

Zbl 1086.01007**Gardies, Jean-Louis****On the modes of existence of mathematical objects. (Du mode d'existence des objets de la mathématique.)** (French)

Problèmes et Controverses. Paris: Librairie Philosophique J. Vrin. 152 p. EUR 24.00 (2004).

The present work consists of six chapters, the first five of which function rather as mental preparation than as foundation for the conclusions drawn in the last one. Irrespective of that role, these chapters are interesting in themselves.

Chapters 1 and 2 discuss Euclidean geometry and arithmetic as essentially dealing with first-order objects. This characteristic was already pointed out by Jakob Klein in 1936, but Gardies makes several important further observations in particular (1) that the contorted definition of the equality of ratios in *Elements V* provides a first-order definiens for what is essentially a second-order definiendum. Further, (2) that the numbers of *Elements VII* and *IX*, defined as “[classes of congruent] collections of units”, are not properly of the first order; but they are treated as if they were, as legitimated a posteriori by a principle enunciated by Russell and Whitehead [here retranslated from the French], “we may replace the type of individuals by any other type, provided we make corresponding changes for all other types”. However, one may doubt that the latter proviso is thoroughly respected by “the Euclidean author” (Gardies’ cautious expression), given that the passage “[classes of congruent]” has to be inserted into Euclid’s definition, a difficulty which Gardies does not consider. In any case, the price to be paid for keeping both disciplines at the first order is that they cannot properly communicate; on the whole, Euclid takes care to pay this price, even though (as Gardies points out) *Elements X* does combine geometry and arithmetic.

Chapter III investigates Descartes’ *Géométrie* from a related point of view; most important is the analysis of book II, whose curves (by way of the underlying functions and equations with several unknowns which ipso facto become variables) are higher-order objects.

Chapter IV is a semiotic discussion of the relation between the mainly phonetic writing of ancient mathematics and the ideographic writing of post-Viète-Descartes mathematics, leading to an exploration of the possibility of operating with such ideographic representatives for mathematical objects whose existence (or mere possibility) has not (yet) been ascertained.

Chapter V discusses axiomatics as a genre containing several species, distinguishing these types:

1. Cases like Euclidean geometry based on the full set of Hilbert axioms, where the only basis for truth is syntactic.
2. Cases like propositional calculus, where truth can alternatively be decided on a semantic base (truth tables, in the example).
3. Cases like predicate calculus and arithmetic, where the approach through syntax is by necessity incomplete with respect to their semantic reach.
4. Incomplete axiomatizations like group theory applying to widely differing structures, derivable by reduction from complete type-1 axiomatizations of these.

Chapter VI takes up the problem of existence, confronting the (supposedly “abusive”) view that, e.g., everything beyond the natural integers is a human/mathematicians construct with the (supposedly true) insight that also higher-order objects really exist. The argument is made abstractly, even allusively, but follows this example: If the natural numbers exist, then so does the set of pairs (p, q) of such numbers, as well as the equivalence classes between such sets corresponding to the set of integers; and so on, for rational, real and complex numbers as well as functions of these. If unrestricted, this approach leads directly to all the familiar set-theoretical paradoxes. Gardies does not

point that out, but he has clearly seen it, since he later restricts himself to a “countable infinity of modes of existence” (p. 137), specifying afterwards (p. 138) that there is “at least the same infinity of possible modes of existence as the modes of existential quantification admitted a priori by the constitutive grammatical rules of the theory of types”. Since not all variants of the theory of types eliminate all the paradoxes, it is not clear whether Gardies natural-language restriction is really sufficient; on the other hand, the reference to possible modes of existence appears to come close to an admission that the infinity of levels is only a potential infinity, vindicating thus Dedekindian, perhaps even Kroneckerian constructivism.

Though thus not fully convincing in all respects, the book should make stimulating reading for anyone interested in its topic(s).

Jens Høyrup (Roskilde)

Keywords : Axiomatics; Theory of types; Euclid; Descartes

Classification :

- *00A30 Philosophy of mathematics
- 03A05 Philosophical and critical
- 01A20 Greek or Roman mathematics
- 01A45 Mathematics in the 17th century
- 01-02 Research monographs (history)