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ALCHEMY AND MATHEMATICS
Technical knowledge subservient to ancient γνῶσις

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A parallel – a model

In [1966], Leo Oppenheim analyzed two cuneiform texts containing chemical recipes, one written for the library of Nebuchadnezzar I (1124–1103 BCE) but copied from an older text, the other from Ašurbanipal's library and thus copied in the seventh century BCE. Correlating them with "the sudden appearance of technically very sophisticated glass containers all over the Ancient Near East – Egypt included" and a number of other technical texts he concluded that "sometimes in the second half of the second millennium B.C. the traditional, unwritten technology of Mesopotamia must have clashed on a rather broad front with a new technology of alien origin" (p. 35).

We need not go into details with this part of the argument. What is important for the present purpose is a passage on pp. 41*f*:

> Two well known and early Greek manuscripts with chemical instructions, the Papyrus X of Leiden, and the very closely related[1] Papyrus Graecus Holmiensis, dating from about the end of the third century A.D., contain a number of prescriptions which parallel to an astonishing degree both types of chemical instructions in cuneiform discussed above. These two papyri, both originating in Egypt – most likely in Thebes – seem to have survived the systematic destruction of all manuscripts dealing with alchemy, the making of gold and silver, allegedly ordered in 290 A.D. by the Emperor Diocletian. It is mainly on these two papyri that the claim of Egypt, Hellenistic and pre-Hellenistic, as the mother country of chemistry and alchemy is based. In fact, the technological traditions which find expression in these Greek papyri from Egypt are attested already in Mesopotamia in the 13th and 7th centuries B.C. as shown by the fragments of clay tablets discussed in the first part of this article. This I will try to demonstrate by comparing the topical ranges of both sources and certain characteristic phrases which recur in them.
>
> The respectively 101 and 150 recipes of the Leiden and Stockholm papyri deal with three main topics: methods of imitating precious metals, i.e. gold, silver, and electron, by making alloys whose color and polish is to resemble these metals; methods of coloring (or otherwise changing) stones to give them the appearance of precious stones; and, lastly, producing purple dyes of many shades. [...] the last two sections of the Nineveh fragment (7th century B.C.) and the Babylon fragment (12th century B.C.) correspond respectively to the first two topics of the papyri.

Both the tablets and the papyri point out that the fake precious metals cannot be detected, and both emphasize that the recipes should not be revealed but kept as

[1] According to Robert Multhauf [1966: 97], the two papyri are not only closely related but two parts of the same jeweller's recipe book that accidentally have landed in two different libraries.

craft secrets, apparently for economical purposes.[2]

After further description of the contents of the texts Oppenheim continues thus (pp. 44*f*:

> It is well known that the great transformation brought about by the influence of Greek philosophy on the basically practical chemistry of the Ancient Near East took place in Egypt even before the date of the two Greek papyri which I have discussed. The fundamentally utilitarian intent of augmenting (the Greek texts speak of "doubling" and "tripling") metals, making alloys look like precious metals, and "manufacturing" expensive precious stones underwent a fateful change. In an entirely novel mood and on the wings of a new "theoretical" approach, new technical methods were devised in a quest for the "transmutation" of metals – of course, in the direction of economically preferred combinations of "atoms". Mystically oriented interpretations, magic and theurgic practices, and astrology combined with philosophy to accomplish the shift in outlook which separates the chemistry of the papyri from the alchemy of the Alexandrinian tradition. [...].
>
> The few fragments of clay tablets contribute to the history of chemistry – and that of science – the realization that the pre-alchemistic phase extended over as long a period as the alchemistic. The former appears now as rich in variety and as "international" as the latter, as well as scientific in character in the sense that experiments were made and results recorded and kept from the middle of the second millennium B.C.

In the traditional division of ancient alchemy into three phases as summarized in [Stolzenberg 1999: 3 n. 1],

i alchemy as an art (technique without a theory)
ii philosophical alchemy (technique joined with theory)
iii alchemy as a mystical religion aimed at salvation,

the papyri in question are thus late representatives of the first phase. Phase 2 had begun much earlier, at least with the pseudo-Democritean *Physica et mystica*, perhaps to be dated c. 200 BCE – see [Multhauf 1966: 93–101], where the relation of this work to the Leiden and Stockholm papyri is also discussed. The opening of the third phase is linked to Zosimos of Alexandria (fl. c. 300 CE?);[3] in his works, "alchemy was placed at the service of a mystical religion aimed at *gnosis*, but its practical dimension remained a central concern"; for his immediate successors, on the other hand, "alchemy became primarily a speculative endeavor centered on the exposition of texts" [Stolzenberg 1999: 3].

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[2] The cuneiform tablets are evidently deluxe copies written for the royal libraries; actually, even the two papyri seem to be deluxe specimens [Multhauf 1966: 96].

[3] At least in the sense that his precursors are unidentified (Bolos / pseudo-Democritos, Maria the Jewess) or legendary (Hermes); but already the title *Physica et mystica* shows that the marriage between Greek natural philosophy and proto-Gnosticism had begun centuries before Zosimos's time.

Is this really a parallel and a model?

Alchemy is not my topic, nor certainly a specialty of mine. The preceding prelude was meant to suggest that *part* of the development of Greek mathematics, or certain steps in this development, may perhaps be understood through the model provided by alchemy.

Or, to put it differently: Nobody is scandalized if alchemy is no Greek invention. Its standing is not very high among the prophets of a particular "Western Rationality" – "from PLATO to NATO", in the title of a recent book [Gress 1998]; the touristic sympathizers of "mystically oriented interpretations, magic and theurgic practices", on the other hand, are mostly quite pleased to see it as an expression of "Oriental Wisdom". Reasoned mathematics, on the other hand, is preferentially seen by friend and foe alike as a product of ancient Greek culture; what came before in Egypt and Mesopotamia is generally considered as "art (technique without a theory)" by those who understand the difference.[4]

But Greek alchemy and Greek mathematics developed inside the same cultural framework; no culture is certainly simple, and this does not imply that everything within both expresses the same mind-set or *Zeitgeist*. But the cultures of mathematics and alchemy were also not simple, and it would not be strange if certain strains of one were similar to certain strains of the other.[5]

The strains on which I shall concentrate are those where knowledge of the properties of matter or mathematical objects was seen as a road to *gnosis*, Wisdom (strains which are *not* restricted to gnosticism[6]). I shall build the argument on a few selected mathematical topics.

Summation of series

In the cuneiform tablet AO 6484[7] (a mixed anthology text dated to the early second century BCE, and thus from the Seleucid epoch), we find among other things summations of series "from 1 to 10". In obv. 1–2, $1+2+...+2^9$ is found, in obv. 3–4 $1+4+...+10^2$ is determined.

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4 It is characteristic that [Joseph 1991] on the "non-European roots of mathematics" is mostly applauded by newspaper reviewers who do not know the difference between a computation, a rule and a theorem, and for whom the particular category of "reasoned" mathematics is therefore meaningless.

5 Another parallel is offered by astrology. As discovered by Francesca Rochberg [1988], that Late Babylonian astrology had the same classes of benefic and malefic planets as Ptolemy; even though chronology does not exclude a borrowing, other considerations makes this highly unlikely. However, it is no less unlikely that Ptolemy's explanation of these characteristics of the planets in the *Tetrabiblos* via Greek natural philosophy and spherical cosmology should have been part of the lore that Greek astrology took over from Babylonia.

6 Theon of Smyrna's characterization (*Expositio* I, introduction, ed. [Dupuis 1892: 20]) of philosophy as an "initiation to the truly sacred and revelation of true mysteries" and the relation of this statement to Plato's more equivocal description of the original founders of mysteries as genuine philosophers (*Phaedo* 69C–D, ed. [H. N. Fowler 1914: 241]) may remind of Plato's ambiguous and Theon's unambiguous appurtenance to this current.

7 Ed. Neugebauer in [MKT I, 96–99].

The latter follows the formula

$$Q_{10} = \sum_{i=1}^{10} i^2 = (1\cdot\frac{1}{3}+10\cdot\frac{2}{3})\cdot 55 ,$$

which can be interpreted as a special case of the formula

$$Q_\text{n} = \sum_{i=1}^{n} i^2 = (1\cdot\frac{1}{3}+n\cdot\frac{2}{3})\cdot T_i , \quad \text{where} \quad T_n = \sum_{i=1}^{n} i .$$

The determination of the factor $1\cdot\frac{1}{3}+n\cdot\frac{2}{3}$ is described in precise detail; we may therefore be fairly sure that the unexplained number 55 was found as T_{10} in an earlier problem of the original text from which the anthology has borrowed its two summations.

This may be impressive, but being totally isolated within the cuneiform tradition it is not very informative. Whether the idea of taking precisely 10 members in both cases is a quirk of the author or in agreement with a more general pattern cannot be decided. Since 10 occurs in one other problem as a freely chosen parameter we cannot even decide whether there is a particular connection between this number and summations.

Or rather, we cannot be sure before we take a look at P. British Museum 10520,[8] a Demotic papyrus probably of early Roman date. It begins by stating that “1 is filled up twice to 10”, that is, by asking for the sums

$$T_{10} = \sum_{i=1}^{10} i \quad \text{and} \quad P_{10} = \sum_{i=1}^{10} T_i$$

and answering from the correct formulae

$$T_n = \frac{n^2+n}{2} \quad \text{and} \quad P_n = (\frac{n+2}{3})\cdot(\frac{n^2+n}{2}) .$$

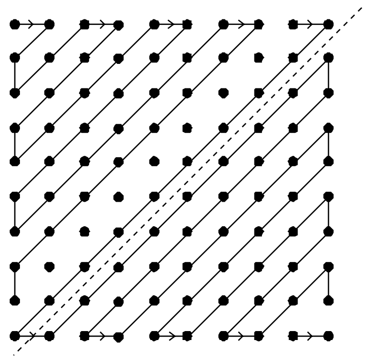

Figure 1. 10×10 arranged as a “race-course”.

This does not overlap with the series dealt with in AO 6484, but the four summations are sufficiently close in style to be reckoned as members of a single cluster. Moreover, the cuneiform formula for Q_n follows from the Demotic formula for P_n when combined with the observation that $i^2 = T_i+T_{i-1}$.

If we look at the formulae for T_n, P_n and Q_n together it is noteworthy, firstly, that the latter two are expressed in terms of the former (represented by the number 55); secondly, that T_n is *not* found as the product of mean value and number of terms, as normal in most cultures.

In algebraic symbolism, the formula is easily derived from $n^2 = T_n+T_{n-1}$, which gives

$$n^2+n = T_n+T_{n-1}+n = T_n+T_n, \text{ and thus } T_n = \frac{1}{2}\cdot(n^2+n) .$$

This was evidently not the way it was derived in Antiquity, but structurally it corresponds to an observation made by Iamblichos in his commentary to Nicomachos’s *Introduction*[9],

[8] Ed., trans. [Parker 1972].

[9] Ed. [Pistelli 1975: 75^{25-27}], cf. [Heath 1921: 113*f*].

and by various modern editors and commentators to Greek arithmetical writings[10] – namely that 10×10 laid out as a square and counted "in horse-race" as shown in Figure 1 shows that 10×10 =
(1+2+...+9)+10+(9+...+2+1), whence 10×10+10 = $2T_{10}$.

Given the peculiar character of the formula, there is little reason to doubt that the Neopythagorean observation and the Seleucid-Demotic formulae are linked. Since both sources for the latter postdate Euclid, they might in principle have borrowed a result obtained by early Greek arithmeticians (whether Pythagoreans or not). However, the total absence from the same texts of anything else which reminds of Greek theoretical mathematics makes such a borrowing unlikely. Independent adoption of the same type of Greek material in Egypt and in Mesopotamia is also hard to imagine, given the general virtual absence of such borrowings from both the Seleucid cuneiform and the Demotic tradition.

Another piece of evidence for the connection also speaks against a Greek invention. The determination of

$$Q_{10} = 1^2+2^2+...+10^2 \quad \text{as} \quad (1\cdot\frac{1}{3}+10\cdot\frac{2}{3})\cdot\sum_{i=1}^{10} i$$

also turns up in the pseudo-Nicomachean *Theologumena arithmeticae* (X.64, ed. [de Falco 1975: 86], trans. [Waterfield 1988: 115]), in a quotation from the mid–third-century bishop and computist Anatolios of Alexandria (in a passage dealing with the many wonderful properties of the number 55). Anatolios, however, gives the sum in abbreviated form, as "sevenfold" $\sum_{i=1}^{10} i$, that is, in a form from which the correct Seleucid formula cannot be derived; this in itself does not prove that earlier Greek arithmeticians did not know better; but it shows that the Seleucid-Demotic cluster cannot derive from the form in which the formula was known to Anatolios. In addition, the absence of the formula from any earlier Greek source derived from the theoretical or Pythagorean tradition (including Theon of Smyrna and Nicomachos) suggests that the learned Anatolios has picked it up elsewhere.

All in all, the only argument in favour of a Greek theoreticians' discovery of the formulae is that their shape points with high certainty toward a derivation or proof based on ψῆφοι, and only if this observation is combined with the axiom that no mathematics not inspired by the Greeks can have been based on proofs. If this axiom is given up, we may conclude the other way around: that (heuristic) proofs based on pebbles were no Pythagorean invention but part of the heritage which the Greeks adopted from the cultures of the Near East – most likely from that practitioners' melting pot of which the various shared themes and formulae of Seleucid (or older Babylonian) and Demotic mathematics bear witness.[11] Since Epicharmos Fragment B 2 [Diels 1951: I, 196; earlier than c. 475 BCE] refers to the representation of an odd number ("or, for that matter, an even number") by a collection of ψῆφοι as

[10] The diagram described by Iamblichos is identical with what we find in J. Dupuis's edition of Theon of Smyrna's *Expositio* [1892: 69 n. 14] and in Ivor Bulmer Thomas's commentary to an excerpt from Nicomachos [1939: 96 n. a].

[11] In another connection I have argued that the determination of an inner height in a scalene triangle by means of an algebraic analogue of *Elements* II.13 must be a similar borrowing which inspired both a generalization to external heights and a reformulation of both as "extended Pythagorean theorem"; see [Høyrup 1997: 81–85].

something trivially familiar, the adoption must be placed no later than the early phase of Pythagoreanism. It can be a coincidence, but the ever-recurrent summation until precisely ten suggest that even the sacred Pythagorean ten may have been a borrowing – whether sacred before it was borrowed or not cannot be decided on this basis. If the two numbers ten have independent origins, the coincidence will at least have pleased the Pythagoreans, as we see from the *Theologumena*.

Ψῆφος arithmetic was mostly used for other purposes than the summation of series – the Epicharmos fragment refers to the "doctrine of odd and even", apart from which the figurate numbers constitute its most conspicuous application. The Seleucid-Demotic material suggest that even the Near Eastern predecessors of the Greeks had used it to argue about triangular and square numbers and the corresponding pyramid numbers P_n and Q_n; since these turn up together (and always together with the sum $\sum_{i=1}^{n} i^3 = T_n^{\ 2}$) in Indian sources and in al-Karajī's *Fakhrī*,[12] it is a fair assumption that these were all dealt with before the borrowing took place; the absence of higher polygonal numbers from all these sources (of which the Indian sources, Āryabhaṭa as well as Bhāskara I and Brahmagupta, are more systematic than can be expected from the random fragments of surviving clay tablets and papyrus) indicates that these are Greek theoretical extrapolations.[13]

All in all we find the same three stages as in the transformation of alchemy: Near Eastern practitioners' knowledge (an "art", but in this case certainly a reasoned art) being adopted into incipient "philosophical mathematics", and later taken over by currents that were more interested in hidden meanings than in principles and proofs and tended to forget or dilute the technical insights.

This certainly did not happen to everything in "philosophical" mathematics. Neopythagorean and related currents did try to appropriate certain insights of theoretical geometry for allegorical ("mystical") use;[14] but theoretical geometry was already too complex for the mystagogues in the outgoing fifth century (whether Theodoros's work on the production of irrational magnitudes or Hippocrates's lunules). On its part, the literature oriented toward Wisdom abounds in references to bits of mathematical knowledge that did not enter the canon of high theory. As we shall see, *some* of these bits may also have their roots in the Near Eastern precursor civilizations.

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[12] See [Clark 1930: 37] (Āryabhata), [Colebrooke 1817: 290–294] (Brahmagupta), [Colebrooke 1817: 51–57] (Bhāskara II), and [Woepcke 1853: 61] (*Fakhrī*).

[13] In their turn taken over by other practitioners, namely the Roman agrimensors, who mistook these inhomogeneous expressions for area determinations of regular polygons; thus for instance Epaphroditus & Vitruvius Rufus, ed. [Bubnov 1899: 534–545]. No Near Eastern surveyor (nor Hero) would have made this mistake, and there is thus little doubt that it came from mistaken Greek theory (maybe a reminiscence from the teaching of liberal-arts arithmetic). The side of the regular polygons in the same treatise, on the other hand, is invariably 10, which seems to be a heritage from the Near Eastern tradition – see [Høyrup 1997: 91].

[14] See for example the use of *Elements* II.14 in Plutarch, *Quaestiones conviviales* VII.2.3, as a metaphor for the imposition of divine (or social) order on unruly matter (or masses).

Side-and-diagonal numbers

One such bit is the reference to the algorithm for side-and-diagonal numbers. It is first described by Theon of Smyrna (*Expositio* I.XXXI, ed. [Dupuis 1892: 70–74], but according to his own statement in agreement with Pythagorean traditions without any addition whatsoever (book II, the introduction). It is also habitually assumed that Plato's reference to "a hundred numbers determined by the rational diameters of the pempad lacking one in each case" (*Republic* 546C, trans. [Shorey 1930: II, 247]) shows him to be familiar with the same algorithm. Actually, all it shows for certain is that he was familiar with the use of 7 as an (approximate) value value for the diagonal in a square with side 5.[15] In any case, another discussion of the algorithm is found in Proclos's commentary to the passage in question from the *Republic*[16]. Finally, Proclos's commentary to *Elements* I contains an oblique but unmistakeable reference to the topic[17] and speaks of it as σύνεγγυζ, "proximate".

Nesselmann [1842: 230] discusses the topic in terms of indeterminate analysis; closer to what we may expect to have been the original reasoning, Hultsch ([1900], and in [Kroll 1899: II, 396*f*]) proves that if s and d are the side and diagonal of a square, then the same holds for $S = s+d$ and $D = 2s+d$ – leaving to experience and common-sense reflection the observation that iteration of the process from values s_1 and d_1 not fulfilling the condition $d^2 = 2s^2$ leads to convergence of the ratio $d_n{}^2 : s_n{}^2$ toward 2 : 1.[18] In Kroll's opinion, the proof was based on *Elements* II.10, which he supposed to be of Pythagorean origin.

We have no direct evidence for the existence of either rule or algorithm in earlier times, but several pieces of indirect evidence suggest that the rule was known and the algorithm used to procure approximate values for the side and diagonal of squares.[19] Two pieces that go together are the Old Babylonian tablets YBC 7289 and YBC 7243 [MCT, 43, 136], both of which use the value 1°24′51″10‴ for the ratio between the square diagonal and side. Neugebauer and Sachs noticed that this value is the sixth step in an alternating iteration by

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15 Heath [1926: I, 399] and others read the "lacking one" as a reference to the fact that 7^2 is lacking 1 compared to the square on the true (irrational) diameter in the square with side 5, which corresponds to an essential feature of the sequence of approximations produced by the algorithm (see imminently). Actually, as pointed out to me by Marinus Taisbak (personal communication), Plato's point is rather that the number 48 (the number which is required) is lacking one with regard to the "number on the rational diameter 7" (and 2 with regard to that on the irrational diameter *dynamei*, as Plato goes on). This is indeed also Proclos's explanation, cf. Hultsch in [Kroll 1899: II, 407].

16 Ed. [Kroll 1899: II, 24*f*]; cf. discussion in [Vitrac 1990: 351*f*].

17 Ed. [Friedlein 1873: 427[21–23]], trans. [Morrow 1972: 339].

18 For simplicity we may designate the theorem proved by Hultsch the "side-and-diagonal rule", in order to distinguish it from the algorithm producing numbers whose ratio converges toward √2:1 via the iteration $d_{n+1} = d_n+2s_n$, $s_{n+1} = d_n+s_n$

19 Nothing in the sources suggests that anybody before the Greeks had noticed or been interested in the fact that the value of $2s_n{}^2-d_n{}^2$ alternates between +1 and –1 (or between $+n$ and $-n$ if we do not start from the ratio 1:1.

arithmetic and harmonic means,[20] of which the fourth step is the value 1°25′ (= $^{17}/_{12}$) that also turns up in cuneiform sources. They regarded the "fact that both values for √2 are links of the same chain [as] a rather strong argument in support" of their reconstruction.

Unfortunately, as shown by David Fowler and Eleanor Robson [1998], the calculations require repeated divisions by sexagesimally irregular numbers; if we try to approximate by regular divisors (in agreement with what we know about Babylonian computational techniques), the reconstruction no longer yields the numbers that it should.

Strangely enough, a different iteration also links both values (as the fourth and the sixth step), namely the one using side-and-diagonal numbers, (1,1), (3,2), (7,5), (*17,12*), (41,29), (99,70), (*239,169*), ... Transforming the latter into a sexagesimal place value number only requires one division by an acceptably simple irregular number (already mid-third millennium tablets exhibit division by 7 and 33, and one of the two extant Old Babylonian tablets listing approximate reciprocals for irregular numbers has divided by numbers until 79). All in all, this procedure is more likely to have been used than the one proposed by Neugebauer and Sachs, *if only* we can propose an argument that leads to the rule and which the Old Babylonian calculators might have found.

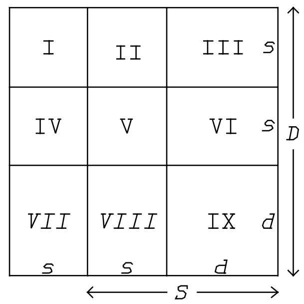

Figure 2. Fibonacci's implicit proof of the side-and-diagonal algorithm.

One argument of the type is used by Leonardo Fibonacci in the *Pratica geometrie* [ed. Boncompagni 1862: 62] in the solution of the problem *D*–*S* = 6, which by simple counting can also be used to prove *Elements* II.10 (see Figure 2); the proof is of a type that is familiar from Seleucid mathematics, and strong arguments can be given that the similarity is based on an actual historical link. From Abū Bakr's *Liber mensurationum*[21] it can be seen that Leonardo presents us with a variation of the problem, and that the traditional form had been *D*–*S* = 4, corresponding to *D* = 14, *S* = 10.[22]

Another proof is based on the construction of a rectangular octagon by superposition of two identical squares (see Figure 3). In spirit it is partially related to some of the diagrams in the Old Babylonian tablet BM 15285 [ed. Robson 1999: 208–217], but the configuration itself is not present; Cantor [1907: 108] refers to it as common in Pharaonic wall painting, but this can hardly be considered as evidence for mathematical reasoning based on it. In the pseudo-Heronian *De mensuris* 52 [ed. Heiberg 1914: 206], however, a reduced version is used for the octagon construction: the oblique square is omitted, but it is used that *AB* = *EC* = *AO* (etc.). To find out this presupposes the same arguments as lead to the side-and-diagonal-rule.

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[20] More likely than this alternation would be the equivalent iteration of the "Heronian" procedure, $\sqrt{n^2+d} \approx n + \frac{d}{2n}$, which can be argued geometrically. This eliminates half of the steps from the Neugebauer-Sachs procedure, but leaves the relevant ones.

[21] So far known only in a Latin translation due to Gherardo da Cremona [ed. Busard 1968], whence the Latin title.

[22] Abū Bakr gives a correct solution *S* = 4+√32 (as can be derived from the side-and-diagonal rule); whether the problem had originally circulated together with the sham solution *S* = 10, *D* = 14 cannot be decided.

The reduced construction also turns up in Abū'l-Wafā'’s *Book on What is Necessary from Geometric Construction for the Artisan* as problem VII.xxii [ed., Russian trans. Krasnova 1966: 93]; in the *Geometria incerti auctoris* no. 55 [ed. Bubnov 1899: 360*f*]; and in Roriczer’s fifteenth-century *Geometria deutsch* [ed. Shelby 1977: 119*f*], whose treatment of circles exhibits links reaching back to the Old Babylonian epoch. Roriczer’s *Wimpergbüchlein* [ed. Shelby 1977: 108*f*] makes use of the superimposed squares and shows (though this is not the topic) that Roriczer knew some of their relevant properties. The superimposed squares producing the regular octagon are found as an illustration to the determination of its area (via the octagonal number! – cf. note 13) in Epaphroditus & Vitruvius Rufus [ed. Cantor 1875: 212, Fig. 40[23]]. According to Hermann Kienast (personal communication) they can also be seen to have been used in the ground plan of the Athenian “Tower of the Winds” from the first century BCE. All in all there is thus no doubt that both constructions were known by the practical geometers of the classical age if not with certainty in earlier epochs; the distribution of the references to and traces of the construction, as well as the absence of hints to it in sources based on the theoretical tradition, make it unlikely that the idea originated among Greek theoretical mathematicians – including the Pythagorean *mathematikoi*.

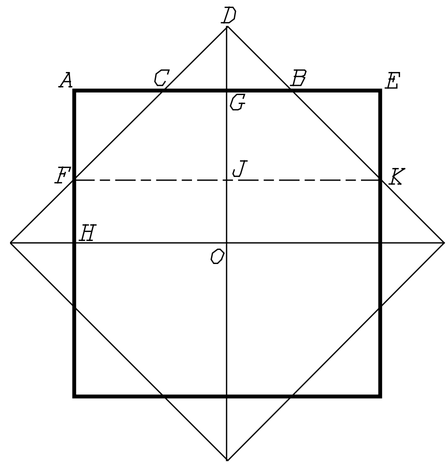

Figure 3. A regular octagon produced by superimposed squares.

A final piece which may perhaps belong to the puzzle is the proof of *Elements* II.10, the diagram of which is nothing but the section of Figure 3 designated by the letters *KEBGCDD* (but in the general case without the specific ratio between *GB* and *BE*). The proposition states that $\square CE+\square BE = 2(\square EG+\square BG)$, which is obviously fulfilled when $\square EB = 2\square BG$, $\square CE = 2EG$, as in the case of the superimposed squares.

We may summarize these observations. Firstly, both Old Babylonian approximations to $\sqrt{2}$ are easily explained if we presuppose that the side-and-diagonal–number algorithm was used, whereas the only alternative explanation proposed so far turns out to be unsatisfactory. Secondly, several implicit heuristic proofs of the rule[24] can be found in sources that point directly to the Near Eastern practitioners’ tradition. Finally, the proof of *Elements* II.10 (the content of which was familiar at least since the early second millennium BCE) may perhaps have been inspired by the configuration of superimposed squares – whereas the proof ideas of II.1-8 seem to be as old as the knowledge they contain (and thus to go back to c. 2000 BCE; cf. below), those of II.9–10 are so different in style from anything we know from the second millennium that they are likely to be relatively recent.

There is no reason to doubt the testimony of Theon as to the Pythagorean character of his source; whether this means that Pythagoreans were the first to investigate the topic “theoretically” can probably not be decided. So much is certain, however, that all sources

[23] The text is also in [Bubnov 1899: 539], but the diagram is omitted.

[24] We should evidently not look for proofs of (the convergence of) the algorithm – proofs of convergence where this convergence is not obvious belong to much later times.

which we possess link mathematics with Wisdom; if mathematicians with less esoteric orientations had once worked on the topic, they seem to have lost all interest in epochs from which sources survive.[25] None of our explicit sources – that is, neither Theon nor Proclos – show convincingly to know the "principles and causes" behind the algorithm;[26] as in the case of alchemy and pyramid numbers, the merger with esoterism entailed a loss of technical substance.[27]

Geometrical riddles

In other contexts – most thoroughly in [Høyrup 1998] – I have argued that a collection of quasi-algebraic geometrical riddles circulated in Western Asia since the earliest second millennium BCE, where it inspired the creation of Old Babylonian school algebra around 1800 BCE; that a characteristic new (more extensive) variant turns up in Seleucid and Demotic sources, which leaves traces also in Greco-Latin "low" mathematics and in Mahāvīra's ninth-century *Gaṇita-sāra-sangraha* (in this respect probably a witness of adoption in the early Jaina period); and that a cross-bread of the early group and the Seleucid-Demotic extension survives in the Arabic world at least until the late Middle Ages, whence it influences Leonardo Fibonacci's *Liber abbaci* and various Italian *abbaco* writings.

The riddles mostly treated of the (measurable) sides and areas of squares and rectangles, and presupposed the sides to be "broad lines", that is, to be provided with a virtual breadth of one length unit, which permitted meaningful additive and subtractive operations involving both line segments and areas.[28] They would deal with entities that were "really there": *the* area of a rectangle or a square (or *the* areas of two squares), *the* side or *all four* sides of a square, *the* side, *both* sides or *all four* sides of a rectangle ("both" sides in the early phase, "all four" in the Seleucid-Demotic-Jaina type). Solutions proceeded by means of "naive" cut-and-paste operations, that is, operations that could be "seen" immediately to lead to a correct solution.

Greek theoretical geometry did not operate with measured entities, and its lines were not broad but explicitly defined as "lengths without breadth". For this reason, it could only be interested in the homogeneous types, and in other types only in so far as they could be made homogeneous by application of a real breadth replacing the virtual breadth of the broad lines (cf. *Elements* II.2–3). But the problems and techniques from the original cluster that *could* be saved under these conditions inspired that strain in Greek theoretical geometry which since Zeuthen has been termed "geometric algebra"; more precisely, *Elements* II.1–10 seem to be "critiques" of the traditional procedures and solutions, establishing *why* and *under which*

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25 Heath [1926: I, 398] speaks of the equations $2x^2–y^2 = \pm 1$ as "a problem of indeterminate analysis which received much attention from the ancient Greeks". Given the actual sources, this is most charitably characterized as a case of poetic license.

26 Proclos comes closest, referring to the Euclidean proposition; but this alone is not sufficient without a specification of the ratio between the line segments involved.

27 Heath [1926: 399] proves the alternation of excess and deficit algebraically and believes (perhaps betrayed by the simplicity of the calculation in this modern formalism) that it was "no doubt omitted because it was well known". "No doubt" another case of poetic license.

28 See [Høyrup 1995] on this notion of "broad lines" and its role in various practical geometries.

conditions these were justified.

There is no reason to proceed with this topic; what is interesting for our present purpose is a riddle type that was *not* adopted into theoretical geometry: to find a rectangle or square whose fours sides equalled the area.

We do not find it in the Platonic corpus; what is offered there is the explanation why it was not taken seriously by the theoreticians. The passage from the *Laws* which speaks of "approximate" commensurability deals precisely with this question: the Greeks should be ashamed, it explains, because they are not aware that a length and a surface (or a volume) are neither absolutely nor approximately commensurate – nor *a fortiori* equal. This is not likely to be an explicit reference to the riddle in question, only to the practice of regarding surfaces and volumes as broad lines and thick surfaces,[29] that is, not to notice the fundamental difference between the three kinds of "feet" by which all are measured. But it shows that the category of Greeks who in Plato's opinion ought to be ashamed (but were not so) would be fully willing to regard the riddle as meaningful.

Neither Hero nor the authors of the pseudo-Heronian *Geometrica* collections belong to the category. Plutarch, however, suggests that "the Pythagoreans" (whoever that term referred to in his days) did. According to his *Isis et Osiris*, chapter 42[30],

> Selon les mythes égyptiens, la mort d'Osiris survint un 17, le jour où il devient tout à fait visible que la pleine lune a accompli son temps. Les Pythagoriciens, pour cette raison, appellent ce jour "Interposition" et ils abominent absolument le nombre dix-sept. Tombant en effet entre le carré 16 et l'oblong 18, seuls nombres plans à avoir leur périmètre égal à leur aire, 17 s'interpose entre eux, les sépare, interrompt leur progression (de raison 9/8 [the whole tone/JH]) et détermine des intervalles inégaux.

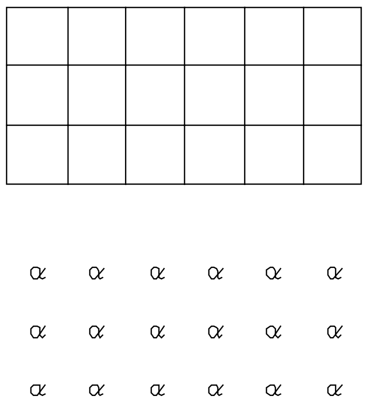

Figure 4. 6×3 represented as a surface and as a surface number.

The text, it is true, speaks of surface numbers, not surfaces, which might make us believe that it refers to a ψῆφος representation of numbers, in which case the counting of the total number of calculi and of those on

[29] "Thick surfaces" are presupposed in Babylonian metrology, which measures volumes in area units, that is, presupposing a standard thickness (of one cubit); they also turn up in certain Italian *abbaco* treatises, where volumes are measured in "braccia quadre".

It is not quite certain that this was the concept which Plato reproached his countrymen to hold. It is possible that they thought instead in terms of "thick lines"; these turn up in the Egyptian Reisner Papyrus I, where volumes are occasionally measured in length units, that is, in terms of lengths provided with a standard cross-section of one square cubit – see [Clagett 1999: 265].

[30] Ed., trans. [Froidefond 1988: 214*f*]. This passage must be the one to which Heath [1921: I, 96] tells to have found a reference in a letter from Sluse to Huygens without being able to locate it.

the perimeter is meaningful; but then the statement is false – cf. Figure 4, bottom.[31] There is no doubt that both Plutarch and those Pythagoreans whom he refers to thought of the upper configuration and its square counterpart.

I know of two other references to the equality of square perimeter and area, both found in the same work: *Theologumena arithmeticae* II.11 and IV.29[32] – once under the dyad and once under the tetrad. In the former passage, the fact that in smaller squares the perimeter is larger than the area and in larger squares the perimeter is smaller explains why 16 is "a sort of mean between larger and lesser"; the second, taken over from Anatolios, explains that 4 "is called 'justice', since the square which is based on it is equal to the perimeter"; both observations refer to themes that fit early as well as later Pythagorean currents – and, in general, fit the metaphorical use of mathematics in the service of Wisdom. So does Plutarch's account of the matter.

Possibly, we should not speak of a "loss of technical substance" due to the merger with esoterism in this case, but rather of exclusion of the more technical stuff from the very beginning; our lack of reliable information about the role of the Pythagoreans in the development leading to *Elements* II (and about the epoch and identity of the Pythagoreans that might have been involved) prevents us from knowing. After Archytas' and Plato's time, however, the outcomes of "initial exclusion" and "subsequent loss" become indistinguishable.

Concluding remarks

Many of the "recreational" problems which we know from mediaeval problem collections are likely to have circulated already in classical Antiquity. The "purchase of a horse" seems to be referred to in the *Republic* (333B–C) and is in any case present in Diophantos's *Arithmetica* (I.xxvi–xxv) in undressed but still unmistakeable form, along with other intricate first-degree problems. Iamblichos's account of "Thymaridas's bloom"[33] also suggest that Pythagoreans from Plato's times were interested in this kind of problems (quite apart from the structural similarity to the "horse-" and similar problems, the sequence of numerical parameters in the problem is indeed characteristic of the "recreational" style). It also demontrates that this knowledge survived into the third century CE within an environment that found it worthwhile to remember Thymaridas's name.

[31] The use of α's instead of dots follows Theon of Byzantium; it emphasizes that each really stands for a unit, a μονάς.

[32] Ed. [de Falco 1975: $11^{11–13}$, $29^{6–10}$], trans. [Waterfield 1988: 44, 63].

[33] Ed. [Pistelli 1975: 62–67], cf. [Heath 1921: 94–96].

But beyond this it makes clear that such complicated matters did not fit the purpose of constructing esoteric or moral metaphors. It is no accident that the passage from the *Republic* on the wedding number is meant to tell that the mathematics required for keeping the ideal republic stable is too tangled and will necessarily lead to errors; the notorious obscurity of the passage is likely to have been intentional and meant to illustrate this difficulty *ad oculos*. The tendency of Neopythagorean and other esoteric currents to retain only the simpler mathematical metaphors and observations and to discard too technical matters will have been in full agreement with their project (as was the gradual elimination of technical chemical detail from alchemy after Zosimos).

This project was neither scientific nor technical; we may like or not like the project, we may see it as an alternative or correction to the project of the "Greek miracle" as embodied by Herodotos, Archimedes, Aristotle and Galen (and many others), or as a deplorable decay or a defection from the enlightenment ideals of the early philosophers; but reproaching the late alchemists or Iamblichos that they threw out good science and retained only the refuse amounts to judging the means they used for their purpose instead of the purpose itself. To a first view this might seem to correspond to judging Archimedes from the primitive theology of the cattle problem (the wholly transcendental divine as an owner of livestock!).

There is a difference, however. Archimedes never pretended to make theology when asking for the cattle of the Sun God; but like their modern counterparts, the ancient pursuers of Wisdom often pretended to have higher insights even in the matters which they neglected or had not really understood (one prophet from our own century, we remember, studied physics for one year without passing any examination; having become a spiritual leader he proclaimed himself an "atomic physicist"). The only way to demonstrate that their aim *was* different not only from that of recent science but also from that of an Archimedes or an Aristotle may be analysis of the means by which they pursued this aim; when this is taken into account we will be forced to realize that analysis of the technical shortcomings of the esoterics is no illegitimate export of the "science wars" to a territory where they have nothing to do but a legitimate tool for a historical understanding of the actors themselves that goes beyond their proclamations; it may also be a legitimate defense against certain uses of ancient Wisdom in the contemporary debate.

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