“OXFORD” AND “CREMONA”: ON THE RELATION BETWEEN TWO VERSIONS OF AL-KHWARIZMI’S *ALGEBRA*

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I. The starting point

In a number of previous publications[1] I have approached the prehistory of algebra up to the final fixation of the subject in written systematic treatises by al-Khwārizmī and ibn Turk in the early 9th century (C.E.). The outcome of these investigations can be briefly summarized as follows:

The branch of Old Babylonian mathematics normally identified as "algebra" was no rhetorical algebra of the kind known from the Islamic and European Medieval period (and from Diophantos). It did not deal with known and unknown *numbers* represented by words or symbols. Strictly speaking it did not deal with numbers at all, but with measurable line segments. Some of its problems were thus really concerned with inverted mensuration geometry (e.g., to find the dimensions of a rectangular field, when the area and the excess of the length over the width are given); others *represented* unknown non-geometrical entities by line segments of unknown but measurable length (e.g., a pair of numbers from the table of reciprocals whose difference is given to be 7, and which is represented by the dimensions of a rectangle of area 60, in which the length exceeds the width by 7).

Correspondingly, the *operations* used to define and solve these problems were not arithmetical but concrete and geometrical. The texts, indeed, distinguish two different "additive" operations: *joining*—e.g., a complementary square to a gnomon; and *adding* measuring numbers arithmetically. two different "subtractive" operations: *removing* a part, the inverse of "joining"; and *comparing* two different entities. And finally no

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[1] Among which the following:
- (1990), presenting in depth the comparative philological analysis of Old Babylonian "algebraic" texts.
- (1989), a concise overview of the same subject-matter, discussing also some of the general implications for our understanding of early "algebra".
- (1986) and (1990b), presenting the evidence that Abū Bakr's *Liber mensurationum* builds on a continuation of the Old Babylonian "cut-and-paste"-tradition, and that al-Khwārizmī's geometrical proofs of the rules of *al-jabr* are inspired from the same source.
- (1990a), investigating the nature of that kind of practitioners' tradition which appears to connect the mathematicians of the early Islamic period with the Babylonian calculators.
- (1987), discussing *inter alia* the specific character of Islamic mathematics as a synthesis between Greek mathematics and such "sub-scientific" traditions.

less than four different "multiplicative" operations: the *arithmetical multiplication* of number by number; the *computation* of a concrete magnitude, e.g. from an argument of proportionality; the *construction* of a rectangle; and the *concrete repetition* of an entity, e.g., the repetition 9 times of a square as a 3×3-square.

The geometrical conceptualizations are reflected in geometrical techniques. The central technique for the solution of mixed second-degree problems is the partition and reorganization of figures (one might speak of a "cut-and-paste" technique). So, the rectangle referred to in the above examples is cut and reorganized as a gnomon, and a complementary square (of area $3^1/_2 \times 3^1/_2$) is joined to it, yielding a greater square of area $60+12^1/_2=72^1/_2$ (cf. Figure 2, which shows the principle). Non-normalized and certain other complex problems are treated by means of a technique of "scaling" (which can be considered a change of unit in one or both directions of the plane). In all cases, the geometry involved can be characterized as "naïve": The operations are seen immediately to yield the correct result (as *we* see, immediately and without further reflection, a=7 to follow from a+2=9=7+2); the texts contain no separate, formal proofs, for instance of Euclidean type.

This "naïve geometry" is fairly similar to the proofs given by al-Khwārizmī in his *Algebra* that the rules used to solve mixed second-degree problems are correct. Another, presumably roughly contemporary text demonstrates that the similarity can hardly be accidental. A *Liber mensurationum*—written by an otherwise unidentified Abū Bakr and only known from a Latin translation due to Gherardo da Cremona (ed. Busard 1968)—contains in its first half a large number of quasi-geometrical, quasi-algebraic problems (finding the side of a square when the sum of the area and the side is known; finding length and width of a rectangle when the area and the excess of length over width are given; etc.). These are solved in two ways: Secondarily by means of *aliabra*—evidently *al-jabr* as known from al-Khwārizmī, rhetorical reduction to standard *māl-jiḏr*-problems and solution of these by means of standard algorithms; but primarily by means of what seems to be a naïve-geometrical cut-and-paste technique, carrying perhaps the name *augmentatio et diminutio* (possibly *al-jam' wa'l-tafrīq* in Arabic, as I have suggested on earlier occasions; but cf. contrary evidence below).

Abū Bakr's treatise does not contain the complete gamut of Old Babylonian "algebra". It is restricted to what looks most as surveyors' riddles: Combinations of the area and the side / all four sides / the diagonal / both diagonals, of squares / rectangles / rhombs. For this reason, Abū Bakr has no use for the Old Babylonian "scaling" technique; everything can be done by cut-and-paste style manipulation of figures.

The character of the transmission link connecting the Old Babylonian epoch with the early Islamic period is made clear by a number of observations: through Abū Bakr's inclusion of the problems in a treatise dealing purportedly with mensuration; through the mathematical contents and the riddle character of the problems; and through a description of the favourite techniques of practical geometers given by Abū'l-Wafā' in his *Book about that which is necessary for artisans in geometrical construction* (transl. Krasnova 1966: 115): When asked to find a square equal to three (identical) smaller squares they would present (and only be satisfied with) solutions where the latter were taken apart and put together to form a single square.

Evidently, Abū Bakr's quasi-algebraic problems are of no practical use. They will have been transmitted since the Babylonian Bronze Age in what I suggest be called a "sub-scientific tradition", within an environment of practical geometers (surveyors, architects, master builders, and the like) not for practical use but as "recreational" problems[2]—probably connected to the training of apprentices.

Diophantos had already drawn some of his problems from such sub-scientific specialists' traditions[3], and it is a reasonable assumption that Greek theoretical mathematics started in part as critical reflection upon the ways of sub-scientific mathematical practice. But these sources were never acknowledged, and Greek mathematics did not integrate sub-

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[2] The established name of this *genre* can justly be regarded a misnomer: In traditional culture, "recreational" problems (and riddles in general) do not serve as *recreation:* Their purpose is agonistic (cf. Ong 1982: 44). In particular, mathematical and other profession-specific riddles have the function of fortifying professional identity and pride: "I have laid out a square field; its four sides, taken together with the area, was 140. Tell me, if you are an accomplished surveyor, the length of the side!".

[3] In particular a large number of problems from Book I of his *Arithmetica*—see Høyrup 1990c: 17ff.

scientific mathematics as a total body, nor was its aim (Hero and a few others apart) to provide practitioners with better methods. The integration of practical mathematics (as carried by the sub-scientific traditions) with theoretical mathematics (as inherited from the Greeks), was a specific accomplishment of the early Islamic culture.

One expression of this process of synthetization is precisely al-Khwārizmī's *Algebra*. *Al-jabr* itself will have been one such sub-scientific tradition, of whose prehistory nothing is known[4], but which will probably have been carried by notarial and commercial calculators. The basic technique of the geometrical proofs will have been borrowed from the surveyors' tradition; the idea that proofs should be supplied, and the way to formulate them in writing by means of lettered diagrams, will have been taken from Greek mathematics.

II. The original intention of the present investigation

Another expression of the drive toward synthetization is Abū Bakr's treatise. Here, the process is the reverse of that performed by al-Khwārizmī: The basic topic is the surveyors' tradition; but it is elucidated by means of the alternative method offered by *al-jabr*. Together with the drive toward conceptual and methodological renewal, however, Abū Bakr's treatise

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[4] The only things we know are:

– that there must be a source—al-Khwārizmī presupposes that the name of the discipline and the meaning of certain fundamental terms are already familiar, and he tells that he has been asked by the 'Abbaside Khalif al-Ma'mūn to write a concise treatise on the subject—not the thing a ruler (or anybody else) would ask for if the subject did not exist already;

– and that this source can be neither Greek nor Indian *scientific* mathematics—as argued cogently by the proponents of Indian and Greek roots, respectively.

The only possibility left is thus that of an anonymous tradition—which, considering the relatively esoteric character of second-degree problems in a world where even the multiplication table was not common knowledge, must have been some kind of specialists' tradition. Certain terminological considerations (not least the use of the term *root*) suggests affinities with the Indian area. Others, however, show connection to the Mediterranean region. One possibility does not exclude the other; it is quite conceivable that the trading community interacting along the Silk Road will have carried certain algebraic techniques to everywhere between China and the Mediterranean, as it demonstrably diffused certain "recreational" problems in the whole area reached by its activity.

presents definite archaic features.

One of these is what may be called the “rhetorical structure” of the text. The normal format of Old Babylonian was as follows: *“If somebody has said to you: [statement]. You, by your method: [procedure]”*. The statement would be formulated in the past tense, first person singular (“I have made...”), with one exception—the excess of one length over the other would be told as a neutral fact in the present tense (“the length exceeds the width by ...”). The procedure would be told in the present tense, second person singular, alternating with the imperative; quotations from the statement justifying particular steps would be introduced by the phrase “because he has said”. All these features recur in Abū Bakr’s text, together with certain others of the same descent.

This astonishing agreement between a Latin text and cuneiform tablets antedating it by 3000 years suggest that the precise wording of the Arabic text might disclose further details on the character of the transmitting tradition. In the absence of the Arabic version of the treatise it might even be possible, so it would seem, to make use of Gherardo’s translation for this purpose. Gherardo, indeed, was an extremely conscientious translator (cf. also Lemay 1978: 175f)—probably one of the most accurate translators of scientific and philosophical texts of all times. Since he also translated al-Khwārizmī’s *Algebra* (ed. Hughes 1986), it might therefore be possible to find his particular Latin equivalences for Arabic terms. If these could be argued to be transferred from one translation to the other, we might get access to certain terminological features of the *Liber mensurationum*.

This was what I intended to attempt and to contribute at the present symposium. As I set out to compare Gherardo’s Latin version with the published Arabic text of al-Khwārizmī’s *Algebra,* however, the two turned out to differ so strongly precisely in the essential chapter (the geometrical demonstrations) that reliable conclusions appeared to be out of sight. Instead, however, Gherardo’s text turned out to reflect to an astonishing extent the process through which al-Khwārizmī constructed this part of his treatise, and thereby also to demonstrate that the Arabic manuscript used for all editions and translations[5] is the outcome of a process of

[5] Oxford, Bodleian I CMXVIII, *Hunt*. 214/I, folios 1–34. I used Rosen’s edition supported by Rozenfeld’s Russian translation (1983) (Rosen’s English translation is too free to be relied

stylistic normalization and thus not identical with al-Khwārizmī's original text—significantly farther removed from it (at least at certain points) than the manuscript used by Gherardo for his translation.

The results of this investigation are thus what I am going to present in the following, together with the meagre conclusions which can all the same be drawn concerning my original question.

III. Gherardo's version

I am not going to present a full stylistic and structural comparison of Gherardo's text and the published Arabic text. For good reasons, in fact: I do not read Arabic, and thus have to restrict myself to what can be done by means of dictionary and grammar[6], supported to some extent by Rozenfeld's fairly yet not fully literal Russian translation. I shall hence focus on a specific stylistic feature, which turns out to be significant.

The format of Abū Bakr's surveyors' riddles (a format which goes back, we remember, to Old Babylonian times) was presented above: "Somebody" says, "I have done". In order to solve this problem, "You do ...". This reflects a tradition where teaching takes the form of inculcation of rules and procedures (whether reasoned or acquired through rote learning).

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upon for my present purpose). Page-references to the Oxford Arabic text refer to the Arabic pages in Rosen 1831.

Only in the very last moment, and only owing to the kind assistance of Professor Essaim Laabid, Marrakesh, did I get hold of a xerox of the Cairo edition (Mušarrafah & Aḥmad 1939), which is also based on the Oxford manuscript. I checked all passages of relevance for the following, but found no disagreements which affect the conclusions (cf. also Gandz's discussion of the character of Rosen's errors—1932: 61–63). The major disagreements which turned up concerned the diagrams, where *both* editions proved deficient when compared with a reproduction from the manuscript facing Mušarrafah & Aḥmad 1939: 24. Rosen omits most of the numbers which label lines and areas in the diagrams; Mušarrafah & Aḥmad, e.g., do not distinguish *alif* from *mīm*, with the result that one diagram carries two of the latter but none of the former. Since letters are important for my argument but numbers not, I have chosen to reproduce Rosen's diagrams.

All English translations from the Arabic, the Latin and the Russian are mine.

[6] My main aids have been Wehr's dictionary (1961), Brockelmann's grammar (1960), and Souissy's doctoral dissertation on Arabic mathematical terminology (1968). I apologize for the wrong vocalizations which I will certainly have committed in the following.

Modern mathematics, on the other hand, is mostly presented in the first person plural mixed up with an impersonal third person, passive or active present or future tense, “We construct”, “The line is drawn”, “the value will be”, etc. The latter format is already found in Greek mathematical texts (even though the Greek mathematicians often speak in the first person singular).

Unlike Abū Bakr, al-Khwārizmī does not stick to a single format. But his choice in particular chapters is not random. Nor is the choice of grammatical person always identical in the Oxford Arabic text and in Gherardo’s version. The variations of this pattern is what provides me with my main evidence.

It is evidently legitimate to ask whether even as meticulous a translator as Gherardo would really respect such minor grammatical shades in a translation. After all, his purpose was to transmit scientific knowledge and not Arabic grammatical gradations—and he did cut down two full pages (1–2) of Arabic text, containing the praise of God and the dedicatory letter, to the single phrase “After the praise and exaltation of God he says”[7].

Inside the translation, however, even grammatical shades turn out to be respected. This is confirmed by one of the chapters which has *not* been submitted to stylistic normalization in the Arabic version, the one on multiplication of composite expressions (Oxford Arabic pp. 15–19, Gherardo pp. 241–243). The chapter contains a large number of examples, some of them purely numerical and given neutrally, “if it is ten diminished by one times ten diminished by two”, others algebraic and set forth by a “somebody”, e.g., “And if he has said, ten and thing times its equal”[8]. All the way through the chapter, the forms agree—and in the single case where the Arabic text uses the passive tense, this is also done by

[7] “Hic post laudem dei et ipsius exaltationem inquit” (Hughes 1986: 233 line I,4). (All page references to Gherardo’s text in the following refer to this edition).

[8] Oxford Arabic p. 16, last line (*wa’in qāla ...*), Gherardo p. 242, line 37 (*Quod si dixerit: "Decem et res in decem et rem"*).

Strictly speaking one might claim that even the purely numerical examples are preceded by a reference to a “somebody”—*viz* the one which inaugurates the whole chapter. Still, this does not change the fact that the two types are treated differently.

Gherardo[9]. No doubt, then, that Gherardo took care to render Arabic grammatical details as closely as possible in Latin[10]; we may confidently trust him as a witness of the forms used in *his* Arabic original, even when they differ from ours—in particular, of course, because the deviations turn out to be systematic, which they would not be if resulting from occasional nodding.

Apart from this chapter on multiplication, we shall have to look at three different passages, which demonstrate systematic variations in usage and as regards the relations between the two versions of the text: the presentation of the rules used to solve the mixed equations; their geometrical proofs; and the chapter on addition and subtraction of composite expressions. When adequate, other than grammatical considerations will be made appeal to. For the moment, we shall concentrate on Gherardo's text.

The rules

The chapter containing the rules (pp. 234–236) starts off by presenting the three composite modes in non-personal format, "treasures[11] and roots

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[9] Oxford Arabic p. 16, line 6 from bottom (*f'iḏā qīla laka*); Gherardo p. 242, line 31 (*Cumque tibi dictum fuerit*).

[10] We may compare this with the two modern translations. Rosen (English pp. 21–27) misses the distinction between numerical and algebraic examples completely; Rozenfeld respects it in full, but renders both active and passive forms as "they have said" (*skažut*), judging (rightly, I suppose) the distinction to be a mere stylistic whim.

[11] "Treasure" renders Latin *census* and Arabic *māl*. This translation is to be preferred to the conventional "square", which is misleading for several reasons. Firstly, "square" possesses geometrical connotations, which were only to be associated with *māl* in later times—indeed by those generations who had learned their algebra from al-Khwārizmī. The customary translation therefore makes a fool of al-Khwārizmī when he takes great pain to explain that a geometrical square represents the *māl*. Secondly, the *algebraic* understanding of "square" is also misleading: The square is the second power of the unknown, and no unknown in its own right. This, again, makes a fool of al-Khwārizmī (and quite a few modern scholars *have* considered him lacking in mathematical consequence on this account) when, after finding the root (*jiḏr*), he also finds the *māl*. Thirdly, speaking of the *māl* as a second power of the unknown makes us believe that the *root* is meant as the *root of the equation*—once again a meaning only taken on by the term as a consequence of al-Khwārizmī's work. To al-Khwārizmī, the *root* is simply the *square root* of the *māl*.

That the *māl* is considered a basic and not a derived unknown is born out by the rather frequent use of the term to designate the unknown in a first degree problem as a *māl*—e.g., in one of the monetary problems from al-Karajī's *Kāfī* (ed., transl. Hochheim 1878: iii, 14), are made equal to number" etc. Then each of them is exemplified in personalized style, and followed by a rule:

> But treasures and roots which are made equal to number are as if you say, "a treasure and ten roots are made equal to thirty nine dragmas". Whose meaning is this: from which treasure, to which is added the equal of ten of its roots, will be collected a totality which is thirty nine? Whose rule is that you halve the roots, which in this question are five. So multiply them with themselves, and from them arise twenty five. Add to these thirty nine, and they will be sixty four. Whose root you take, which is eight [...][12].

This succinct rule for the normalized case of the first composite mode is followed by a more discursive and explanatory exposition of the reduction of non-normalized cases to normal form. In this occurs one of the two grammatical first persons of the chapter:

> It is therefore needed that two treasures be reduced to one treasure. But now we know that one treasure is the half of two treasures. Therefore reduce everything which is in the question to its half [...].

The other turns up in the concluding passage:

> These are thus the six modes [three simple and three composite—JH], which we mentioned in the beginning of this book of ours. And we have also already explained them and said what the modes were of those in which the roots are not halved [i.e., in the simple modes—JH]. Whose rules and necessities we have shown in the preceding. That, however, which is necessary on the halving of the roots in the three other sections we have put down with the verified sections. Now, however, for each section we make a figure [*forma* / *ṣūrah*], through which the cause of the halving shall be found.

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and in the bulk of first-degree problems contained in the *Liber augmenti et diminutionis* (ed. Libri 1838: I, 304ff; Libri's commentary, it is true, misses the point completely, demonstrating *ad oculos* the dangers of the conventional translation).

[12] Census autem et radices que numero equantur sunt sicut si dicas: "Census et decem radices equantur triginta novem dragmis." Cuius hec est significatio: ex quo censu cui additur equale decem radicum eius aggregatur totum quod est triginta novem. Cuius regula est ut medies radices que in hac questione sunt quinque. Multiplica igitur eas in se et fiunt ex eis viginti quinque. Quos triginta novem adde, et erunt sexaginta quattuor. Cuius radicem accipias que est octo [...] (p. 234, lines B.5–11).

The proofs

As we shall see below, this may be what al-Khwārizmī intended at first. In all known versions of the text, however, he presents us with *two* diagrams for the case "treasure and roots made equal to number".

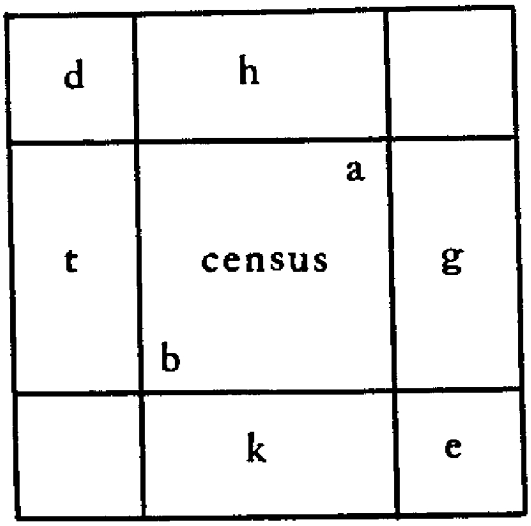

Figure 1: Treasure and roots made equal to number (A)
(Hughes 1986: 237; Rosen 1831: 10 (Arabic))

The first of these is peculiar in several ways. As in those Greek mathematical works which will have been known to al-Khwārizmī at least from his colleagues in the *House of Wisdom*, it is lettered—but several letters label whole rectangles and not points[13]. Moreover, it does not *halve* the number of roots, so as to represent the 10 roots by two rectangles of length 5 and *R* (*R:* the root); it divides 10 into 4 times $2^1/_2$, and represents the 10 roots by four rectangles $2^1/_2 \times R$.

All the other diagrams follow the respective rules closely, halving the number of roots and manipulating the corresponding rectangles and a quadrate of unknown dimensions so as to permit a quadratic completion:

13 In *Elements* II, 5–8, it is true, a notation occurs which at first looks similar: the designation of a gnomon by means of three letters (ed. Heiberg 1883: I, 130–140). But at closer inspection the similarity turns out to be misleading, as the letters mark points on a circular arc going through the three quadrangles from which the gnomon is composed.

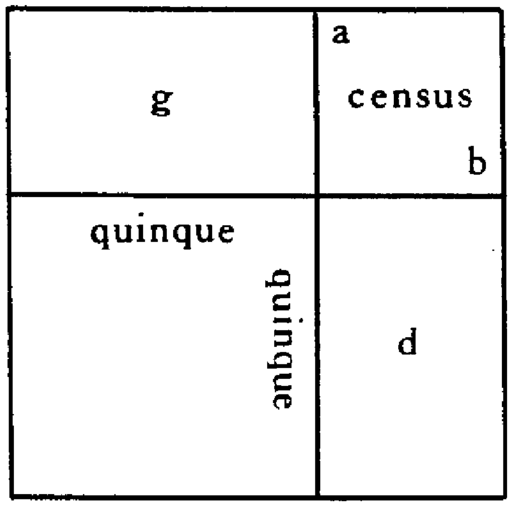

Figure 2: Treasure and roots made equal to number (B)
(Hughes 1986: 238; Rosen 1831: 11 (Arabic))

The alternative diagram for the first case labels whole rectangles by single letters, as does the main diagram; the others, to the contrary, follow the normal Greek (and, as it was to become, the normal Arabic) pattern.

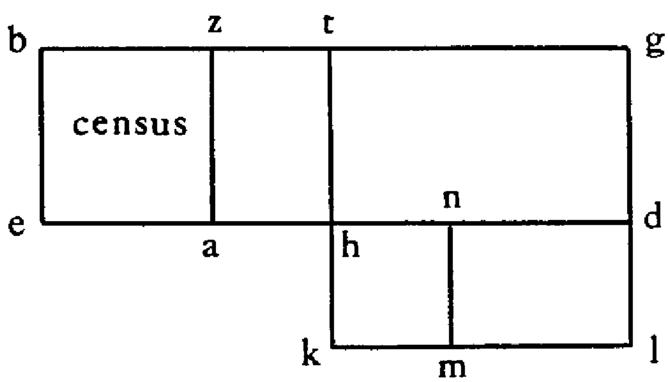

Figure 3: Treasure and number made equal to roots
(Hughes 1986: 239; Rosen 1831: 13 (Arabic))

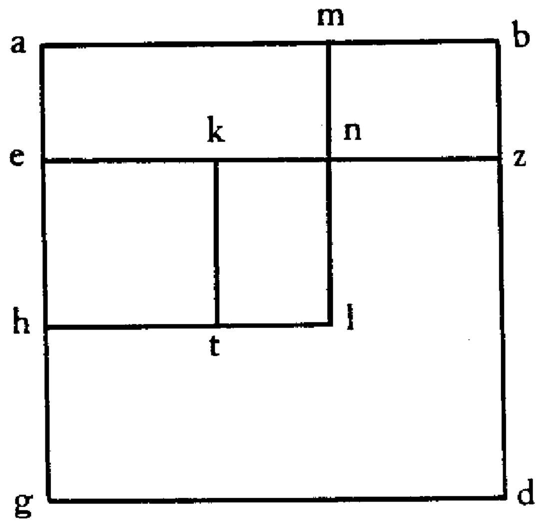

Figure 4: Roots and number made equal to treasure
(Hughes 1986: 240; Rosen 1831: 15 (Arabic))

When we turn our attention to the grammatical person used in the text, differentiations will be observed which follow another pattern. The first proof of the first case starts out in the first person singular (future tense): *accipiam*, *faciam*. Then comes an argument that *we have known* (*scivimus*) a certain surface to have a certain numerical value—*viz* from the statement of the problem; from that point onwards, everything with one exception continues in the first person plural (*addiderimus*, *nos novimus*, *minuam*, *mediamus*, *multiplicamus*, *addimus*, *compleatur nobis*, *sufficit nobis*). The style of the whole argument is discursive and almost colloquial:

> I take [...]. Now we know [...]. If now we add [...]. But we have found out [...]. Therefore one of its sides is its root, which is eight. I shall therefore subtract

in references to what *we* know or want to be done, and when performing arithmetical operations on the already existing diagrams (this rule, it will be observed, does not fit the second proof, and only fits the first proof completely in a specific interpretation to which we shall return below). The last proof also contains a reference to "the three roots and four which I indicated for you" (*quos tibi nominavi*).

Both the third and the fourth proof give a rather discursive explanation of the purpose of the construction of the diagram, i.e., of the way the squares and rectangles of the diagrams represent the given treasure, roots and number. Even this makes their style different from that of the second proof.

Addition and subtraction of composite expressions

The proofs of the rules for solving the mixed second-degree equations were borrowed by numerous mathematical authors in later centuries, Arabic as well as Latin. But they are not the only geometrical proofs offered by al-Khwārizmī. After the chapter on multiplication of binomials comes another on "aggregation and diminution", which first gives some examples of addition and subtraction of binomials and trinomials, promising an explanation by means of a figure in the end of the chapter, and then proceeds to exemplified rules for the multiplication of roots by integers and their reciprocals and for the multiplication and division of a root by another root. In the end of the chapter (pp. 245–247) the promised proofs are brought—two proofs by means of diagrams and one rhetorical, because the diagram attempted by al-Khwārizmī has turned out to "make no sense".

The promise is stated in the first person singular, and the rules and examples set forth in the second person singular ("You should know that if you want to take half the root of a treasure, you should multiply [...]" (244, line 18); "if you want to divide the root of nine by the root of four, divide nine by four [...]" (244, lines 32f). The choice of grammatical person in the geometrical proofs agrees with the main style of the previous ones: Making the constructions in the first person singular, but using the first person plural when "we" wish to do something, when "we" see, and when arithmetical operations are performed on the basis of diagrams which are already at hand.

IV. The Oxford text

As told above, the manuscript which has been used for the modern editions and translations differs from the one which Gherardo must have used. It does so in several ways, of which I shall concentrate on two.

Let us first apply the standard methods for comparing classical geometrical texts: The agreement / disagreement between the letterings of diagrams and in the structures of proofs. Already at this simple level, indeed, the relation between the two manuscripts can be seen to differ from chapter to chapter.

Starting from behind, the diagrams used for the addition of binomials exhibit optimal agreement: *alif*-a, *bā'*-b, *jīm*-g, *dāl*-d, *ḥā'*-h, *zāy*-z, *hā'*-e. If we observe that the labels of the four rectangles in Figure 1 can be freely interchanged, the same agreement is seen in the first proof of the case "treasure and roots made equal to number" (with the supplementary correspondences *ṭā'*-t, *kāf*-k). The alternative diagram in the Oxford manuscript contains two letters *rā'* and *ḥā'* with no counterparts in Gherardo's version (cf. Figure 2), and the texts differ correspondingly: Where Gherardo only refers to "the quadrate of the greater surface" (p. 238, lines 42f), the Oxford manuscript has "the greater surface, which is the surface *rh*"[14]. Apart from that, the letters agree according to the same scheme of correspondences. So they do in every respect in the case "roots and number made equal to treasure" (Figure 4; since the Oxford manuscript omits many diacritical dots, the correspondence *rā'*-z (Rosen) cannot be distinguished safely from the correspondence *zāy*-z (Mušarrafah & Aḥmad)).

In the diagram for the case "a treasure and twenty-one made equal to ten roots", on the other hand, only 3 out of 12 letters agree. Remarkable differences will also be found in the progression of the proofs, together with significant similarities.

One of these demonstrates that one of the proofs is made on the basis of the other, and not independently. This is an idiosyncratic didactical explanation that if in a quadrate "a side is multiplied by one, the outcome

[14] *al-saṭḥ al-a'ẓam al-ḏī huwa saṭḥ RH*, if I read it correctly (p. 11, line 1).

is one root; and if by two, two of its roots”[15]. Another similarity, coupled with a deviation, shows Gherardo’s source to be better than the Oxford manuscript. Gherardo explains (p. 238, lines 51f) his rectangle *ga* to be 21; nothing similar is found in the Oxford version; but at a later point *both* texts refer to this value as already known[16]. The Oxford text is thus the result of a revision—a *Verschlimmbesserung*, indeed.

Gherardo’s proof only leads to one of the two solutions (namely 3); the Oxford proof ends by also giving the solution 7. Alas, the diagram only fits the case where the root is smaller than 5 (unless we accept that line segments may have negative lengths, which was certainly not intended). While Gherardo’s proof errs by incompleteness, the Oxford version commits a genuine mathematical mistake. The person responsible for this minor blunder, however, cannot be the editor who is responsible for the changed lettering and for the omitted identification of the rectangle *ga* (Gherardo’s lettering) as 21; this follows from a comparison with Robert of Chester’s translation[17]. Two hands, at least (one working before and one after the Oxford manuscript family branched off from Robert’s family, and none of them too competent) will have been active in recasting the Oxford version of this particular proof.

Similarly, the Oxford proof of the case “three roots and four made equal to a treasure” has been tinkered with: it omits Gherardo’s observation that the area of *al* is $6^1/_4$ (p. 240, lines 101f), and changes one passage through and through (p. 14, lines 3–7). All other proofs, on the other hand, agree completely in mathematical structure, apart from one or two brief omissions.

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[15] Gherardo, p. 238, lines 54–56; Oxford Arabic p. 11, lines 4–3 from bottom.

[16] According to the Oxford text (p. 12, line 8), “it has already become clear” (*qad kāna tabayyana*?).

[17] Ed. Hughes 1989: 39–41. Robert has the same lettering as Gherardo, except that he interchanges the correspondences of *hā* and *ḥā* and makes *kāf* correspond to *c*. He has the same diagram as Gherardo (and, lettering apart, the Oxford version)—the supplementary diagram found in Karpinski’s edition (1915: 85) has been added by Scheubel. He also tells the area of *ga* to be 21. But like the Oxford version, Robert gives the double solution in spite of his diagram, in words which come too close to those of the Oxford version to be independent; Robert also agrees with this version in omitting erroneously from his description the drawing of *ht* (Gherardo’s lettering).

The other approach is through the use of grammatical persons. If, once again, we start from behind, the proofs concerned with the addition of binomials on the whole follow the same system as Gherardo: Use of the first person singular for constructions, and of the first person plural for what “we” know or want to do, and for arithmetical argumentation on the already existing diagram[18].

The proofs of the “rules” for mixed second-degree equations, on the other hand, exhibit a much more even picture than Gherardo’s text. *No* first person singular and *no* imperatives are to be found: all are replaced by the first person plural. The only exception is the mistaken insertion “proving” the double solution in the case “treasure and number made equal to roots”, which makes use of an invariable “you”.

In the chapter on the multiplication of composite expressions, we remember, the Oxford text agreed with Gherardo in the use of grammatical person. The same holds for the definition of the six cases, for the exposition of the rules, and in the chapters containing algebraic problems. Apart from the chapters on proofs, indeed, the two versions only diverge in this respect at two places—and that, curiously enough, “the other way round”.

One of the places is where al-Khwārizmī rounds off the presentation of the six modes and their “rules” and enters his geometrical demonstrations. The Oxford text (p. 8, line 11–16) speaks in the first person singular (“in the first part of my book”, “I have made clear”, etc.) and stays in teh role of an author speaking to his reader (“the square which you seek”—p. 8, 2 last lines). Gherardo, as quoted above, speaks (p. 236, lines 67–72) in the plural (“in the beginning of this book of ours”; “we have shown”; “the treasure which we want to know”). The other place is when al-Khwārizmī tells that he has attempted a geometrical proof for the addition of trinomials, but that the result was unsatisfactory (Oxford version p. 24 lines 5–7, Gherardo p. 247, lines 93–93). In both places, the author steps forward as the author of the whole book. Most plausibly, Gherardo has felt it appropriate to follow normal Latin style precisely in these places; there is no reason to believe that his Arabic manuscript

[18] Only one exception will be obsered: to Gherardo’s *secabo*, “I shall cut off” (p. 246, line 83), however, corresponds the plural *qata‘anā* (p. 22, line 9).

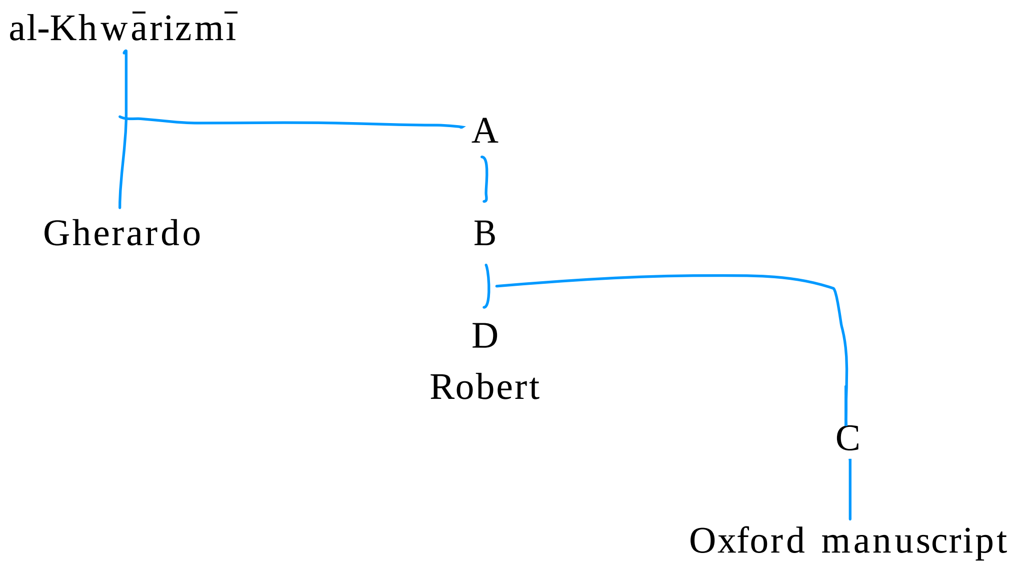
differed from the Oxford manuscript in the two passages in question.

V. Conclusions concerning al-Khwārizmī

In all other places, however, we must prefer Gherardo's choice of grammatical person to the Oxford choice. If al-Khwārizmī had written his demonstrations of the rules for the mixed equations in an invariable first person plural, Gherardo (or anybody between him and al-Khwārizmī) would have had no reason to introduce the systematic distinctions which are found in *his* version. Nor would mere sloppiness on Gherardo's (or an intermediate copyist's) part have produced anything resembling a *system*. The divergent uses of grammatical person in the two versions must therefore (apart from the two Gherardo passages in "author's plural") be explained as deviations of the Oxford version from al-Khwārizmī's original text, produced by somebody aiming at stylistic normalization or in any case following his own stylistic preferences while rewriting—but since normalization has taken place even in proofs where mathematical substance is copied faithfully, intentional rectification of style seems to be involved.

This rectification, as we have seen, only affects the geometrical proofs of the rules for mixed equations but not the proofs concerned with the addition of binomials (nor other matters, indeed); comparison with Robert of Chester's translations, furthermore, tells that it has taken place before his times. We cannot trust Robert's own grammatical choices, it is true[19]. But since the insertion on the double solution, which was known to Robert, has escaped that grammatical normalization which has affected its surroundings, the normalization must precede the insertion, which must precede Robert's translation. The stemma will have to be something like this:

[19] It is thus no powerful argument that Robert mostly uses the first person plural. This might easily be a consequance of his own stylistic feelings—even the insertion on the double solution is, indeed, formulated first person plural throughout. In general, it should be remembered, Robert of Chester is a less literal translator as Gherardo, and would, for instance, reduce "it is obvious to us" to a mere "it is obvious".



Here, A represents the grammatical normalization and B the mistaken addition on the double solution. C corresponds to the changed lettering in Figure 4. Omissions from the proofs take place both in the region A–B and in the vicinity of C.

It is noteworthy that "A" only submitted the first set of geometrical demonstrations to his stylistic treatment. Evidently, he must have found the other demonstrations uninteresting or superfluous—a view which was shared by others[20].

Starting from the above conclusion, *viz* that Gherardo's text can be regarded as a faithful reflection of al-Khwārizmī's own use of grammatical person, we may make some further inferences concerning al-Khwārizmī's working method. The use of the "somebody", the "I" and the "you", as pointed out above, belongs with the sub-scientific traditions drawn upon by al-Khwārizmī. When presenting rules, problems and solutions / methods borrowed from these, he takes over their format, even when the words are actually his own.

In proofs, however, his ways are different—and, as a matter of fact, uneven. The principal system, as we remember, was that constructions were told in the first person singular, while intentions, insights and arithmetical argumentation from existing diagrams were told in the first person plural.

[20] The proofs concerned with the addition of binomials are omitted by Robert of Chester and thus, in all probability, by his original (say, by "D"); and they were not taken over by Abū Kāmil or other later writers on algebra.

There were, however, two exceptions to this rule, both to be found in the proofs concerned with the case "treasure and roots made equal to number". Firstly, in the first proof the outer segments of the side of the larger square are subtracted by a *minuam* (p. 237, line 23). Secondly, the second proof employs the plural consistently, apart from the slip where a "sub-scientific imperative" steals in.

The first exception may not really be one. The first proof, indeed, is the one most obviously taken over from the sub-scientific cut-and-paste tradition[21]; within this tradition, however, the subtraction in question would be a real, geometrical removal, and thus one of those constructive steps which al-Khwārizmī tells in the first person singular in other places.

The other exception, however, is indubitable. It looks, indeed, as a first step toward that stylistic normalization which was carried through by "A". The context is the alternative proof. The best explanation of its anomalous style seems to be that it has been written after the other proofs. It could have crept in during an early revision of the text performed by somebody else, familiar perhaps with ibn Turk's similar proof (Sayili 1962: 145f—ibn Turk, as a matter of fact, also speaks in the first person plural). But the way rectangles are labeled by only one letter reminds too much of al-Khwārizmī's first proof to make the intervention of a foreign hand plausible. It is more likely that al-Khwārizmī first prepared a text containing one diagram, and one proof, for each case; this, indeed, is what is promised in the preceding passage; at some later moment, perhaps after discussion with more grecophile colleagues at the House of Wisdom he inserted another diagram and proof somewhat closer to *Elements* II, 6, expressing himself in a somewhat different style[22].

This and other questions may be answered more definitively through further philological work on the text. One thing, however, should then be remembered: Since Gherardo's translation is (as far as it goes) closer to the original than the Oxford version, no investigation of al-Khwārizmī's

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21 See my (1990a: 80 and note 61).

22 It appears that this conjectural "later moment" must be considerably later: In a newly located, better manuscript of the Latin translation of al-Khwārizmī's algorism, which refers to the *Algebra* as an earlier work, al-Khwārizmī still makes use of the first person singular (Menso Folkerts, private communication).

Algebra should be made without attentive consideration of this Latin version, all modern editions and translations being based on the Oxford manuscript. Robert's less literal translation is not to be relied upon to the same extent; but even Robert may provide us with important supplementary evidence.

Since the Oxford text appears to be the outcome several deliberate attempts at revision, it would be obvious to get behind it by taking other Arabic manuscripts of the work into account[23]. But even the published texts—Oxford and Latin verions—might provide many clues. After all, the present paper was based only on very few textual parameters, which turned out to yield unexpected quantities of information. Other parameters—vocabulary, grammar, structure of the exposition—might yield more.

VI. Conclusions concerning Gherardo and the *Liber mensurationum*

The lack of agreement between Gherardo's source and the Oxford version thwarted my original project: To find the Arabic terminology used by Abū Bakr in the *Liber mensurationum*. Still, the chapters of al-Khwārizmī's *Algebra* which have been least tinkered with in the Oxford version provide some bits of information.

Most important is probably that one of the main uses of *aggregare* in the translation of al-Khwārizmī cannot possibly fit its use in the translation of Abū Bakr. Recurrent in the latter are phrases like "I have aggregated the side and the area [of a square]" and "I have aggregated its four sides and its area" (Busard 1968: 87). A survey of the use of the term in the translation of al-Khwārizmī, from the beginning through the second set of geometrical demonstrations, gives 7 correspondences to *balaġa*, "to reach", "to amount to", together with derivations from this root; 9 to *jamaʿa*, "to gather", "to put together", and to derived forms (most indeed to *ajtamaʿa* (VIII), "to be / come together", and concentrated in the chapter on addition of binomials); one instance falls in a passage which has been

[23] Three are mentioned by Sezgin (1974: 240, 401).

changed in the Oxford version; one, finally, expands a passage where this version only has a *kāna*, "to be / occur", but where *balaġa* might have been used, and may thus have been used in the original text. Of course, *jamaʿa*, would fit the use of the term in the *Liber mensurationum;* but *balaġa* would certainly not.

Two other additive terms from the *Liber mensurationum* are *adiungere* and *addare*. Both are also found in Gherardo's translation of al-Khwārizmī, the relatively rare *adiungere* mostly where the Oxford version has *jamaʿa*[24], the more frequent *addare* corresponding to *zāda*, "to increase", "to augment".

The obvious conclusions to draw from these observations are negative: Even though he took great care to be precise, Gherardo made no attempt to establish a one-to-one correspondence between Arabic and Latin terms used within a single work[25]. *A fortiori*, whatever terminological correspondences we may establish *within* a particular translation cannot be transferred without the greatest circumspection to other translations. Even if the Arabic original used by Gherardo in his translation of al-Khwārizmī had been at hand, it would have been difficult to carry through my original project, perhaps impossible. Still, one observation can be made: even though my previous conjectural identification of Abū Bakr's *augmentatio et diminutio* with *al-jamʿ wa'l-tafrīq* is not directly excluded by the equivalence *jamaʿa-aggregare*, it is certainly not substantiated.

[24] In one instance, Gherardo's *adiungare* (p. 238, line 50) corresponds to an Arabic *wasala*, "to connect", "to join", "to attach" in the Oxford edition (to judge from the printed editions, the Oxford manuscript has a meaningless *nsm*—Rosen 1831: 11 line 7). But since this falls in the proof of the second case, which was emended both mathematically and stylistically, no firm conclusion follows.

[25] Probably for good reasons; if his translations were to be used by others, he was constrained to respect, or at least compromise with, the conceptual boundaries of current Latin usage. Evidently, these differed strongly from those of the Arabic.

Even in his choice of grammatical form, he was of course constrained by the difference between the two languages. One of his strategies to circumvent the problem was touched at above: When an Arabic perfect was too obviously not a preterit, Gherardo would choose the Latin future tense to demarcate it from the implicitly imperfect present tense.

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