibn Ibrāhīm al-Uqlīdisī written in Damascus in the Year 341 (A.D. 952/53),* Translated and Annotated, Dordrecht, Reidel, 1978.

A. S. SAIDAN, "The *Takmila fi'l-Ḥisāb* of al-Baghdādī", in D. A. KING & G. SALIBA (eds), *From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E. S. Kennedy, Annals of the New York Academy of Sciences,* Volume 500, New York, New York Academy of Sciences, 1087, pp. 437-443.

G. A. SALIBA, "The Meaning of al-jabr wa'l-muqābalah", *Centaurus,* 17, 1972-73, pp. 189-204.

A. SAYILI, "Abdülhamid ibn Türk'ün katişik denklemlerde mantikî zaruretler adli yazisi ve zamanin cebri (Logical Necessities in Mixed Equations by ʿAbd al Ḥamîd ibn Turk and the Algebra of his Time)", *Publications of the Turkish Historical Society,* Series VII, No 41, Ankara, Türk Tarih Kurumu Basimevi, 1962 Reprint 1985.

J. SESIANO, 1987. "A Treatise by al-Qabīṣī (Alchabitius) on Arithmetical Series",in D. A. KING & G. SALIBA (eds), *From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E. S. Kennedy, Annals of the New York Academy of Sciences,* Volume 500, New York, New York Academy of Sciences, 1987, pp. 483-500.

S.X. SIRAŽDINOV (éd.), *Muxammad ibn Musa al-Xorezmi Matematičeskie traktaty ,* Taškent, Izdatel'stvo «FAN» Uzbekskoj CCP, 1983 .

D. SOUBEYRAN, "Textes mathématiques de Mari", *Revue d'Assyriologie* , 78, 1984, pp. 19-48.

E. WIEDEMANN, "Aufsätze zur arabischen Wissenschaftsgeschichte, I-II. Mit einem Vorwort und Indices herausgegeben von Wolfdietrich Fischer", *Collectanea,* VI/1-2, Hildesheim & New York, Georg Olm 1970.

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"'Oxford' and 'Cremona': On the Relations between two Versions of al-Khwārizmī's *Algebra*". *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1991 nr. 1.

»OXFORD« AND »CREMONA«
ON THE RELATION BETWEEN TWO VERSIONS OF AL-KHWĀRIZMĪ'S *ALGEBRA*

By JENS HØYRUP

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3rd Maghrebian Symposium on the History of Arabic Mathematics
Alger, 1-3 December 1990

In memory of
J. A. Bundgaard, Niels Ferlov and Helge Poulsen
who once opened up the world of grammar to me

And to Barnabas B. Hughes
editor of the Latin al-Khwārizmī

I. The starting point

In a number of previous publications[1] I have approached the prehistory of algebra up to the final fixation of the subject in written systematic treatises by al-Khwārizmī and ibn Turk in the early 9th century (C.E.). The outcome of these investigations can be briefly summarized as follows:

The branch of Old Babylonian mathematics normally identified as »algebra« was no rhetorical algebra of the kind known from the Islamic and European Medieval period (and from Diophantos). It did not deal with known and unknown *numbers* represented by words or symbols. Strictly speaking it did not deal with numbers at all, but with measurable line segments. Some of its problems were thus really concerned with inverted mensuration geometry (e.g., to find the

[1] Among which the following:

— (1990), presenting in depth the comparative philological analysis of Old Babylonian »algebraic« texts.

— (1989), a concise overview of the same subject-matter, discussing also some of the general implications for our understanding of early »algebra«.

— (1986) and (1990b), presenting the evidence that Abū Bakr's *Liber mensurationum* builds on a continuation of the Old Babylonian »cut-and-paste«-tradition, and that al-Khwārizmī's geometrical proofs of the rules of *al-jabr* are inspired from the same source.

— (1990a), investigating the nature of that kind of practitioners' tradition which appears to connect the mathematicians of the early Islamic period with the Babylonian calculators.

— (1987), discussing *inter alia* the specific character of Islamic mathematics as a synthesis between Greek mathematics and such »sub-scientific« traditions.

dimensions of a rectangular field, when the area and the excess of the length over the width are given); others *represented* unknown non-geometrical entities by line segments of unknown but measurable length (e.g., a pair of numbers from the table of reciprocals whose difference is given to be 7, and which is represented by the dimensions of a rectangle of area 60, in which the length exceeds the width by 7).

Correspondingly, the *operations* used to define and solve these problems were not arithmetical but concrete and geometrical. The texts, indeed, distinguish two different »additive« operations: *joining*—e.g., a complementary square to a gnomon; and *adding* measuring numbers arithmetically. two different »subtractive« operations: *removing* a part, the inverse of »joining«; and *comparing* two different entities. And finally no less than four different »multiplicative« operations: the *arithmetical multiplication* of number by number; the *computation* of a concrete magnitude, e.g. from an argument of proportionality; the *construction* of a rectangle; and the *concrete repetition* of an entity, e.g., the repetition 9 times of a square as a 3×3-square.

The geometrical conceptualizations are reflected in geometrical techniques. The central technique for the solution of mixed second-degree problems is the partition and reorganization of figures (one might speak of a »cut-and-paste« technique). So, the rectangle referred to in the above examples is cut and reorganized as a gnomon, and a complementary square (of area $3^1/_2$ $3^1/_2$) is joined to it, yielding a greater square of area $60+12^1/_2=72^1/_2$ (cf. Figure 2, which shows the principle). Non-normalized and certain other complex problems are treated by means of a technique of »scaling« (which can be considered a change of unit in one or both directions of the plane). In all cases, the geometry involved can be characterized as »naïve«: The operations are seen immediately to yield the correct result (as *we* see, immediately and without further reflection, a=7 to follow from a+2=9=7+2); the texts contain no separate, formal proofs, for instance of Euclidean type.

This »naïve geometry« is fairly similar to the proofs given by al-Khwārizmī in his *Algebra* that the rules used to solve mixed second-degree problems are correct. Another, presumably roughly contemporary text demonstrates that the similarity can hardly be accidental. A *Liber mensurationum*—written by an otherwise unidentified Abū Bakr and only known from a Latin translation due to Gherardo da Cremona (ed. Busard 1968)—contains in its first half a large number of quasi-geometrical, quasi-algebraic problems (finding the side of a square when the sum of the area and the side is known; finding length and width of a rectangle when the area and the excess of length over width are given; etc.). These are solved in two ways: Secondarily by means of *aliabra*—evidently *al-jabr* as known from al-Khwārizmī, rhetorical reduction to standard *māl-jiḏr*-problems and solution of these by means of standard algorithms; but primarily by means of what seems to be a naïve-geometrical cut-and-paste technique, carrying perhaps the name *augmentatio et diminutio* (possibly *al-jamʿ wa'l-tafrīq* in Arabic, as I have suggested on earlier occasions; but cf. contrary evidence below).

Abū Bakr's treatise does not contain the complete gamut of Old Babylonian »algebra«. It is restricted to what looks most as surveyors' riddles: Combinations of the area and the side/all four sides/the diagonal/both diagonals, of squares/rectangles/rhombs. For this reason, Abū Bakr has no use for the Old Babylonian »scaling« technique; everything can be done by cut-and-paste style manipulation of figures.

The character of the transmission link connecting the Old Babylonian epoch with the early Islamic period is made clear by a number of observations: through Abū Bakr's inclusion of the problems in a treatise dealing purportedly with mensuration; through the mathematical contents and the riddle character of the problems; and through a description of the favourite techniques of practical geometers given by Abū'l-Wafāʾ in his *Book about that which is necessary for artisans in geometrical construction* (transl. Krasnova 1966: 115): When asked to find a square equal to three (identical) smaller squares they would present (and only be satisfied with) solutions where the latter were taken apart and put together to form a single square.

Evidently, Abū Bakr's quasi-algebraic problems are of no practical use. They will have been transmitted since the Babylonian Bronze Age in what I suggest be called a »sub-scientific tradition«, within an environment of practical geometers (surveyors, architects, master builders, and the like) not for practical use but as »recreational« problems[2]—probably connected to the training of apprentices.

Diophantos had already drawn some of his problems from such sub-scientific specialists' traditions[3], and it is a reasonable assumption that Greek theoretical mathematics started in part as critical reflection upon the ways of sub-scientific mathematical practice. But these sources were never acknowledged, and Greek mathematics did not integrate sub-scientific mathematics as a total body, nor was its aim (Hero and a few others apart) to provide practitioners with better methods. The integration of practical mathematics (as carried by the sub-scientific traditions) with theoretical mathematics (as inherited from the Greeks), was a specific accomplishment of the early Islamic culture.

One expression of this process of synthetization is precisely al-Khwārizmī's *Algebra*. *Al-jabr* itself will have been one such sub-scientific tradition, of whose prehistory nothing is known[4], but which

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[2] The established name of this *genre* can justly be regarded a misnomer: In traditional culture, »recreational« problems (and riddles in general) do not serve as *recreation:* Their purpose is agonistic (cf. Ong 1982: 44). In particular, mathematical and other profession-specific riddles have the function of fortifying professional identity and pride: »I have laid out a square field; the area, taken together with its four sides, was 140. Tell me, if you are an accomplished surveyor, the length of the side!«.

[3] In particular a large number of problems from Book I of his *Arithmetica*—see Høyrup 1990c: 17ff.

[4] The only things we know are:
— that there must be a source—al-Khwārizmī presupposes that the name of the discipline and the meaning of certain fundamental terms are already familiar, and he tells that he has been asked by the ᶜAbbaside Khalif al-Maʾmūn to write a concise treatise on the subject—not the thing a ruler (or anybody else) would

will probably have been carried by notarial and commercial calculators. The basic technique of the geometrical proofs will have been borrowed from the surveyors' tradition; the idea that proofs should be supplied, and the way to formulate them in writing by means of lettered diagrams, will have been taken from Greek mathematics.

II. The original intention of the present investigation

Another expression of the drive toward synthetization is Abū Bakr's treatise. Here, the process is the reverse of that performed by al-Khwārizmī: The basic topic is the surveyors' tradition; but it is elucidated by means of the alternative method offered by *al-jabr*. Together with the drive toward conceptual and methodological renewal, however, Abū Bakr's treatise presents definite archaic features.

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ask for if the subject did not exist already;
— and that this source can be neither Greek nor Indian *scientific* mathematics— as argued cogently by the proponents of Indian and Greek roots, respectively.

The only possibility left is thus that of an anonymous tradition—which, considering the relatively esoteric character of second-degree problems in a world where even the multiplication table was not common knowledge, must have been some kind of specialists' tradition. Certain terminological considerations (not least the use of the term *root*) suggests affinities with the Indian area. Others, however, show connection to the Mediterranean region. One possibility does not exclude the other; it is quite conceivable that the trading community interacting along the Silk Road will have carried certain algebraic techniques to everywhere between China and the Mediterranean, as it demonstrably diffused certain »recreational« problems in the whole area reached by its activity.

One of these is what may be called the »rhetorical structure« of the text. The normal format of Old Babylonian was as follows: *»If somebody has said to you: [statement]. You, by your method: [procedure]«*. The statement would be formulated in the past tense, first person singular (»I have made...«), with one exception—the excess of one length over the other would be told as a neutral fact in the present tense (»the length exceeds the width by ...«). The procedure would be told in the present tense, second person singular, alternating with the imperative; quotations from the statement justifying particular steps would be introduced by the phrase »because he has said«. All these features recur in Abū Bakr's text, together with certain others of the same descent.

This astonishing agreement between a Latin text and cuneiform tablets antedating it by 3000 years suggest that the precise wording of the Arabic text might disclose further details on the character of the transmitting tradition. In the absence of the Arabic version of the treatise it might even be possible, so it would seem, to make use of Gherardo's translation for this purpose. Gherardo, indeed, was an extremely conscientious translator (cf. also Lemay 1978: 175f)—probably one of the most accurate translators of scientific and philosophical texts of all times. Since he also translated al-Khwārizmī's *Algebra* (ed. Hughes 1986), it might therefore be possible to find his particular Latin equivalences for Arabic terms. If these could be argued to be transferred from one translation to the other, we might get access to certain terminological features of the *Liber mensurationum*.

This was what I intended to attempt and to contribute at the present symposium. As I set out to compare Gherardo's Latin version with the published Arabic text of al-Khwārizmī's *Algebra*, however, the two turned out to differ so strongly precisely in the essential chapter (the geometrical demonstrations) that reliable conclusions appeared to be out of sight. Instead, however, Gherardo's text turned out to reflect to an astonishing extent the process through which al-Khwārizmī constructed this part of his treatise, and thereby also to demonstrate that the Arabic manuscript used for all editions and translations[5] is the outcome of a process of stylistic normalization and thus not identical with al-Khwārizmī's original text—significantly farther removed from it (at least at certain points) than the manuscript used by Gherardo for his translation.

The results of this investigation are thus what I am going to present in the following, together with the meagre conclusions which can all the same be drawn concerning my original question.

[5] Oxford, Bodleian I CMXVIII, *Hunt*. 214/I, folios 1-34. I used Rosen's edition supported by Rozenfeld's Russian translation (1983) (Rosen's English translation is too free to be relied upon for my present purpose). Page-references to the Oxford Arabic text refer to the Arabic pages in Rosen 1831.

Only in the very last moment, and only owing to the kind assistance of Professor Essaim Laabid, Marrakesh, did I get hold of a xerox of the Cairo edition (Mušarrafah & Aḥmad 1939), which is also based on the Oxford manuscript. I checked all passages of relevance for the following, but found no disagreements which affect the conclusions (cf. also Gandz's discussion of the character of Rosen's errors—1932: 61-63). The major disagreements which turned up concerned the diagrams, where *both* editions proved deficient when compared with a reproduction from the manuscript facing Mušarrafah & Aḥmad 1939: 24—cf. Figure 3. Rosen omits most of the numbers which label lines and areas in the diagrams; Mušarrafah & Ahmad, e.g., do not distinguish *alif* from *mīm*, with the result that one diagram carries two of the latter but none of the former. Since letters are important for my argument but numbers not, I have chosen to reproduce Rosen's diagrams.

All English translations from the Arabic, the Latin and the Russian are mine.

III. *Gherardo's version*

I am not going to present a full stylistic and structural comparison of Gherardo's text and the published Arabic text. For good reasons, in fact: I do not read Arabic, and thus have to restrict myself to what can be done by means of dictionary and grammar[6], supported to some extent by Rozenfeld's fairly yet not fully literal Russian translation. I shall hence focus on a specific stylistic feature, which turns out to be significant.

The format of Abū Bakr's surveyors' riddles (a format which goes back, we remember, to Old Babylonian times) was presented above: »Somebody« says, »I have done«. In order to solve this problem, »You do ...«. This reflects a tradition where teaching takes the form of inculcation of rules and procedures (whether reasoned or acquired through rote learning). Modern mathematics, on the other hand, is mostly presented in the first person plural mixed up with an impersonal third person, passive or active present or future tense, »We construct«, »The line is drawn«, »the value will be«, etc. The latter format is already found in Greek mathematical texts (even though the Greek mathematicians often speak in the first person singular).

Unlike Abū Bakr, al-Khwārizmī does not stick to a single format. But his choice in particular chapters is not random. Nor is the choice of grammatical person always identical in the Oxford Arabic text and

[6] My main aids have been Wehr's dictionary (1961), Brockelmann's grammar (1960), and Souissy's doctoral dissertation on Arabic mathematical terminology (1968). I apologize for the wrong vocalizations which I will certainly have committed in the following.

in Gherardo's version. The variations of this pattern is what provides me with my main evidence.

It is evidently legitimate to ask whether even as meticulous a translator as Gherardo would really respect such minor grammatical shades in a translation. After all, his purpose was to transmit scientific knowledge and not Arabic grammatical gradations—and he did cut down two full pages (1-2) of Arabic text, containing the praise of God and the dedicatory letter, to the single phrase »After the praise and exaltation of God he says«[7].

Inside the translation, however, even grammatical shades turn out to be respected. This is confirmed by a chapter which has *not* been submitted to stylistic normalization in the Arabic version, the one on multiplication of composite expressions (Oxford Arabic pp. 15-19, Gherardo pp. 241-243). The chapter contains a large number of examples, some of them purely numerical and given neutrally, »if it is ten diminished by one times ten diminished by two«, others algebraic and set forth by a »somebody«, e.g., »And if he has said, ten and thing times its equal«[8]. All the way through the chapter, the forms agree—and in the single case where the Arabic text uses the passive tense, this is also done by Gherardo[9]. No doubt, then, that Gherardo took care to render Arabic grammatical details as closely as possible in Latin[10]; we may confidently trust him as a witness of

[7] »Hic post laudem dei et ipsius exaltationem inquit« (Hughes 1986: 233 line I,4). (All page references to Gherardo's text in the following refer to this edition).

[8] Oxford Arabic p. 16, last line (*wa'in qāla* ...), Gherardo p. 242, line 37 (*Quod si dixerit: "Decem et res in decem et rem"*).

Strictly speaking one might claim that even the purely numerical examples are preceded by a reference to a »somebody«—*viz* the one which inaugurates the whole chapter. Still, this does not change the fact that the two types are treated differently.

[9] Oxford Arabic p. 16, line 6 from bottom (*f'iḏā qīla laka*); Gherardo p. 242, line 31 (*Cumque tibi dictum fuerit*).

[10] We may compare this with the two modern translations. Rosen (English pp. 21-27) misses the distinction between numerical and algebraic examples completely; Rozenfeld respects it in full, but renders both active and passive forms as »they

the forms used in *his* Arabic original, even when they differ from ours—in particular, of course, because the deviations turn out to be systematic, which they would not be if resulting from occasional nodding.

Apart from this chapter on multiplication, we shall have to look at three different passages, which demonstrate systematic variations in usage and as regards the relations between the two versions of the text: the presentation of the rules used to solve the mixed equations; their geometrical proofs; and the chapter on addition and subtraction of composite expressions. When adequate, other than grammatical considerations will be made appeal to. For the moment, we shall concentrate on Gherardo's text.

The rules

The chapter containing the rules (pp. 234-236) starts off by presenting the three composite modes in non-personal format, »treasures[11] and roots are made equal to number« etc. Then each of

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have said« (*skažut*), judging (rightly, I suppose) the distinction to be a mere stylistic whim.

[11] »Treasure« renders Latin *census* and Arabic *māl*. This translation is to be preferred to the conventional »square«, which is misleading for several reasons. Firstly, »square« possesses geometrical connotations, which were only to be associated with *māl* in later times—indeed by those generations who had learned their algebra from al-Khwārizmī. The customary translation therefore makes a fool of al-Khwārizmī when he takes great pain to explain that a geometrical square represents the *māl*. Secondly, the *algebraic* understanding of »square« is also misleading: The square is the second power of the unknown, and no unknown in its own right. This, again, makes a fool of al-Khwārizmī (and quite a few modern scholars *have* considered him lacking in mathematical consequence on this account) when, after finding the root (*jiḏr*), he also finds the *māl*. Thirdly, speaking of the *māl* as a second power of the unknown makes us believe that the *root* is meant as the *root of the equation*—once again a meaning only taken on by the term as a consequence of al-Khwārizmī's work. To al-Khwārizmī, the *root* is simply the *square root* of the *māl*.

That the *māl* is considered a basic and not a derived unknown is born out by the rather frequent use of the term to designate the unknown in a first degree them is exemplified in personalized style, and followed by a rule: »But treasures and roots which are made equal to number are as if you say, 'a treasure and ten roots are made equal to thirty nine dragmas'. Whose meaning is this: from which treasure, to which is added the equal of ten of its roots, will be collected a totality which is thirty nine? Whose rule is that you halve the roots, which in this question are five. So multiply them with themselves, and from them arise twenty five. Add to these thirty nine, and they will be sixty four. Whose root you take, which is eight [...]«[12].

This succinct rule for the normalized case of the first composite mode is followed by a more discursive and explanatory exposition of the reduction of non-normalized cases to normal form. In this occurs one of the two grammatical first persons of the chapter: »It is therefore needed that two treasures be reduced to one treasure. But now we know that one treasure is the half of two treasures. Therefore reduce everything which is in the question to its half [...]«. The other turns up in the concluding passage: »These are thus the six modes [three simple and three composite—JH], which we mentioned in the beginning of this book of ours. And we have also already explained them and said what the modes were of those in which the roots are not halved [i.e., in the simple modes—JH]. Whose rules and necessities we have shown in the preceding. That, however, which is necessary

problem as a *māl*—e.g., in one of the monetary problems from al-Karajī's *Kāfī* (ed., transl. Hochheim 1878: iii, 14), and in the bulk of first-degree problems contained in the *Liber augmenti et diminutionis* (ed. Libri 1838: I, 304ff; Libri's commentary, it is true, misses the point completely, demonstrating *ad oculos* the dangers of the conventional translation).

[12] »Census autem et radices que numero equantur sunt sicut si dicas: 'Census et decem radices equantur triginta novem dragmis.' Cuius hec est significatio: ex quo censu cui additur equale decem radicum eius aggregatur totum quod est triginta novem. Cuius regula est ut medies radices que in hac questione sunt quinque. Multiplica igitur eas in se et fiunt ex eis viginti quinque. Quos triginta novem adde, et erunt sexaginta quattuor. Cuius radicem accipias que est octo [...]« (p. 234, lines B.5-11).

on the halving of the roots in the three other sections we have put down with the verified sections. Now, however, for each section we make a figure [*forma/ṣūrah*], through which the cause of the halving shall be found«.

The proofs

As we shall see below, this may be what al-Khwārizmī intended at first. In all known versions of the text, however, he presents us with *two* diagrams for the case »treasure and roots made equal to number«.

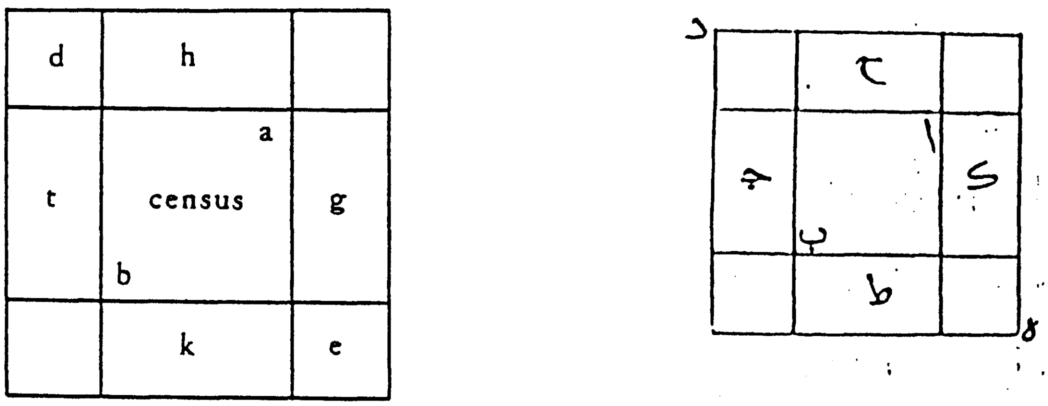

Figure 1: Treasure and roots made equal to number (A) (Hughes 1986: 237; Rosen 1831: 10 (Arabic))

The first of these is peculiar in several ways. As in those Greek mathematical works which will have been known to al-Khwārizmī at least from his colleagues in the *House of Wisdom,* it is lettered—but several letters label whole rectangles and not points[13]. Moreover,

[13] In *Elements* II, 5-8, it is true, a notation occurs which at first looks similar: the designation of a gnomon by means of three letters (ed. Heiberg 1883: I, 130-140). But at closer inspection the similarity turns out to be misleading, as the letters marks point on a circular arc going through the three quadrangles from which the gnomon is composed.

it does not *halve* the number of roots, so as to represent the 10 roots by two rectangles of length 5 and *R* (*R:* the root); it divides 10 into 4 times $2^1/_2$, and represents the 10 roots by four rectangles $2^1/_2$ *R*.

All the other diagrams follow the respective rules closely, halving the number of roots and manipulating the corresponding rectangles and a quadrate of unknown dimensions so as to permit a quadratic completion:

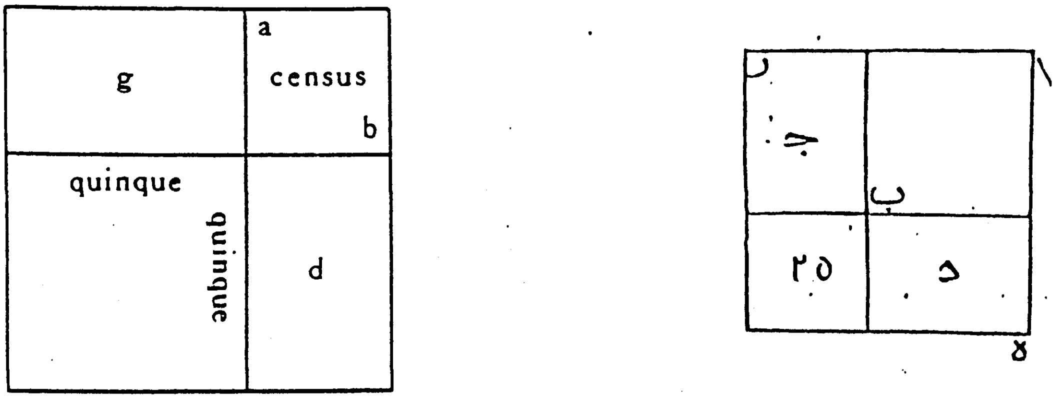

Figure 2: Treasure and roots made equal to number (B)
(Hughes 1986: 238; Rosen 1831: 11 (Arabic))

The alternative diagram for the first case labels whole rectangles by single letters, as does the main diagram; the others, to the contrary, follow the normal Greek (and, as it was to become, the normal Arabic) pattern.

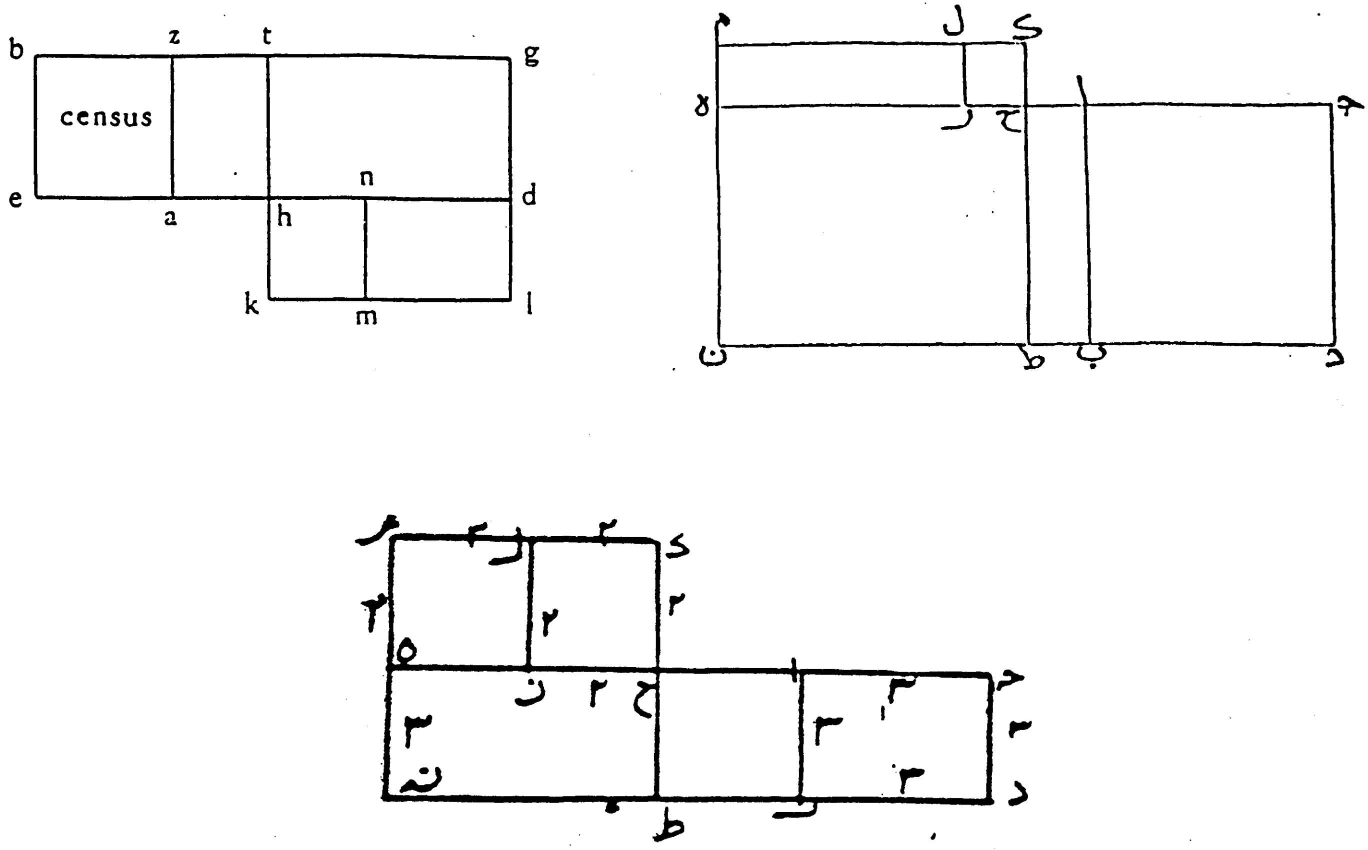

Figure 3: Treasure and number made equal to roots
(Hughes 1986: 239; Rosen 1831: 13 (Arabic); manuscript, reproduced from Mušarrafah & Aḥmad (eds) 1939: (facing) 24)

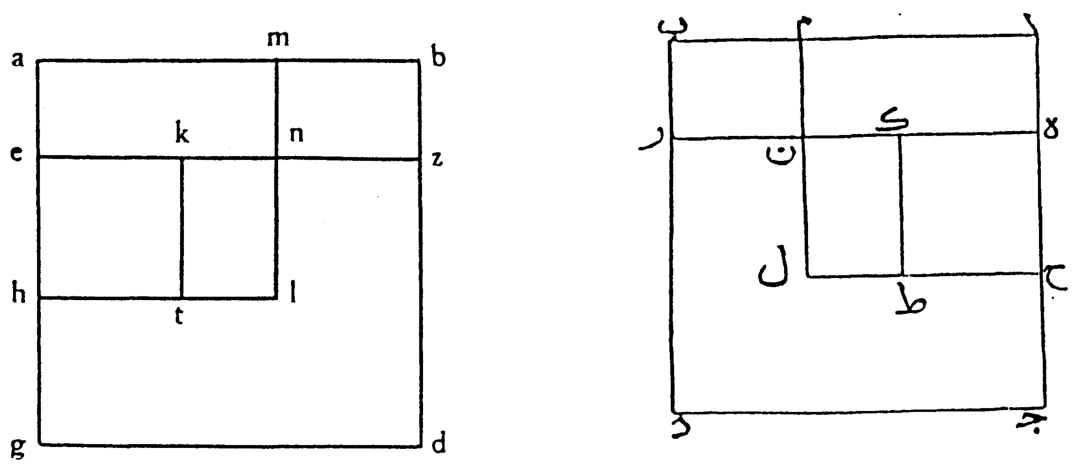

Figure 4: Roots and number made equal to treasure
(Hughes 1986: 240; Rosen 1831: 15 (Arabic))

When we turn our attention to the grammatical person used in the text, differentiations will be observed which follow another pattern. The first proof of the first case starts out in the first person singular (future tense): *accipiam, faciam*. Then comes an argument that *we have known* (*scivimus*) a certain surface to have a certain numerical value—*viz* from the statement of the problem; from that point onwards, everything with one exception continues in the first person plural (*addiderimus, nos novimus, minuam, mediamus, multiplicamus, addimus, compleatur nobis, sufficit nobis*). The style of the whole argument is discursive and almost colloquial: »I take [...]. Now we know [...]. If now we add [...]. But we have found out [...]. Therefore one of its sides is its root, which is eight. I shall therefore subtract [...]. However, we have only halved the ten roots [...] in order that the larger figure may be completed for us with that which was lacking for us in the four corners [...].

The alternative proof for the first case is formulated in the first person plural all the way through, except for one phrase »take its root«. It is also more concise and formal in style, and one might believe that al-Khwārizmī has left behind certain initial pedagogical and stylistic habits and completed a shift to formal writing in the plural. This guess, however, is contradicted by the proofs of the last two cases. Both of these start out by describing the construction process in the first person singular, and both of them afterwards make a partial shift to the plural; the plural seems to be used in references to what *we* know or want to be done, and when performing arithmetical operations on the already existing diagrams (this rule, it will be observed, does not fit the second proof, and only fits the first proof completely in a specific interpretation to which we shall return below). The last proof also contains a reference to »the three roots and four which I indicated for you« (*quos tibi nominavi*).

Both the third and the fourth proof give a rather discursive explanation of the purpose of the construction of the diagram, i.e., of the way the squares and rectangles of the diagrams represent the given treasure, roots and number. Even this makes their style different from that of the second proof.

Addition and subtraction of composite expressions

The proofs of the rules for solving the mixed second-degree equations were borrowed by numerous mathematical authors in later centuries, Arabic as well as Latin. But they are not the only geometrical proofs offered by al-Khwārizmī. After the chapter on multiplication of binomials comes another on »aggregation and diminution«, which first gives some examples of addition and subtraction of binomials and trinomials, promising an explanation by means of a figure in the end of the chapter, and then proceeds to exemplified rules for the multiplication of roots by integers and their reciprocals and for the multiplication and division of a root by another root. In the end of the chapter (pp. 245-247) the promised proofs are brought—two proofs by means of diagrams and one rhetorical, because the diagram attempted by al-Khwārizmī has turned out to »make no sense«.

The promise is stated in the first person singular, and the rules and examples set forth in the second person singular (»You should know that if you want to take half the root of a treasure, you should multiply [...]« (244, line 18); »if you want to divide the root of nine by the root of four, divide nine by four [...]« (244, lines 32f). The choice of grammatical person in the geometrical proofs agrees with the main style of the previous ones: Making the constructions in the first person singular, but using the first person plural when »we« wish to do something, when »we« see, and when arithmetical operations are performed on the basis of diagrams which are already at hand.

IV. The Oxford text

As told above, the manuscript which has been used for the modern editions and translations differs from the one which Gherardo must have used. It does so in several ways, of which I shall concentrate on two.

Let us first apply the standard methods for comparing classical geometrical texts: The agreement/disagreement between the letterings of diagrams and in the structures of proofs. Already at this simple level, indeed, the relation between the two manuscripts can be seen to differ from chapter to chapter.

Starting from behind, the diagrams used for the addition of binomials exhibit optimal agreement: *alif*-a, *bāʾ*-b, *jīm*-g, *dāl*-d, *ḥāʾ*-h, *zāy*-z, *hāʾ*-e. If we observe that the labels of the four rectangles in Figure 1 can be freely interchanged, the same agreement is seen in the first proof of the case »treasure and roots made equal to number« (with the supplementary correspondences *tāʾ*-t, *kāf*-k). The alternative diagram in the Oxford manuscript contains two letters *rāʾ* and *hāʾ* with no counterparts in Gherardo's version (cf. Figure 2), and the texts differ correspondingly: Where Gherardo only refers to »the quadrate of the greater surface« (p. 238, lines 42f), the Oxford manuscript has »the greater surface, which is the surface *rh*«[14] . Apart from that, the letters agree according to the same scheme of correspondences. So they do in every respect in the case »roots and number made equal to treasure« (Figure 4; since the Oxford manuscript omits many diacritical

[14] *al-saṭḥ al-aʿẓam al-ḏī huwa saṭḥ RH*, if I read it correctly (p. 11, line 1).

dots, the correspondence *rā'*-z (Rosen) cannot be distinguished safely from the correspondence *zāy*-z (Mušarrafah & Aḥmad)).

In the diagram for the case »a treasure and twenty-one made equal to ten roots«, on the other hand, only 3 out of 12 letters agree. Remarkable differences will also be found in the progression of the proofs, together with significant similarities.

One of these demonstrates that one of the proofs is made on the basis of the other, and not independently. This is an idiosyncratic didactical explanation that if in a quadrate »a side is multiplied by one, the outcome is one root; and if by two, two of its roots«[15]. Another similarity, coupled with a deviation, shows Gherardo's source to be better than the Oxford manuscript. Gherardo explains (p. 238, lines 51f) his rectangle *ga* to be 21; nothing similar is found in the Oxford version; but at a later point *both* texts refer to this value as already known[16]. The Oxford text is thus the result of a revision—a *Verschlimmbesserung*, indeed.

Gherardo's proof only leads to one of the two solutions (namely 3); the Oxford proof ends by also giving the solution 7. Alas, the diagram only fits the case where the root is smaller than 5 (unless we accept that line segments may have negative lengths, which was certainly not intended). While Gherardo's proof errs by incompleteness, the Oxford version commits a genuine mathematical mistake. The person responsible for this minor blunder, however, cannot be the editor who is responsible for the changed lettering and for the omitted identification of the rectangle *ga* (Gherardo's lettering) as 21; this follows from a comparison with Robert of Chester's translation[17]. Two

[15] Gherardo, p. 238, lines 54-56; Oxford Arabic p. 11, lines 4-3 from bottom.

[16] According to the Oxford text (p. 12, line 8), »it has already become clear« (*qad kāna tabayyana*?).

[17] Ed. Hughes 1989: 39-41. Robert has the same lettering as Gherardo, except that he interchanges the correspondences of *hā* and *ḥā* and makes *kāf* correspond to *c*. He has the same diagram as Gherardo (and, lettering apart, the Oxford version)—the supplementary diagram found in Karpinski's edition (1915: 85) has been added by Scheubel. He also tells the area of *ga* to be 21. But like the Oxford

hands, at least (one working before and one after the Oxford manuscript family branched off from Robert's family, and none of them too competent) will have been active in recasting the Oxford version of this particular proof.

Similarly, the Oxford proof of the case »three roots and four made equal to a treasure« has been tinkered with: it omits Gherardo's observation that the area of *al* is $6^1/_4$ (p. 240, lines 101f), and changes one passage through and through (p. 14, lines 3-7). All other proofs, on the other hand, agree completely in mathematical structure, apart from one or two brief omissions.

The other approach is through the use of grammatical persons. If, once again, we start from behind, the proofs concerned with the addition of binomials on the whole follow the same system as Gherardo: Use of the first person singular for constructions, and of the first person plural for what »we« know or want to do, and for arithmetical argumentation on the already existing diagram[18].

The proofs of the »rules« for mixed second-degree equations, on the other hand, exhibit a much more even picture than Gherardo's text. *No* first person singular and *no* imperatives are to be found: all are replaced by the first person plural. The only exception is the mistaken insertion »proving« the double solution in the case »treasure and number made equal to roots«, which makes use of an invariable »you«.

In the chapter on the multiplication of composite expressions, we remember, the Oxford text agreed with Gherardo in the use of grammatical person. The same holds for the definition of the six cases, for the exposition of the rules, and in the chapters containing algebraic

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version, Robert gives the double solution in spite of his diagram, in words which come too close to those of the Oxford version to be independent; Robert also agrees with this version in omitting erroneously from his description the drawing of *ht* (Gherardo's lettering).

[18] Only one exception will be obesred: to Gherardo's *secabo*, »I shall cut off« (p. 246, line 83), however, corresponds the plural *qataʿanā* (p. 22, line 9).

problems. Apart from the chapters on proofs, indeed, the two versions only diverge in this respect at two places—and that, curiously enough, »the other way round«.

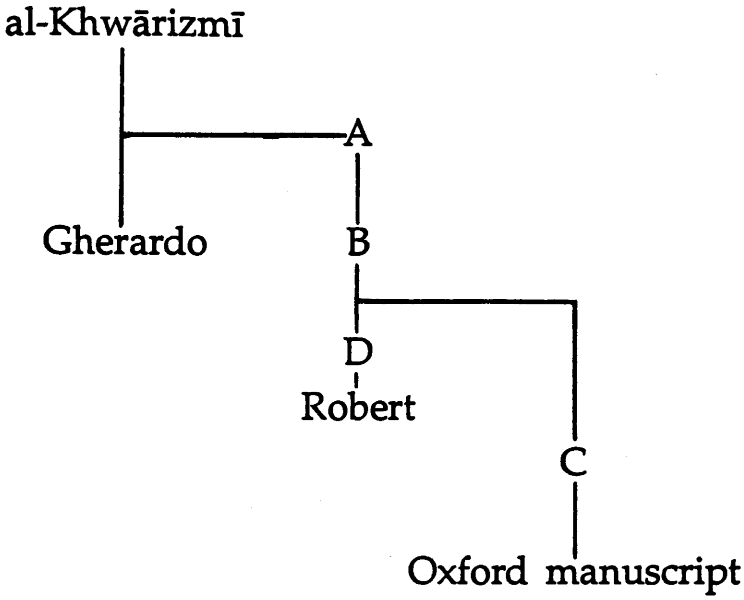
One of the places is where al-Khwārizmī rounds off the presentation of the six modes and their »rules« and enters his geometrical demonstrations. The Oxford text (p. 8, line 11-16) speaks in the first person singular (»in the first part of my book«, »I have made clear«, etc.) and stays in the role of an author speaking to his reader (»the square which you seek«—p. 8, 2 last lines). Gherardo, as quoted above, speaks (p. 236, lines 67-72) in the plural (»in the beginning of this book of ours«; »we have shown«; »the treasure which we want to know«). The other place is when al-Khwārizmī tells that he has attempted a geometrical proof for the addition of trinomials, but that the result was unsatisfactory (Oxford version p. 24 lines 5-7, Gherardo p. 247, lines 93-93). In both places, the author steps forward as the author of the whole book. Most plausibly, Gherardo has felt it appropriate to follow normal Latin style precisely in these places; there is no reason to believe that his Arabic manuscript differed from the Oxford manuscript in the two passages in question.

V. Conclusions concerning al-Khwārizmī

In all other places, however, we must prefer Gherardo's choice of grammatical person to the Oxford choice. If al-Khwārizmī had written his demonstrations of the rules for the mixed equations in an invariable first person plural, Gherardo (or anybody between him and al-Khwārizmī) would have had no reason to introduce the systematic distinctions which are found in *his* version. Nor would mere sloppiness on Gherardo's (or an intermediate copyist's) part have produced anything resembling a *system*. The divergent uses of grammatical person in the two versions must therefore (apart from the two Gherardo passages in »author's plural«) be explained as deviations of the Oxford version from al-Khwārizmī's original text, produced by somebody aiming at stylistic normalization or in any case following his own stylistic preferences while rewriting—but since normalization has taken place even in proofs where mathematical substance is copied faithfully, intentional rectification of style seems to be involved.

This rectification, as we have seen, only affects the geometrical proofs of the rules for mixed equations but not the proofs concerned with the addition of binomials (nor other matters, indeed); comparison with Robert of Chester's translations, furthermore, tells that it has taken place before his times. We cannot trust Robert's own grammatical choices, it is true[19]. But since the insertion on the double solution, which was known to Robert, has escaped that grammatical normalization which has affected its surroundings, the normalization must precede the insertion, which must precede Robert's translation. The stemma will have to be something like this:

[19] It is thus no powerful argument that Robert mostly uses the first person plural. This might easily be a consequance of his own stylistic feelings—even the insertion on the double solution is, indeed, formulated first person plural throughout. In general, it should be remembered, Robert of Chester is a less literal translator as Gherardo, and would, for instance, reduce »it is obvious to us« to a mere »it is obvious«.



Here, A represents the grammatical normalization and B the mistaken addition on the double solution. C corresponds to the changed lettering in Figure 4. Omissions from the proofs take place both in the region A-B and in the vicinity of C.

It is noteworthy that »A« only submitted the first set of geometrical demonstrations to his stylistic treatment. Evidently, he must have found the other demonstrations uninteresting or superfluous—a view which was shared by others[20].

Starting from the above conclusion, *viz* that Gherardo's text can be regarded as a faithful reflection of al-Khwārizmī's own use of grammatical person, we may make some further inferences concerning al-Khwārizmī's working method. The use of the »somebody«, the »I« and the »you«, as pointed out above, belongs with the sub-scientific traditions drawn upon by al-Khwārizmī. When presenting rules, problems and solutions/methods borrowed from these, he takes over their format, even when the words are actually his own.

[20] The proofs concerned with the addition of binomials are omitted by Robert of Chester and thus, in all probability, by his original (say, by »D«); and they were not taken over by Abū Kāmil or other later writers on algebra.

In proofs, however, his ways are different—and, as a matter of fact, uneven. The principal system, as we remember, was that constructions were told in the first person singular, while intentions, insights and arithmetical argumentation from existing diagrams were told in the first person plural. There were, however, two exceptions to this rule, both to be found in the proofs concerned with the case »treasure and roots made equal to number«. Firstly, in the first proof the outer segments of the side of the larger square are subtracted by a *minuam* (p. 237, line 23). Secondly, the second proof employs the plural consistently, apart from the slip where a »sub-scientific imperative« steals in.

The first exception may not really be one. The first proof, indeed, is the one most obviously taken over from the sub-scientific cut-and-paste tradition[21]; within this tradition, however, the subtraction in question would be a real, geometrical removal, and thus one of those constructive steps which al-Khwārizmī tells in the first person singular in other places.

The other exception, however, is indubitable. It looks, indeed, as a first step toward that stylistic normalization which was carried through by »A«. The context is the alternative proof. The best explanation of its anomalous style seems to be that it has been written after the other proofs. It could have crept in during an early revision of the text performed by somebody else, familiar perhaps with ibn Turk's similar proof (Sayılı 1962: 145f—ibn Turk, as a matter of fact, also speaks in the first person plural). But the way rectangles are labeled by only one letter reminds too much of al-Khwārizmī's first proof to make the intervention of a foreign hand plausible. It is more likely that al-Khwārizmī first prepared a text containing one diagram, and one proof, for each case; this, indeed, is what is promised in the preceding passage; at some later moment, perhaps after discussion with more grecophile colleagues at the House of Wisdom he inserted

[21] See my (1990a: 80 and note 61).

another diagram and proof somewhat closer to *Elements* II, 6, expressing himself in a somewhat different style[22].

This and other questions may be answered more definitively through further philological work on the text. One thing, however, should then be remembered: Since Gherardo's translation is (as far as it goes) closer to the original than the Oxford version, no investigation of al-Khwārizmī's *Algebra* should be made without attentive consideration of this Latin version, all modern editions and translations being based on the Oxford manuscript. Robert's less literal translation is not to be relied upon to the same extent; but even Robert may provide us with important supplementary evidence.

Since the Oxford text appears to be the outcome several deliberate attempts at revision, it would be obvious to get behind it by taking other Arabic manuscripts of the work into account[23]. But even the published texts—Oxford and Latin verions—might provide many clues. After all, the present paper was based only on very few textual parameters, which turned out to yield unexpected quantities of information. Other parameters—vocabulary, grammar, structure of the exposition—might yield more.

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[22] It appears that this conjectural »later moment« must be considerably later: In a newly located, better manuscript of the Latin translation of al-Khwārizmī's algorism, which refers to the *Algebra* as an earlier work, al-Khwārizmī still makes use of the first person singular (Menso Folkerts, private communication).

[23] Three are mentioned by Sezgin (1974: 240, 401).

VI. Conclusions concerning Gherardo and the *Liber mensurationum*

The lack of agreement between Gherardo's source and the Oxford version thwarted my original project: To find the Arabic terminology used by Abū Bakr in the *Liber mensurationum*. Still, the chapters of al-Khwārizmī's *Algebra* which have been least tinkered with in the Oxford version provide some bits of information.

Most important is probably that one of the main uses of *aggregare* in the translation of al-Khwārizmī cannot possibly fit its use in the translation of Abū Bakr. Recurrent in the latter are phrases like »I have aggregated the side and the area [of a square]« and »I have aggregated its four sides and its area« (Busard 1968: 87). A survey of the use of the term in the translation of al-Khwārizmī, from the beginning through the second set of geometrical demonstrations, gives 7 correspondences to *balaġa*, »to reach«, »to amount to«, together with derivations from this root; 9 to *jamaᶜa*, »to gather«, »to put together«, and to derived forms (most indeed to *ajtamaᶜa* (VIII), »to be/come together«, and concentrated in the chapter on addition of binomials); one instance falls in a passage which has been changed in the Oxford version; one, finally, expands a passage where this version only has a *kāna*, »to be/occur«, but where *balaġa* might have been used, and may thus have been used in the original text. Of course, *jamaᶜa*, would fit the use of the term in the *Liber mensurationum*; but *balaġa* would certainly not.

Two other additive terms from the *Liber mensurationum* are *adiungere* and *addare*. Both are also found in Gherardo's translation of al-Khwārizmī, the relatively rare *adiungere* mostly where the Oxford version has *jamaᶜa*[24], the more frequent *addare* corresponding to *zāda,* »to increase«, »to augment«.

The obvious conclusions to draw from these observations are negative: Even though he took great care to be precise, Gherardo made no attempt to establish a one-to-one correspondence between Arabic and Latin terms used within a single work[25]. *A fortiori,* whatever terminological correspondences we may establish *within* a particular translation cannot be transferred without the greatest circumspection to other translations. Even if the Arabic original used by Gherardo in his translation of al-Khwārizmī had been at hand, it would have been difficult to carry through my original project, perhaps impossible. Still, one observation can be made: even though my previous conjectural identification of Abū Bakr's *augmentatio et diminutio* with *al-jamᶜ wa'l-tafrīq* is not directly excluded by the equivalence *jamaᶜa-aggregare,* it is certainly not substantiated.

[24] In one instance, Gherardo's *adiungare* (p. 238, line 50) corresponds to an Arabic *waṣala,* »to connect«, »to join«, »to attach« in the Oxford edition (to judge from the printed editions, the Oxford manuscript has a meaningless *nṣm*—Rosen 1831: 11 line 7). But since this falls in the proof of the second case, which was emended both mathematically and stylistically, no firm conclusion follows.

[25] Probably for good reasons; if his translations were to be used by others, he was constrained to respect, or at least compromise with, the conceptual boundaries of current Latin usage. Evidently, these differed strongly from those of the Arabic.

Even in his choice of grammatical form, he was of course constrained by the difference between the two languages. One of his strategies to circumvent the problem was touched at above: When an Arabic perfect was too obviously not a preterit, Gherardo would choose the Latin future tense to demarcate it from the implicitly imperfect present tense.

BIBLIOGRAPHY

Brockelmann, Carl, 1960. *Arabische Grammatik*. 14. Auflage—besorgt von Manfred Fleischhammer. Leipzig: Harrassowitz.

Busard, H. L. L., 1968. "L'algèbre au moyen âge: Le »Liber mensurationum« d'Abû Bekr". *Journal des Savants*, Avril-Juin 1968, 65-125.

Gandz, Solomon (ed.), 1932. "The Mishnat ha Middot, the First Hebrew Geometry of about 150 C.E., and the Geometry of Muhammad ibn Musa al-Khowarizmi, the First Arabic Geometry <c. 820>, Representing the Arabic Version of the Mishnat ha Middot. A New Edition of the Hebrew and Arabic Texts with Introduction, Translation and Notes". *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen* **2**.

Heiberg, J. L. (ed., transl.), Euclidis *Elementa*. 5 vols. Leipzig: Teubner, 1883-1888.

Hughes, Barnabas B. (ed.), 1986. "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*: A Critical Edition". *Mediaeval Studies* **48**, 211-263.

Hughes, Barnabas B. (ed.), 1989. Robert of Chester's Latin translation of al-Khwārizmī's *Al-jabr*. A New Critical Edition. (Boethius. Texte und Abhandlungen zur Geschichte der exakten Naturwissenschaften, 14). Wiesbaden: Franz Steiner.

Hochheim, Adolph (ed., transl.), 1878. *Kafî fîl Hisâb (Genügendes über Arithmetik)* des Abu Bekr Muhammed ben Alhusein Alkarkhi. 3 vols. Halle: Louis Nebert.

Høyrup, Jens, 1986. "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra". *Erdem* **2** (Ankara), 445-484.

Høyrup, Jens, 1987. "The Formation of »Islamic Mathematics«. Sources and Conditions". *Science in Context* **1**, 281-329.

Høyrup, Jens, 1989. "Zur Frühgeschichte algebraischer Denkweisen". *Mathematische Semesterberichte* **36**, 1-46.

Høyrup, Jens, 1990. "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". *Altorientalische Forschungen* **17**, 27-69, 277-369.

Høyrup, Jens, 1990a. "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* **28**, 63-86.

Høyrup, Jens, 1990b. "»Algèbre d'*al-ğabr*« et »algèbre d'arpentage« au neuvième siècle islamique et la question de l'influence babylonienne". *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1990 nr. 2. To appear in *Actes du Colloque »D'Imhotep à Copernic«*, Université Libre de Bruxelles, Novembre 1989.

Høyrup, Jens, 1990c. "Sub-scientific Mathematics: Undercurrents and Missing Links in the Mathematical Technology of the Hellenistic and Roman World". *Filosofi og videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints* 1990 nr. 3. To appear in *Aufstieg und Niedergang der römischen Welt*, II vol. 37,3.

Karpinski, Louis Charles (ed., transl.), 1915. *Robert of Chester's Latin Translation of the Algebra of al-Khowarizmi*. (University of Michigan Studies, Humanistic Series, vol. 11). New York. Reprint in L. C. Karpinski & J. G. Winter, *Contributions to the History of Science*. Ann Arbor: University of Michigan, 1930.

Krasnova, S. A. (ed., transl.), 1966. "Abu-l-Vafa al-Buzdžani, *Kniga o tom, čto neobxodimo remeslenniku iz geometričeskix postroenij*", pp. 42-140 *in* A. T. Grigor'jan & A. P. Juškevič (eds), *Fiziko-matematičeskie nauki v stranax vostoka*. Sbornik statej i publikacij. Vypusk I (IV). Moskva: Izdatel'stvo »Nauka«.

Lemay, Richard, 1978. "Gerard of Cremona". Pp. 173-192 *in Dictionary of Scientific Biography*, vol. XV. New York: Scribner.

Libri, Guillaume, 1838. *Histoire des mathématiques en Italie*. 4 vols. Paris, 1838-1841. Reprint Hildesheim: Georg Olms, 1967.

Mušarrafah, ᶜAlī Muṣṭafā, & Muḥammad Mursī Aḥmad (eds), 1939. al-Khwārizmī, *al-Kitāb al-muḫtasar fī ḥisāb al-jabr wa'l-muqābalah*. Cairo. Reprint 1968.

Ong, Walter J., 1982. *Orality and Literacy. The Technologizing of the World*. London & New York: Methuen.

Rosen, Frederic (ed., transl.), 1831. The *Algebra* of Muhammad ben Musa, Edited and Translated. London: The Oriental Translation Fund.

Rozenfeld, Boris A. (perevod), 1983. Muhammad ibn Musa al-Xorezmi, *Kratkaja kniga ob isčislenii algebry i almukabaly*, pp. 20-81, 118-142 *in* S. X. Siraždinov (ed.), Muxammad ibn Musa al-Xorezmi *Matematičeskie traktaty*. Taškent: Izdatel'stvo »FAN« Uzbekskoj CCP.

Sayılı, Aydın, 1962. *Abdülhamid ibn Türk'ün katışık denklemlerde mantıkî zaruretler adlı yazısı ve zamanın cebri (Logical Necessities in Mixed Equations by ʿAbd al Ḥamîd ibn Turk and the Algebra of his Time)*. (Publications of the Turkish Historical Society, Series VII, N° 41). Ankara: Türk Tarih Kurumu Basımevi.

Sezgin, Fuat, 1974. *Geschichte des arabischen Schrifttums*. V. Mathematik bis ca. 430 H. Leiden: Brill.

Souissi, Mohamed, 1968. *La langue des mathématiques en arabe*. Thèse principale pour le Grade de Docteur d'État, Université de Paris, Faculté des lettres et des sciences humaines. Tunis: Université de Tunis.

Wehr, Hans, 1961. *A Dictionary of Modern Written Arabic*. Edited by J. Milton Cowan. Wiesbaden: Harrassowitz / London: Allen & Unwin.

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JENS HØYRUP

The Formation of "Islamic Mathematics"
Sources and Conditions

In memory of George Sarton

The Argument

The development of autonomous theoretical science is often considered a "Greek miracle." It is argued in the present paper that another "miracle," necessary for the creation of modern science, took place for the first time in the Islamic Middle Ages, viz. the integration of (still autonomous) theory and (equally autonomous) practice.

The discussion focuses on the mathematical disciplines. It starts by investigating the plurality of traditions which were integrated into Islamic mathematics during its formation, emphasizing practitioners' "sub-scientific" traditions, and shows how these were synthesized in a way virtually unknown in earlier cultures. A discussion of the sociocultural roots of this specific synthesis concludes that a major role was played in the earlier period by the combination of fundamentalist convictions characteristic of Islam – that the most humble daily activity is directly responsible to the highest ontological level, while conversely this highest level is concerned with the humblest ranks of daily existence – with the absence of an institutionalized "Church" able to monopolize the interpretation of the mutual bond of the divine and the everyday levels.

As the institutions of learning crystallized around the turn of the millennium, the integrative attitude to theory and practice was fixated institutionally; the latter process is discussed, first with the example of the *madrasah* institution as the carrier of an arithmetical textbook tradition, and second with that of the bond between astronomy and theoretical geometry.

Contents

I. Introducing the Problem

When the "history of science" in prehistoric or Bronze Age societies is described, one normally finds a description of the technologies and of the practical knowledge which these technologies presuppose. This state of our art reflects perfectly the state of the arts in these societies: they present us with *no specific, socially organized, and systematic search for and maintenance of cognitively coherent knowledge* concerning the natural or practical world – i.e., with nothing like our own scientific endeavor.

The ancestry of that specific endeavor is customarily traced back to the "Greek Miracle," the rise of pure theory, well described by Aristotle (*Metaphysica,* 981^b 14–982^a 1, trans. Ross [1928] 1972):

> At first he who invented any art whatever that went beyond the common perceptions of man was naturally admired by men, not only because there was something useful in the inventions, but because he was thought wise and superior to the rest. But as more arts were invented, and some were directed to the necessities of life, others to recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered [. . .].
> So [. . .], the theoretical kinds of knowledge [are thought] to be more of the nature of Wisdom than the productive.

This passage establishes the fundamental distinction between "theoretical" and "productive" knowledge, between "art" and "science," and thus the break with those earlier traditions where knowledge beyond the useful was carried precisely by the same groups that possessed the greatest amount of useful knowledge.[1] A fairly complete social and cognitive separation of the two is also inherent, though not fully explicit. Even if obvious deviations from this ideal can be found in several ancient Greek scientific authors (some of whom we shall mention below), Aristotle's

[1] I have discussed this relation at some depth for the case of Old Babylonian mathematics in my paper of 1985. A short but striking illustration for the case of Egypt is supplied by the opening phrase of the Rhind Mathematical Papyrus, the mainly utilitarian contents of which are presented as "accurate reckoning of entering into things, knowledge of existing things all, mysteries . . . secrets all" (trans. Chace et al. 1929, plate 1; cf. the similar translation in Peet 1923, 33).

discussion can be regarded as a fair description of the prevailing tendency throughout Greek antiquity.

On the other hand, it is definitely not adequate as a description of modern or contemporary attitudes to the relationship between science and technology (which we are often disposed to regard as "applied science"[2]). We are separated from the Bronze Age organization of knowledge not only by a "Greek Miracle," but also by at least one later break, leading to the acknowledgment of the practical implications of theory. Customarily we locate this break in the late Renaissance, and regard Francis Bacon as a pivotal figure.

One aim of the present paper is to show that the break took place earlier, in the Islamic Middle Ages, which first came to regard as a fundamental epistemological premise that problems of social and technological practice can (and should) lead to scientific investigation, and that scientific theory can (and should) be applied in practice. Alongside the "Greek Miracle" we shall hence have to reckon an "Islamic Miracle." A second aim is to trace the circumstances that made medieval Islam produce this "miracle."

I shall not pursue these two aims in broad generality, which would be beyond my competence. Instead, I shall concentrate on the case of the mathematical sciences. I shall do so not as a specialist in Islamic mathematics,[3] but as a historian of mathematics with a reasonable knowledge of the mathematical cultures connected to that of medieval Islam, basing myself on a fairly broad reading of Arabic sources in translation. What follows is hence a tentative outline of a synthetic picture as it suggests itself to a neighbor looking into the garden of Islam; it should perhaps best be read as a set of questions to the specialists in the field, formulated by an interested outsider.

From this point of view, the mathematics of the Islamic culture[4] appears to differ from its precursors by a wider scope and a higher degree of integration. It took up the full range of interests of all the mathematical traditions and cultures with which it came in contact, "scientific" as well as "subscientific" (a concept which I discuss below); furthermore, a significant number of Islamic mathematicians mastered and worked on the whole gamut from elementary to advanced mathematics (for which reason they tended to see the former *vom höheren Standpunkt aus* [from a higher

[2] I shall not venture into a discussion of this conception, which is probably no better founded than its Aristotelian counterpart.

[3] This status is barred to me already because my knowledge of Arabic is restricted to some elements of basic grammar and the ability to use a dictionary. Indeed, the only Semitic language I know is the simple Babylonian of mathematical texts.

[4] I use the term "culture" as it is done in cultural anthropology. Consequently, the Sabian, Jewish, and Christian minorities which were integrated into Islamic society were all participants in the "Islamic culture," in *dār al-Islām*.

Similarly, "Islamic mathematics" is to be read as an abbreviation for "mathematics of the Islamic culture," encompassing contributions made by many non-Muslim mathematicians. I have avoided the term "Arab mathematics" not only because it would exclude Persian and other non-Arab mathematicians but also (and especially) because Islam and not the Arabic language must be considered to be the basic unifying force of the "Islamic culture." Cf. below, chapter X.

vantage point], to quote Felix Klein). Even if we allow for large distortions in our picture of Greek mathematics, due to the schoolmasters of late antiquity (cf. Toomer 1984, 32), similar broad views of the essence of mathematics appear to have been rare in the mature period of Greek mathematics; those who approached it tended to miss either the upper end of the scale (like Hero) or its lower part (like Archimedes and Diophantos, the latter with a reserve for his lost work on fractions, the *Moriastica*).

The "Greek Miracle" would not have been possible had it not been for the existence of antecedent intellectual source traditions. If we restrict ourselves to the exact sciences, nobody will deny that Egyptian and Babylonian calculators and astronomers supplied much of the material (from Egyptian unit fractions to Babylonian astronomical observations) that was so radically transformed by the Greek mathematicians.[5] It is equally certain, however, that the Egyptian and Babylonian cultures had never been able to perform this transformation, which was brought about by specific social structures and cultural patterns present in the Greek polis.[6] Similarly, if we want to understand the "miracle" of Islamic mathematics, and to trace its unprecedented integration of disciplines and levels, we must also look for both the sources that supplied the material to be synthesized and the forces and structures in the culture of Islam that caused and shaped the transformation – the "formative conditions."

II. Scientific Source Traditions: The Greeks

A dichotomy between "scientific" and "subscientific" source traditions was introduced above. I shall return to the latter and discuss why they must be taken more seriously than is normally done. First, however, I shall concentrate on the more conspicuous *scientific* sources, starting with the most conspicuous of all to medieval Islamic lexicographers as well as to modern historians of science:[7] Greek mathematics.

That this source was always regarded as having paramount importance can be seen, e.g., from the *Fihrist (Catalogue)* written by the tenth-century Baghdad court librarian al-Nadīm.[8] The section on mathematics and related subjects contains the names and known works of 35 pre-Islamic scholars. Of these 21 are Greek

[5] Even the protophilosophical cosmogonies that precede and announce the rise of Ionian natural philosophy are now known to make use of Near Eastern material (see Kirk, Raven, and Schofield 1983, 7–74, passim).

[6] A stimulating discussion of the formative conditions for the rise of philosophy is Vernant 1982. An attempt to approach specifically the rise of scientific mathematics is offered in my paper of 1985.

[7] For the same reason, I shall treat this part of the subject with great brevity, mentioning only what is absolutely necessary for what follows. A detailed account of the transmission of works of individual Greek authors will be found in *GAS*, V:70–190.

[8] A recent translation based on all available manuscripts is Dodge 1970. Chapter 7, section 2, dealing with mathematics, and the mathematical passages from section 1, dealing with philosophy, were translated from Flügel's critical edition (1872, based on a more restricted number of manuscripts) by Suter (1892; supplement 1893).

mathematicians (including writers on harmonics, mathematical astronomy, and mathematical technology). All the others deal with astrology (in the narrowest sense, it appears) and Hermetic matters (4 of these belong to the Greco-Roman world, 6 to the Assyro-Babylonian orbit, and 4 are Indians). Thus not a single work on mathematics written by a non-Greek, pre-Islamic scholar was known to our tenth-century court librarian,[9] who would certainly be in good position to know anything there was to know.

Central to the Greek tradition, as it was taken over by the Islamic world, were the *Elements* and the *Almagest.* Together with these belonged, however, the "Middle Books," the *Mutawassiṭāt:* the "Little Astronomy" of Autolycos, Euclid, Aristarchos, Hypsicles, Menelaos, and Theodosios; the Euclidean *Data* and *Optics* and some Archimedean treatises.[10] Even Apollonios and a number of commentators on Euclid, Ptolemy, and Archimedes (Pappos, Hero, Simplicios, Theon, Proclos, Eutocios) belong to the same cluster.[11]

Somewhat less central are the Greek arithmetical traditions, whether Diophantos or the Neopythagorean current as presented by Nicomachos (or by the arithmetical books of the *Elements,* for that matter). Still, all the works in question were of course translated; further work on Diophantine ideas by al-Karajī and others is well attested to;[12] and even though Nicomachean arithmetic was, according to Ibn Khaldūn, "avoided by later scholars" as "not commonly used [in practice]" (*Muqaddimah* VII:19, trans. Rosenthal 1958, III:121), it inspired not only Thābit (Nicomachos's translator), but other scholars too.[13] Finally, the treatment of the subject in encyclopedic works demonstrates familiarity with the concepts of Pythagorean and Neopythagorean arithmetic.[14]

Also somewhat peripheral – *yet less peripheral than they were for those Byzantine scholars whose selection of works to be studied and hence to survive created our image of Greek mathematics* – are the subjects that we might characterize tentatively as

[9] Comparison with other chapters in the *Catalogue* demonstrates that the lopsided selection is not due to any personal bias of the author.

[10] A full discussion is given by Steinschneider (1865), a brief summary by Sarton (1931, 1001f.).

[11] This can be compared with the list of works that al-Khayyāmī presupposes as basic knowledge in his *Algebra:* The *Elements* and the *Data,* Apollonios's *Conics* I–II, and (implicit in the argument) the established algebraic tradition (trans. Woepcke 1851, 7). The three Greek works in question constitute an absolute minimum, we are told.

[12] See Woepcke's introduction to and selections from al-Karajī's *Fakhrī* (1851, 18–22 and passim); Sesiano 1982, 10–13; and Anbouba 1979, 135.

[13] On Thābit's investigation of "amicable numbers," see Hogendijk 1985, or Woepcke's translation of the treatise (1852). Two later treatises on theoretical arithmetic were also translated by Woepcke (1861), one anonymous and one by Abū Jaʿfar al-Khāzin (a Sabian like Thābit). Among recent publications on the subject, works by Anbouba (1979) and Rashed (1982; 1983) can be mentioned.

[14] So, al-Fārābī's *Iḥsha' al-ʿulūm* (*De scientiis,* trans. Palencia 1953, 40); al-Khuwārizmī's *Mafātiḥ al-ʿulūm* (translation of the section on arithmetic in Wiedemann 1970, I:411–28; theoretical arithmetic is treated amply on pp. 411–18); and the encyclopedic part of the *Muqaddimah,* (trans. Rosenthal 1958, III:118–21).

The treatment in the encyclopedias is remarkably technical. In itself it seems highly probable that late Hellenistic Hermeticism, and Sabian, Jabirian, and Ismaʿīlī numerology would mix with "speculative" arithmetic. To judge from the encyclopedias, however, any inspiration from that quarter has remained without consequence for the contents of the subject *when understood as mathematics.* Cf. also below, chapterXVI.

"technological mathematics" (al-Fārābī speaks of ʿ*ilm al-ḥiyal* "science of artifices" [Palencia 1953, Arabic p. 73]) and its cognates: optics and catoptrics, the "science of weights," and nonorthodox geometrical constructions (geometry of movement, geometry of fixed compass opening). They are well represented e.g. in works by Thābit, the Banū Mūsā, Qusṭā ibn Lūqā, ibn al-Haytham, and Abū'l-Wafa'; but detailed discussions would lead us too far astray from our topic.

III. Scientific Source Traditions: India

The way al-Nadīm mentions the Indians indicates how the Indian inspiration must have looked from the Islamic end of the transmission line, even though he misses (and is bound to miss) essential points. Indian mathematics, when it reached the Caliphate, had, according to all available evidence, become anonymous: Indian trigonometry was adopted via Siddhantic astronomical works and *zījes* (astronomical tables with elements of theory) based fully or in part on Indian sources.[15] Islamic algebra was untouched by Indian influence – which would in all probability not have been the case if the Islamic mathematicians had had direct access to great Indian authors like Āryabhata and Brahmagupta.[16]

Below the level of direct scientific import, some influence of Indian algebra is plausible. This is indicated by the metaphorical use of *jidhr* ("root," "stem," "lower end," "stub," etc.) for the first power of the unknown. Indeed, this same metaphor (which can hardly be considered self-evident, especially not in a rhetorical, non-geometric algebra – cf. below, chapter VI) is found already around 100 B.C. in India (Datta and Singh 1962, II:169). In all probability, however, this borrowing was made via practitioners' subscientific transmission lines, to which we shall return below; furthermore, the ultimate source for the term need not have been Indian.

Apart from trigonometry, the main influence of Indian mathematics is the use of "Hindu numerals." If the Latin translation of al-Khwārizmī's introduction of the system is to be believed (and it probably should be[17]), he only refers it to "the Indians." So does Severus Sebokht in the mid-seventh century (fragment published in Nau 1910). The earliest extant "algorism" in Arabic, that of al-Uqlīdisī from the mid-

[15] See e.g. *GAS*, V: 191ff.; Pingree 1973; and Pingree, *DSB*, IV: 555f. The *Zīj al-sindhind*, the Sanskrit astronomical treatise translated with the assistance of al-Fazārī around 773 A.D. was mainly built upon the methods of Brahmagupta's *Brāhmasphuṭasiddhānta*, but influence from the *Āryabhaṭiya* is present. The original authors had become unknown in the process.

[16] The discrepancy between the advanced syncopated algebra of the Indians and the rhetorical algebra of al-Khwārizmī was already noticed by Léon Rodet (1878). This observation remains valid even if his supplementary claim (viz. that al-Khwārizmī's method and procedures are purely Greek and identical with those of Diophantos [p. 95]) is unacceptable.

Al-Khwārizmī can be considered a key witness: he is one of the early Islamic workers on astronomy, and mainly oriented towards the *Zīj al-sindhind*, with some connection to the Pahlavi *Zīj al-Shāh* and (presumably) to Hellenistic astronomy (cf. Toomer, *DSB*, VII:360f.). As we shall see when discussing his treatise on mensuration, he would recognize and acknowledge Indian material when using it.

[17] The translation conserves the traditional Islamic invocation of God (see Vogel 1963, 9), which would in all probability have been cut out before credit-giving references were touched (as it was cut out in both Gherardo of Cremona's and Robert of Chester's translations of al-Khwārizmī's *Algebra* – see the editions in Hughes 1986, 233 and Karpinski 1915, 67, or the quotations in Høyrup 1985a, 39, n. 58.

tenth century, is no more explicit. Most of its references are to "scribes" or "people of this craft" – evidently, local users of the technique are thought of: there is no explicit reference to the origin of the craft more precise than "Indian reckoners" (trans. Saidan 1978, 45, 104, 113). In addition, the dust-board so essential for early "Hindu reckoning" was known under a Persian, not an Indian, name: *takht* (ibid., 351). Finally, the methods of indeterminate equations and combinatorial analysis (both of which are staple goods in Indian arithmetic textbooks) are not found with the early Islamic expositions of Hindu reckoning (even though examples of indeterminate equations can be found in textbooks based on finger reckoning).[18] So the Islamic introduction of Hindu reckoning can hardly have been based on direct knowledge of "scientific" Indian expositions of arithmetic. Like trigonometry, it appears to derive from contact with practitioners using the system.

Some inspiration for work on the summation of series may have come from India. Apart from the chessboard problem (to which we shall return below), the evidence is not compelling, and proofs given by al-Karajī and others may be of Greek as well as Indian inspiration.

The two scientific source traditions were mainly tapped directly through translations from Greek and Sanskrit. To some extent, however, the mathematics of Indian astronomy found its way through Pahlavi, while elementary Greek astronomy may have been diffused through both Pahlavi and Syriac.[19] Neither of these secondary channels of transmission appears to have been scientifically creative, and they should probably be counted among the scientific source traditions only insofar as we distinguish "scientific" (e.g. astronomical and astrological) practice from "subscientific" practice (that of surveyors, builders, calculators, etc.).

IV. Subscientific Source Traditions: Commercial Calculation

This brings us to the problems of subscientific sources, which we may initially approach through an example. The last chapter in al-Uqlīdisī's arithmetic is entitled "On Doubling One, Sixty-Four Times" (*Kitāb al-Fuṣūl*... IV:32, trans. Saidan 1978, 337). This is the chessboard problem, to which al-Khwārizmī had dedicated a treatise,[20] and whose appurtenant tale is found in various Islamic writers from the ninth century onward.[21] Al-Uqlīdisī, however, states that "this is a question many

[18] Cf. Saidan 1978, 14. On the later (and probably independent) origin of Islamic combinatorial analysis, see Djebbar 1981, 67ff.

[19] See Pingree 1963, 241ff.; 1973, 34. Through the same channels, especially through the Sabians, some late Babylonian astronomical lore may have been transmitted (cf. the Babylonian sages mentioned in the *Fihrist*). Still, the integration of Babylonian results and methods into Greek as well as Indian astronomy makes it impossible to distinguish any possible direct Babylonian contributions.

In principle, nonastronomical Greek mathematics *may* also have been conveyed through Syriac learning. There is, however, absolutely no evidence in favor of this hypothesis (cf. below, chapter XI).

[20] According to a remark in the third part of Abū Kāmil's *Algebra* (Jan Hogendijk, personal communication).

[21] Relevant passages from al-Yaʿqūbī and al-Khāzinī are translated in Wiedemann 1970, I:442–53.

people ask. Some ask about doubling one 30 times, and others ask about doubling it 64 times," thereby pointing to a wider network of connections. In the mid-twelfth century, Bhāskara II asks about 30 doublings in the *Līlāvāti* (trans. Colebrook 1817, 55); so does Problem 13 in the Carolingian collection *Propositiones ad acuendos juvenes,* ascribed to Alcuin (ed. Folkerts 1978). A newly published cuneiform tablet from the eighteenth century B.C.[22] contains the earliest extant version of the problem. Like the chessboard problem, it deals with grain, and its 30 doublings correspond to the 30 cases of a current game board; but like the Carolingian problem, it avoids speaking of "doubling" or "multiplication by 2," telling instead that "to each grain/soldier comes another one."

The problem belongs in the category of "recreational problems," defined by Hermelink as "problems and riddles which use the language of everyday but do not much care for the circumstances of reality" (Hermelink 1978, 44), to which we may add the further observation that an important aspect of the "recreational" value of the problems in question is a funny, striking, or even absurd deviation from these circumstances. With good reason, Stith Thompson includes the chessboard doublings in his *Motif-Index of Folk Literature* (Thompson 1975, V:542 [Z 21.1]). From a somewhat different perspective, we may see recreational mathematics as a "pure" outgrowth of practical mathematics (which, in the premodern era, means computation). It does not seek mathematical truth or theory; instead, it serves to display virtuosity.[23]

Other recreational problems share the widespread distribution of the repeated doublings. Shared problem types (and sometimes shared numbers) and similar or common dress connect the arithmetical epigrams in Book XIV of the *Anthologia Graeca,*[24] Ananias of Shirak's arithmetical collection from seventh-century Armenia (trans. Kokian 1919),[25] the Carolingian *Propositiones,* part of the ancient Egyptian Rhind Papyrus, and ancient and medieval problem collections from India and China.[26] They turn up without fancy dress in Diophantos's *Arithmetica,* and recur in medieval Islamic, Byzantine, and Western European problem collections. The pattern looks very much like the distribution of folktales (even to the extent that

22 Published in Soubeyran 1984, 30; discussion and comparison with the Carolingian problem and the chessboard problem in Høyrup 1986, 477f.

23 In my paper of 1985, I use the same distinction between theoretical aim and display of virtuosity in a sociological discussion of the different cognitive and discursive styles of Greek and Babylonian mathematics. Even the difference between the arithmetical books VII–IX of the *Elements* and Diophantos's *Arithmetica* is elucidated by the same dichotomy; truly, Diophantos has theoretical insight into the methods he uses, but his presentation is still shaped by the origin of his basic material in recreational mathematical riddles.

We observe that the complex of practical and recreational mathematics can (structurally and functionally) be regarded as a continuation of the Bronze Age organization of knowledge (cf. above, chapter I). The two complexes were, however, separated by a decisive gap in social prestige – comparable to the gap between the Homeric bard and a medieval peasant telling stories in the tavern.

24 The epigrams were edited around 500 A.D. by Metrodoros.

25 It should be observed that Ananias had studied in the Byzantine Empire, and that parts of the collection appear to come from the Greek orbit.

26 A detailed discussion would lead us too far astray. A wealth of references will be found in Tropfke/Vogel et al. 1980, passim.

Diophantos's adoption of the material can be seen as a parallel to the literate adoption of folktale material). The geographical distribution is also roughly congruent with that of the Eurasian folktale (viz., "from Ireland to India" [Thompson 1946, 13ff.]). This, however, can be regarded only as a parallel, not as an explanation. First of all, the recreational problems cover an area stretching into China, beyond the normal range of Eurasian folktales;[27] second, mathematics can be entertaining only in an environment that knows something about the subject. The predominant themes and techniques of the problems in question point to the community of traders and merchants interacting along the Silk Road, the combined caravan and sea route reaching from China to Cadiz.[28]

"Oral mathematics" is rarely encountered in vivo in the sources. Like folktales before the age of folklorists, it has normally been worked up by those who took the care to write it down, adoption entailing adaptation.[29] Generally they were mathematicians, who at least arranged the material systematically, and perhaps gave alternative or better methods for solution or supplied a proof. In a few cases, however, they added a description of the situation in which they found the material. So Abū Kāmil in the preface to his full mathematical treatment of the indeterminate problem of "the hundred fowls,"[30] which he describes as

> a particular type of calculation, circulating among high-ranking and lowly people, among scholars and among the uneducated, at which they rejoice, and which they find new and beautiful; one asks the other, and he is then given an approximate and only assumed answer, they know neither principle nor rule in the matter ("Book of Rare Things in Calculation"; German trans. Suter 1910, 100; my English J.H.).

A similar aggressive description of reckoners who

> strain themselves in memorising [a procedure] and reproduce it without knowledge or scheme [and others who] strain themselves by a scheme in which they hesitate, make mistakes, or fall in doubt,

is given by al-Uqlīdisī in connection with the continued doubling problem raised by "many people"(trans. Saidan 1978, 337).

It is precisely this situation that has distorted the approach to the subscientific traditions. The substratum was anonymous and ubiquitous, and its procedures

[27] This is illustrated beautifully by the chessboard problem and its framework tale. The *motif* turns up in the Chinese as well as in the "Eurasian" domain; the Chinese *tale,* however, is wholly different, dealing with a peasant and the wages of his servant, determined as the successively doubled harvests from one grain of rice (Thompson 1975, V:542, Z 21.1.1).

[28] It is worth noting that two arithmetical epigrams from the *Anthologia graeca* deal with the Mediterranean extensions of the route: XIV:121 with the land route from Rome to Cadiz, and XIV:129 with the sea route from Crete to Sicily.

[29] The Carolingian *Propositiones* appear to form an exception. The editor of the collection (Alcuin?) was obviously not more competent as a mathematician than the practitioners who supplied the material.

[30] Its distribution (from ancient China and India to Aachen) is described in Tropfke/Vogel et al. 1980, 614–16.

deserve the designation of recipes rather than methods. Every mathematician inspired by it had to employ his own techniques to solve the common problems (or at least translate the recipes into his own theoretical idiom): Diophantos would use rhetorical algebra, the Chinese *Nine Chapters on Arithmetic* would manipulate matrices, and the *Liber abaci* would find the answer by means of proportions. We should not ask, as is commonly done, whether Diophantos (or the Greek arithmetical environment) was the source of the Chinese or vice versa. There was *no specific source: the ground was wet everywhere.*

Besides supplying problems and procedures, the community of merchants and bookkeepers appears to have provided Islamic mathematics with two of its fundamental arithmetical techniques: the peculiar system of fractions and the "finger reckoning."

The system of fractions is built up by means of the series of "principal fractions" – ½, ⅓, . . . , ⅒ (the fractions that possess a name of their own in the Arabic language) – and their additive and multiplicative combinations (described e.g. in Saidan 1974, 368; Juschkewitsch 1964, 197ff.; Youschkewitch *DSB*, I:40). The system has been ascribed to Egyptian influence and to independent creation within the territory of the Eastern Caliphate.[31] It turns out, however, that some Old Babylonian texts use similar expressions, e.g. "the third of X, and the fourth of the third of X" for $\frac{5}{12}X$.[32] So we are really confronted with an age-old system and at least a common Semitic usage; but, for example, the formulation of Problem 37, of the Rhind Papyrus suggests in fact a common *Hamito*-Semitic usage, which had already provided the base on which the Egyptian scribes developed their unit fraction system around the turn of the second millennium B.C.[33] Since "fractions of fractions" are also used occasionally in the Carolingian *Propositiones*,[34] they appear to have spread over the whole Near East and Roman Empire in late antiquity, and thus to have been well-rooted in the commercial communities throughout the region covered by the Islamic expansion; further evidence of this is their use in arithmetical textbooks written for merchants, accounting officials, etc. in the earlier Islamic period (cf. below, chapter XIV).

The use of "principal fractions" and "fractions of fractions" appears to coincide with that of "finger reckoning," another characteristic method of Islamic elementary

[31] Saidan (1974, 358) among others proposes Egyptian influence. Youschkevitch *DSB*, I:40) quotes M. I. Medevoy for the suggestion of independence.

[32] "Sha-lu-ush-ti 20 ú ra-ba-at sha-lu-ush-ti ú-ṭe₄-tim" – MLC 1731, rev. 34–35, in A. Sachs 1946, 205 (the whole article deals with such phenomena). Expressions in the same vein are encountered in the tablet YBC 4652, nos. 19–22 (in Neugebauer and Sachs 1945, 101).

[33] The problem in question is of typical "riddle" or "recreational" character: "Go down I times 3 into the *hekat*-measure, ⅓ of me is added to me, ⅓ of ⅓ of me is added to me, ⅑ of me is added to me; return I, filled am I" (the "literal translation," Chace et al. 1929, plate 59). It can thus be seen as a witness of a current, more or less popular usage. A close analysis of the (very few) instances of rudimentary unit fraction notation from the Old Kingdom (from the twenty-fourth century B.C.) suggests that they stand midway between this original usage and the fully developed system (space and relevance does not permit further unfolding of the argument).

[34] Nos. 2, 4, and 40 deal with *medietas medietatis* (and nos. 2 and 40 further with *medietas [huius] medietatis*, i.e., with ½ of ½ of ½); no. 3 treats of *medietas tertii*. All four problems are of the same type as no. 37 of the Rhind Papyrus, cf. note 33.

mathematics. It was referred to as *ḥisāb al-Rūm wa'l-ʿArab* (calculation of the Byzantines and the Arabs) (Saidan 1974, 367 [misprint corrected]): a system related to that used in medieval Islam had been employed in ancient Egypt.[35] Various ancient sources refer to the symbolization of numbers by means of the fingers (Menninger 1957, II:11–15), without describing, it is true, the convention employed. But since the very system used in Islam is described around 700 A.D. in Northumbria by Bede (*De temporum ratione,* cap. I, ed. Jones 1943, 179–81), who would be familiar with descendants of ancient methods rather than with the customs of Islamic traders, we may safely assume that all three systems were identical.

V. Subscientific Source Traditions: Practical Geometry

So far we have dealt with what appears to have been a more or less shared tradition for practitioners of bookkeeping and commercial arithmetic (*ḥisāb,* to use the Arabic term). Another group possessing a shared tradition (for practical geometry) comprised surveyors, architects, and "higher artisans."[36]

In the case of this subscientific geometry, we can follow how the process of mathematical synthesis had begun long before the Islamic era. Indeed, various ancient civilizations had had their specific practical geometries. The (partly) different characters of Egyptian and Babylonian practical geometry have often been noted.[37] The melting pots of the Assyrian, Achaemenid, Hellenistic, Roman, Bactrian, and Sassanian empires mixed them up completely;[38] and through the Heronian corpus

[35] See the cubit rod reproduced in Menninger 1957, II:23.

[36] Once again, the evidence for a shared tradition is found in Islamic sources – e.g., Abū'l-Wafā''s *Book on What is necessary from Geometric Construction for the Artisan,* trans. Krasnova 1966.

[37] E.g., different ways to find the area of a circle; the Babylonian treatment of irregular quadrangles and of the bisection of trapezia, and the absence of both types of problem in Egypt.

[38] Thus, the Demotic Papyrus Cairo JE 89127–30, 137–43 (third century B.C.) has replaced the excellent Egyptian approximation of the area of a circle (equivalent to $\pi = {}^{256}/_{81} \approx 3.16$) with the much less satisfactory Babylonian and biblical value $\pi = 3$ (Parker 1972, 40f., problems 32–33). The same value is also taken over by pre-Heronian Greco-Egyptian practical geometry, cf. Pap. Gr. Vind. 19996 as published by Vogel and Gerstinger (1932, 34). A formula for the area of a circular segment which is neither correct nor near at hand for naive intuition is used in the Demotic papyrus mentioned above (no. 36); in the Chinese *Nine Chapters on Arithmetic,* it is used in nos. I, 35–36, and made explicit afterwards (trans. Vogel 1968, 15). Hero, finally, ascribes it to "the Ancients" (οἱ ἀρχαῖοι – *Metrika* I:xxx [ed. Schöne 1903, 72]) while criticizing it (cf. the discussion in van der Waerden 1983, 39f., 174).

The Babylonian calculation of the area of the circle, which is inferior to the Middle Kingdom Egyptian method, was probably an improvement over early Greek and Roman practitioners' methods: Polybios and Quintilian both tell us that most people incorrectly measured the area of a figure by its periphery (and Thucydides himself uses this measure – see Eva Sachs 1917, 174). Precisely the same method turns up in the Carolingian *Propositiones,* nos. 25 and 29, which find the area of one circle as that of the isoperimetric square, and of another as that of an isoperimetric rectangle.

To complete the confusion, the *Propositiones* find the area of all nonsquare quadrangles (rectangles and trapezoids alike) by means of the "surveyors' formula" for the irregular quadrangle (semisum of lengths times semisum of widths). This formula is employed in old Babylonian tablets. It was used by surveyors in Ptolemaic Egypt (see Cantor 1875, 34f.), and it turns up in the pseudo-Heronian *Liber geeponicus* (ibid., 43). It was not used by Hero, nor by the Roman agrimensors (nor, it appears, in Seleucid Babylonia); but it turns up again in the eleventh-century Latin compilation *Boethii geometria altera* II:xxxii (ed. Folkerts 1970, 166). In the eleventh century A.D., Abū Manṣūr ibn Ṭāhir al-Baghdādī ascribes the formula to "the Persians" (Anbouba 1978, 74), but al-Khwārizmī (who does not use it) had probably seen it in the Hebrew *Mishnat ha-Middot* II:1 (ed. Gandz 1932, 23), or eventually in some lost prototype for that work.

some Archimedean and other improvements were infused into the practitioners' methods and formulas.[39] This mixed and often disparate type of calculatory geometry was encountered locally by the mathematicians of Islam, who used it as a basic material while criticizing it, just as they encountered, used, and criticized the practices of commercial and recreational arithmetic.

VI. Algebra and Its Alternative

As pointed out in chapter III, Islamic algebra was probably not inspired by Indian scientific algebra. Detailed analysis of a number of sources suggests instead a background in the subscientific tradition – or, indeed, in two different subscientific traditions. I have published the arguments for this elsewhere (Høyrup 1986),[40] and here shall therefore present only the results of the investigation briefly.

Al-jabr was performed by a group of practitioners engaged in *ḥisāb* (calculation) and spoken of as *ahl al-jabr* (algebra people) or *aṣḥāb al-jabr* (followers of algebra). The technique was purely rhetorical; a central subject was the reduction and resolution of quadratic equations – the latter by means of standardized algorithms (analogous to the formula $x = \frac{1}{2}b + \sqrt{((\frac{1}{2}b)^2 + c)}$ for solving the equation $x^2 = bx + c$, etc.) unsupported by arguments, with the rhetorical argument reserved for reduction. Part of the same practice (but possibly not understood as covered by the term *al-jabr*) was the rhetorical reduction and solution of first-degree problems.

As argued in chapter III, part of the characteristic vocabulary suggests a subscientific (but probably indirect) connection with India. An ultimate connection to Babylonian algebra is also inherently plausible, but not demonstrated by any clear-cut evidence; in any case the path that may have led from Babylonia to the early

[39] So the value $\pi = 22/7$, represented by Hero as a simple approximation (*Metrika* I:26, ed. Schöne 1903, 66), is taken over by Roman surveying (Columella and Frontinus, see Cantor 1875, 90, 93f.) and stands as plain truth in Latin descendants of the agrimensor tradition (e.g. *Boethii geometria altera* II:xxxii, ed. Folkerts 1970, 166). The *Mishnat ha-Middot* (II:3 ed. Gandz 1932, 24) presents the matter in the same way. So does al-Khwārizmī in the parallel passage of his *Algebra,* but in the introductory remark he represents the factor $3\frac{1}{7}$ as "a convention among people without mathematical proof"(ed. ibid. pp. 69 and 81f.) – thereby telling us that he considered at least that section of the *Mishnat ha-Middot* (or its prototype) representative of a general subscientific environment.

Other Heronian improvements are the formula for the area of a triangle and his better calculation of the circular segment, which turn up in various places (see, e.g., Cantor 1875, 90, reporting Columella, and *Mishnat ha-Middot* V, ed. Gandz 1932, 47ff.).

[40] The essential sources involved in the argument are al-Khwārizmī's *Algebra* (ed. and trans. Rosen 1831); the extant fragment of ibn Turk's *Algebra* (ed. and trans. Sayili 1962); Thābit's Euclidean *Verification of the Problems of Algebra through Geometrical Demonstrations* (ed. and trans. Luckey 1941); Abū Kāmil's *Algebra* (ed. and trans. Levey 1966); the *Liber mensurationum* written by some unidentified Abū Bakr and known in a Latin translation due to Gherardo of Cremona (ed. Busard 1968); and Abraham bar Ḥiyya's (Savasorda's) *Ḥibbur ha-meshiḥah we-tishboret* (*Collection on Mensuration and Partition;* Latin translation *Liber embadorum,* ed. Curtze 1902).

On one point, my paper of 1986 should be corrected. On p. 472 I quote Abū Kāmil for a distinction between "arithmeticians" (*ba'alei ha-mispar,* in the Hebrew translation [Levey 1966, 95], i.e., "masters of number" and "calculators" (*yinhagu ha-ḥasbanim,* in the Hebrew translation [Levey 1966, 97], "those who pursue calculation"). The distinction turns out to be absent from the Arabic facsimile edition of the work (ed. Hogendijk 1986) (Jan Hogendijk, personal communication).

medieval Middle East must have been tortuous, as the methods employed in the two cases are utterly different.

The last statement is likely to surprise, since Babylonian algebra is normally considered to be built either on standardized algorithms or on oral rhetorical techniques. A detailed structural analysis of the terminology and of the distribution of terms and operations inside the texts shows, however, that this view does not hold water; furthermore, it turns out that the only interpretation of the texts that makes sense is geometric – the texts have to be read as (naive, nonapodictic) constructional prescriptions, really dealing, as they seem to do when read literally, with (geometric) squares, rectangles, lengths, and widths (all considered as *measured* entities); they split, splice, and combine figures so as to obtain a figure with known dimensions in a truly analytic though completely heuristic way. Only certain problems of the first degree (if any) are handled rhetorically, and no problems are solved by standard algorithms.[41]

This is quite certain for Old Babylonian algebra (c. seventeenth century B.C.), where the basic problems are thought of as dealing with rectangles, and are solved by naive-geometric "analysis." A few tablets dating from the Seleucid period and written in the Uruk environment of astronomer-priests contain second-degree problems too. They offer a more ambiguous picture. Their garb is geometric, as is the method (though rather synthetic than analytic); but the geometric procedure is obviously thought of as *an analogy to a set of purely arithmetical relations* between the unknown magnitudes.

Having worked intensively with Babylonian texts for some years, I was utterly amazed to discover accidentally their peculiar rhetoric (characterized by fixed shifts between present and past tense, and between the first, second, and third person singular) in the medieval Latin translation of a *Liber mensurationum* written by an unidentified Abū Bakr. The first part of this *misāḥa* (surveying) text contains a large number of problems similar to those known from the Old Babylonian tablets: A square plus its side is 110; in a rectangle, the excess of length over width is 2; and the sum of the area and the four sides is 76; etc. The problems, furthermore, are first solved in a way strikingly reminiscent of the Old Babylonian methods (although the matter is obscured by the absence of a number of figures alluded to in the text); a second solution employs the usual rhetorical reductions and solutions by means of standard algorithms. The second method is spoken of as *aliabra,* evidently a transliteration of *al-jabr*. The first usually goes unlabeled, being evidently the standard method belonging to the tradition; in one place, however, it is spoken of as "augmentation and diminution" – apparently the old splicing and splitting of figures.

A precise reading of the text in question leaves no reasonable doubt that its first part descends directly from the Old Babylonian "algebra" of measured line segments

[41] The arguments for this are, as any structural analysis, complex, and impossible to repeat in the present context. A brief sketch is given in my paper of 1986, 449–56). A detailed but fairly unreadable presentation is given in my preliminary paper (1985b). Another detailed but more accessible exposition is now available as preprint (Høyrup 1987).

(the second part contains real mensuration in agreement with the Alexandrinian tradition). Once this is accepted as a working hypothesis, a number of other sources turn out to give meaningful evidence. The geometrical "proofs" of the standard algebraic algorithms given by al-Khwārizmī and ibn Turk will have been taken over from the parallel naive-geometric tradition and not from Greek geometry; Thābit's Euclidean proof of the same matter is therefore really *something different,* which probably explains his silence with regard to these hitherto presumed predecessors; etc.

Especially interesting is Abū'l-Wafā''s report of a discussion between (Euclidean) geometers on one hand and surveyors and artisans on the other (*Book on What is Necessary from Geometric Construction for the Artisan,* X:xiii, trans. Krasnova 1966:115). He refers to the "proofs" used by the latter in questions concerned with the addition of figures; these proofs turn out to be precisely the splitting and splicing used by Abū Bakr and in the Old Babylonian texts. This confirms a suspicion already suggested by the appearance of the "algebra" of measured line segments in a treatise on mensuration: this "algebra" belonged to the practitioners engaged in subscientific, practical geometry, and was hence a tradition of surveyors, architects, and higher artisans. *Al-jabr,* on the other hand, was carried by a community of calculators, and was considered part of *ḥisāb,* as Abū Kāmil seems to tell us (cf. note 40).

VII. Reception and Synthesis

In pre-Islamic times these different source traditions had already merged to some extent. The development of a syncretic practical geometry was discussed above, and the blend of (several sorts of) very archaic surveying formulas with less archaic recreational arithmetic in the *Propositiones ad acuendos juvenes* was also touched upon.[42] Still, merging, and especially critical and creative merging, was not the dominant feature.

From the ninth century onwards, however, it came to be the dominant feature of Islamic mathematics. The examples are too numerous to be listed, but a few illustrations may be given.

A modest example is the geometric chapter of al-Khwārizmī's *Algebra.* As shown by Solomon Gandz (1932, discussion and the two texts), it is very closely related to the Hebrew *Mishnat ha-Middot,* which is a fair example of pre-Islamic syncretic practical geometry, or at least a very faithful continuation of that tradition.[43]

[42] True, the *Propositiones* are not pre-Islamic according to chronology. Still, they show no trace of Islamic influence, and were collected in an environment where mathematical development was to all evidence extremely slow. We can safely assume that most of the mathematics of the *Propositiones* was already present (if not necessarily collected) in the same region by the sixth century A.D.

[43] In his English summary, Sarfatti (1968, ix) claims that Arabic linguistic influence "although not evident prima facie, underlies [the] mathematical terminology" of the *Mishnat ha-Middot.* If this is true, the work must be dated in the early Islamic period. The main argument of the book is in Hebrew, and I am thus unable to evaluate its force – but since Arabic and Syriac (and other Aramaic) technical terminologies are formed in analogous ways, and since no specific traces of Arabic terms are claimed to be present, it does not seem to stand on firm ground.

Al-Khwārizmī's version of the same material is not very different; but before treating of circular segments, precisely as done in the *Mishnat ha-Middot,* he says that the ratio 3⅐ between perimeter and diameter of a circle "is a convention among people without mathematical proof." (In the *Mishnat ha-Middot* the value stands as undiscussed truth.) He goes on to inform us that the Indians "have two other rules," one equivalent to $\pi = \sqrt{10}$ and the other to $\pi = 3.1416$.[44] Finally he gives the exact value of the circular area as the product of semiperimeter and semidiameter, together with a heuristic proof.[45] So not only are the different traditions brought together, we are also offered a sketchy critical evaluation of their merits.

If the whole of al-Khwārizmī's *Algebra* is taken into account, the same features become even more obvious. The initial presentation of the *al-jabr* algorithms is followed by their geometric justification, by reference to figures inspired by the "augmentation-and-diminution" tradition, but which are more synthetic in character, and for the sake of clear presentation make use of Greek-style letter formalism (Rosen 1831, 16–20).[46] A little further on, the author attempts his own extrapolation of the geometrical technique in order to prove the rules of rhetorical reduction.

The result is still somewhat eclectic, especially in the chapter on geometry. Comparison and critical evaluation do not amount to real synthesis. But in the work in question, and still more in the complete oeuvre of the author, an effort to make more than random collection and comparison of traditions is clearly visible. Soon after al-Khwārizmī, furthermore, other authors wrote more genuinely synthetic works. One example, in the same field as al-Khwārizmī's naive-geometric proofs of the *al-jabr* algorithms, is Thābit's treatise on the "verification of the rules of *al-jabr*" by means of *Elements* II.5–6 (ed., trans. Luckey, 1941). Another, in the field of practical geometry, is Abū'l-Wafā''s *Book on What is Necessary from Geometric Construction for the Artisan,* where methods and problems of Greek geometry (including, it now appears, Pappos's passage on constructions with restricted and constant compass opening [see Jackson 1980]) and Abū'l-Wafā''s own mathematical ingenuity are used to criticize and improve upon practitioners' methods, but where the practitioners' perspective is also kept in mind as a corrective to otherworldly theorizing.[47] These examples could be multiplied ad libitum. Those already given,

[44] The former is an ancient Jaina value, the latter is given by Āryabhaṭa – see Sarasvati Amma 1979, 154.

[45] Arabic text and translation in Gandz (1932, 69f.); *ahl al-handasa* ("surveyors" and later "geometers," translated "mathematicians" by Gandz) corrected to *ahl al-hind* ("people of India") in agreement with Anbouba 1978, 67.

[46] The whole technique of the proofs has normally been taken to be of purely Greek inspiration, partly because of the letter formalism, partly because neither the Old Babylonian naive-geometric technique nor its early medieval descendant was known.

[47] Russian trans. by A. Krasnova (1966). Interesting passages include chapter I, on the instruments of construction, and X.i and X.xiii, which discuss the failures of the artisans as well as the shortcomings of the (too theoretical) geometers. Consideration of practitioners' needs and requirements is also reflected in the omission of all proofs.

Though more integrative than al-Khwārizmī's *Algebra,* Abū'l-Wafā''s work is not completely free of traces of eclecticism. This is most obvious in the choice of grammatical form, which switches unsystematically between a Greek "we" and the practitioners' "If somebody asks you . . . , then you [do so and so]."

however, will suffice to show that the Islamic synthesis was more than bringing together methods and results from the different source traditions; it included an explicit awareness of the difference between the perspectives of theoreticians and practitioners, and of the legitimacy of both, as well as acknowledgment of the possible relevance and critical potentiality *of each* when applied to results, problems, or methods belonging to the other. While the former aspect of the synthesizing process, though much further developed than in other ancient or medieval civilizations, was not totally unprecedented, the latter *was* exceptional (cf. also below, chapter XIII).

Apart from a violent cultural break and an ensuing cultural flowering (which of course explains much, but unspecifically), what accounts for the creative assimilation, reformulation, and (relative) unification of disparate legacies as the "mathematics of the Islamic world"? And what accounts for the *specific* character of Islamic mathematics as compared with Greek or medieval Latin mathematics, for example?

VIII. "Melting Pot" and Tolerance

I shall not pretend to give anything approaching an exhaustive explanation. Instead, I shall point to some factors which appear to be important, and possibly fundamental.

On a general level, the "melting-pot effect" was an important precondition for what came about. Within a century after the *Hegira,* the whole core area of medieval Islam had been conquered, and in another century or so the most significant strata of the Middle Eastern population were integrated into the emerging Islamic culture.[48] This – and also the movements of individual scholars as well as those of larger population groups, especially toward the Islamic center in Baghdad – broke down earlier barriers between cultures and isolated traditions and offered the opportunity for "cultural learning." The religious and cultural tolerance of Islam was important here. Muslims were of course aware of the break in history marked by the rise of Islam; in the field of learning a distinction was maintained between *awāʾil,* "pristine" (i.e., pre-Islamic), and Muslim/Arabic "science" (i.e.,*ʿilm,* a term roughly corresponding to the Latin *scientia* and perhaps better rendered as "field of knowledge"). Since, however, the latter realm encompassed only religious (including legal), literary, and linguistic studies (see ibn Khaldūn, MuqaddimahVI. 9:passim, esp. VI. 9, trans. Rosenthal 1958, II – III, esp. II:436–39; and Nasr 1968, 63f.), the complex societal setting of learning in the mature Islamic culture prevented the development of Greek-like contempt for "barbarians." Furthermore, the rise of persons whose

[48] According to Bulliet's counting of names (1979), the majority of the Iranian population was converted around 200 A.H. (816 A.D.), while the same point was reached in Iraq, Syria, and Egypt some fifty years later. The socially (and scientifically!) important urban strata (artisans, merchants, religious and state functionaries) were predominantly Muslim some eighty years earlier (cf. also Waltz 1981). An examination of the possible correlation of Bulliet's geographical distinctions with the emergence of local Islamic scholarly life would probably be rewarding.

roots were in *different* older elites to high positions in the Caliphate may have precluded the sort of cultural exclusiveness that came to characterize Latin Christianity during *its* phase of learning from other cultures (where, even when most open to Islamic and Hebrew scholarship, it took over only translations and practically no scholars, and showed little interest in translating works with no relation to the "culturally legitimate" Greco-Roman legacy).[49]

Whatever the explanation, Islam remained freer of ethnocentrism and culturocentrism than many other civilizations.[50] Due to this tolerance, the intellectual and cognitive barriers that were molten down in the melting pot were not immediately replaced by new barriers, which would have nullified the positive effects of the cultural recasting. It also permitted Islamic learning (both in its initial phase and later) to draw on the service of Christians, Jews, and Sabians, and on the Muslims rooted in different older cultures – for examples Māshā'āllāh the Jew, Thābit the heterodox Sabian, Ḥunayn ibn Isḥāq the Nestorian, Qusṭā ibn Lūqā the Syrian Christian of Greek descent, Abū Maʿshar the heterodox, pro-Pahlavi Muslim from the Hellenist-Indian-Chinese-Nestorian-Zoroastrian contact point at Balkh, and ʿUmar ibn al-Farrukhān al-Ṭabarī the Muslim from Iran. In later times, the conversion from Judaism of al-Samaw'al late in life appears to have been unrelated to his scientific career.[51] Still later, the presence of Chinese astronomers collaborating with Muslims at Hulagu's observatory at Maragha underscores the point (Sayili 1960, 205–7).

Still, "melting-pot effect" and tolerance were only preconditions – "material causes," in a quasi-Aristotelian sense. This leaves open the other aspect of the question: Which "efficient" and "formal" causes made medieval Islam scientifically and mathematically creative?

IX. Competition?

It has often been claimed that the early-ninth-century awakening of interest in *awā'il* (pre-Islamic) knowledge[52] "must be sought in the new challenge which Islamic society faced" through the "theologians and philosophers of the religious minorities within the Islamic world, especially the Christians and Jews" in "debates carried on in cities like Damascus and Baghdad between Christians, Jews, and Muslims," the last being "unable to defend the principles of faith through logical arguments, as could the other groups, nor could they appeal to logical proofs to demonstrate the truth of the tenets of Islam" (formulations of Nasr 1968, 70). One problematic feature of this thesis is that, according to O'Leary (1949, 142) who is otherwise close

[49] I have discussed this particular culturocentrism in my paper of 1985a, 19–25 and passim. One of the rare fields where it is not clearly felt appears to be mathematics, where the requirements of mathematical astronomy, "Hindu reckoning," and commercial arithmetic and algebra in general may have opened a breach of relative tolerance.

[50] This is of course not to say that it remained totally free. The conquering Arabs, e.g., felt ethnically superior to others, as conquerors have always done. The lack of ethnocentrism is only relative.

[51] Cf. the biographies in *DSB*.

[52] The beginnings of astronomical interests in the late eighth century is different, as it is bound up with practical interests in astrology. The same applies to the very early interest in medicine – cf. *GAS*, III:5.

to the idea of stimulation through intellectual competition, "we have very little evidence of philosophical or theological speculation in Syria [including Damascus] under the ʿUmayyad dynasty." Another serious challenge to it is the fact that Islamic learning advanced well beyond the level of current Syriac learning in a single bound (not least in mathematics). Here, the cases of the translators Ḥunayn ibn Isḥāq and Thābit should be remembered. Both began their translating activity (and Thābit his entire scientific career) in the wake of the ʿAbbasid initiative. Most of their translations and other writings were in Arabic; their work in Syriac was clearly secondary, and included none of their mathematical writings or translations (unless some minor work be hidden among Thābit's "various writings on astronomical observations, Arabic and Syriac"; Suter 1900, 36). Ḥunayn's translations show us that by the early ninth century A.D. the Syrian environment was almost as much in need of broad Platonic and Aristotelian learning as the Muslims (cf. Rosenfeld and Grigorian, *DSB* XIII:288–95; Anawati and Iskandar, *DSB* XV:230–49; cf. also below, chapter XI). True, a Provençal-Hebrew translation, dated 1317 A.D., of an earlier Mozarabic treatise claims that an Arabic translation of Nicomachos's *Introduction to Arithmetic* was made from the Syriac before 822 A.D.[53] This testimony, even if reliable, is however of little consequence: an understanding of scientific mathematics is neither a necessary condition for interest in Nicomachos nor a consequence of even profound familiarity with his *Introduction.* Equally little can be concluded from the existence of a (second-rate) Syriac translation of Archimedes' *On the Sphere and Cylinder,* since it may well have been prepared as late as the early ninth century and thus have been a spin-off from the ʿAbbasid wave of translation (*GAS* V:129).[54]

A quest for intellectual competitiveness will thus hardly do, and definitely not as the *sole* explanation of the scientific and philosophical zeal of the early-ninth-century ʿAbbasid court and its environment. Similarly, the ʿAbbasid adoption of many institutions and habits from the Sassanian state and court, and the concomitant peaceful reconquest of power by the old social elites,[55] may explain the use of astrologers in the service of the court, e.g. at the famous foundation of Baghdad. But

[53] The Mozarabic work is a paraphrase of Nicomachos written by a mid-tenth century Andalusian bishop Rabīʿ ibn Yaḥya. In his preface, Rabīʿ refers to commentaries that al-Kindī should have made to a translation from Syriac. The evidence can hardly be considered compelling; on the other hand, *some* Arabic translation antedating Thābit's must have existed. See Steinschneider 1896, 352 and *GAS,* V:164f.

[54] Positive evidence that Syriac learning was close to mathematical illiteracy is found in a letter written by Severus Sebokht around 662 (trans. Nau 1910, 210–14). In this letter, the Syrian astronomer par excellence of the day quotes the third-century astrologer Bardesanes extensively and is full of contempt for those who do not understand the clever argument – which is in fact nothing but a mathematical blunder, as enormous as it is elementary.

[55] "The people will become subject to the people of the East and the government will be in their hands," as expressed by the contemporary Māshā'allāh in his *Astrological History* (trans. Kennedy and Pingree 1971, 55). Or, in Peter Brown's modern expressive prose (1971, 201): "Khusro I had taught the *dekkans*, the courtier-gentlemen of Persia, to look to a strong ruler in Mesopotamia. Under the Arabs, the *dekkans* promptly made themselves indispensable. They set about quietly storming the governing class of the Arab empire. By the middle of the eighth century they had emerged as the backbone of the new Islamic state. It was their empire again: And, now in perfect Arabic, they poured scorn on the refractory Bedouin who had dared to elevate the ways of the desert over the ordered majesty of the throne of the Khusros."

it does not explain why Islamic astrologers were not satisfied with the *Zīj al-Shah,* (the astronomical tables prepared for the Sassanian Shah Yazdijird III) with its connection to the past of the reborn Sassanian elite. An attempt to be *still more splendid* than the Sassanian Shah could explain that the Caliph made his scholars tap Siddhantic and Ptolemaic astronomy directly from the sources, instead of using only the second-hand digests known from the Pahlavi astrologers (Pingree 1973, 35). Continuity or revival of elites and general cultural patterns are, however, completely incapable of explaining the sudden *new* vigor of scientific culture in the Irano-Iraqi area – a truly qualitative jump. In particular, they are unable to tell us why a traditional interest in the astronomic-astrological applications of mathematics should suddenly lead to an interest in mathematics per se – not to speak of the effort toward synthesizing separate traditions.

X. Institutions or Sociocultural Conditions?

Sound sociological habit suggests that one look for explanations at the institutional level. But there is a serious problem to this otherwise reasonable "middle-range" approach to the problem (to use Robert K. Merton's expression[56]): the institutions of Islamic learning were still in their swaddling clothes in the early ninth century, if born at all. In this age of fluidity and fundamental renewal, Islamic learning formed its institutions quite as much as the institutions formed the learning.[57] In order to get out of this closed circle of pseudo-causality we will have to ask why institutions became shaped the way they did. The explanations should hold for the entire core area of medieval Islam and at the same time be specific for this area.

Two possibilities suggest themselves: Islam itself, which was shared as a cultural context even by non-Muslim minorities and scholars; and the Arabic language. Language can be ruled out without hesitation. The flexibility of Semitic languages (especially the rich verbal system with its complex of generalized aspect and voice, and its vast array of corresponding nominal derivatives) makes them well suited both to render accurately foreign patterns of thought and to serve as the basis for the development of autochthonous philosophical and scientific thought. But Syriac and

[56] See his polemical defense of a "middle-range theory" whose abstractions are "close enough to observed data to be incorporated in propositions that allow empirical testing" against such precocious total systems whose profundity of aims entails triviality in the handling of all empirical details (Merton 1968, 39–72, especially the formulations on pp. 39 and 49).

[57] As "institutions" of learning in the widest sociological sense (i.e., socially fixed patterns of rules, expectations, and habits) one can mention the short-lived "House of Wisdom" at al-Ma'mūn's court, together with kindred libraries; the institutions of courtly astronomy and astrology and, more generally, the ways astronomy and astrology were generally practiced; that traditional medical training which made medicine almost a monopoly of certain families (cf. Anawati and Iskandar relating Abī Uṣaybiʿa, in *DSB,* XV:230); the fixed habits and traditions of other more or less learned practical professions; the gatherings of scholars; the fixed form in which science could already be found in Byzantium; the mosque as a teaching institution (the *madrasah* was developed only much later); and practical and theoretical management of Islamic jurisprudence, including the transmission of *ḥadīth* [jurisprudentially informative traditions on the doings and sayings of the Prophet]. Of only one of these institutions – Byzantine science – can it be claimed that it was really fixed by the early ninth century. Cf. Nasr 1968, 64–88; Makdisi 1971; and Watt and Welch 1980, 235–50.

other Aramaic dialects are no less Semitic than Arabic; they had been shaped and honed for philosophical use over centuries, and much of the Arabic terminology was in fact modeled upon the Syriac (see Pines 1970, 782). The Arabic tongue was an adequate medium for what was about to happen, but it replaced another medium that was just as adequate. Language, then, cannot be the explanation; Islam remains.

Of course, the explanation need not derive from Islam regarded as a system of religious teachings; what matters is Islam as a specific integrated social, cultural, and intellectual complex. From this complex some important factors can be singled out.

XI. Practical Fundamentalism

One factor is the very character of the complex as an integrated structure – i.e., those implicit fundamentalist claims of Islam that have most often been discussed with relation to Islamic law as

> the totality of God's commands that regulate the life of every Muslim in all its aspects; it comprises on an equal footing ordinances regarding worship and ritual, as well as political and (in the narrow sense) legal rules, details of toilet, formulas of greeting, table-manners, and sick-room conversation (Schacht 1974, 392).

True, religious fundamentalism in itself has normally had no positive effects on scientific and philosophical activity, and it has rarely been an urge toward intellectual revolution. In ninth-century Islam, however, fundamentalism was confronted with a complex society in transformation. Religious authority was not segregated socially as a "Church": *ḥadīth* ("traditions," cf. note 57) and Islamic jurisprudence in general were the province mainly of persons engaged in practical life, be it handicraft, trade, secular teaching, or government administration (Cohen 1970). Furthermore, a jealous secular power did its best to restrain the inherent tendencies of even this stratum to get the upper hand.[58]

Fundamentalism, combined with the practical involvement of the carriers of religious authority, may have expressed itself in the recurrent tendency in Islamic thought to regard "secular" knowledge (*scientia humana,* in the medieval Latin sense) not as an alternative to Holy Knowledge (*kalām al-dīn,* "the discourse of Faith,"[59] *scientia divina*) but as a way to it, and even to contemplative truth, *ḥāl* (*sapientia,* in yet another Latin approximate parallel). Psychologically, it would be

[58] There were at least two (tightly coupled) good reasons for this jealousy. First, traditionists and jurisprudents might easily develop into a secondary center of power; second, they might inspire, participate in, or strengthen popular risings, which were already a serious problem for the ʿAbbasid caliphs.

The destruction of the Baghdad "House of Learning" (*Dār al-ʿilm* – "Residence of Knowledge" would perhaps be a more precise translation) in a Sunni riot in 1059 A.D. shows what could be the fate even of scholarly institutions when religious fervor and social anger combined. See Makdisi 1961, 7f.

[59] The expression is quoted from the ninth-century *Mutakallim* al-Jāḥiẓ via Anton Heinen (1978, 64), who sums up (p. 57) his point of view in the formula *"Knowledge (ʿilm) = kalām al-dīn + kalām al-falsafah,"* the latter term meaning "the discourse of philosophy," i.e., secular theoretical knowledge.

almost impossible to regard a significant part of the activity of "religious personnel" as irrelevant to their main task,[60] or as directly irreverent (as illustrated by the instability generated by scandalous popes, etc., and leading to the Reformation. In Islam, the ways of the ʿUmayyad caliphs provided as effective a weapon for *Khārijite* radicals and for the ʿAbbasid takeover as that given to Anabaptists and to Lutheran princes by the Renaissance popes). That integration of science and religious attitude was not just a *Mutakallimūn*'s notion, but was shared to a certain extent by active mathematicians, is apparent from the ever-recurring invocation of God at the beginning and end of their works (and the references to Divine assistance interspersed throughout the text of some works.[61]

The legitimization of scientific interests through the connection to a religion that was fundamentalist in its theory and bound up with social life in its practice may also have impeded the complete segregation of pure science from the needs of daily life without preventing it, however, from rising above these needs. This could lead not only to the phenomenon that even the best scientists would occasionally be concerned with the most practical and everyday applications of their science,[62] but also to a general appreciation that theory and practice belong together naturally (cf. above, chapter VII, and especially below, chapter XIII).

The plausibility of this explanation can be tested against some parallel cases. One of these is that of Syriac learning, which belonged in a religious context with similar fundamentalist tendencies. Syrian Christianity, however, was carried by a church, i.e., by persons who were segregated from social practice in general, and Syriac learning was carried by these same persons. The custodial and noncreative character of Syriac learning looks like a sociologically trite consequence of this situation; as Pines (1970, 783) explains:

> pre-Islamic monastic Syriac translations appear to have been undertaken mainly to integrate for apologetic purposes certain parts of philosophy, and perhaps also of the sciences, into a syllabus dominated by theology. In fact great prudence was exercised in this integration; for instance, certain portions of Aristotle were judged dangerous to faith, and banned.

[60] In a similar way, practical charity, the management of ritual and sacraments, and religious teaching are understood as necessarily belonging together in Christian environments where the church (and eventually the same priest) takes care of all of them.

[61] One work containing such copious references to God is Abū Bakr's *Liber mensurationum,* which was discussed above. Normally the invocations were abridged or left out in the Latin translations (not least in Gherardo's translations); in this case, however, they have survived because of their position within the body of the text (while the compulsory initial invocation is deleted).

Of course, routine invocation is no indicator of deep religious feeling. What matters is that the invocation *could* develop into a routine, and that it was thus considered *a matter of course* or *decorum* even in mathematical texts. You may perhaps persevere in an activity which you fear is unpleasant to God; but if so, you rarely invite him explicitly (routinely or otherwise) to pay heed or assist you in your sins.

[62] Among the numerous examples I shall mention Abū'l-Wafā''s *Book on What is Necessary from Geometric Construction for the Artisan* (trans. Krasnova 1966), which was discussed above; al-Uqlīdisī's *Arithmetic,* the mathematical level of which suggests that the author must have been beyond the rank-and-file; and ibn al-Haytham's works on the determination of the *qiblah* (the direction toward Mecca) and on commercial arithmetic (nos. 7 and 10 in Abī Uṣaybiʿa's list, trans. Nebbia 1967, 187f., cf. Rozenfeld 1976, 75).

Other interesting cases are found in twelfth- and thirteenth-century Latin Christianity. Particularly close to certain ninth-century Islamic attitudes is Hugh of Saint-Victor, the teacher and rationalistic mystic from the Paris school of Saint Victor (cf. Chenu 1974). He was active during the first explosive phase of the new Latin learning (like the Islamic late eighth and early ninth century A.D. the phase during which the *Elements* were translated) in a school that was profoundly religious and at the same time bound up with the life of its city. The sociological parallels with ninth-century Baghdad are striking. Also striking are the parallel attitudes toward learning. In the propaedeutic *Didascalicon,*[63] Hugh pleads for the integration of the theoretical "liberal arts" and the practical "mechanical arts"; his appeal "learn everything, and afterwards you shall see that nothing is superfluous"[64] permits the same wide interpretation as the Prophet's saying "seek knowledge from the cradle to the grave";[65] and Hugh considers Wisdom (the study of which is seen in I.iii as "friendship with Divinity") to be a combination of moral and theoretical truth and practitioners' knowledge.[66] One can hardly come closer to al-Jāḥiẓ's "formula," as quoted in note 59.

Hugh, however, was an exception already in his own century. The established church, as represented by the eminently established Bernard of Clairvaux, fought back; even the later Victorines demonstrated through their teaching that a socially segregated ecclesiastical body is not compatible with a synthesis of religious mysticism, rationalism, and an open search for all-encompassing knowledge. As a consequence, the story of twelfth- and thirteenth-century Latin learning can on the whole be read as a tale of philosophy as potentially subversive knowledge, of ecclesiastical reaction, and of a later final synthesis in which the "repressive tolerance" of Dominican domesticated Aristotelianism blocked the future development of learning.[67] It is also a tale of segregation between theoretical science and practitioners' knowledge:

[63] I use the edition in *PL* 176, col. 739–952. A recent English translation is Taylor 1961 (it should be noted that the chapters are numbered somewhat differently in the two versions).

[64] *Omnia disce, videbis postea nihil esse superfluum* (*Didascalicon* VI:iii). Strictly speaking, "everything" is "everything in sacred history," since this is the subject of the chapter; but in the argument Hugh's own play as a schoolboy with arithmetic and geometry, his acoustical experiments and his all-devouring curiosity are used as parallel illustrative examples.

[65] Quoted from Nasr 1968, 56. In this case, as in that of Hugh, the intended meaning of "knowledge" is probably not quite as wide as a modernizing reading might assume. This, however, is less important than the open formulation and the optimism about the religious value of knowledge that was read into it, and against which the opponents of *awā'il* knowledge had to fight (cf. Goldziher 1915, 6) – for centuries only with limited success.

A major vehicle for the high evaluation of knowledge in medieval Islam (be it knowledge in the narrow sense, viz. knowledge of God's "Uncreated Word," i.e., of the Koran, and of the Arabic language), and at the same time a virtual medium for the spread of a high evaluation of knowledge in a more general sense, was the establishment of education on a large scale in Koran schools and related institutions. (I am grateful to Jan Hogendijk for reminding me of this point, which I had not mentioned in the first version of this paper.)

A wealth of anecdotes illustrating an almost proverbial appreciation of knowledge (from the level of elementary education to that of genuine scholarship) will be found in Tritton 1957, 27f. and passim.

[66] I.ix. "Practitioners' knowledge" translates *scientia mechanica,* while "moral truth" renders *intelligentia practica/activa.*

[67] This picture is of course unduly schematized. The subject is dealt with in somewhat greater detail in my paper of 1985, 32–38.

lay theoretical knowledge gained a subordinated autonomy, but only by being cut off from the global world view[68] and concomitantly also from common social practice. This entailed loss of that unsubordinated, mutually fecundating integration with practical concerns that was a matter of course in Islam.[69]

XII. Variations of the Islamic Pattern

Above, Islam was regarded as an undifferentiated whole or a broad average. Such generalizations have their value. Still, looking at specific religious currents or theological schools might give us some supplementary insight into the mechanisms coupling faith, learning, and social life, or at least give the picture the shades of life.

One current in which the coupling between the "discourse of philosophy" and the "discourse of Faith" was strong was the *muʿtazila*. According to earlier interpretations, admittedly, the muʿtazilite attitude to philosophy should have been as distorted as that of Syrian monasticism;[70] but from Heinen's recent analysis it appears that the *muʿtazila* in general did not derive the Syro-Christian sort of intellectual censorship on philosophy from the theological aims of *kalām* (cf. above, note 59). Under al-Maʾmūn, who used muʿtazilism in his political strategy, the inherent attitudes of this theological current were strengthened by the ruler's interest in clipping the wings of those traditionalists whose fundamentalism would lead them to claim supreme authority, concerning knowledge first of all, but implicitly also in the moral and political domain (cf. above, note 58 and main text there). Among the *Ismāʿīlī* there was an equally strong (or stronger) acceptance of the relevance of *awāʾil* knowledge for the acquisition of Wisdom, even though the choice of disciplines was different from that of the *mutakallimūn* (cf. the polemic quoted in note 70): to judge from the *Ikhwān al-ṣafāʾ*, Neoplatonic philosophy, Harranian astrology, and the Hermeticism of late antiquity were central subjects (cf. Marquet, *DSB*, XV:249–51); but the curriculum of the *Ismāʿīlī al-Azhar madrasah* in Cairo included philosophy, logic, astronomy, and mathematics (Fakhry 1969, 93), the central subjects of

[68] The situation is expressed pointedly by Boethius de Dacia in his beautiful *De eternitate mundi* (ed. Sajo 1964, 46 and passim), when he distinguishes the truth of natural philosophy (*veritas naturalis*) from "Christian, that is genuine, truth" (*veritas christianae fidei et etiam veritas simpliciter*).

[69] When integration was needed by a group of practitioners, as in thirteenth- and fourteenth-century astronomy, the need was satisfied by means of simplifying compendia, in striking contrast to the development in Islam – cf. below, chapter XVI. Latin science, when applied, was subordinated, and hence not fertilized by the interaction with the questions and perspectives of practice; for this reason, on the other hand, applications were bound to remain on the level of common sense. Cf. Beaujouan 1957, especially the conclusion.

[70] So, according to Albert Nader, "les muʿtazila touchent à la sphère physique avec des mains conduites par des regards dirigés vers une sphère métaphysique et morale: la raison cherchant à concilier les deux sphères (1956, 218, quoted from Heinen 1978, 59). As pronounced enemies of the *Muʿtazila*, the tenth-century Ismāʿīlī *Ikhwān al-ṣafāʾ* are still more emphatical, claiming that the *Muʿtazila* "die medizinische Wissenschaft für Unnütz, die Geometrie als zur Erkenntnis des wahren Wesens der Dinge unzuständig halten, die Logik und die Naturwissenschaften für Unglauben und Ketzerei und ihre Vertreter für irreligiöse Leute erklären" (IV:95; quoted from Goldziher 1915, 25).

ancient science. The same broad spectrum of religiously accepted interests (including also Jabīrian alchemy) can be ascribed to the Shi'ite current in general. In the Ashᶜarite reaction to muᶜtazilism, and in later Sunna, the tendency was to emphasize fundamentalism and to reject non-Islamic philosophy, or at least to deny its relevance for Faith. Accordingly, the curriculum of the Sunni Nizamiyah *madrasah* in Baghdad included, alongside the traditionalist disciplines (religious studies, Arabic linguistics, and literature), only "arithmetic and the science of distributing bequests" (Fakhry 1969, 93; the latter being in fact a "subdivision of arithmetic," as ibn Khaldūn explains[71]).

In the long run, muᶜtazilism lost out to Sunnism, and in the very long run the dominance of traditionalist Sunnism (and of an equally traditionalist Shi'ism) was probably one of the immediate causes for Islamic science's loss of vigor. During the Golden Age, however, when institutionalization was still weaker than a vigorous and multidimensional social life, the attitudes of the formulated theological currents reflected rather than determined ubiquitous dispositions (cf. also above, note 65). Muᶜtazilism was only the most clear-cut manifestation of more general tendencies, and the reversal of muᶜtazilite policy in 849 did not mean the end of secular intellectual life in the Caliphate nor of the routine expression of religious feelings in the opening and closing sentences of scholarly works. Furthermore, through Sufi learning, and in the person of al-Ghazzālī, secular knowledge gained a paradoxical new foothold – more open, it is true, to quasi-Pythagorean numerology than to cumulative and high-level mathematics,[72] but still able to encourage more serious scientific and mathematical study. It appears that the Sufi mathematician ibn al-Bannā' did in fact combine mathematical and esoteric interests (Renaud 1938); and even though al-Khayyāmī's Sufi confession may be suspected of not reflecting his inner opinions as much as his need for security (Youschkevitch and Rosenfeld, *DSB,* VII:330; Kasir 1931, 3f.), his claim that mathematics can be part of "wisdom"[73] must either have been an honest conviction or (if it was meant to ensure his safety) have had a plausible ring in contemporary ears. At the same time, the two examples show that the role of Gnostic sympathies was only one of *external* inspiration: ibn al-Bannā''s works are direct (and rather derivative) continuations of the earlier mathematical and astronomical tradition (see Vernet, *DSB,* I:437f.), and al-Khayyāmī's treatise was written as part of a running tradition of metamathematical commentaries on the *Elements,* and in direct response to ibn al-Haytham. Gnostic sympathies might lead scholars to approach and go into the mathematical traditions, but it did

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71 *Muqaddimah* VI:19, trans. Rosenthal 1958, III:127. It will also be remembered that "inheritance calculation" occupies just over one half of al-Khwārizmī's *Algebra* (pp. 86–174 in Rosen's 1831 translation).

72 The similarity with the *Ismāᶜīlī* orientation is clear; according to ibn Khaldūn, good reasons for such similarity exist through the close relations between the early Sufis and "Neo-Ismâᶜîlîyah Shîᶜah extremists" (*Muqaddimah* VI:16, trans. Rosenthal 1958, III:92).

The paradoxical (or at least quite vacillating) attitude of the mature al-Ghazzālī toward mathematics is illustrated through a number of quotations in Goldziher 1915, passim.

73 The claim is even given emphasis by a somewhat clumsy repetition in the introduction to his "Discussion of Difficulties of Euclid" (trans. Amir-Móez 1959, 276).

not transform the traditions, nor did it influence how work was carried out within traditions.

XIII. The Importance of General Attitudes: The Mutual Relevance of Theory and Practice

Analysis according to specific religious currents is hence not to be taken too literally when the shaping of mathematics is concerned. As long as religious authority was not *both* socially concentrated and segregated *and* in possession of scholarly competence (as it tended to be in thirteenth-century Latin Christianity), the attitudes of even dominant religious currents and groups could influence the internal development and character of learning only indirectly, by influencing overall scholarly dispositions and motivations. (What they could do directly, without fulfilling these two conditions, was to strangle rational scholarship altogether – such things happened; but they affected the pace and ultimately the creativity of Islamic science, which is a different issue.) If scholars could find a place in an institution under princely protection (a "library with academy," i.e., *Dār al-ᶜilm* and the like, or an observatory) or covered by a religious endowment (*madrasah,* hospital, etc.), the absence of a centralized and scholarly competent church permitted them to work in relative intellectual autonomy, as long as they stayed within the limits defined by institutional goals:[74] princes, at least, were rarely competent to interfere with learning by more subtle and precise means than imprisonment or execution.[75] The general attitude – that mathematics qua knowledge was religiously legitimate and perhaps even a way to Holy Knowledge, and that conversely the Holy was present in the daily practice of this world – could mold the disposition of mathematicians to the goals of their discipline; but even a semi-Gnostic conception of rational knowledge as a step toward Wisdom appears not to have manifested itself as a direct claim on the subjects or methods of actual scientific work – especially not as a claim to abandon traditional subjects or methods.

Accordingly, in Islamic mathematical works explicit religious references are normally restricted to the introductory dedication to God, the corresponding clause at the end of the work, and perhaps passing remarks invoking his assistance for understanding the matter or mentioning his monopoly on absolute knowledge. Apart from that, the texts are as secular as Greek or medieval Latin mathematics. It

[74] In a case like the Sunni *Nizamiyah madrasah* in Baghdad, the institutional goals were of course already quite restricted. They would permit you to teach *al-jabr* but not Apollonios.

Formally, the situation was thus not very different from that of Syriac monastic learning. The Syriac learned monk, too, had to stick to the institutional goal of Church and monastery. Materially, however, the difference was all-important, because the institutional goal of the Church included the defense of an already established theological opinion. We may think of the difference between the obligation to *teach biology instead of sociology* and the *prohibition to teach anything except creationist biology.*

[75] Cf. the anecdotes about Hulagu and Naṣīr al-Dīn al-Ṭūsī reported by Sayili (1960, 207), and the story of the closing of the Istanbul observatory when its astrological predictions had proved catastrophically wrong (ibid., 291–93). At most, the ruler was able to make cuts in a program that was too ambitious for his taste, or to close an institution altogether.

is impossible to tell whether the divine dedications in Qusṭā ibn Lūqā's translation of Diophantos[76] are interpolations by a Muslim copyist, or were written by the Christian translator with reference to a different God: they are quite external to the rest of the text. The ultimate goals of the activity were formulated in a different way from what we find in Greek texts (*when* a formulation is found at all in Greek texts; but cf. the initial quotation from Aristotle). The Islamic mathematician would (irrespective of his personal creed) not be satisfied by staying at the level of immediate practical necessity, he would go beyond these and produce something higher, viz., principles, proofs, and theory; nor would he, however, feel that any theory, however abstract, was *in principle* above application,[77] or that the pureness of genuine mathematics would be polluted by possible contact with daily needs. Several examples were discussed above, in chapter VII (al-Khwārizmī, Abū'l-Wafā') and in chapter XI, note 62 (al-Uqlīdīsī, ibn al-Haytham). An example involving a nonmathematician (or rather a philosopher-not-primarily-mathematician) is al-Fārābī's chapter on *ʿilm al-ḥiyal,* the "science of artifices" or "of high-level application" (see above, end of chapter II). We should of course not be surprised to find the science of *al-jabr wa'l-muqābalah* included under this heading; algebra *was* already a high-level subject when al-Fārābī wrote. But even if we build our understanding of the subject on Abū Kāmil's treatise, we might well feel entitled to wonder at seeing it intimately connected to the complete ancient theory of surd ratios, including both that given by Euclid in *Elements* X and "that which is not given there" (my translation from the Spanish of Palencia 1953, 52). More expressive than all this, however, is the preface to al-Bīrūnī's trigonometrical treatise *Kitāb istikhrāj al-awtār fī'l-dā'irah bi-khawāṣṣ al-khaṭṭ al-munḥanī al-wāqiʿ fīhā* (*Book on finding the Chords in the Circle* ...), which I quote in extensive excerpts from Suter's German translation (1910a, my English translation – emphasis added):

> You know well, God give you strength, for which reason I began searching for a number of demonstrations proving a statement due to the ancient Greeks concerning the division of the broken line in an arbitrary circular arc by means of the perpendicular from its center, and which passion I felt for the subject [...], so that you reproached [?] me my preoccupation with these chapters of geometry, not knowing the *true essence of these subjects, which consists precisely in going in each matter beyond what is necessary.* If you would only, God give you strength, observe the aims of geometry, which consist in determining the mutual relation between their magnitudes with regard to quantity, and [if you would only observe] that it is in this way that one reaches knowledge of the magnitudes of *all things measurable and ponderable found between the center of the world and the ultimate limits of*

[76] Openings of Books IV, V, VI, and VII, and the closing formula of the work, trans. Sesiano 1982, 87, 126, 139, 156, 171, or Rashed 1984, III:1, IV:1, 35, 81, 120. In the last of these places, the praise that ends the work is followed by the date of copying, which is again followed by another praise of God and a blessing of the Prophet, in a way that suggests (through comparison with other treatises with Muslim author and Muslim copyist), but does not prove, that the first praise goes back to Qusṭā himself.

[77] "In principle," for of course much theory went unapplied in practice, and theory was developed regardless of possible application.

> *perception through the senses.* And if you only knew that by them [the geometrical magnitudes] are meant the [mere] forms, detached from matter [. . .]. Whatever way he [the geometer] may go, through exercise will he be *lifted from the physical to the divine teachings,* which are little accessible because of the difficulty to understand their meaning, because of the subtility of their methods and the majesty of their subject, and because of the circumstance that not everybody is able to have a conception of them, *especially not the one who turns away from the art of demonstration.* You would be right, God give you strength, to reproach me, had I neglected to search for these ways [methods], and used my time for something where an easier approach would suffice; or if the work had *not arrived at the point which constitutes the fundament of astronomy,* that is to the calculation of the chords in the circle and the ratio of their magnitude to that supposed for the diameter [. . .].
> Only in God the Almighty and All-wise is relief!

By going beyond the limits of immediate necessity and by cultivating the abstract and apodictic methods of his subject, the geometer hence worships God – but only on the condition that (like God, we may add) he cares for the needs of everyday astronomy.

Al-Bīrūnī's formulation is unusually explicit, perhaps reflecting an unusually explicit awareness of current attitudes and their implications. Normally these attitudes stand out most obviously in comparison with texts of similar purpose or genre from neighboring cultures. Particularly gratifying in this respect is the field of technical literature. As mentioned in note 69 in the case of astronomy, the prevailing tendency in Latin learning was to opt for the easy way by means of simplifying, nonapodictic compendia. Most illustrative are the various Anglo-Norman treatises on estate management. One such treatise was compiled on the initiative of (or even by) Robert Grosseteste;[78] yet it contains nothing more than common-sense and rules of thumb. Not only was the semiautonomous arena granted to rational philosophical discussion in the thirteenth-century compromise not to encroach on *sacred* ground; neither should it divert the attention of *practical* people and waste their time. In contrast, a handbook on "commercial science," written by one Shaykh Abū'l-Faḍl Jaʿfar ibn ʿAlī al-Dimishqī sometime between 870 and 1174 A.D., combines general economic theory (the distinction between monetary, movable, and fixed property) and Greek political theory with systematic description of various types of goods and with good advice on prudent trade.[79] Knowledge of the fine points of trade was, just

[78] Several of the treatises were edited by Oschinsky (1971). Grosseteste's involvement is discussed pp. 192ff.; cf. the texts, pp. 388–409.

[79] A discussion and a partial translation of the treatise is given by Ritter (1916). The treatise is a contrast not only to thirteenth-century European handbooks on prudent management, but also to Greek common-sense deliberations like Hesiod's *Works and Days* or Xenophon's *Oeconomica.* A similar contrast is obvious if we compare Ovid's *Ars amoris* or the pseudoscholarly treatises to which it gave rise in the Latin Middle Ages with the development of regular sexology in Islam. Closer to mathematics, we may compare Villard de Honnecourt's very unscholarly reference to *figures de lart de iometrie* (*Sketchbook,* ed. Hahnloser 1935, Taf. 38) with the serious study of Euclidean geometry by Islamic architects (see Wiedemann 1970, I:114).

like mathematics or any other coherently organized, systematic field of knowledge, considered a natural part of an integrated world view covered by *al-Islām*. In agreement with the basic (fundamentalist but still non institutionalized) pattern of this world view, the theoretical implications of applied knowledge were no more forgotten than were the possible practical implications of theory.

The use of theory to improve on practice looks like a fulfillment of the ancient Heronian and Alexandrinian project.[80] It had not been totally inconsequential: the acceptance of $\pi = 22/7$ was discussed above; "Archimedes' screw" and Alexandrinian military and related techniques were also reasonably effective (Gille 1980). On the whole, however, the project had proved beyond the forces of ancient science and of ancient society – for good reasons, we may assume, since fruitful application of theory presupposes a greater openness to the specific problems and perspective of practitioners than was current in Greek science.[81] Thus, as was claimed in the introductory chapter, the *systematic* theoretical elaboration of applied knowledge was a specific creation of the Islamic world. It was already seen in the early phase of Islamic mathematics, when the traditions of "scientific" and "subscientific" mathematics were integrated. The great synthetic works in the vein of al-Khwārizmī's *Algebra* or Abū'l-Wafā''s *Book on What is Necessary from Geometric Construction for the Artisan* were discussed above (chapter VII), as was their occasionally eclectic character. One step further was taken in cases where a problem taken from the subscientific domain was submitted to theoretical investigation on its own terms (i.e., not used only as inspiration for an otherwise independant investigation, as when Diophantos takes over various recreational problems and undresses them in order to obtain pure number-theoretical problems). Thābit's Euclidean "verification of the rules of *al-jabr*" was mentioned above (chapter VII), and Abū Kāmil's preface to his investigation of the recreational problem of the "hundred fowls" was quoted in chapter IV.[82] A final and decisive step occurred when the results of theoretical investigation were adopted in and transmitted through books written for practitioners. A seeming first adoption of Thābit's *Verification* is found in Abū Kāmil's *Algebra* (trans. Levey 1966: 34–36). At closer inspection, however, the reference to Euclid is rather ornamental, and the Euclidian proposition referred to is "demonstrated" by naive geometry. But in Abraham bar Ḥiyya's (Savasorda's) *Collection on Mensuration and Partition*, ~~where Euclid is not even mentioned by name,~~ the actual argument can be understood only by somebody knowing his Euclid by heart.[83]

[80] Cf. Hero's introductions to the *Metrika* and *Dioptra* (ed. Schöne 1903, 2ff. and 188ff.).

[81] We may remember Benjamin Farrington's observation that "it was not [. . .] only with Ptolemy and Galen that the ancients stood on the threshold of the modern world. By that late date they had already been loitering on the threshold for four hundred years. They had indeed demonstrated conclusively their inability to cross it" (1969, 302).

[82] A parallel case is ibn al-Haytham's investigation of the "purchase of a horse" (partially translated in Wiedemann 1970, II:617–19). This treatise too opens with a polemic against practitioners who do not justify their procedures.

[83] Latin (and German) translation in Curtze 1902, 38, 40. It should be observed that Abraham's text is meant most practically. It is in the same tradition as Abū Bakr's *Liber mensurationum*, cf. above, chapter VI.

XIV. The Institutionalized Cases: (1) *Madrasah* and Arithmetical Textbooks

The two most interesting cases of infusion of theory into an inherently practical mathematical tradition appear to be coupled not only to the general dispositions of Islamic culture, but also to important institutions that developed in the course of time. The first institution I shall discuss is the writing of large-scale, reasoned arithmetical textbooks. Al-Uqlīdisī's work was already discussed above (chapter IV and note 62); in his introductory commentary to the translation Saidan (1978, 19–31) describes a number of other works that have come down to us.[84] In the early period, two largely independent types can be described: the "finger-reckoning type" and the "Hindi type," using verbal and Hindu numerals, respectively. The two most important finger-reckoning books are those of Abū'l-Wafā' (description in Saidan 1974) and of al-Karajī (trans. Hochheim 1878). The two earliest extant Hindi books are those of al-Khwārizmī (extant only in Latin translation) and al-Uqlīdisī. Among lost works on the subject from the period between the two, al-Kindī's treatise in four sections can be mentioned.[85] Later well-known examples are Kūshyār ibn Labbān's explanation of the system for astronomers (trans. Levey and Petruck 1965), and al-Nasawī's for accounting officials.[86]

After the mid-eleventh century, it becomes difficult to distinguish two separate traditions. While al-Nasawī, examining around 1030 A.D. earlier treatises on his subject, would still (according to his preface) restrict the investigation to Hindi books, his contemporary ibn Ṭāhir inaugurated an era where the traditions were combined, and wrote a work presenting "the elements of hand arithmetic and the chapters of *takht* [dust board, i.e. Hindi – J.H.] arithmetic," together with "the methods of the people of arithmetic" (apparently his section 6 on Greek theoretical arithmetic) and the "arithmetic of the zīj" (sexagesimal fractions).[87]

The same combination is found again in the Maghrebi arithmetical tradition as we know it from works of al-Ḥaṣṣar (reported extensively in Suter 1901), ibn al-Bannā' (ed. and trans. Souissy 1969), al-Umawī,[88] and al-Qalaṣādī,[89] and through ibn Khaldūn's report.[90] This tradition is interesting in several respects, not least for its

[84] In the following, I follow Saidan's typology.

[85] Listed in al-Nadīm's *Fihrist*, trans. Dodge 1970, 617. It is not clear whether the "Introduction to Arithmetic, five sections," also mentioned there, is a finger-reckoning treatise, a commentary on Nicomachos (in whom he was certainly interested), or a combination of these two.

[86] To be precise, the Arabic treatise that has come down to us was written for the Būyid vizier Sharaf al-Mulūk; but we must assume that the author stayed close to the earlier treatise written in Persian, of which he himself speaks in the preface. See the translation of the preface in Woepcke 1863, 492–95, and Saidan, *DSB*, IX:614. The discussion of the treatise in Suter 1906 covers only the brief section dealing with the extraction of roots.

[87] Quoted from Saidan 1978, 24 (pp. 24–29 present an extensive abstract of the whole work).

[88] See Saidan 1978a and idem, *DSB*, XIII:539f. Al-Umawī taught in Damascus, but he came from the West, where he had been taught, and he brought its methods to the East.

[89] His "rather extended and rich summary" of arithmetic (as he describes it himself in the introduction) was translated by Woepcke (1859).

[90] *Muqaddimah* VI:19, trans. Rosenthal 1958, III:122f. Ibn Khaldūn had himself been taught by a disciple of ibn al-Bannā' (see Vernet, *DSB*, I:437). A systematic investigation of certain aspects of Maghrebi mathematics has been undertaken by Djebbar (1981).

systematic development of arithmetical and algebraic symbolism,[91] the former of which was taken over by Leonardo Fibonacci in the *Liber abaci*.[92]

The early writers of major arithmetic textbooks appear to have been relatively independent of one other. During the initial phase of synthesis, they collected, systematized, and reflected upon current methods and problems of one or the other group of practitioners, and they also used earlier treatments that were accessible as books. The first source appears to have been used in al-Uqlīdisī's work, while al-Nasawī quotes the written works he has consulted. The integration of finger and Hindi reckoning, on the other hand, appears to depend upon the more continuous teaching tradition of the *madrasah*. Already ibn Ṭāhir is reported to have taught at the mosque (Saidan, *DSB*, XV:9), and, as mentioned above (chapter XII), the only non-"traditional" subject permitted at the Baghdad *Nizamiyah madrasah* was arithmetic. In the Maghreb tradition, ibn al-Bannā' was taught and himself became a teacher of mathematics and astronomy at the *madrasah* in Fez (Vernet, *DSB*, I:437). This makes it inherently plausible that even al-Ḥaṣṣar, upon whose works he commented, had relations with the *madrasah* (at the very least his works must have been used there). Ibn al-Bannā''s network of disciples also appears to cohere through the social network of *madrasah* learning. Al-Qalaṣādī (as a writer of commentaries to ibn al-Bannā') must be presumed to belong to the same context, and in fact he tells himself that his arithmetic is written as a manual for the brightest among his students (trans. Woepcke 1859, 231; cf. also Saidan, *DSB*, XI:229f.). Finally, even al-Umawī was active as a teacher in Damascus. The theoretical elevation of the subject of arithmetic was hence not only a product of the general dispositions of Islamic culture; according to all evidence it was also mediated by the *madrasah*, which in this respect came to function as an embodiment and as an institutional means of fixation of these same attitudes.

That theoretical elevation of this practical subject requires a specific explanation becomes evident when we compare the Islamic tradition with the fate of its "Christian" offspring: the *Liber abaci*, which carried the elevation of practical arithmetic to its apex. This is not to say that Fibonacci's book was a cry in the desert. Its algebra influenced scholarly mathematics in the fourteenth century, e.g., Jean de Murs (see G. l'Huillier 1980, passim); it is also plausible that it inspired Jordanus de Nemore.[93] Part of the material was also taken over by the Italian "abacus schools" for merchant youth. The scholars, however, took over only specific problems and ideas, and the abacus teachers only the more elementary, practically oriented facets of the work. Western Europe of the early thirteenth century had no institution that could appreciate, digest, and continue Fibonacci's work. Only in the fifteenth century do similar

[91] A general account is given in Djebbar 1981, 41–54. The explanation given by ibn al-Bannā''s commentator ibn Qunfudh is translated in Renaud 1944, 44–46. Woepcke (1854) deals mainly with al-Qalaṣādī's symbolism.

[92] Compare Fibonacci's various complicated fractions (ed. Boncompagni 1857, 24) with the similar forms in Djebbar (1981, 46f.).

[93] Another possibility is that Fibonacci was drawing on Jordanus for the revised edition written to Michael Scot – see my paper (1985a, 7f.).

orientations turn up once again – apparently not without renewed relations with the Islamic world.[94]

XV. The Institutionalized Cases: (2) Astronomy and Pure Geometry

The other case of an entire tradition integrating theoretical reflection and investigation into a branch of practical mathematics is offered by astronomy. Because of its ultimate connection to astrology, astronomy was itself a practical discipline;[95] the *mathematics* of astronomy was, of course, practical even when astronomy itself happened to be theoretical.

In the Latin thirteenth through fifteenth centuries, this practical aim of the mathematics of astronomy led to a reliance on compendia, as was observed above (note 69). In Islam, however, astrology was the occasion for the continuing creation of new zījes and for the stubborn investigation of new planetary models. Islam was not satisfied with using good established models like the *Zīj al-Shah* and the *Zīj al-Sindhind* in the way the Latin late Middle Ages went on using the *Theorica planetarum* and the *Toledan Tables* for centuries.

Astronomy can even be seen to have been the main basis for mathematical activity in medieval Islam. This appears from even the most superficial collective biography of Islamic mathematicians. Their immense majority is known to have been active in astronomy.[96] Since astronomy (together with teaching at levels up to that of the *madrasah,* which hardly required anything more advanced than large arithmetic and mensuration textbooks), was the most obvious way for a mathematician to earn a living, one is forced to conclude that the astronomer's career involved quite serious work on mathematics, and perhaps serious work *in* mathematics.

This is also clear from al-Nayrīzī's introductory explanation to his redaction of the al-Ḥajjāj version of the *Elements:* here it is stated that "the discipline of this book is

[94] An early example is a Provençal arithmetic written c. 1430. A certain affinity to Islamic traditions is suggested by an initial invocation of God, of Mary his mother, and of the patron saint of the city (see Sesiano 1984 – the invocation is on pp. 29–31). Later examples are Chuquet's *Triparty* and Luca Pacioli's *Summa de arithmetica.* The *Triparty* is said (in its first line) to be divided into three parts "a lonneur de la glorieuse et sacree trinite" (ed. Marre 1880, 593) – perhaps a jocular reference to familiar invocations in related treatises? In any case, the same author's *Pratique de géometrie* (ed. H. L'Huillier 1979) has no religious introduction.

[95] Two other practical aims for astronomy can also be mentioned: finding the *qiblah,* and fixing the prayer times. None of them called for astronomy of such sophistication as developed around the princely observatories.

[96] Among those referred to above, only Abū Kāmil, al-Karajī, al-Samaw'al and al-Qalaṣādī stand out as exceptions. Al-Samaw'al, however, is at least known to have written a refutation of astrology, involving both mathematical arguments and knowledge of observations (Anbouba, *DSB,* XII:94). Strictly speaking, even ibn Turk, al-Uqlīdisī, al-Ḥaṣṣar, and al-Umawī might also be counted as exceptions, since no astronomical works from their hand are known. However, our knowledge of these scholars is so restricted that they fall outside all attempts at statistical analysis.

A last important mathematician who appears definitely not to have been an astronomer is Kamāl al-Dīn, whose important work concentrates on optics (cf. Suter 1900,.159, no. 389; Rashed, *DSB,* VII:212–19).

an introduction to the discipline of Ptolemy's *Almagest*."[97] Later, the same connection was so conspicuous that the Anglo-Norman writer John of Salisbury could observe, in 1159, that "demonstration," i.e., the use of the principles expounded in Aristotle's *Posterior Analytics,* had

> practically fallen into disuse. At present demonstration is employed by practically no one except mathematicians, and even among the latter has come to be almost exclusively reserved to geometricians. The study of geometry is, however, not well-known among us, although this science is perhaps in greater use in the region of Iberia and the confines of Africa. For the peoples of Iberia and Africa employ geometry more than do any others; they use it as a tool in astronomy. The like is true of the Egyptians, as well as some of the peoples of Arabia (*Metalogican* IV, vi, quoted from McGarry's translation 1971, 212).

So it was not only the factual matter of the *Elements* that was reckoned part of the astronomical curriculum. According to the rumors that had reached John of Salisbury (via the translators?), the geometry of astronomy was also concerned with the metamathematical aspects and problems of the *Elements.*

In the initial eager and all-devouring phase of Islamic science (say until al-Nayrīzī's time, i.e., the early tenth century A.D.), the general positive appreciation of theoretical knowledge may well have laid the foundation both for the extension of astrology into the realm of high-level theoretical astronomy and for the extension of astronomy into that of theoretical mathematics. Down-to-earth sociology of the astronomers' profession may be a supplementary explanation of the continuation of the *first* tradition: the importance of the court astronomer (and, where it existed, of the court observatory) could only increase if astronomy was a difficult and inaccessible subject. But even if this common-sense sociology is correct, it is not clear why intricacy should be obtained via the integration of metamathematics, the difficulties of which would only be known to the astronomer himself, and which would therefore hardly impress his princely employer. Why then should the integration have survived for so long?

It appears, once more, that the original positive appreciation of (mathematical) theoretical knowledge was materialized institutionally, in a relatively fixed curriculum for the study of astronomy. This curriculum began (as stated by al-Nayrīzī) with the *Elements,* and ended with the *Almagest.* In between came the *Mutawassiṭāt,* the "Middle Books" (cf. above, chapter II).

It is not clear to what degree this established curriculum was developed at different times. A full codification of the corpus of Middle Books is known only from the Naṣirean canon (Steinschneider 1865, 467 and passim; Nasr, *DSB,* XIII:509) and the precise delimitation of the concept may have varied with time and place. Most remarkable are perhaps the indications that books I–II of the *Conics* may also have been considered normal companions of the *Elements* in the times of ibn al-Haytham (see Sesiano 1976, 189) and al-Khayyāmī (cf. note 11 above). It appears, however,

[97] My translation from the Latin of Besthorn and Heiberg (1893, I:7).

that Ḥunayn ibn Isḥāq made a translation of the "Little Astronomy" which already served the purpose, and that Thābit had a similar concept (Steinschneider 1865, 464 and 457, resp.).[98] Al-Nayrīzī, too, appears to have a fixed curriculum in mind.

So from the ninth century A.D. onward it appears that astronomical practice and interest kept the focus upon pure and metatheoretical geometry not only because of a vague and general appreciation of the importance of theoretical knowledge,[99] but also because of the institutional establishment of this appreciation. This does not imply that the long series of investigations of the fundamental problems of the *Elements* were all made directly (or just presented) as astronomical *prolegomena;* the opposite is evident both in al-Khayyāmī's *Discussion of Difficulties in Euclid* (cf. note 73) and in Thābit's two proofs of the parallel postulate (both translated in Sabra 1968). Other metatheoretical investigations, however, were expressly written for recensions (*taḥrīr*) of the *Elements* for the introductory curriculum of astronomy; this is the case of Muḥyi'l-Dīn al-Maghribī's and Naṣīr al-Dīn al-Ṭūsī's proofs of the same postulate (Sabra 1969, 14f., and 10, note 59 resp.). With very few exceptions, the authors of such metamathematical commentaries appear to have been competent in mathematical astronomy.[100]

XVI. A Warning

The above might look like a claim that the global character and all developmental trends of Islamic mathematics can be explained in terms of one or two simple formulas. Of course this is not true. Without going into details, I shall point to one development that is puzzlingly different from those discussed above: magic squares.[101] Their first occurrences in Islam are in the Jabirian corpus, in the *Ikhwān al-Ṣafā'*, and (according to Abī Uṣaybiʿa) in a lost treatise by Thābit. Various Islamic authors ascribe the squares to the semilegendary Apollonios of Tyana,[102] or even to Plato or Archimedes. An origin in classical antiquity is, however, highly improbable: a passage in Theon of Smyrna's *On the Mathematical Knowledge Which is Needed to Read Plato* is so close to the idea that he would certainly have mentioned it had he

[98] The Greek "Little Astronomy" was to form the backbone of the *mutawassiṭāt* even in the Naṣirean canon, where, however, Euclid's and Thābit's *Data* and Archimedes' *Measurement of the Circle, On the Sphere and Cylinder* and *Lemmata* are included, together with some other works.

[99] Such general attitudes, too, remained effective – they are expressed in the praise of Archimedes' *Lemmata* formulated by al-Nasawī, who speaks of the "beautiful figures, few in number, great in *utility*, on the fundaments of geometry, in the highest degree of *excellency* and *subtility*" (quoted from Steinschneider 1865, 480; emphasis added). Cf. also al-Bīrūnī as quoted in chapter XIII.

[100] This includes all those mentioned above, as well as others mentioned in Sabra 1969 (Qayṣar ibn Abī'l-Qāsim, Yūḥannā al-Qass, al-Jawharī) and Folkerts 1980 (which, besides some of the same, mentions al-Māhānī), – with the ill-documented al-Qass as a possible exception (Suter 1900, no. 131).

[101] A good and fairly recent overview of magic squares in Islam is Cammann (1969); but see also Ahrens (1916); Bergsträsser (1923); Hermelink (1958); Sesiano (1980; 1981); and Sarton (1927, 1931 and index references to "magic squares").

[102] This is the most widespread assumption, judging from the *Fihrist* (trans. Dodge 1970, 733).

heard about it (ed. Dupuis 1892, 1966);[103] but neither he nor any other ancient author gives the slightest hint in that direction. On the other hand, an origin in late Hellenistic or Sabian Hermeticism is possible, though still less probable than diffusion along the trade routes from China, where magic squares had long been known and used. This doubt notwithstanding, it is obvious that the subject was soon correlated with Hermeticism and Ismāʿīlī and related ideas. At least one mathematician of renown took up the subject – ibn al-Haytham, whose omnivorous habits we have already met on several occasions. Some progress also took place, from smaller toward larger squares and toward systematic rules for the creation of new magic squares. On the whole, however, the subject remained isolated from general mathematical investigations and writings. The exceptional character of ibn al-Haytham's work is revealed by an observation by an anonymous later writer on the squares, that "I have seen numerous treatises on this subject by crowds of people. But I have seen none which speaks more completely about it than Abū ʿAlī ibn al-Haytham" (quoted from Sesiano 1980, 188). The treatise just mentioned combines the subject with arithmetic progressions; but integration into larger arithmetic textbooks or treatises seems not to have occurred. Thus we see that Islamic mathematics did not integrate *every* subject into its synthesis. Instead, magic squares appear to have maintained their intimate connection to popular superstition and illicit sorcery.[104]

It is not plausible that the exclusion of magic squares from the mathematical mainstream can be explained by any inaccessibility to theoretical investigation; other subjects went into the arithmetic textbook tradition even though they were known only empirically and not by demonstration.[105] So the exclusion of magic squares from honest mathematical company must rather be explained by cultural factors: perhaps the subject did not belong inside the bundle of recognized subdisciplines that had been constituted during the phase of synthesis; or its involvement with magic and sorcery made it a nonmathematical discipline;[106] or the involvement of mainstream mathematicians with practically oriented social strata made them keep away from a subject (be it mathematics or not) involved with *ṣūfī* and other esoteric (or even outspokenly heretical) currents. I shall not venture into any definite evaluation of these or other hypotheses (even though ibn Khaldūn makes me prefer the second),

[103] The passage shows the square $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

[104] This is clearly the point of view of ibn Khaldūn in the *Muqaddimah,* wherever he approaches the subjects of talismans, letter magic, and magic squares (which mostly go together). In one place he also claims that a work based on such things "most likely [. . .] is incorrect, because it has no scientific basis, astrological and otherwise" (III:52, trans. Rosenthal 1958, I:224).

[105] So al-Karajī's summation of square numbers in the *Fakhrī* (see the paraphrase in Woepcke 1853, 60).

[106] Ibn Khaldūn does not mention the subject at all during his discussion of arithmetic (*Muqaddimah* VI:19, trans. Rosenthal 1958, III:118–29). Like amicable numbers (once investigated mathematically by Thābit but now mentioned only as a talisman producing love) it is relegated to the chapters on magic and sorcery (VI:27–28, trans. Rosenthal 1958, III:156–227). (The silence on the subject of amicable numbers is all the more striking, since the circle of Maghrebi mathematicians was in fact interested in that subject; cf. Rashed 1983, 116f.).

but conclude only that the place of magic squares in the culture of medieval Islam is *not* explainable in the same terms as the synthesis, the integration of practical mathematics and theoretical investigation, the development of the arithmetic textbook tradition, or the interest in the foundations of geometry. No culture is simple.

XVII. The Moral of the Story

The above is not a complete delineation of medieval Islamic mathematics; nor was it meant to be. The purpose was to demonstrate that Islamic mathematics possessed certain features not present in any earlier culture (but shared with early modern science), and to trace their causes. I hope that I have succeeded in demonstrating the existence of these features, and hence of an "Islamic Miracle" just as necessary for the rise of our modern scientific endeavor as its Greek namesake, and to have offered at least a partial explanation of what happened.

This leaves us with a question of a different order: was the integration of theory and advanced practice in Renaissance and early modern Europe a legacy from Islam, or was it an independent but parallel development?

Answering this question involves us in the recurrent difficulty of diffusionist explanations. "Miracles" and other cultural patterns *cannot* be simply borrowed: they can only inspire developments inside the receiving culture. Even a piece of technology can be borrowed only if the recipient is ready for it. The experience of cargo cults shows the degree to which the recipient determines the outcome of even a seemingly technological inspiration, and investigations of any process of cultural learning will show us radical reinterpretations of the original message. (We may ask whether Charlemagne's identification of the Palace school of Aachen with the resurrection of Athenian philosophy was less paradoxical than any cargo cult.)

We know the eagerness with which the European Renaissance tried to learn from ancient Rome and Greece; and we know the enormous extent to which the social and cultural conditions of Europe made it misunderstand the message. In contrast, no serious effort was made to understand the cultural messages of the Islamic world; on the contrary, great efforts were invested to prove that *such messages were morally wrong*. We can therefore be confident that no general cultural patterns or attitudes (including the attitudes toward rational knowledge and technology) were borrowed wholesale by Christian Europe. Nor was there any significant borrowing of institutions,[107] including those institutions that embodied the attitudes to knowledge. The only way Renaissance and early modern Europe could learn from the "Islamic Miracle" was through acquaintance with its *products*, i.e., through scholarly works and technologies that it had produced or stamped. Because they were received in a society that was already intellectually and technologically ready to make an analogous leap, part of the "Islamic message" could be apprehended even through this

[107] A few exceptions, e.g. in commercial law, can be found. But the difference between the Maghrebi arithmetic-textbook tradition and the Italian abacus school shows that even the institutions of *commercial education* could not be transferred.

channel. Primarily, however, Renaissance Europe developed its new integrative attitudes to rational and technological knowledge on its own; transfers were only of secondary importance.

This conclusion does not make the Islamic miracle irrelevant to the understanding of modern science. *First* two relatively independent developments of analogous but otherwise historically unprecedented cultural patterns should make us ask whether similar effects were not called forth by similar causes. Here the sources of Islamic and Renaissance mathematics were of course largely identical (not least because Christian Europe supplemented the meager direct Greco-Roman legacy with translations from such Arabic works accessible in Spain, i.e. mainly works dating from the ninth century). These sources had, however, not been able to produce the miracle by themselves before the rise of Islam. Were there then any shared "formative conditions" that help us explain the analogous transformation of the source material?

Probably the answer is "yes." It is true that High Medieval Christianity in the West had been dominated by a powerful ecclesiastical institution; moreover, after the twelfth century it could hardly be claimed to be fundamentalist. Yet precisely during the critical period (the period of Alberti, Ficino, Bruno, and Kepler) the fences of the Thomistic synthesis broke down, and rational knowledge came to be thought of both as a way to ultimate truths concerning God's designs and to radical improvements of practice. At the same time, the ecclesiastical institution lost much of its force, both politically and in relation to the conscience of the individual; if anything, however, religious feelings were stronger than in the thirteenth century. It would therefore not be astonishing if patterns like the noninstitutionalized practical fundamentalism of ninth-century Islam could be found among Renaissance scholars and higher artisans. It would also be worthwhile to reflect once more in this light upon the "Merton thesis" concerning the connections between Puritanism, social structure, and science.[108]

Second, the entire investigation should make us aware that there are no privileged heirs to the cultural "miracles" of the past. It is absurd to claim that "science, as we know it and as we understand it, is a specific creation of the Greco-Occidental world" (Castoriadis 1986, 264; my translation). First, Greek "science" was radically different from "science as we know it and as we understand it." Second, with relation to science (and in many other respects, too), it is no better (and no worse) to speak of a "Greco-Occidental" than of a "Greco-Islamic" world, and not much better to claim a "Greco-Occidental" than an "Islamo-Occidental" line of descent.

In times more serene than ours, these points might appear immaterial. If Europe wants to descend from ancient Greece and to be her heir par excellence, then why not let her believe it? Our times are, however, not serene. The "Greco-Occidental" particularity always served (and serves once again in many quarters) as a moral justification of the behavior of the "Occident" toward the rest of the world, going

[108] A supplementary approach might compare the institutions of "courtly science" and the patterns of princely protection in the two settings.

together with anti-Semitism, imperialism, and gunboat diplomacy. In theory it might be different, and the occidentalist philosopher just quoted finds it "unnecessary to specify that no "practical" or "political" conclusions should be drawn" from "our" privileged place in world history (ibid., 263, n. 3). It is, alas, *not* unnecessary to recall Sartre's observation that the "intellectual terrorist practice" of liquidation "in the theory" may all too easily end up expressing itself in *physical liquidation of those who do not fit the theory* (Sartre 1960, 28).

As Hardy once said, "a science is said to be useful if its development tends to accentuate the existent inequalities in the distribution of wealth, or more directly promotes the destruction of human life." The ultimate goal of the present study has been to undermine a "useful" myth on science and its specifically "Greco-Occidental" origin – whence the dedication to a great humanist.

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To state that the author remains responsible for all errors is not just a matter of academic decorum; given that I come to the subject as an outsider, it expresses an obvious matter of fact.

Bio-Bibliographical Cues

Based mainly upon *DSB; GAS; GAL;* Suter 1900; Sarton 1927; Sarton 1931; and Dodge 1970. All dates A.D.

ʿAbbasid dynasty. Dynasty of caliphs, in effective power in Baghdad from 749/50 to the later tenth–earlier eleventh century, formally until 1258.

Abī Uṣaybiʿa. 1203/4–1270. Syrian lexicographer and physician.

Abū Bakr. Early ninth century? Otherwise unknown author of the *Liber mensurationum.*

Abū Kāmil. Fl. late ninth and/or early tenth century. Egyptian mathematician.

Abū Maʿshār. 787–886. Eastern Caliphate. Astrologer.

Abū'l-Wafāʾ. 940–997/8. Eastern Caliphate. Mathematician and astronomer.

Al-Bīrūnī. B. 973, d. after 1050. Astronomer, mathematician, historian, and geographer from Khwārezm.

Al-Fārābī. Ca. 870–950. Eastern Caliphate. Philosopher.

Al-Fazārī. Fl. second half of the eighth century. Eastern Caliphate. Astronomer.

Al-Ghazzālī. 1058–1111. Eastern Caliphate. Theologian.

Al-Ḥajjāj. Fl. late eighth to early ninth centuries. Eastern Caliphate. Translator of the *Elements* and of the *Almagest.*

Al-Ḥaṣṣār. Fl. twelfth or thirteenth century, probably in Morocco. Mathematician.

Al-Jāḥiẓ. Ca. 776–868/9. Iraq. Muʿtazilite theologian; zoologist.

Al-Jawharī. Fl. ca. 830. Eastern Caliphate. Mathematician and astronomer.

Al-Karajī. Fl. ca. 1000. Eastern Caliphate. Mathematician.

Al-Khayyāmī. 1048(?)–1131(?). Iran. Mathematician, astronomer, philosopher.

Al-Khāzin. D. 961/71. Iran. Mathematician and astronomer.

Al-Khazinī. Fl. ca. 1115–1130. Iran. Astronomer, theoretician of mechanics and instruments.

Al-Khuwārizmī. Fl. ca. 980. Khwārezm. Lexicographer.

Al-Khwārizmī. Late eighth to mid-ninth centuries. Eastern Caliphate. Mathematician, astronomer, geographer.

Al-Kindī. Ca. 801 to ca. 866. Eastern Caliphate. Philosopher, mathematician, astronomer, physician, etc.

Al-Māhanī. Fl. ca. 860 to ca. 880. Eastern Caliphate. Mathematician, astronomer.

Al-Maʾmūn. 786–833. ʿAbbasid caliph 803–833, ardent muʿtazilite, patron of *awāʾil* learning.

Al-Nadīm. Ca. 935–990. Baghdad. Librarian, lexicographer.

Al-Nasawī. Fl. 1029–1044. Khurasan, Eastern Caliphate. Mathematician.

Al-Nayrīzī. Fl. early tenth century. Eastern Caliphate. Mathematician, astronomer.

Al-Qalaṣādī. 1412–1486. Spain, Tunisia. Mathematician, jurisprudent.

Al-Samawʾal. Fl. mid-twelfth century. Iraq, Iran. Mathematician, physician.

Al-Umawī. Fl. fourteenth century. Spain, Damascus. Mathematician.

Al-Uqlīdisī. Fl. 952/3. Damascus. Mathematician.

Al-Yaʿqūbī. Fl. 873–891. Eastern Caliphate. Shi'ite historian and geographer.

Anania of Shirak. Fl. first half of the seventh century. Armenia. Mathematician, astronomer, geographer, etc.

Banū Mūsā ("Sons of Mūsā"). Three brothers, ca. 800 to ca. 875. Baghdad. Mathematicians, translators, organizers of translation.

Bar Ḥiyya (Savasorda) Abraham. Fl. before 1136. Hispano-Jewish philosopher, mathematician, and astronomer.

Ḥunayn ibn Isḥāq. 808–873. Eastern Caliphate. Physician, translator.

Ibn al-Bannāʾ. 1256–1321. Morocco. Mathematician, astronomer.

Ibn al-Haytham. 965 to ca. 1040. Iraq, Egypt. Mathematician, astronomer.

Ibn Khaldūn. 1332–1406. Maghreb, Spain, Egypt. Historian, sociologist.

Ibn Qunfudh. D. 1407/8. Algeria. Jurisprudent, historian. Commentator to ibn al-Bannāʾ.

Ibn Ṭāhir. D. 1037. Iraq, Iran. Theologian, mathematician, etc.

Ibn Turk. Early ninth century. Turkestan. Mathematician.

Ikhwān al-Ṣafāʾ ("Epistles of the Brethren of Purity"). Tenth-century *Ismāʿīlī* encyclopaedic exposition of philosophy and sciences.

Kamāl al-Dīn. D. 1320. Iran. Mathematician, mainly interested in optics.

Kushyār ibn Labbān. Fl. ca. 1000. Eastern Caliphate. Astronomer, mathematician.

Māshāʾallāh. Fl. 762 to ca. 815. Iraq. Astrologer.

Muḥyi'l-Dīn al-Maghribī. Fl. ca. 1260 to 1265. Syria and Iran. Mathematician, astronomer, astrologer.

Naṣīr al-Dīn al-Ṭūsī. D. ca. 1214. Iran. Astronomer, mathematician.

Qayṣar ibn Abī'l-Qāsim. 1178–1251. Egypt, Syria. Jurisprudent, mathematician, technologist.

Qusṭā ibn Lūqā. Fl. 860 to 900. Eastern Caliphate. Physician, philosopher, translator.

Rabīʿ ibn Yaḥya. Fl. tenth century? Bishop at Elvira.

Severus Sebokht. Fl. mid-seventh century. Syrian bishop. Astronomer, commentator on philosophy.

Thābit ibn Qurra. 836–901. Eastern Caliphate. Mathematician, astronomer, physician, translator, etc.

ʿUmar ibn al-Farrukhān. 762–812. Eastern Caliphate. Astronomer, astrologer.

ʿUmayyad dynasty. Dynasty of caliphs, 661–750.

Yūḥannā al-Qass. Fl. first half of tenth century? Mathematician, translator of mathematics.

Bibliography and Abbreviations

The following abbreviations are used throughout the text and bibliography:

DSB: Dictionary of Scientific Biography: Editor in Chief: Charles Coulston Gillispie, vols. I–XVI. New York: Charles Scribner's Sons, 1970–1980.

GAL: Carl Brockelmann, *Geschichte der Arabischen Literatur.* Band 1–2, Supplementband 1–3. Berlin: Emil Fischer, 1898, 1902; Leiden: Brill, 1937, 1938, 1942.

GAS: Fuat Sezgin, *Geschichte des arabischen Schrifttums.* Vols. I–IX. Leiden: Brill, 1967–1984.

PL: Patrologiae cursus completus, series latina, accurante J. P. Migne. 221 vols., Paris 1844–64.

Ahrens, W., 1916. "Studien über die "magischen Quadrate" der Araber," *Islam* 7:186–250.

Amir-Móez, Ali R., 1959. "*Discussion of Difficulties in Euclid,* by Omar Ibn Abrahim al-Khayyāmī (Omar Khayyam), translated," *Scripta Mathematica* 24:275–303.

Anawati, G. C. and A. Z. Iskandar. "Ḥunayn ibn Isḥāq," *DSB,* vol. XV, pp. 230–49.

Anbouba, Adel, 1978. "Acquisition de l'algèbre par les Arabes et premiers développements. Aperçu général," *Journal for the History of Arabic Science* 2:66–100.

——, 1979. "Un traité d'Abū Jaʿfar sur les triangles rectangles numériques," *Journal for the History of Arabic Science* 3:134–56 (Arabic: pp. 178-57).

——, "Al-Samawʾal," *DSB,* vol. XII, pp. 91–95.

Anthologia graeca, see *The Greek Anthology*

Beaujouan, Guy, 1957. *L'interdépendance entre la science scolastique et les sciences utilitaires,* les Conférences du Palais de la Découverte, série D no. 46, Paris: Université de Paris.

Bergsträsser, G., 1923. "Zu den magischen Quadraten," *Islam* 13: 227–35.

Besthorn, R. O., and J. L. Heiberg, ed., 1893. *Codex Leidensis 399, 1. Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii.* Arabice et latine ediderunt notisque instruxerunt R. O. Besthorn et J. L. Heiberg, vols. I–III (1893–1932), Copenhagen: Gyldendalske Boghandel.

Boncompagni, Baldassare, ed., 1857. *Scritti di Leonardo Pisano matematico del secolo decimoterzo. I. Il "Liber abaci" di Leonardo Pisano,* Roma: Tipografia delle Scienze Matematiche e Fisiche.

Brown, Peter, 1971. *The World of Late Antiquity,* London: Thames and Hudson.

Bulliet, Richard W., 1979. *Conversion to Islam in the Medieval Period: An Essay in Quantitative History,* Cambridge, Mass. and London: Harvard University Press.

Busard, H. L. L., 1968. "L'algèbre au moyen âge: Le *Liber mensurationum* d'Abû Bekr," *Journal des Savants,* avril–juin 1968, 65–125.

Cammann, Schuyler, 1969. "Islamic and Indian Magic Squares," *History of Religions* 8:181–209, 271–99.

Cantor, Moritz, 1875. *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst. Eine historisch-mathematische Untersuchung,* Leipzig: Teubner.

Castoriadis, Cornelius, 1986. *Domaines de l'homme (Les carrefours du labyrinthe II),* Paris: Seuil.

Chace, Arnold Buffum, Ludlow Bull, and Henry Parker Manning, 1929. *The Rhind Mathematical Papyrus.* II. Photographs, Transcription, Transliteration, Literal Translation. Oberlin, Ohio: Mathematical Association of America.

Chenu, Marie-Dominique, O.P., 1974. "Civilisation urbaine et théologie. L'école de Saint-Victor au XII siècle," *Annales. Economies, sociétés, civilisations* 29:1253–63.

Cohen, H. J., 1970. "The Economic Background and the Secular Occupations of Muslim Jurisprudents and Traditionists in the Classical Period of Islam (until the Middle of the Eleventh Century)," *Journal of the Economic and Social History of the Orient* 13:16–59.

Colebrooke, H. T., ed., trans., 1817. *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara,* London: John Murray. Reprint Wiesbaden 1973: Martin Sändig.

Curtze, Maximilian, ed., 1902. *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance,* Abhandlungen zur Geschichte der mathematischen Wissenschaften, vols. 12–13, Leipzig: Teubner.

Datta, Bibhutibhusan and Avadhesh Narayan Singh, [1935–38], 1962. *History of Hindu Mathematics. A Source Book,* Parts I and II, Bombay: Asia Publishing House. First ed. Lahore: Motilal Banarsidass.

Djebbar, A., 1981. *Enseignement et recherche mathématiques dans le Maghreb des XIIIe–XIVe siècles (étude partielle),* Publications mathématiques d'Orsay, 81-02, Orsay: Université de Paris-Sud.

Dodge, Bayard, ed., trans., 1970. *The "Fihrist" of al-Nadīm. A Tenth-Century Survey of Muslim Culture.* 2 vols., Records of Civilization. Sources and Studies, no. 43, New York and London: Columbia University Press.

Dupuis, J., ed., trans., 1892. *Théon de Smyrne, philosophe platonicien, "Exposition des connaissances mathématiques utiles pour la lecture de Platon." Traduite pour la première fois du grec en français,* Paris: Hachette. (Includes Hiller's critical edition of the Greek text.)

Fakhry, Majid, 1969. "The Liberal Arts in the Medieval Arabic Tradition from the Seventh to the Twelfth Centuries," in *Arts libéraux et philosophie au Moyen Age. Actes du Quatrième Congrès International de Philosophie Médiévale, 27 août–2 Septembre 1967,* Montréal: Institut d'Etudes Médiévales; Paris (1969): Vrin, pp. 91–97.

Farrington, Benjamin, [1944–49], 1969. *Greek Science,* Harmondsworth, Middlesex: Penguin.

Folkerts, Menso, 1970. *"Boethius" Geometrie II. Ein mathematisches Lehrbuch des Mittelalters,* Wiesbaden: Franz Steiner.

——, 1978. "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes.* Überlieferung, Inhalt, Kritische Edition," *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften* 116, Band 6, Abhandlung (Vienna).

—— , 1980. "Probleme der Euklidinterpretation und ihre Bedeutung für die Entwicklung der Mathematik," *Centaurus* 23:185–215.

Gandz, Solomon, ed., 1932. "The Mishnat ha-Middot, the First Hebrew Geometry of about 150 C.E., and the Geometry of Muhammad ibn Musa al-Khowarizmi, the First Arabic Geometry (c. 820), Representing the Arabic Version of the Mishnat ha-Middot. A New Edition of the Hebrew and Arabic Texts with Introduction, Translation and Notes," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen* 2.

Gille, Bertrand, 1980. *Les mécaniciens grecs. La naissance de la technologie,* Paris: Seuil.

Goldziher, Ignácz, 1915. "Die Stellung der alten islamischen Orthodoxie zu den antiken Wissenschaften," *Abhandlungen der Preußischen Akademie der Wissenschaften. Philosophisch-historische Klasse* 1915 no. 8, 1916, Berlin.

Hahnloser, H. R., ed., 1935. *Villard de Honnecourt. Kritische Gesamtausgabe des Bauhüttenbuches ms. fr 19093 der Pariser Nationalbibliothek,* Wien: Anton Schroll.

Harvey, E. Ruth. "Qusṭā ibn Lūqā al-Baʿlabakkī," *DSB,* vol. XI, pp. 244–46.

Heinen, Anton M., 1978. "Mutakallimūn and Mathematicians," *Islam* 55:57–73.

Hermelink, Heinrich, 1958. "Die ältesten magischen Quadrate höherer Ordnung und ihre Bildung," *Sudhoffs Archiv* 42:199–217.

—— , "Arabic Recreational Mathematics as a Mirror of Age-Old Cultural Relations Between Eastern and Western Civilizations," in *Proceedings of the First International Symposium for the History of Arabic Science, April 5–12, 1976,* ed. Ahmad Y. al-Hassan, Ghada Karmi, and Nizar Namrum, 1978. Vol. II, *Papers in European Languages,* Aleppo: Aleppo University, Institute for the History of Arabic Science, pp. 44–52.

Hochheim, Adolph, ed., trans., 1878. *Kâfî fîl Hisâb (Genügendes über Arithmetik) des Abu Bekr Muhammed ben Alhusein Alkarkhî.* 3 vols. Halle: Louis Nebert.

Hogendijk, Jan P., 1985. "Thābit ibn Qurra and the Pair of Amicable Numbers 17296, 18416," *Historia Mathematica* 12:269–273.

—— , ed., 1986. Abū Kāmil Shujāʿ ibn Aslam (Second half ninth century A.D.), *The Book of Algebra. Kitāb al-Jabr wa l-muqābala.* Publications of the Institute for the History of Arabic-Islamic Science, Series C: Facsimile Editions, vol. 24, Frankfurt a. M.: Institute for the History of Arabic-Islamic Science.

Høyrup Jens, 1985. "Varieties of Mathematical Discourse in Pre-Modern Socio-Cultural Contexts: Mesopotamia, Greece, and the Latin Middle Ages," *Science and Society* 49:4–41.

—— , 1985a. "Jordanus de Nemore, 13th Century Mathematician: An Essay on Intellectual Context, Achievement, and Failure," preprint, April 15, 1985, Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science. Revised version to appear in *Archive for History of Exact Sciences.*

——, 1985b. *Babylonian Algebra from the View-Point of Geometrical Heuristics. An Investigation of Terminology, Methods, and Patterns of Thought,* Second, slightly corrected printing. Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science.

——, 1986. "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra," *Erdem* (Ankara) 5 (2):445–84.

——, 1987. "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought," *Filosofi og Videnskabsteori på Roskilde Universitets center,* Third Series: *Preprints and Reprints,* no. 2.

Hughes, Barnabas, O.F.M., 1986. "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr:* A Critical Edition," *Mediaeval Studies* 48:211–63.

Jackson, D. E. P., 1980. "Toward a Resolution of the Problem of τὰ ἑνὶ διαστήματι γραφόμενα in Pappus' Collection Book VIII," *The Classical Quarterly,* New Series 30:523–33.

Jones, Charles W., ed., 1943. Bedae *Opera de temporibus,* The Mediaeval Academy of America, Publication no. 41, Cambridge, Mass.: The Mediaeval Academy of America.

Juschkewitsch, A. P., 1964. *Geschichte der Mathematik im Mittelalter,* Leipzig: Teubner 1st ed. (in Russian), 1961.

Karpinski, Louis Charles, ed., trans., 1915. *Robert of Chester's Latin Translation of the Algebra of al-Khowarizmi,* University of Michigan Studies, Humanistic Series, vol. 11, New York. Reprinted in Karpinski L. G. and J. G. Winter, 1930. *Contributions to the History of Science,* Ann Arbor: University of Michigan.

Kasir, Daoud S., ed., trans., 1931. *The Algebra of Omar Khayyam,* Dissertation, Faculty of Philosophy, Columbia University. New York: Bureau of Publications, Teachers College, Columbia University.

Kennedy, E. S. and David Pingree, 1971. *The Astrological History of Māshā'allāh,* Cambridge, Mass.: Harvard University Press.

Kirk, G. S., J. E. Raven, and M. Schofield, 1983. *The Presocratic Philosophers. A Critical History with a Selection of Texts,* 2nd ed., Cambridge: Cambridge University Press.

Kokian, P. Sahak, ed., trans., 1919. "Des Anania von Schirak arithmetische Aufgaben," *Zeitschrift für die deutsch-österreichischen Gymnasien* 69 (1919–1920):112–17.

Krasnova, S. A., ed., trans., 1966. "Abu-l-Vafa al-Buzdžani, *Kniga o tom, čto neobxodimo remeslenniku iz geometričeskix postroenij,*" in *Fiziko-matematičeskie nauki v stranax vostoka,* ed. A. T. Grigor'jan and A. P. Juškevič 1966: Sbornik statej i publikacij. Vypusk I (IV), Moskva: Izdatel'stvo "Nauka," pp. 42–140.

Levey, Martin, ed., trans., 1966. *The "Algebra" of Abū Kāmil, "Kitāb fī al-jābr [sic] wa'l-muqābala," in a Commentary by Mordechai Finzi,* Hebrew Text, Translation, and Commentary with Special Reference to the Arabic Text, Madison: University of Wisconsin Press.

Levey, Martin and Marvin Petruck, ed., trans., 1965. *Kūshyār ibn Labbān, "Principles of Hindu Reckoning." A Translation with Introduction and Notes of the "Kitāb fī usūl hisāb al-hind,"* Madison and Milwaukee: University of Wisconsin Press.

L'Huillier, Ghislaine, 1980. "Regiomontanus et le *Quadripartitum numerorum* de Jean de Murs," *Revue d'Histoire des Sciences et de leurs applications* 33:193–214.

L'Huillier, Hervé, ed., 1979. *Nicolas Chuquet, "La géométrie". Première géométrie algébrique en langue française (1484). Introduction, texte et notes,* Paris: Vrin.

Luckey, Paul, 1941. "Thābit b. Qurra über den geometrischen Richtigkeitsnachweis der Auflösung der quadratischen Gleichungen," *Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse. Berichte* 93:93–114.

McGarry, Daniel D., ed., trans., [1955] 1971. *The "Metalogicon" of John of Salisbury. "A Twelfth-Century Defence of the Verbal and Logical Arts of the Trivium,"* Translated with an Introduction and Notes, Gloucester, Mass.: Peter Smith.

Makdisi, George, 1961. "Muslim Institutions of Higher Learning in Eleventh-Century Baghdad," *Bulletin of the School of Oriental and African Studies* (London) 24:1–56.

—— , 1971. "Law and Traditionalism in the Institutions of Learning in Medieval Islam," in *Theology and Law in Islam,* ed. G. E. von Grünebaum, Wiesbaden: Harrasowitz, pp. 75–88.

Marquet, Yves. "Ikhwān al-Ṣafā'," *DSB,* vol. pp. XV, pp. 249–51.

Marre, Aristide, ed., 1880. "Le Triparty en la science des nombres par Maistre Nicolas Chuquet Parisien," *Bulletino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche* 13:593–659, 693–814.

Menninger, Karl, 1957. *Zahlwort und Ziffer. Eine Kulturgeschichte der Zahl. I. Zählreihe und Zahlsprache. II. Zahlschrift und Rechnen,* 2. neubearbeitete und erweiterte Auflage, Göttingen: Vandenhoeck and Ruprecht.

Merton, Robert K., 1968. *Social Theory and Social Structure,* enlarged edition, New York and London: The Free Press.

Nader, A. N., 1956. *Le système philosophique des Muʿtazila (Premiers penseurs de l'Islam),* Beyruth. (Quoted from A. M. Heinen, 1978.)

Nasr, Seyyed Hossein, 1968. *Science and Civilization in Islam,* Cambridge, Mass.: Harvard University Press.

—— , "Al-Ṭusī, ... Naṣīr al-Dīn," *DSB,* vol. XIII, pp. 508–14.

Nau, F., 1910. "Notes d'astronomie syrienne," *Journal Asiatique* 10, série 16:209–28.

Nebbia, G., 1967. "Ibn al-Haytham nel millesimo anniversario della nascita," *Physis* 9:165–214.

Neugebauer, O. and A. Sachs, 1945. *Mathematical Cuneiform Texts,* American Oriental Series, vol. 29, New Haven, Conn.: American Oriental Society.

O'Leary, De Lacy, 1949. *How Greek Science Passed to the Arabs,* London: Routledge and Kegan Paul. Reprinted 1964.

Oschinsky, Dorothea, 1971. *Walter of Henley and Other Treatises on Estate Management and Accounting,* Oxford: Oxford University Press.

Palencia, Angel Gonzales, ed., trans., 1953. *Al-Fārābī, "Catálogo de las ciencias." Edición y traduccion Castellana,* segunda edición, Madrid: Consejo Superior de Investigaciones Científicas, Instituto Miguel Asín.

Parker, Richard A., 1972. *Demotic Mathematical Papyri,* Providence and London: Brown University Press.

Paton, W. R., trans., 1918. *The Greek Anthology,* vol. V, Loeb Classical Library, vol. 86, London: Heinemann. Reprinted 1979.

Peet, T. Eric, 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058.* Introduction, Transcription, Translation and Commentary, London: University Press of Liverpool.

Pines, Shlomo, 1970. "Philosophy [in Islam]," in *The Cambridge History of Islam,* ed. P. M. Holt, Ann K. S. Lambton, and Bernard Lewis, 2 vols., Cambridge: Cambridge University Press, vol. II, pp. 780–823.

Pingree, David, 1963. "Astronomy and Astrology in India and Iran," *Isis* 54:229–46.

—— , 1973. "The Greek Influence on Early Islamic Mathematical Astronomy," *Journal of the American Oriental Society* 93:32–43.

——, "Abū Maʿshar," *DSB,* vol. I, pp. 32–39.

——, "Al-Fazārī," *DSB,* vol. IV, pp. 555–56.

——, "Māshāʾallāh," *DSB,* vol. IX, pp. 159–62.

——, "ʿUmar ibn al-Farrukhān al-Ṭabarī," *DSB,* vol. XIII, pp. 538–39.

Rashed, Roshdi, 1982. "Matériaux pour l'histoire des nombres amiables et de l'analyse combinatoire," *Journal for the History of Arabic Science* 6:209–11 (French introduction), 212–78 (Arabic texts).

——, 1983. "Nombres amiables, parties aliquotes et nombres figurés aux XIIIème et XIVème siècles," *Archive for History of Exact Sciences* 28:107–47.

—— , ed., trans., 1984. *Diophante, "Les Arithmétiques," tômes III (livre IV), IV (livres V, VI, VII).* Texte établi et traduit, Paris: "Les Belles Lettres."

——, "Kamāl al-Dīn," *DSB,* vol. VII, pp. 212–19.

Renaud, H. P. J., 1938. "Ibn al-Bannâ de Marrakech, ṣûfî et mathématicien (XIIIe–XIVe s. J. C.)," *Hespéris* 25:13–42.

——, 1944. "Sur un passage d'ibn Khaldun relatif à l'histoire des mathématiques," *Hespéris* 31:35–47.

Ritter, Hellmut, 1916. "Ein arabisches Handbuch der Handelswissenschaft," *Islam* 7:1–91.

Rodet, Léon, 1878. "L'algèbre d'al-Khârizmi et les méthodes indienne et grecque," *Journal Asiatique,* 7e série 11:5–98.

Rosen, Frederic, ed., trans., 1831. *The "Algebra" of Muhammad ben Musa, Edited and Translated,* London: The Oriental Translation Fund.

Rosenfeld, B. A. and A. T. Grigorian. "Thābit ibn Qurra," *DSB,* vol. XIII, pp. 288–95.

Rosenthal, Franz, ed., trans., 1958. *Ibn Khaldûn, "The Muqaddimah." An Introduction to History,* Translated from the Arabic, 3 vols, London: Routledge and Kegan Paul.

Ross, W. D., ed., trans., [1928] 1972. *The Works of Aristotle. Translated into English.* VIII: *Metaphysica,* second ed., Oxford: The Clarendon Press.

Rozenfeld, Boris A., 1976. "The List of Physico-Mathematical Works of Ibn al-Haytham Written by Himself," *Historia Mathematica* 3:75–76.

Sabra, A. I., 1968. "Thābit ibn Qurra on Euclid's Parallels Postulate," *Journal of the Warbourg and Courtauld Institutes* 31:12–32.

—, 1969. "Simplicius' Proof of Euclid's Parallels Postulate," *Journal of the Warburg and Courtauld Institutes* 32:1–24.

Sachs, Abraham J., 1946. "Notes on Fractional Expressions in Old Babylonian Mathematical Texts," *Journal of Near Eastern Studies* 5:203–14.

Sachs, Eva, 1917. *Die fünf platonischen Körper. Zur Geschichte der Mathematik und der Elementenlehre Platons und der Pythagoreer,* Philologische Untersuchungen, 24. Heft, Berlin: Weidmannsche Buchhandlung.

Saidan, Ahmad S., 1974. "The Arithmetic of Abū'l-Wafā'," *Isis* 65:367–75.

—, ed., trans., 1978. *The "Arithmetic" of al-Uqlīdisī. The Story of Hindu-Arabic Arithmetic as Told in "Kitāb al-Fuṣūl fī al-Ḥisāb al-Hindi" by Abū al-Ḥasan Aḥmad ibn Ibrāhīm al-Uqlīdisī written in Damascus in the Year 341 (A.D. 952/53).* Translated and Annotated, Dordrecht: Reidel.

—, 1978a. "Number Theory and Series Summations in Two Arabic Texts." in *Proceedings of the First International Symposium for the History of Arabic Science, April 5–12 1976,* ed. Ahmad Y. al-Hassan, Ghada Karmi, and Nizar Namrum, 1978, vol. II, Aleppo: Aleppo University, Institute for the History of Arabic Science, pp. 145–163.

—, "Al-Baghdadī, Abu Manṣūr ... ibn Ṭāhir ... ," *DSB,*, vol. XV, 9–10.

—, "Al-Nasawī," *DSB,* vol. IX, pp. 614–15.

—, "Al-Qalaṣādī," *DSB,* vol. XI, pp. 229–30.

—, "Al-Umawī," *DSB,* vol. XIII, pp. 539–40

Sajo, Géza, ed., 1964. *Boetii de Dacia "Tractatus de eternitate mundi,"* Quellen und Studien zur geschichte der Philosophie, IV, Berlin: De Gruyter.

Sarasvati Amma, T. A., 1979. *Geometry in Ancient and Medieval India,* Delhi: Motilal Banarsidass.

Sarfatti, Gad, 1968. *Mathematical Terminology in Hebrew Scientific Literature of the Middle Ages,* (In Hebrew, English abstract.), Jerusalem: The Magnes Press/The Hebrew University.

Sarton, George, 1927. *Introduction to the History of Science. I. From Homer to Omar Khayyam,* Carnegie Institution of Washington, Publication 376, Baltimore: William and Wilkins.

—, 1931. *Introduction to the History of Science. II. From Rabbi ben Ezra to Roger Bacon,* Carnegie Institution of Washington, Publication 376, Baltimore: William and Wilkins.

Sartre, Jean-Paul, 1960. *Critique de la raison dialectique* précédé de *Questions de méthode,* tôme I, *Théories des ensembles pratiques,* Paris: Gallimard.

Sayılı, Aydin, 1960. *The Observatory in Islam and its Place in the General History of the Observatory,* Publications of the Turkish Historical Society, Series VII, no. 38, Ankara: Türk Tarih Kurumu Basımevi.

—— , ed., trans., 1962. *Abdülhamid ibn Türk'ün katışık denklemlerde mantıkı zaruretler adlı yazısı ve zamanın cebri (Logical Necessities in Mixed Equations by ʿAbd al Ḥamîd ibn Turk and the Algebra of his Time)* Publications of the Turkish Historical Society, Series VII, no. 41, Ankara: Türk Tarih Kurumu Basımevi. Reprinted 1985.

Schacht, Joseph, 1974. "Islamic Religious Law," in *The Legacy of Islam,* 2nd ed., ed. Joseph Schacht and C. E. Bosworth, Oxford: Oxford University Press, pp. 392–405.

Schöne, Hermann, ed., trans., 1903. *Herons von Alexandria "Vermessungslehre" und "Dioptra"* (Griechisch und deutsch), Heronis Alexandrini Opera quae supersunt omnia, vol. III, Leipzig: Teubner.

Sesiano, Jacques, 1976. "Un Mémoire d'Ibn al-Haytham sur un Problème arithmétique solide," *Centaurus* 20:189–215.

—— , 1980. "Herstellungsverfahren magischer Quadrate aus islamischer Zeit (I)," *Sudhoffs Archiv* 64:187–96.

——, 1981. "Eine arabische Abhandlung über die Bildung der magischen Quadrate," Photocopy of manuscript, distributed at the XVIth International Congress of the History of Science, Bucharest.

——, ed., trans., 1982. *Books IV to VII of Diophantos's "Arithmetica" in the Arabic Translation Attributed to Qusṭā ibn Lūqā,* Sources in the History of Mathematics and Physical Sciences 3, New York: Springer.

—— , 1984. "Une arithmétique médiévale en langue provençale," *Centaurus* 27:26–75.

Soubeyran, Denis, 1984. "Textes mathématiques de Mari," *Revue d'Assyriologie* 78:19–48.

Souissi, Mohamed, ed., trans., 1969. *Ibn al-Bannāʾ, "Talkhīṣ aʿmāl al-ḥisāb,"* Texte établi, annoté et traduit, Tunis: L'Université de Tunis.

Steinschneider, Moritz, 1865. "Die 'mittleren' Bücher der Araber und ihre Bearbeiter," *Zeitschrift für Mathematik und Physik* 10: 456–98.

—— , 1896. "Die arabischen Übersetzungen aus dem Griechischen. Zweiter Abschnitt: Mathematik," *Zeitschrift der Deutschen Morgenländischen Gesellschaft* 50:161–219, 337–417.

Suter, Heinrich, 1892. "Das Mathematiker-Verzeichniss im Fihrist des Ibn Abî Jaʿḳûb an-Nadîm," *Zeitschrift für Mathematik und Physik* 37 (supplement):1–87.

——, 1893. "Nachtrag zu meiner Uebersetzung des Mathematikerverzeichnisses im Fihrist des Ibn Abî Jaʿḳûb an-Nadîm," *Zeitschrift für Mathematik und Physik* 38:126–27.

—— , 1900. "Die Mathematiker und Astronomen der Araber und ihre Werke," *Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* 10.

——, 1901. "Das Rechenbuch des Abû Zakarîjâ el-Ḥaṣṣar," *Bibliotheca Mathematica* 3, Folge 2 (1901–1902):12–40.
——, 1906. "Über das Rechenbuch des Alî ben Aḥmed el-Nasawî," *Bibliotheca Mathematica* 3, Folge 7 (1906–1907):113–19.
——, 1910. "Das Buch der Seltenheiten der Rechenkunst von Abū Kāmil al-Miṣrī," *Bibliotheca Mathematica* 3, Folge 11 (1910–1911):100–20.
——, 1910a. "Das Buch der Auffindung der Sehnen im Kreise von Abū'l-Raihan Muh. el-Bīrūnī. Übersetzt und mit Kommentar versehen," *Bibliotheca Mathematica* 3, Folge 11 (1910–1911):11–38.
Taylor, Jerome, 1961. *The "Didascalicon" of Hugh of St. Victor. A Medieval Guide to the Arts.* Translated from the Latin with an Introduction and Notes, Records of Civilization. Sources and Studies, no. 64, New York and London: Columbia University Press.
The Greek Anthology, with an English translation by W. R. Paton, vol. V., Loeb Classical Library, vol. 86, London 1918: Heinemann. Reprinted 1979.
Thompson, Stith, 1946. *The Folktale,* New York: The Dryden Press.
——, 1975. *Motif-Index of Folk-Literature. A Classification of Narrative Elements in Folktales, Ballads, Myths, Fables, Mediaeval Romances, Exempla, Fabliaux, Jest Books and Local Legends,* 6 vols., revised ed., London: Indiana University Press.
Toomer, G. J., 1984. "Lost Greek Mathematical Works in Arabic Translation," *The Mathematical Intelligencer* 6(2):32–38.
——, Al-Khwārizmī," *DSB,* vol. VII, pp. 358–65.
Tritton, A. S., 1957. *Materials on Muslim Education in the Middle Ages,* London: Luzac and Co.
Tropfke, J./Kurt Vogel, et al., 1980. *Geschichte der Elementarmathematik.* 4. Auflage. Band 1: *Arithmetik und Algebra.* Vollständig neu bearbeitet von Kurt Vogel, Karin Reich and Helmuth Gericke, Berlin and New York: W. de Gruyter.
van der Waerden, B. L., 1983. *Geometry and Algebra in Ancient Civilizations,* Berlin: Springer Verlag.
Vernant, Jean-Pierre, 1982. *The Origins of Greek Thought,* Ithaca, N. Y.: Cornell University Press.
Vernet, J., "Ibn al-Bannā'," *DSB,* vol. I, pp. 437f.
Vogel, Kurt, ed., 1963. *Mohammed ibn Musa Alchwarizmi's Algorismus. Das früheste Lehrbuch zum Rechnen mit indischen Ziffern.* Nach den einzigen (lateinischen) Handschrift (Cambridge Un. Lib. Ms. Ii.6.5) in Faksimile mit Transkription und Kommentar herausgegeben, Aalen: Otto Zeller.
——, ed., trans., 1968. *Chiu chang suan shu. Neun Bücher arithmetischer Technik. Ein chinesisches Rechenbuch für den praktischen Gebrauch aus der frühen Hanzeit (202 v. Chr. bis 9 n. Chr.),* Ostwalds Klassiker der Exakten Wissenschaften. Neue Folge, Band 4, Braunschweig: Friedrich Vieweg and Sohn.
Vogel, Kurt, and H. Gerstinger, 1932. "Eine stereometrische Aufgabensammlung im Pap. Gr. Vind. 19996," *Mitteilungen aus der Papyrussammlung der Nationalbibliothek in Wien,* N.S. 1, *Griech. literar. Papyri 1,* Wien, 11–76.

Waltz, James, 1981. "Review of *Conversion to Islam in the Medieval Period: An Essay in Quantitative History* by R. W. Bulliett, 1979," *Speculum* 56:360–62.

Watt, W. Montgomery and Alford T. Welch, 1980. *Der Islam.* I. *Mohammed und die Frühzeit – Islamisches Recht – Religiöses Leben,* Die Religionen der Menschheit, Band 25,1, Stuttgart: W. Kohlhammer.

Wiedemann, Eilhard, 1970. *Aufsätze zur arabischen Wissenschaftsgeschichte.* 2 vols. Mit einem Vorwort und Indices herausgegeben von Wolfdietrich Fischer, Collectanea VI/1–2, Hildesheim and New York: Georg Olm.

Woepcke, Franz, ed., trans., 1851. *"L'Algèbre" d'Omar Alkhayyâmî, publiée, traduite et accompagnée d'extraits de manuscrits inédits,* Paris: Benjamin Duprat.

—, 1852. "Notice sur une théorie ajoutée par Thâbit ben Korrah à l'arithmétique spéculative des Grecs," *Journal Asiatique,* 4e série 20:420–29.

— , 1853. *Extrait du Fakhrî, traité d'algèbre par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; précédé d'un mémoire sur l'algèbre indéterminé chez les Arabes,* Paris: L'Imprimerie Impériale.

—, 1854. "Recherches sur l'histoire des sciences mathématiques chez les Orientaux, d'après des traités inédits arabes et persans. Premier article. Notice sur des notations algébriques employées par les Arabes," *Journal Asiatique,* 5e série 4:348–84.

—, 1859. "Traduction du traité d'arithmétique d'Aboûl Haçan Alî Ben Mohammed Alkalçâdî," *Atti dell'Accademia Pontificia de' Nuovi Lincei* 1858–59, (Roma) 12:230–75, 399–438.

— , 1861. "Traduction d'un fragment anonyme sur la formation des triangles rectangles en nombres entiers, et d'un traité sur le même sujet par Aboû Dja'far Mohammed Ben Alhoçaïn," *Atti dell'Accademia Pontificia de' Nuovi Lincei* 1860–61, (Roma) 14:221–27, 241–69, 301–24, 343–56.

—, 1863. 'Mémoires sur la propagation des chiffres indiens," *Journal Asiatique,* 6e série 1:27–79, 234–90, 442–529.

Youschkevitch, A. P., 1964. See Juschkewitsch, A. P., 1964.

—, "Abū'l-Wafā'," *DSB,* vol. I, pp. 39–43.

Youschkevitch, A. P. and B. A. Rosenfeld. "Al-Khayyāmī," *DSB,* vol. VII, pp. 323–34.

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