The algorithm concept – tool for historiographic interpretation or red herring?

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Abstract. With starting point in Donald Knuth's paper "Ancient Babylonian Algorithms", and using the algebraic reading of pre-Modern mathematical texts as a parallel, the paper discusses the relevance of the algorithm concept, on one hand as an analytical tool for the understanding and comparison of mathematical procedures, on the other as a possible key to how pre-Modern reckoners *thought* their mathematics and to how they thought *about* it.

Keywords: pre-Modern mathematics; algorithm concept as an historiographic analytical tool; algebra as a historiographic analytical tool.

A "red herring": the smoked herring drawn across the trail of the fox in order to distract the hounds and make the hunt last longer

To August Ziggelaar
on occasion of his eighty years' birthday, 17 January 2008

A parallel but preceding issue

When the Rhind Mathematical Papyrus – the most important single source for ancient Egyptian mathematics – was first published by August Eisenlohr in 1877, he interpreted some of the calculations of the text by means of that kind of equation algebra which in his times was currently taught in school. In 1880, Moritz Cantor followed him in the first edition of the first volume of his *Vorlesungen über Geschichte der Mathematik*,[1] making it thereby (if any specific excuse was needed) the canonical way to read the text. It remained so in spite of the well-argued objections formulated by Léon Rodet already in 1881 (with the conclusion [24: 205] that "when studying the history of a science, exactly as when one wants to obtain something, one should `rather ask God himself than his saints'"[2]). In his third edition, Cantor [3: 76] refers to Rodet's objections and alternative interpretation through the method of a "single false position" (p. 76),[3] but sees no genuine difference.

[1] Still in the third edition from 1907 [3: 74].

[2] My translation, as everywhere in the following when no translator is identified.

[3] The method may be illustrated on the problem by which Fibonacci introduces the method in the *Liber abbaci* [2: 173]: $^1/_4$ and $^1/_3$ of a tree are underground, and this part is 21 palms. We posit

Eric Peet, in his new edition of the Rhind Papyrus [20: 60], characterizes the matter as "not one of essence but of form".

Egyptian mathematics was not alone in this situation. In 1886, H. G. Zeuthen published *Die Lehre von den Kegelschnitten im Altertum*, arguing that in *Elements* II.1–10 the ancient Greek geometers possessed "what one may call a *geometric algebra*, since on one hand, like algebra, its deals with general magnitudes, irrational as well as rational, on the other uses other means than ordinary language in order to make its procedures intelligible and impress them on memory" [29: 7]. What Zeuthen had in mind was obviously a much more modern kind of algebra than what Eisenlohr had thought of, and the assertion is rather unobjectionable *if* Zeuthen's whole explanation is taken into account.[4] But it was not, and the resulting conventional wisdom of twentieth-century historiography was that the ancient Greek mathematicians had algebra *without qualification*, "dressed up" as geometry but algebra in "mathematical essence".

Algebra was also the obvious interpretational tool when "Babylonian algebra" was discovered and deciphered in the years around 1930 – and in subsequent decades it was taken for the very truth, as historians and historically interested mathematicians read the commentary and popularizations of the "saints" (i.e., of Neugebauer, van der Waerden and others) – see [7].

Cautious objections against the existence of a "Babylonian algebra" were raised by Michael Mahoney in 1971 [17], based however on a definition of algebra which excluded everything written before Viète, and therefore perhaps not very relevant for historians interested, e.g., in al-Khwārizmī's or Fibonacci's algebra. A famous clash in 1975–1978 between Sabetai Unguru [26], B. L. van der Waerden [27], Hans Freudenthal [5] and André Weil [28] at least made it clear that the status of Greek "geometric algebra" was under discussion.

In mild form, the association of large areas of pre-Modern mathematics to algebra is reflected in the characterization of problems as "equations". An illustrative example chosen at random (that is, from a book which I happened to review recently) is the statement that a twelfth-century *Liber augmentis et diminutionis* shows "how linear equations with one unknown or systems of linear equations with two unknowns may be solved with the help of the rule of double false position"[5] [4: I, 5]. This also illustrates why

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a length for the tree, of which the fractions can be taken conveniently – most obviously 12. $^1/_4$+ $^1/_3$ of 12 palms are 7 palms – but we need 21 palms. Therefore the initial guess should be multiplied by $^{21}/_7 = 3$.

The method can also be used for homogeneous problems of (for instance) the second degree; then the scaling factor is the square root of the error factor.

The Rhind Papyrus only uses the method for first-degree problems, but elsewhere in the Pharaonic mathematical corpus it is applied to homogeneous problems of the second degree (to find the sides of a rectangle from their ratio and the area).

4 Admittedly, soon afterwards Zeuthen [29: 12] expresses *Elements* II.1–10 as algebraic equations dealing with a, b, c, ... – but then he explains that these must be understood as statements about lines and rectangles.

5 That is, making two guesses and finding the correct value from the two errors that arise by means of a calculation which follows the principle of the "alligation rule" (though the latter link

some historians object to the automatic algebraic reading. One problem of the treatise runs as follows [16: I, 326]:

> Somebody traded with a quantity of money, and this quantity was doubled for him. From this he gave away two dragmas, and traded with the rest, and it was doubled for him. From this he gave away four dragmas, after which he traded with the rest, and it was doubled. But from this he gave away six dragmas, and nothing remained for him.

Actually, the treatise solves this problem (and many others) not only through application of the "double false position" but also by stepwise reverse calculation and by means of what the treatise calls its *regula*, the formulation and solution of a first-degree equation in which the unknown initial quantity is called a *thing* and treated exactly as an *x*. Seeing *the problem itself* simply as "an equation" misses the need for what Viète following Pappos called "zetetics", the *formulation* of the problem as an equation – and, in the present case, masks that zetetics is no automatic process, since the problem may as well be translated for instance into a system of three equations with three unknowns (the successive amounts traded with).

A translation of a literary text always identifies that which the reader is supposed *not* to know – the words of the foreign language – within a framework which the reader is supposed to know. In cases where the semantic structures of the two languages are different, it is sometimes possible for the translator to make a choice depending on local semantics without telling the reader – in a classic example, translating English "wood" into German "Holz" if the material is thought of, and into "Wald" if the "wood" refers to many trees growing together. If an English pun is involved, an explanatory note is needed for the German reader.

Such a note is, *mutatis mutandis*, what Zeuthen gave. His reference to "algebra" was a tool for making his readers understand how the theorems from *Elements* II were used. *Applied thus*, the reference to the reader's notion of algebra was hence a fruitful as well as legitimate explanatory tool – and even a way to make the reader reflect upon his own notion of algebra.

Zeuthen's followers forgot the note, and many of those who explained Egyptian and Babylonian mathematics as "algebra" never thought of making similar notes. Thereby "algebra" became a red herring, distracting from analysis of what goes on in the ancient texts and what went on in the mind of its carriers instead of elucidating it.

Seeing historical texts through algorithms

In recent decades, it has become customary to appeal to the algorithm concept, mostly as an alternative, at times as a supplement to "algebra". The precedent invites us to ask whether this is a new and better tool or another red herring?

The first publication to use the notion of algorithms as a tool to understand what goes on in historical texts was probably Donald Knuth's "Ancient Babylonian Algorithms"

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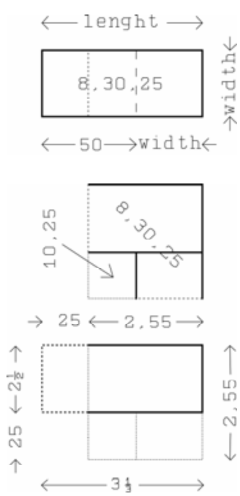
is never made). In mathematical principle, we may see the method as a linear interpolation, and some medieval mathematicians indeed provided a corresponding geometric proof.

from 1972 [15]. He did not see algorithms as an alternative way to explain Babylonian mathematics but states indeed (p. 622) that the

> Babylonian mathematicians [...] were adept at solving many types of algebraic equations. But they did not have an algebraic notation that is quite as transparent as ours; they represented each formula by a step-by-step list of rules for its evaluation, i.e. by an algorithm for computing that formula. In effect, they worked with a "machine language" representation of formulas instead of a symbolic language.

There are at least three layers in this. *Firstly*, that the algorithm is a prescription for finding a result – it provides neither the idea behind the procedure nor any proof of its correctness, and cannot do that (on this level of mathematics) as long as everything is understood as a prescribed sequence of abstract numerical operations – as, *secondly*, was Knuth's understanding of the mathematical texts, based on the translations and the interpretation of the time [7]. Only the abstract understanding of the numbers of the texts as devoid of ontological reference allows us to consider them as elements of a "machine language". *Thirdly*, that an "algorithm" is a "step-by-step list of rules"; this may seem uncontroversial – but see below, note 15.

Knuth gives this illustration (from the tablet BM 85200+VAT 6599 #24[6]). What is at stake is to find the length and the width of the base of a cistern, whose volume is given (in the usual transcription of sexagesimal numbers) as 27;46,40 (meaning 27+ $^{46}/_{60}$ + $^{40}/_{3600}$), and whose depth is 3;20, given that the length exceeds the width by 0;50. I conserve Knuth's parenthetical explanations:



A (rectangular) cistern.
The height is 3,20, and a volume of 27,46,40 has been excavated.
The length exceeds the width by 50. (The object is to find the length and the width.)
You should take the reciprocal of the height, 3,20, obtaining 18.
Multiply this by the volume, 27,46,40, obtaining 8,20. (This is the length times the width; the problem has been reduced to finding x and y, given that x–y = 50 and xy = 8,20. A standard procedure for solving such equations, which occurs repeatedly in Babylonian manuscripts, is now used.)
Take half of 50 and square it, obtaining 10,25.
Add 8,20, and you get 8,30,25. (Remember that the radix point position always needs to be supplied. In this case, 50 stands for $^{5}/_{6}$ and 8,20 stands for $8\,^{1}/_{3}$, taking into account the sizes of typical cisterns!)
The square root is 2,55.
Make two copies of this, adding (25) to the one and subtracting from the other.
You find that 3,20 (namely $3\,^{1}/_{3}$) is the length and 2,30 (namely $2\,^{1}/_{2}$) is the width.
This is the procedure.

[6] Knuth translates freely from the translation in [19: I, 198, 205]. Revised transliteration and retranslation in agreement with recent insights in [9: 146].

We observe that until the beginning of the "standard procedure", the numbers are not ontologically abstract (in other words, deprived of semantics), not "machine language" but intrinsically also an explanation – knowing that the volume is the product of base and height, we understand that division of the volume by the height (which the Babylonians performed as a multiplication by its reciprocal) must give the base.

What Knuth could not know in 1972 is that the "standard procedure" refers to a sequence of geometric cut-and-paste operations – shown here alongside the prescription. His "square root" is thus the side of a square, and the "two copies" (the text actually says "posit it twice") are the two sides which meet in a corner. What Knuth renders "adding (25) to one and subtracting from the other" (actually "join to one, remove from one") is a recurrent ellipsis for a sub-sub-procedure in which the half-excess is joined to one side and removed from the other – often *first* removed and *only afterwards* – because *the same* line segment is involved and therefore has to be at disposition – joined to the other side. Even this part therefore is not written in "machine language" but semantically loaded; the inherent references to the geometric diagram[7] which is manipulated provides a justification of the procedure which is just as adequate as the one that follows from our manipulations of an algebraic equation.[8]

Removal of the reference to the "machine language", a misunderstanding induced by the translation into modern arithmetical language, does not prevent us from speaking of the prescription as an "algorithm": it still consists of a "step-by-step list of rules". However, as Knuth points out (p. 674), he only finds "straight-line calculations, without any branching or decision-making involved. In order to construct algorithms that are really non-trivial from a computer-scientist's point of view, we need to have some operations that affect the flow of control". The closest he gets is the reading of a text with repetition as an expanded macro-iteration.

He *might* have pointed to that use of an embedded sub-routine which he observes in the text he quotes. This feature of the Babylonian texts was explored in some depth by Jim Ritter [23]. Ritter centred the discussion on the tablet Str. 368,[9] which has the same embedded sub-routine as the example discussed by Knuth – with one small difference. Instead of performing the bisection within the subroutine, the main procedure omits a previous doubling that should produce the number to be bisected. The same pairwise cancellation of operations, one inside and the other outside the sub-routine, is found in other texts. The algorithmic interpretation can of course be saved (we may just speak of two related but different sub-routines) – but the two-level algorithmic interpretation can still be seen to be only a formalization of the sequence of operations, and not to cover that

7 These references are visible in the terminology, which is only rendered inadequately by Knuth. The Old Babylonian mathematical terminology (that is, the terminology of the earlier second millennium BCE, the period from which most mathematical texts stem) distinguishes two different "additive operations", two different "subtractions", two different "halves", and no less than four "multiplications" (one of which is not a genuine multiplication but a rectangle construction).

8 Karine Chemla has repeatedly used the formulation that the text is "algorithm and proof in one". For the whole geometric interpretation of the procedure, see for instance [9].

9 Transliteration and translation in [19: I, 311*f*].

insight from which the sequence of steps is planned – which would not astonish Knuth, cf. above.

In what Knuth regarded as the trivial sense, Babylonian mathematical texts – more precisely, the "procedure texts"[10] – can certainly be understood as consisting of algorithms. The texts teach by means of paradigmatic examples, that is, by means of steps in sequence; the ontological identifications of the entities which are operated on ("the height", "the volume", etc.) just show that the algorithm is not a purely numerical one; occasional explanatory remarks ("because he has said that ...", referring to the statement) we may understand as "comment fields".

In this sense, however, even a Euclidean construction (say, *Elements* I,1, "On a given line segment to construct an equilateral triangle", ed. [6: I, 10]) can be read as an algorithm, with the only difference that the comments field (here a proof) follows after the completion of the algorithmic prescription ("With centre A and distance AB to draw the circle ΒΓΔ ..."; and with centre B and distance BA to draw ..."). Even this is a trivial linear algorithm, even though it may be applied as a sub-routine in other constructions (thus already in *Elements* I.2).[11] We may legitimately ask whether a conceptual tool which can be applied so widely is really informative (but the answer will probably depend on taste rather than on arguments).

Greek mathematics is certainly more than geometrical construction, and the "comment fields" of constructional propositions attach these to the general endeavours of theory and demonstration. On the other hand, the concentration on paradigmatic examples was not a Babylonian monopoly. Knuth (p. 676) already refers to the ancient Egyptians and to Indian and Chinese mathematics (rightly, indeed, with the only difference that Indian and Chinese sources regularly state their "algorithms" in the abstract before giving the paradigmatic examples); and the list need not stop there. If the preponderant use of (branch-free) algorithms characterizes these types of mathematics, should we not expect it also to characterize the way their carriers understood mathematics?

Old Babylonian and late medieval texts allow us to reach at least a partial answer to this question. Before we turn to the carriers' perspective, however, we shall take up a final aspect of the use of the algorithm notion as a historiographical tool.

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10 Beyond these texts, which describe the procedure to be followed in problem solutions, the corpus of mathematical texts encompasses "catalogues" listing only problem statements (at times with indication of the solution), mathematical tables and tablets containing only numerical calculations.

11 More interesting embedding is present in ancient Greek geometry at the level of the formulaic language, as discussed by Germaine Aujac [1] and particularly by Reviel Netz [18: 127–167]. But this has hardly anything to do with algorithms, it only shows that the notion of embedding is interesting on its own – cf. [8].

Algorithmic analysis

In order to distil from a text problem its "mathematical substance" (and thus to decide if and why the procedure is adequate), some kind of formalization is often needed. To take a simple example, the "rule of three":[12]

> 7 *tornesi* are worth 9 *parigini*.[13] Say me, how much will 20 *tornesi* be worth? Do thus, the thing that you want to know is that which 20 *tornesi* will be worth. And the not similar (thing) is that which 7 *tornesi* are worth, that is, they are worth 9 *parigini*. And therefore we should multiply 9 *parigini* times 20, they make 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $^5/_7$. And 25 *parigini* and $^5/_7$ will 20 *tornesi* be worth. And thus the similar computations are done.

At first we replace 7 by a, 9 by b and 20 by c, and then we say that c *tornesi* are worth $^{(cb)}/_a$ *parigini*. We might also have argued that 20 is $^{20}/_7$ times as much as 7, and the value of the 20 *tornesi* hence $(^{20}/_7)\cdot 9$ *parigini*, that is, $(^c/_a)\cdot b$ (this method was called "by ratio" by the Arabic mathematician Ibn Thabāt [21: 43] around 1200 and preferred by some Arabic mathematicians).

If read as computational prescriptions (that is, as straight-line algorithms, "first multiply ...then divide ..." respectively "first divide ... then multiply"), these formulae are quite adequate. The danger is, however, that they are read as algebraic formulae, in which case the reader might believe that the two methods are identical – which is clearly a bad approach to historical texts, since it conflates an opaque procedure (the rule of three) and a transparent one. The frequent references in general histories of mathematics to the presence of the rule of three in Babylonian and Egyptian mathematics shows that the mere possibility of translating into algebraic formulae suffices to produce the mistake.

At times, moreover, even a literal reading of an algebraic formula does not allow an unambiguous reconstruction of the computational procedure which it expresses – and thus not to decide whether two texts actually use the same procedure. For instance, we may look at this problem from the Late Babylonian tablet BM 34568:[14]

> The diagonal and the length I have accumulated: 9. 3 the width. What the length and the diagonal. Since you do not know,
> 9 steps of 9, 81, and 3 steps of 3, 9. 9 from 81 you lift:
> remaining 72. 72 steps of ½ you go: 36. 9 steps of what
> may I go so that 36 (is produced)? 9 steps of 4 you go: 36. 4 the length.
> 4 from 9 you lift: remaining 5. 5 the diagonal.

To render this procedure by the line

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12 I quote from Jacopo da Firenze's *Tractatus algorismi* from 1307, ed., trans. [12: 237].

13 "Tornesi" are minted in Tours, "parigini" in Paris.

14 I use the translation in [9: 393], but replace the sexagesimal place value numbers with decimal ones.

$$l \text{ is found as } \frac{\tfrac{1}{2}\cdot([d+l]^2-w^2)}{d+l}, \quad d \text{ as } (d+l)-l$$

(as done in [10: 13] apart from a missing fraction line in the print) is only adequate because $d+l$ is a given number; if it had been calculated, the formula would not tell whether it was calculated twice in the formula for l or once, and saved.

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Translation	*Numerical Algorithm*	*Symbolical Algorithm*
[1] Method of calculating a quantity,		
calculated $1\,\overline{2}$ times	$1\,\overline{2}$	D_1
together with [2] 4	4	D_2
and it has come to 10.	10	D_3
What is the quantity that says it?		
[3] Then you calculate the difference of these 10 to these 4. Then 6 results.	$10 - 4 = 6$	(1) $D_3 - D_2$
[4] Then you divide 1 by $1\,\overline{2}$. Then $\overline{\overline{3}}$ results.	$1 : 1\,\overline{2} = \overline{\overline{3}}$	(2) $1 : D_1$
[5] Then you calculate $\overline{\overline{3}}$ of these 6 Then 4 results.	$\overline{\overline{3}} \cdot 6 = 4$	(3) $(2) \cdot (1)$
Behold it is 4 (the quantity that) [6] says it.		
What has been found by you is correct.		

Annette Imhausen's algorithmic representation of an Egyptian problem [13: 165].

An alternative formalism, able to better grasp the structure and details of complicated calculations for analysis, was proposed by Jim Ritter [23][15] and amply used in adapted shape by Annette Imhausen first in 2002 [13] and next in her dissertation from 2003 [14]: In a three-column scheme, the single steps of the text (in translation), the numerical steps

15 The paper circulated for long before its final publication in 2004. I read it myself in 1997; a preprint [22] appeared in 1998.

It should be added that Jim Ritter's notion of an algorithm is much broader than Knuth's "step-by-step list of rules". He introduces "another, more general level of the algorithm, more general than that of the calculational techniques or that of the arithmetical operations, the level of method of solution, the choice of strategy of solution", and exemplifies this by the method of a single false position which can be seen to underlie several of his examples. This has the apparent advantage of making the carriers' understanding part of the algorithm. As far as I can see, however, the algorithm concept is dissolved by the inclusion of a level which is not linked to the steps of the algorithm (as are the "comments", be they Babylonian or Euclidean) but which is on the other hand common to many algorithms that differ in their steps. Instead of seeing the algorithms used in weather prediction as encompassing the physical theories and differential equations on which they are based, it seems to me to leave more room for analysis to separate the physical and mathematical theories from their implementation in computer algorithms. I shall therefore go on using the usual ("Knuthian") understanding of the term.

and their explanation in symbols stand in parallel. In the symbol column, the outcome of each computation is given a *new* name,[16] which produces an unambiguous trail.

The participants' point of view

So much about the algorithm concept as a tool for historiographic text analysis. We should now return to its possible adequacy as a mirror for the original reckoners' understanding of what they were doing.

Many Old Babylonian procedure texts start the prescription by a phrase "You, by your doing". Is it adequate to read this as a reference to a specific algorithm individualized as such? If so, we might perhaps expect to find occasional references to such algorithms by name.

We do indeed find a few references by name to particular methods. What is striking, however, is that the occurrences of the names show them to point to methods that can be varied, *not* to precise algorithms (not even to what can naturally be interpreted as branched algorithms). One, *makṣarum*/"bundling", refers to the division of a surface (in the actual case, a triangle) or a volume (in the actual case, a cube) into a bundle of smaller surfaces or volumes of the same shape [9: 66, 254]; the other, "the Akkadian [method]" refers to the quadratic completion which we have encountered in the sub-routine discussed above – but it turns up in a procedure of a different and quite peculiar character [9: 194].

This corresponds well to the flexible use of the sub-routine which we discussed above; the Old Babylonian reckoners hence appear to have conceived of their methods as procedures which could be applied flexibly as required by varying contexts, not as a tool-box of fixed algorithms. Only the very standardized set of problems occurring in the texts cause us to find *exactly* the same procedure time and again, and thus giving *us* the impression that fixed algorithms are involved.

I am not aware of the presence of elements in Egyptian mathematical discourse which allow a similar analysis; however, the actual algorithms constructed by Annette Imhausen are often so varied in their details that even they are likely to represent the modern analysis only, not the way the Egyptians understood their mathematical practice.

As far as the Indian and Chinese material is concerned, my inability to read the texts in the original language prevents me from forming a definite opinion; however, the initial abstract formulation of rules which are then followed by examples may suggest that ("trivial") algorithmic thinking was closer to the way Chinese and Indian reckoners thought.

I am much more familiar with the culture of practical arithmetic represented by Leonardo Fibonacci and the Italian and Provencal abbacus treatises of the fourteenth and

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16 It is noteworthy that the same principle was followed by Jordanus of Nemore in the earlier part of the thirteenth century, when he introduced a letter formalism with the purpose of proving the correctness of arithmetical and algebraic theorems (and *not* of making symbolic algebraic calculations).

fifteenth centuries.[17] Within this culture, the word which *might* represent something close to an algorithm is *regula* (*regola*, *reghola*, etc.). It is still reflected in our modern notion of the "rule of three" (the *regola delle tre cose* of the abbacus masters) referred to above. This really looks like an algorithm, and indeed a quite trivial one – but trivial only until we start reading the texts closely. Indeed, if we look at for instance the presentation in Jacopo da Firenze's *Tractatus algorismi* [12: 236–240] we find that it is divided into several cases: all three numbers are integers, one of the first two numbers contains fractions, or both of these do. But the three cases are not treated in parallel – the second and third only tell to multiply adequately by the denominators, leaving it tacitly understood that the rest is as in the first case; although it is not said (and perhaps not precisely conceptualized) it is obvious that the substructure is a less trivial algorithm:

> **IF** all three numbers are integer **GO L**;
> **IF** only one of the former numbers contains a fraction with denominator *p*, multiply both of these by *p*;
> **GO L**;
> **IF** both of them contain fractions, with respective denominators *p* and *q*, multiply both by a common multiple of *p* and *q*;
> **GO L**;
> **L:**
> (multiply and divide)

In the case of the presentation of the "rule of double false position" (see note 5), this structure is even more explicit. Some of the abbacus books, and also Fibonacci, occasionally operate with negative numbers conceptualized as "debts"; but they never do so in the rule of double false. Therefore, the formula to be used depends on whether both guesses turn out to result in an excess (or both in a deficit), or one in an excess, the other in a deficit. The algorithm may not be presented in full – in Barthélemy de Romans' *Compendy de la praticque des nombres* [25: 390] all that is said is thus *plus et plus, meins et meins, sustrayons. Plus et meins, adjoustons* ("excess and excess, deficit and deficit, we subtract. Excess and deficit, we add"). This only describes the initial branching structure, and leaves out the linear part as already known

However, a *regula* is mostly not an algorithm, neither straight-line nor branched. For instance, the *regula* of the *Liber augmentis et diminutionis* (see text around note 5), reappearing as *regula recta* in Fibonacci's *Liber abbaci*, refers to a general and very flexible method: the application of first-degree equation algebra. Several other *regulae* are similarly open-ended; actually, even the rule of three may be adequately but tacitly adapted to problems of inverse proportionality. Application of the algorithm concept thus allows to trace a substructure *in statu nascendi* in the thinking of the abbacus masters; but if they had been asked what they meant by *regola*, the answer would most likely not have made us think of an algorithm.

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[17] Høyrup 2005 gives the reasons that Fibonacci must be seen as an early representative of the same broad mathematical culture as the later abbacus writings and not as the "father" of the abbacus school.

All in all, we may conclude that the algorithm concept, when applied to pre-modern mathematical texts, *may* represent a valid mapping of their procedures – at times useful, at times as trivial as the algorithms which it digs out of the sources. If believed to correspond to the way the early reckoners thought *about* their activity, it is likely to be a red herring (barring perhaps Chinese and Sanskrit texts). If used to trace emerging substructures in the way they *thought* their mathematics it is mostly also misleading – but not always; used with delicacy it may sometimes offer a valuable tool.

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