

**Practitioners – school teachers – “mathematicians”:
The divisions of pre-Modern mathematics and its actors**

Jens Høyrup
jensh@ruc.dk

Contribution to the conference
Writing and Rewriting the History of Science 1900–2000
Les Treilles, 5–11 September 2003

First of all a personal note: I believe – but obviously cannot be sure about a matter of this kind – that my interest in practitioners’ knowledge as an autonomous body goes back to the three years I taught physics at an engineering school some thirty years ago. Among other things I remember one episode which may illustrate the relationship between theoretician’s knowledge and what is often (too often, I would argue) supposed to be its “application”. Two colleagues – say, B and H – planned and held a course in electrodynamics for students of constructional engineering. H had been trained as an engineer himself, while B was a nuclear physicist. They were very good friends, and agreed upon most of what one can agree upon in this world. None the less, H one day complained to me that “B removes a Maxwell equation a year, but nothing changes!” Evidently, merely simplification of high theory was not what was needed in order to bridge the gap between the theoretician’s and the engineering organization of knowledge.

Proto-historiography

Herodotos, followed by numerous other ancient Greek writers until Proclus, maintained that geometry began as (Egyptian) practice, and was later transformed into (Greek) theory; nothing was said by them about theory becoming in its turn the guide for the corresponding practice, although Hero and a few others tried to accomplish something like that (with modest impact outside the realm of war machines).

The standard view of the High Middle Ages – the epoch where the Middle Ages had developed a scientific culture enabling them to form an opinion of their own in the matter and not just repeat what had already been repetition with Isidore – was not very different. In the introduction to the “Adelard III” version of the *Elements* [ed. Busard 2002: 31f] we read that in the case of geometry, as in that of any other skill (*facultas*), *usus* not only preceded theory (*artificium*) but also continues to exist as the *exercitatio* of the skill; the main difference with respect to Antiquity is that the writer – himself certainly an *artifex* – demonstrates to have some interest in the practical *exercitatio*, as could reasonably be expected from someone who had Hugues de Saint-Victor in his intellectual luggage.

Though knowing *the field* of mathematics, perhaps both as theory/*artificium* and as a tool for practice/*an exercitatio*, neither Antiquity nor the Latin Middle Ages was familiar with the figure of the *mathematician* in our sense of the word. At first, a μαθηματικός was a member of the branch of the Pythagorean movement; later in Antiquity, the *mathematicus* would mostly be an astrologer of the “Chaldean” type; the teacher of the mathematical Liberal Arts – the closest we may perhaps come to a *professional* mathematician – would mostly be designated a *geometer*, while a μαθηματικός in the teacher’s garb might teach any liberal or philosophical art. Aristotle does speak about the person who is engaged in mathematical argumentation as a μαθηματικός, but this is a personification of his ideal of epistemological autonomy of

the various fields of knowledge, still no professional role. The Middle Ages often did try to distinguish between the *matematicus*, that is, the astrologer, and the *mathematicus*, the one who practised mathematics; but it would be difficult to find a person who primarily identified himself as a *mathematicus*.

Attitudes begin to change in the Renaissance. In a lecture on the mathematical sciences held in Padua by Regiomontanus in 1463/64 (the introduction to a series of lectures on al-Farghānī, printed by Schönner in 1537, facsimile in [Schmeidler 1972]), everything is seen in a [social as well as metatheoretical] top-down perspective. Mathematics is essentially *theory*, deriving its deserved high prestige, on one hand from its roots in classical Antiquity, on the other from its utility for philosophy and from its civic utility (for courtly pleasure). Much lower merit is ascribed to the applications taught in the *abbaco* school (accounting, surveying, etc.), and next to none to the applications in material production. Whether these low-ranking applications are presumed to derive from theory is not clear.

Regiomontanus was ahead of his times, not only in the sense that he was a better mathematician than any contemporary in the Latin world but also in his attitudes to the character and role of mathematics (attitudes that he could only develop because of his mathematical insights and aptitudes); but a writer who is “ahead of his times” is still bound to his times in many ways. A more mature expression of the conception of the relation between theory and practice that ripened during the later Renaissance is found in Vesalius’s introduction to his *De humani corporis fabrica*.

Vesalius, of course, discusses the medical art, not mathematics. This art, in his opinion, had been almost destroyed by the fact that responsibility for exerting it had been parcelled out into three shares: that of the physician, the one who knows the principles of the art but does not know how to use a knife – or does not dare to lest his social standing might suffer; that of the pharmacist, who at least works under the guidance of the physician (that is, under the control of the medical faculty of universities); and that of the barber or surgeon, ignorant of everything according to Vesalius and therefore unable to do *adequately* that which in fact he does: use his hands. The art can only be restored to its former splendour if the three shares are once again united, and “*the hand*” brought under the control of the *theoretically schooled physician*. In other words: practice – even the dirty practice of cutting and bloodletting – has to become applied science.

Vesalius, as well known, did inaugurate a period of rapidly progressing insights in anatomy. Medicine understood as the art of healing did not keep up with this progress in theory, but Vesalius had some sound justification for his claim. Slightly later we see a similar but stronger claim being made for mathematics by Ramus. Ramus, as is equally well known, wanted to avoid Euclid’s “Platonic error”, the teaching of theory for theory’s sake; but his alternative was an edition of the *Elements* where the proofs had been replaced by explanations of the utility of the single theorem. Theory

should thus, as also requested by Vesalius, reform its mind and discard the mistaken fear of practical utility and dirty hands; but (reformed) theory should govern. In the historical introduction to Ramus's *Scholae mathematicae* [1569: 64f] this view reveals its purely ideological character in the claim that the three famous great discoveries – the magnetic compass, gunpowder and printing – were made in Germany because the mathematician Heinrich von Hessen had been forced to leave Paris in the 1380s and go to Vienna, thus inaugurating the blossoming of German mathematics; he also wonders [1569: 107] that applied mathematics flourishes more in Italy than elsewhere in spite of the modest number of university chairs in mathematics, ignoring the existence of the *abbaco* school institution (*deliberately* ignoring it for sure, just as he deliberately ignores Stiefel from whom he copies wholesale though at the modest level he understands).

In the sixteenth century, the “mathematician” became a recognized social role, not least for those “higher mathematical practitioners” who moved around the Italian courts [Biagioli 1989]; Baldi's majestic *Vite de' matematici* illustrate the development.

What is most charitably characterized as Ramus's pipedream gradually materialized as reality over the next couple of centuries – first by the efforts of *Rechenmeister* like Tartaglia and Faulhaber to appropriate whatever Euclidean and Archimedean knowledge they might need, afterwards in the interplay between these creators of new branches of mixed mathematics and mathematicians with scientific training and engaged in developing useful knowledge, for instance at the request of the Académie des Sciences. In his *Mathematisches Lexikon* from 1716, Christian Wolff recognizes that “*mathesis practica, ausübende Mathematick*” as a category does not coincide with “*mathesis impura sive mixta, angebrachte Mathematick*” – the latter being the application of mathematical understanding to “human life and nature”, whether for the purpose of doing something *or* for obtaining theoretical insight.¹ He adds, however [Wolff 1716: 867], that

It is true that performing mathematics can be learned without reasoning mathematics; but then one remains blind in all affairs, achieves nothing with suitable precision and in the best way, at times it may occur that one does not find one's way at all. Not to mention that it is easy to forget what one has learned, and that that which one has forgotten is not so easily retrieved, because everything depends only on memory. Therefore all master builders, engineers, calculators, artists and artisans who make use of ruler and compass should have learned sufficient reasons for their doings from theory: this would produce great utility for the human race. Since, the more perfect the theory, the more correct will also every performance be.

After the creation École Polytechnique and its nineteenth-century emulations there was no longer any need to repeat this protestation. For pragmatic reasons, Wolff's distinction

¹ Thus, as Wolff observes, “everything in mathematics beyond arithmetic, geometry and algebra [his “pure mathematics”] belongs to accommodated mathematics” [Wolff 1716: 866f]. As everywhere in the following where no other translator is identified, the translation is mine.

between the “practical” and the “mixed” could be discarded – as it was already discarded in the names given by Gergonne and Crelle to their journals, respectively *Annales de mathématiques pures et appliquées* and *Journal für reine und angewandte Mathematik*.

Historiography

Modern historiography of mathematics begins, we might say, with the generations from Montucla and Cossali to Libri and Nesselmann. These were still close to the victory of the “Vesalian” subordination of practice under reformed theory, and furthermore brought up mathematically before the triumphs of the “new” pure mathematics inaugurated by Cauchy, Abel, etc. Finally, they were hungry for sources of any kind. No wonder hence that their attitudes would still have some of their roots in the situation delineated by Wolff. Montucla, when telling [1799: I, 397–402 and *passim*] about Ottoman, Arabic, Persian and Indian mathematics, actually applies what in one of the current meanings of the word can be characterized as an *ethnomathematical* perspective, describing (briefly) teaching practices as well as the uses of mathematics and computation in general social life.

However, the interest in practical mathematics did not die with this generation. When dealing with pre-Modern mathematics, historians like Boncompagni, D. E. Smith, Tropfke, Karpinski and Vogel would still pay much attention to sources that had their roots in practice. At least as a rule, they abstained from using the term “mathematicians” about the originators of what several of them termed “school mathematics” or “elementary mathematics”. Given the sources they relied on,² neither designation was mistaken, but they express a belief in the unity of the mathematical genres that agrees with Wolff’s ideal (and with the perspective of their own times) but not – as I will argue – with the social reality of pre-Modern mathematics.

Around 1930, the perspective changed.³ History of mathematics came to be understood as the history of the *mathematics of mathematicians*, and mathematicians tended to be defined in post-Cauchy-Abel terms. In part that was a consequence of the disappearing interest in European medieval mathematics, on which next to nothing

² Namely, manuscripts and printed works. Montucla, when making his proto-ethnomathematics, had relied on ethnographic informants (diplomats and other travellers), and elsewhere (e.g., [1799; IV, 541]) uses his direct acquaintance with practitioners as supplement to what he can document from written sources. But the historians of mathematics of the following 150 years, like other historians from Ranke’s century, relied on *documents*.

³ Given, for instance, that Vogel lived and worked until 1985 it goes by itself that this statement is an extremely rough approximation to *wie es eigentlich gewesen*, permissible only in the context of an introductory discussion.

was published between 1920 and 1948.⁴ But this explanation from the object of the historian is partial at best: in the 1920s and the early 1930s, the appearance of two good editions of the Rhind Mathematical Papyrus and the publication of the Moscow Mathematical Papyrus spurred some further publication activity; from the late 1920s onwards, the Babylonian mathematical texts were cracked and published, which had a great impact, not least through the acceptance of Neugebauer's thesis about the descent of Greek "geometric algebra" from Babylonian "algebra". In the perspective of the epoch, even Babylonian mathematics came to be understood as the product of "Babylonian mathematicians".⁵ Moreover, even the historiography of early Modern mathematics tended to turn away from the applications of mathematical theory and to concentrate on the "real" mathematics.

Missed opportunities

Two events should be mentioned at this point, not because they affected the historiography of mathematics but rather because it might seem strange that they did not.

The first is the renowned intervention of Soviet scholars at the London Congress of the History of Science in 1931.⁶ Within the historiography of science, Boris Hessen's paper on "The Social and Economic Roots of Newton's *Principia*" was indubitably the one that had the strongest impact. By way of J. D. Bernal's reception and ensuing successful campaign for the implementation of *science policy*, Bukharin's paper on "Theory and Practice from the Standpoint of Dialectical Materialism" and M. Rubinstein's presentation of the "Relations of Science, Technology, and Economics under Capitalism and in the Soviet Union" were probably those that were most consequential.

Hessen's paper was written under conditions which his audience did not know about, and carried a subtle message that it missed.⁷ Bukharin shared Hessen's fate not only in life (both fell victims to Stalin's purges in 1938) but also as regards his London paper. As observed by I. Bernard Cohen, "Bukharin's piece remains impressive today [c. 1989] to a degree that Hessen's is not" [Graham 1993: 141]. But that went largely unnoticed in 1931.

⁴ Most of the few publications that did appear are from Karpinski's hands. If these are excluded, the general absence of interest in medieval Latin and vernacular European mathematics becomes even more striking.

⁵ I do not remember Neugebauer to have employed the expression, but it was used by Thureau-Dangin – e.g., [1938: xxxviii] – and afterwards by various other authors, although most have spoken simply of "the Babylonians".

⁶ The Soviet contributions, printed already at the moment, were reprinted in 1971 in [Science at the Cross Roads].

⁷ See [Graham 1993: 143–151].

Bukharin discusses the relation between theory and practice both from an epistemological and from a sociological point of view. On the first account he emphasized that knowledge comes not from pure observation but from intervention in the world – which may not go beyond what he cites from Marx, Engels and Lenin though certainly beyond what his audience knew about what these authors had said, and which in any case had to wait for Mary Hesse and Thomas Kuhn before it was accepted outside Marxist circles. On the second account – the one that is relevant for my present purpose – he emphasized the complexity and historically conditioned mutability of the relation between knowledge and practice, as well as the changing ways in which different types of knowledge are distributed between carriers with different social roles.

As alluded to, Bukharin's subtleties proved too subtle for the Western audience, and had no impact.⁸ Even Joseph Needham, who later was to make the non-trivial interplay between “clerks and craftsmen” a favourite theme of his, only saw Bukharin's paper as “in its way a classical statement of the Marxist position” [*Science at the Cross Roads*, ix]. Needham instead received his impulse from the second of the above-mentioned events: Zisel's paper from [1942] on “The Sociological Roots of Science” (as well as other papers by the same author).

Recent work on Zisel's *Nachlaß* [Raven & Krohn 2000] shows that this and other papers of his from the same period belong within a larger metatheoretical project that never materialized as such. As it stands and on its own, the paper argues that the discussion about *the* root for the new science of the late sixteenth and the seventeenth century – whether scholastic thought, Humanism, or the knowledge of engineers like Leonardo da Vinci – is mistaken, since it was *the interplay* between natural philosophers in the scholastic tradition, trained Humanists, and “higher artisans” that made possible the breakthrough.

Needham was not the only historian *of science* to be impressed by Zisel's paper, which (like Hessen's article) indeed called forth a number of other publications either taking up the thesis or explicitly arguing against it. Strangely, however, no historian of mathematics seems to have addressed the questions whether Zisel's thesis might apply *mutatis mutandis* to the revolution in early Modern mathematics.⁹ Initially this non-reaction was perhaps not so strange – at the time, and for long, historians of mathematics saw in the most important group of “higher artisans” of relevance for

⁸ They may also have been too subtle for his fellow-countrymen, but until Bukharin's rehabilitation in 1988 these had other reasons not to get too close. For decades, the points of view expressed by the Soviet delegation at the London Congress could only be discussed in the Soviet Union as filtrated through Bernal's not very sophisticated reception.

⁹ At least not before I organized an international workshop on the theme “‘Higher artisans’, Humanism and the University Tradition. The Zisel thesis reconsidered in relation to the Renaissance transformation of mathematics” in 1998 – but even then it did not really happen.

the question (the Italian *abbaco* masters) nothing but not very competent *vulgarisateurs* of Leonardo Fibonacci (if they happened to know at all about their existence); ascribing to such people a stimulating influence was more than could be expected from historians concentrating on the mathematics of (great) mathematicians.¹⁰

No “event” but a process has been the increasing awareness within the history of technology that pre-nineteenth-century technical knowledge, including knowledge leading to technical innovation, cannot be adequately described as “applied science”. Even this process has left fewer traces in the historiography of mathematics than it should perhaps have done.

“Popular” or “sub-scientific”

In spite of the invitations of Bukharin and Zilsel it thus remained common, to the extent the mathematics of medieval and other pre-Modern practitioners was at all taken into account and seen as a different body than that of the “scientific” traditions, to characterize it as “popular” or “folk”. I still did so myself in [1984] in my contribution to the Sarton Centennial Conference when discussing the roots for those aspects of the Islamic mathematical corpus which lexicographers like al-Nadīm do not trace to the Greeks but treat as anonymous traditions or fail to mention.

But evidently neither the use of the “Hindu numerals” nor trigonometry were known at the time by “people” in general; these kinds of supposedly “popular” knowledge had been carried by narrow social groups and was thus certainly *specialists’ knowledge*. In consequence I began speaking of these sources and the traditions to which they belonged as “sub-scientific”, in passing in [Høytrup 1986], with more discussion in [Høytrup 1987]. In [Høytrup 1990a] I elaborated this discussion, emphasizing the oral cultural type of the carrying environment and pointing (i) to the function of (what has come to be misnamed) “recreational problems”¹¹ as “neck riddles” that display appurtenance to a particular craft carrying a particular body of know-how, (ii) to the possibility to use these problems (as eventually adopted into cultures leaving written

¹⁰ Karpinski’s closing commentary [1929: 177] to Jacopo da Firenze’s *abbaco* treatise, though preceding Zilsel’s paper, is characteristic of the attitude that prevailed afterwards:

[the early fourteenth-century] treatise by Jacob of Florence, like the similar [late fifteenth-century] arithmetic of Calandri, marks little advance on the arithmetic and algebra of Leonard of Pisa. The work indicates the type of problems which continued current in Italy during the thirteenth to the fifteenth and even sixteenth centuries, stimulating abler students than this Jacob to researches which bore fruit in the sixteenth century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan and Bombelli.

Obviously, Fibonacci, Jacopo, Calandri and Bombelli belong on *the same* branch, although part of it has undergone some degeneration.

¹¹ More precisely: the problems become “recreational” when adopted into literate culture; the term is only a misnomer in relation to their original function.

sources, thereby becoming properly “recreational”) as index fossils allowing us to trace an oral culture that in the nature of things is not directly documented in writing.¹²

In my [1990a], I still used the term “sub-scientific” about scribal as well as non-literate practitioners’ mathematics, singling out the former type as nothing but a sub-category. Schools – even pre-Modern schools teaching practical mathematics – certainly vary in character, and can be argued to constitute a pluri-dimensional continuum merging gradually into oral apprenticeship teaching on one side; but it is also difficult, even in several pre-Modern settings, to make a totally clean cut between schools teaching for practice and schools teaching “scientific” mathematics.¹³ I would therefore now distinguish between the *sub-scientific* knowledge type, carried by practitioners taught in an apprenticeship network; the “*scholasticized*” or *scribal* practitioners’ knowledge type, communicated in a school by masters whose own genuine practice is that of teaching, not the practical use of the knowledge they teach; and the “*scientific*” or *theory-oriented* type, the one to which historians of mathematics have dedicated most of their efforts – keeping in mind that these are fuzzy categories understood through ideal types functioning as navigational guides rather than classificatory boxes.¹⁴

Applications of the categorization

Networks of categories constitute an instance of formal knowledge (albeit of the most primitive kind). Their utility thus depends on their ability to create order in the tangle of real-world phenomena – those from which they were derived in the first instance through a process of abstraction (that should be the easier but still the obvious first test) as well as others that did not intervene when they were constructed (not necessarily quite as easy). I shall look at one instance of each kind.

When speaking for the first time of a “sub-scientific tradition” in my [1986] I referred to the tradition that linked Old Babylonian “algebra” to the area riddles in Abū Bakr’s *Liber mensurationum*. In my [1990b: 275]) I also voiced a suspicion that the problem BM 13901 #23 (dealing with a square, for which the sum of *the* four sides and the area is given) was “a surveyors’ recreational problem, maybe from a tradition that

¹² Also in the later 1980s, David King investigated the astronomy of Islamic legal scholars and pointed out that it was distinct from the astronomy of mathematicians. He used the term “folk astronomy” but left no doubt that it was the astronomy of the “craft” of legal scholars. See the papers contained in [King 1993].

¹³ See, for late Greco-Roman Antiquity, [Cuomo 2000].

¹⁴ Cf. [Høystrup 1997]. It might be useful to distinguish a fourth type, the “deuteronomic” teaching of theory petrified into and taught in school as a dignified tradition – the shape in which most of the students taught scientific mathematics encountered their Euclid since two thousand years; cf. [Netz 1998]. But since my topic is the relation between mathematical practice and mathematical theory I shall not pursue this theme at present.

was older than – perhaps even a source for – Old Babylonian scribal school ‘algebra’; I also observed the family resemblance of the configuration used in the solution with one of al-Khwārizmī’s proofs. However, at that moment I had to leave both matters there.

Over the following years, being alerted to the stylistic peculiarities that might characterize fresh borrowings from an oral tradition as well as to those that should correspond to transmission within a stable school environment (and being in general stimulated to be sensitive to stylistic detail and not only to so-called “mathematical substance”) I was able (that is at least my own opinion) to put on a firmer footing than done before the claim that Old Babylonian “algebra” and Euclidean “geometric algebra” (both “so-called”) were connected, and to demonstrate also that the geometric riddles of Arabic *misāha* treatises as well as al-Khwārizmī’s geometric proofs for the basic *al-jabr* procedures belonged within the same network. Moreover I could argue (still of course in my own opinion) that the Old Babylonian “algebraic” school discipline built upon original borrowings from the neck riddles of a lay surveyor’s environment, and that this environment and its riddles, not the tradition of scholar-scribes, was responsible for the transmission of the inspiration to later times.

Since I have described this analysis and its outcome at length elsewhere,¹⁵ I shall not go into further detail, but turn instead to a historical phase which I have only started looking seriously at some five or six years ago: the Italian *abbaco* school of the late Middle Ages and the Renaissance, and its relation to Leonardo Fibonacci.

Karpinski, who was one of the first to describe the stylistic peculiarities of an *abbaco* treatise (Jacopo da Firenze’s above-mentioned *Tractatus algorismi* from 1307, in Tuscan in spite of the Latin title, and written in Montpellier), though quite aware of its deviations from what can be read in Fibonacci’s *Liber abbaci*, still appraised its contents as if it was only a station on the road from Fibonacci to Scipione del Ferro (see note 10). At the moment little systematic work had been done on the *abbaco* material,¹⁶ but things did not change even when Gino Arrighi and his pupils had published an appreciable number of manuscripts. Wholly in Karpinski’s vein, Kurt Vogel stated in [1977: 3] that Cowley’s description of the Columbia ms X 511 A1 3 was important because it “filled a lacuna between Leonardo da Pisa’s *Liber abbaci* and Luca Pacioli’s *Summa*”. Even sharper are the formulations of those who have worked most intensely on the material – thus Warren Van Egmond [1980: 7], according to whom all *abbaco* writings “can be regarded as [...] direct descendants of Leonardo’s book”, and Raffaella

¹⁵ Most extensively in [Høystrup 2001] and [Høystrup 2002: 362–417].

¹⁶ Karpinski had described another *abbaco* algebra in [1910], and Elisabeth Buchanan Cowley a whole treatise in [1923]. During the nineteenth century a number of excerpts had been published by Libri, Boncompagni and others, but no coherent descriptions of whole treatises (nor *a fortiori* of the category as such) had appeared.

Franci and Laura Toti Rigatelli, who stated in [1985: 28] that “the abacus schools had risen to vulgarize, among the merchants, Leonardo’s mathematical works”. Quite recently, Elisabetta Ulivi – probably the scholar who has worked most in depth with the social history of the *abbaco* environment – expressed the view that the *abbaco treatises* “were written in the vernaculars of the various regions, often in Tuscan vernacular, taking as their models the two important works of Leonardo Pisano, the *Liber abaci* and the *Practica geometriae*” [2002: 10].

All of these, I would claim, have fallen victims to the “syndrome of The Great Book”, the conviction that every intellectual current has to descend from a *Great Book* that is *known to us* – the same conviction that made those who objected to Neugebauer’s proposed transmission observe that no Greek would have bothered to read the Babylonian clay tablets, and induced many of those who have discussed the possible borrowing of Indian material into Arabic algebra from the premise that borrowings had to come from the writings of an Āryabhata or a Brahmagupta.

Already Karpinski had noticed that Jacopo’s algebra has no problems in common with the *Liber abaci*. Reading of the whole treatise shows it to have no single problem, algebraic or otherwise, in common with the Great Book, but to contain on the other hand numerous problems belonging to classes that are also present in that Book. Some of these belong to the cluster of problems that are found in ancient and medieval sources “from Ireland to India” (in a phrase used by Stith Thompson [1946: 13] to characterize the “European folktale”) – and even in the Chinese *Nine Chapters*. This cluster of problems that usually go together was apparently carried by the community of merchants travelling along the Silk Road¹⁷ and adopted as “recreational problems” by the literate in many places; it is thus a good example of a body of sub-scientific knowledge influencing school knowledge systems in many places and an illustration of the principle that it is impossible to trace the “source” for a particular trick or problem in a situation where “the ground was wet everywhere” [Høyrup 1987: 290].

Other problem types are shared with Fibonacci but not diffused within the larger area (or diffused within a different larger area that may coincide with the Arabic network of sea trade from the Indian Ocean to the Mediterranean). Moreover, Jacopo employs a range of set phrases (“et così se fanno tucte le simile ragioni”, “se ci fosse data alcuna ragione”, etc.) that also turn up copiously in other *abbaco* writings as well as in similar writings from the Provençal-Catalan and the Castilian area¹⁸ – and also, but only at rare occasions, in Fibonacci’s text.

¹⁷ Some of the travelling problems deal precisely with bits of this web of caravan and sea routes extending from China to Cadiz, and no other network (however open-ended) existed that ranged so widely.

¹⁸ See the description of the Pamiers algorism in [Sesiano 1984], Francesc Santcliment’s *Summa de l’art d’aritmética* [ed. Malet 1998] and the Castilian *Arte del algarismo* [ed. Cauneo del Potro & Córdoba de la Llave 2000].

A slightly earlier Umbrian *abbaco* treatise (Florence, Riccardiana ms. 2404, from c. 1290, ed. [Arrighi 1989]) claims in its title to be “according to the opinion” of Fibonacci. Analysis of the text shows this claim to be misleading. Everything basic in the treatise is as different from what we find in the *Liber abbaci* as is Jacopo’s *Tractatus* (and characterized by the presence of the same set phrases); but the writer borrows a number of sophisticated problems from Fibonacci, often demonstrably without understanding even as much as the notation of his source. Obviously, Fibonacci had already become a kind of culture hero (modern historians are not the first to fall victims to the syndrome of The Great book), and the borrowings serve as embellishment beyond the ordinary teaching matters.

From combination of these pieces of evidence it becomes obvious that Jacopo’s as well as the Umbrian treatise refer to an environment spread out in all probability over much of the Romance-speaking Mediterranean region, already in possession of elementary vernacular literacy and probably based in some kind of school teaching similar to the Italian *abbaco* school but with at most tenuous ties to the world of university scholars. It also becomes clear that already Fibonacci had drawn part of his inspiration for the *Liber abbaci* from this environment, whose existence thus antedates 1200 (or at the very least 1228).

Analysis of Jacopo’s algebra chapter and comparison with Arabic algebraic writings suggests that it is ultimately drawn from another level of Arabic algebra than that of the Great Books of al-Khwārizmī, Abū Kāmil, ibn al-Bannā’, etc. It seems likely – but not yet conclusively established – that the just-mentioned school environment was not restricted to the Romance-speaking area but also reached into (and probably came from) a similar environment in the Arabic Mediterranean teaching *mu’āmalāt*-mathematics (even Arabic merchants must have learned their mathematics somewhere, including the use of the rule of three to which already al-Khwārizmī had dedicated the “Chapter on *mu’āmalāt*” of his *Algebra* [ed. Rosen 1831: Arabic 48]). That school in Bejaïa in which Fibonacci [ed. Boncompagni 1857: 1] tell to have spent “some days” learning the *studium abbaci* is likely to have been such a school (the alternative, a mosque school, is not plausible).¹⁹

Though in all probability a descendant of a school environment that had inspired both Fibonacci and Jacopo, the mature Italian *abbaco* school of the fourteenth and

¹⁹ Some of the formulations in Jacopo’s discussion of metrologies are strikingly similar to what we find in Aḥmad ibn Thabāt’s *Reckoners’ Wealth* from c. 1200 (*Ġunyāt al-Ḥussāb*, ed. [Rebstock 1993]), which however both surpasses what it would be reasonable to teach to practical reckoners (e.g., Euclidean geometric definitions) and offers too little training for these; but ibn Thābat was a scholar who taught law as well as *hadīth* and ‘*ilm al-ḥisāb* at the Nizāmiya madrasah [Rebstock 1993: x], and thus wrote a scholarly book about practitioners’ mathematics, no textbook for the training of merchant youth. Apart from his own intellectual pleasure, he may have been motivated by what (for instance) a judge had to understand about all domains of practical computation.

fifteenth century developed characteristics that are not likely to have been present before 1310 – characteristics that appear to have depended on the market competition between *abbaco* masters for jobs and pupils. Both the Umbrian *abbaco* and Jacopo’s treatise make mathematical mistakes from time to time – but they abstain from mathematical fraud. Already within the first two decades after Jacopo’s writing of (what is in all probability) the first Italian vernacular algebra, on the other hand, *abbaco* treatises begin to present blatantly false rules for irreducible equations of the third and fourth degree – not easily unmasked by competitors, however, because the examples are always chosen so as to lead to “solutions” containing radicals. Only at a moment when *abbaco*-trained writers like Luca Pacioli begin moving on the interface between the Humanist-courtly and the scholastic-scholarly areas²⁰ is the fraud exposed – and only then is there space for the genuine del-Ferro solution to contribute to the revolution in mathematics (in good agreement with the Zilsel thesis, we might say).

Italian *abbaco* mathematics is thus not to be understood as an activity bridging one Great Book (the *Liber abaci*) and another one (e.g., Cardano’s *Ars magna*) but as a distinct undertaking, carried neither by scholarly mathematicians nor by a purely oral culture, yet having most of its ultimate roots in an environment of the latter type, and giving eventually important stimuli to the further development of scientific mathematics. I permit myself to claim that the categorization suggested above is fruitful in opening our eyes to evidence in the sources that has so far been overlooked, and thus to understanding better the real historical process. At the same time the example demonstrates that a seemingly simple category (“schools”) covers phenomena of widely different character, held together mainly by being neither orally based nor “scientific” in ambition.²¹

References

- Arrighi, Gino (ed.), 1989. “Maestro Umbro (sec. XIII), *Livro de l’abbecho*. (Cod. 2404 della Biblioteca Riccardiana di Firenze)”. *Bollettino della Deputazione di Storia Patria per l’Umbria* **86**, 5–140.
- Biagioli, Mario, 1989. “The Social Status of Italian Mathematicians, 1450–1600”. *History of Science* **27**, 41–95.
- Boncompagni, Baldassare (ed.), 1857. *Scritti di Leonardo Pisano matematico del secolo decimoterzo*. I. *Il Liber abaci di Leonardo Pisano*. Roma: Tipografia delle Scienze Matematiche e Fisiche.
- Busard, Hubert L. L. (ed.), 2001. *Johannes de Tinemue’s Redaction of Euclid’s Elements, the So-Called Adelard III Version*. 2 vols. (Boethius, 45,1–2). Stuttgart: Franz Steiner.

²⁰ That Luca moves in this zone is quite obvious, e.g., both from his preface to the *De divina proportione* and from his publication of the Campanus version of the *Elements*.

²¹ This point could be sharpened if the *abbaco* school were contrasted, e.g., with the Old Babylonian scribe school, which eliminated mathematical fraud (namely, mock solutions) from its sub-scientific heritage. Analysis of what happens to a specific problem type, e.g., the “hundred fowls”, might highlight the difference between the genuinely sub-scientific and the *abbaco*-school style.

- Caunedo del Potro, Betsabé, & Ricardo Córdoba de la Llave (eds), 2000. *El arte del algarismo*. Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV. (Ms. 46 de la Real Colegiato de San Isidoro de León). Salamanca: Junta de Castilla y León, Consejería de Educación y Cultura.
- Cowley, Elisabeth Buchanan, 1923. "An Italian Mathematical Manuscript", pp. 379–405 in Christabel F. Fiske (ed.), *Vassar Medieval Studies*. New Haven. Not seen.
- Cuomo, Serafina, 2000. *Pappus of Alexandria and the Mathematics of Late Antiquity*. Cambridge: Cambridge University Press.
- Franci, R., & L. Toti Rigatelli, 1985. "Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli". *Janus* 72, 17–82.
- Graham, Loren R., 1993. *Science in Russia and the Soviet Union. A Short History*. (Cambridge History of Science). Cambridge: Cambridge University Press.
- Høyrup, Jens, 1984. "Formative Conditions for the Development of Mathematics in Medieval Islam". Contribution to the George Sarton Centennial, University of Ghent, 14–17 November 1984. Preliminary version. Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science.
- Høyrup, Jens, 1986. "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra". *Erdem* 2 (Ankara), 445–484.
- Høyrup, Jens, 1987. "The Formation of 'Islamic Mathematics'. Sources and Conditions". *Science in Context* 1, 281–329.
- Høyrup, Jens, 1990a. "Sub-Scientific Mathematics. Observations on a Pre-Modern Phenomenon". *History of Science* 28, 63–86.
- Høyrup, Jens, 1990b. "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". *Altorientalische Forschungen* 17, 27–69, 262–354.
- Høyrup, Jens, 1997. "Mathematics, Practical and Recreational", pp. 660–663 in Helaine Selin (ed.), *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*. Dordrecht etc.: Kluwers.
- Høyrup, Jens, 2001. "On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six 'Algebras'". *Science in Context* 14, 85–131.
- Høyrup, Jens, 2002. *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*. (Studies and Sources in the History of Mathematics and Physical Sciences). New York: Springer.
- Karpinski, Louis C., 1910. "An Italian Algebra of the Fifteenth Century". *Bibliotheca Mathematica*, 3. Folge 11 (1910–11), 209–219.
- Karpinski, Louis C., 1929. "The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307". *Archeion* 11, 170–177.
- King, David A., 1993. *Astronomy in the Service of Islam*. (Variorum Collected Studies Series). Aldershot, England/Brookfield, Vt.: Variorum.
- Malet, Antoni (ed.), 1998. Francesc Santcliment, *Summa de l'art d'Aritmètica*. Vic: Eumo Editorial.
- Montucla, J. F., 1799. *Histoire des mathématiques*. 4 vols (III-IV achevés et publiés par Jérôme de la Lande). Paris: Henri Agasse, an VII – an X [1799–1802].
- Netz, Reviel, 1998. "Deuteronomic Texts: Late Antiquity and the History of Mathematics". *Revue d'Histoire des Mathématiques* 4, 261–288.
- Ramus, Petrus, 1569. *Scholarum mathematicarum libri unus et triginta*. Basel: Eusebius Episcopus.
- Raven, Diederick, & Wolfgang Krohn, 2000. "Edgar Zilsel: His Life and Work (1891–1944)", pp. xxi–lxii in Edgar Zilsel, *The Social Origins of Modern Science*, ed. Diederick Raven, Wolfgang Krohn & Robert S. Cohen. (Boston Studies in the Philosophy of Science, 200). Dordrecht etc.: Kluwer.
- Rebstock, Ulrich, 1993. *Die Reichtümer der Rechner (Ġunyāt al-Hussāb) von Ahmad b. Ṭabāt (gest. 631/1234). Die Araber – Vorläufer der Rechenkunst*. (Beiträge zur Sprach- und Kulturgeschichte des Orients, 32). Walldorf-Hessen: Verlag für Orientkunde Dr. H. Vorndran.
- Rosen, Frederic (ed., trans.), 1831. *The Algebra of Muhammad ben Musa*, Edited and Translated. London: The Oriental Translation Fund.

- Schmeidler, Felix (ed.), 1972. Joannis Regiomontani *Opera collectanea*. Faksimiledrucke von neun Schriften Regiomontans und einer von ihm gedruckten Schrift seines Lehrers Purbach. Zusammengestellt und mit einer Einleitung herausgegeben. (Milliaria X,2). Osnabrück: Otto Zeller.
- Science at the Cross Roads*. Papers Presented to the International Congress of the History of Science and Technology [...] 1931. With a New Foreword by Joseph Needham and a New Introduction by P. G. Werskey. London: Frank Cass, 1971.
- Sesiano, Jacques, 1984. "Une arithmétique médiévale en langue provençale". *Centaurus* 27, 26–75.
- Thompson, Stith, 1946. *The Folktale*. New York: The Dryden Press.
- Thureau-Dangin, F., 1938. *Textes mathématiques babyloniens*. (Ex Oriente Lux, Deel 1). Leiden: Brill.
- Ulivi, Elisabetta, 2002. "Benedetto da Firenze (1429–1479), un maestro d'abbaco del XV secolo. Con documenti inediti e con un'Appendice su abacisti e scuole d'abaco a Firenze nei secoli XIII–XVI". *Bollettino di Storia delle Scienze Matematiche* 22, 3–243.
- Van Egmond, Warren, 1980. *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abacus Manuscripts and Printed Books to 1600*. (Istituto e Museo di Storia della Scienza, Firenze. Monografia N. 4). Firenze: Istituto e Museo di Storia della Scienza.
- Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13)*. (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München.
- Wolff, Christian, 1716. *Mathematisches Lexicon*. Leipzig: Joh. Friedrich Gleditschens seel. Sohn. Reprint Hildesheim 1965 (Gesammelte Werke. I. Abteilung: Deutsche Schriften, Band 11).
- Zilsel, Edgar, 1942. "The Sociological Roots of Science". *American Journal of Sociology* 47, 544–562.