Fibonacci – Protagonist or Witness?
Who Taught Catholic Christian Europe about Mediterranean Commercial Arithmetic?

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Paper presented at the workshop
Borders and Gates or Open Spaces?
Knowledge Cultures in the Mediterranean
During the 14th and 15th Centuries
Departamento de Filosofia y Lógica
Universidad de Sevilla
17–20 December 2010

Preprint
26 December 2010

Abstract

Fibonacci during his boyhood went to Bejaïa, learned about the Hindu-Arabic numerals there, and continued to collect information about their use during travels to the Arabic world. He then wrote the *Liber abbaci*, which with half a century's delay inspired the creation of Italian abbacus mathematics, later adopted in Catalonia, Provence, Germany etc.

This piece of conventional wisdom is well known – too well known to be true, indeed. There is no doubt, of course, that Fibonacci learned about Arabic (and Byzantine) commercial arithmetic, and that he presented it in his book. He is thus a witness (with a degree of reliability which has to be determined) of the commercial mathematics thriving in the commercially developed parts of the Mediterranean world. However, much evidence – presented both in his own book, in later Italian abbacus books and in similar writings from the Iberian and the Provençal regions – shows that the *Liber abbaci* did not play a central role in the later adoption. Romance abbacus culture came about in a broad process of interaction with Arabic non-scholarly traditions, interaction at first apparently concentrated in the Iberian region.

Benno Artmann in memoriam

A disclaimer

In these times of rampant publicity and rampant legal complaints, it is not uncommon to run into disclaimers explaining in small print what the wonderful product does *not* promise.

Let me start by stating, in normal font size however, that the following offers *elements* that have to go into a synthetic answer, but too few and too disparate to allow the construction of this synthesis.

Fibonacci's supposed role

In the introduction to Fibonacci's *Liber abbaci*, we read the following:[1]

> After my father's appointment by his homeland as state official in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and, in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days. There, following my introduction, as a consequence of marvelous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business, I pursued my study in depth and learned the give-and-take of disputation.

This can be combined with the prevalent idea about the origin of Italian abbacus mathematics, for instance as expressed recently by Elisabetta Ulivi [2004: 44][2] in her explanation that

> the name "abbacus school" designates those secondary-level schools that were essentially dedicated to practical arithmetic and geometry and were in the tradition of Leonardo Pisano's *Liber abbaci* and *Practica geometriae*.

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[1] I quote Richard Grimm's translation [1976: 100], based on a critical edition of the introduction based on all the manuscripts that contain it. On the point where all the other known manuscripts differ from the one on which Baldassare Boncompagni based his edition [1857] (namely whether Fibonacci only speaks of travels to business places or of business travels), Grimm's text is confirmed by Benedetto da Firenze's quotation of the passage (Siena, Biblioteca degli Intronati, L IV 21, ed. [Arrighi 2004: 156]).

[2] My translation, as everywhere else in the following unless a translator is mentioned.

Similarly, Warren Van Egmond asserted in [1980: 7] that all abbacus writings “can be regarded as [...] direct descendants of Leonardo’s book”.[3]

If the Fibonacci-quotation (in particular in the usual reading, where neither “Greece” nor Provence are noticed) is combined with the opinion expressed by Ulivi and Van Egmond, then it becomes clear that Fibonacci’s *Liber abbaci* was the gate through which practical arithmetic was transmitted from the Arabic world to Italy (and from there to the rest of Christian Europe[4]).

In 1997–98, I discovered that this story is impossible if we look at the specific case of abbacus *algebra*, but apart from that I followed Descartes’ strategy as set forth in the *Discours de la méthode* [ed. Adam & Tannery 1897: VI, 23], to observe the customs and opinions of those among whom I lived until close analysis of the matter would force me to change my observances. When subsequently I *was* forced to do so, I started thinking about the origin of the conventional wisdom.

Part of the explanation is of course that it is easier to look for the lost doorkey within the cone of the street lamp than outside it, in the darkness – to which comes what at another occasion [Høyrup 2003: 10] I have called “the syndrome of the great book”, namely “the conviction that every intellectual current has to descend from a *Great Book* that is *known to us*”.

The only apparently positive evidence comes from a *Livero de l’abbecho* from c. 1300 (Florence, Riccardiana ms. 2404, ed. [Arrighi 1989]) which claims in its first line to be “according to the opinion” of Fibonacci.[5] Close analysis of the treatise [Høyrup 2005] shows, however, that this evidence is fallacious. The text moves on two distinct levels, one elementary, the other advanced. The elementary level corresponds to the curriculum of the abbacus school as we know it from two documents.[6] Here we find the rule of three; metrological shortcuts; exchange and barter; elementary alligation; simple interest and elementary composite

[3] More examples, also drawn from respected colleagues, are quoted in [Høyrup 2005: 24–26] and [Høyrup 2007: 30 n.69].

[4] Here and everywhere in the paper, “Christian Europe” refers narrowly to *Catholic* Christian Europe.

[5] “Quisto ène lo livero de l’abbecho secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa” [ed. Arrighi 1989: 9]. The date of the manuscript is discussed in [Høyrup 2005: 27 n.5, 47 n.37].

[6] One [ed. Arrighi 1967] is from the earlier fifteenth, the other [ed. Goldthwaite 1972: 421–425 n.15] from the early sixteenth century; however, nothing suggests the curriculum to have been reduced in the meantime (nor broadened, for that matter).

interest.[7] As can be seen both from the absence of shared problems and from the way mixed numbers are written, this level is fully independent of Fibonacci – except for a misshaped compromise between the normal writing of mixed concrete numbers and Fibonacci's notation for pure mixed numbers, on which below, note 21. On the other hand, everything on the advanced level is borrowed from the *Liber abbaci* (excepting a final chapter containing mixed sophisticated problems, some of which come from other sources), often demonstrably without understanding.[8] The Fibonacci material thus serves as adornment; it is quite fitting that the copy we possess is a beautiful manuscript on vellum. It follows that the *Liber abbaci* was famous a small century after it was written, and Fibonacci's name a superb embroidered cloak in which the abbacus author in question found it convenient to wrap his book; but also that what this author taught in the abbacus school, and the mathematics he understood, had a different basis.

So far, this concerns a single author (better perhaps, compiler), albeit one of the two earliest abbacus authors whose work has reached us.[9] However, no other abbacus author raises similar claims except a fifteenth-century encyclopedia where the claim is even more misleading,[10] and no other author offers material directly copied from the *Liber abbaci*, except Benedetto da Firenze and a near-contemporary of his, who copy whole sections (the algebra, and chapter 15 part 1, on proportions), but whose own work remains independent of Fibonacci and well within the current abbacus tradition.[11] The situation is slightly different

[7] The curriculum also encompassed the Hindu-Arabic number system with appurtenant calculation, which Fibonacci is often supposed to have brought to Italy. This is absent from the treatise. I shall not discuss Fibonacci's role in this domain, just point out that even here there is no positive evidence that his influence was important

[8] Fibonacci's composite fractions are understood as normal fractions, which implies that the compiler can never have followed those numerous calculations where they occur. The occasional algebraic *cosa* of which Fibonacci makes use when applying the *regula recta* (first-degree algebra) is either skipped, or the role of this "thing" as a representative of an unknown quantity is not understood.

[9] The other early book is the "Columbia algorism", on which below. On the plausible re-dating of the original of this treatise (of which we possess a fourteenth-century copy) to the years 1285–90, see [Høyrup 2007: 31 n. 70].

[10] Vatican, Ottobon. lat. 3307, which presents itself (fol. 1^r) as *Libro di praticha d'arismetrica, cioè fioretti tracti di più libri facti da Lionardo pisano.*

[11] Benedetto's original autograph of his *Praticha d'arismetricha* from 1463 is contained in Siena, Biblioteca degli Intronati, L IV 21; a detailed description is [Arrighi 2004: 129–159];

as regards Fibonacci's *Pratica geometrie*, inasfar as three fourteenth- and fifteenth-century treatises were drawn from it [Hughes 2010].[12] However, normal abbacus geometries borrow nothing directly (and plausible very little indirectly) from Fibonacci.

Then whence? Algebra as initial evidence

This was the negative part of my argument, which invites a search for alternative gates (or even "open spaces").

I shall start where my own exploration began, with algebra. None of the two earliest texts contain any algebra – and the compiler of the *Livero*, as we have seen, does not even know enough about the topic to recognize an algebraic *cosa*. The earliest abbacus algebra is contained in the Vatican manuscript (Vat. lat. 4826) of Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307.[13]

On all accounts (except that it deals with the six fundamental first- and second-degree "cases", but then not only with these, and that it uses the term *censo* for the second power), this algebra differs fundamentally not only from Fibonacci's algebra but also from the Latin translations of al-Khwārizmī – see [Høyrup 2007: 147–159]. It is also very different from Abū Kāmil's algebra, and

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the other "abbacus encyclopedia", slightly later and anonymous (Florence, Bibl. Naz., Palatino 573; also original autograph), is described in detail in [Arrighi 2004: 161–195]. The evidence that both manuscripts are their respective author's original autograph's is presented in [Høyrup 2009a: 28, 34].

It is possible (and even plausible) that both draw their copy of Fibonacci from Antonio de' Mazzinghi's lost *Gran trattato* from the later fourteenth century. But even Antonio's own algebra (as we know it from extracts in the two encyclopediae that were just mentioned) owes nothing to Fibonacci.

[12] Fibonacci's *Pratica* is also used so faithfully in Luca Pacioli's *Summa* [1494] that misprinted letters in Pacioli's diagrams can often be corrected by means of Boncompagni's edition of the *Pratica* [1862]!

[13] The Vatican manuscript can be dated by watermarks to c. 1450. However, stylistic analysis strongly suggests that its algebra belongs together with the rest of the treatise, and that the two manuscripts from which the algebra section is absent are secondary redactions – see [Høyrup 2007: 5–25]. Van Egmond [2009] rejects this conclusion, but with arguments that are refuted by his own earlier publications (and by the sources to which he appeals) – see [Høyrup 2009b]. In any case, the Vatican algebra must belong to the earlier fourteenth century. Moreover, other abbacus writing from the earlier fourteenth century share those of its characteristics that enter in the present argument; for our actual purpose, the identity of its author, and even the question whether it is really the earliest abbacus presentation of algebra, are therefore immaterial.

has only few features in common with al-Karajī's *Kāfī*. Its ultimate root is obviously Arabic algebra. However, its closest Arabic source cannot be any of the erudite treatises that have come down to us; instead, we must think of a mathematical practice in which algebra and commercial calculation were merged.

Moreover, its closest Arabic source cannot be the *immediate* source. Technical works translated directly from the Arabic always contained Arabic loanwords for some of their technical terms; however, no such terminological borrowings are present in the Vatican (or other early abbacus) algebra. The *immediate* source must hence be an environment where algebra was already spoken of in a Romance language. Since Jacopo wrote his treatise in Montpellier (located in Provence, but politically belonging to the Aragon-Catalan kingdom), this environment can reasonably be assumed to have been situated somewhere in the Ibero-Provençal area.

That observation brings to mind Fibonacci's claim to have also learned about the use of the Hindu-Arabic numeral system *in Provence*, a claim that mostly goes unnoticed. Indeed, if fifteenth-century Provencal mathematics of the abbacus type took its inspiration from Italy, and the Italians had their practice from Fibonacci, what could Fibonacci have learned in Provence? One at least of the premises for this paradox has to be given up.

One early manuscript of the *Liber abbaci*[14] contains another reference to an unexpected locality: according to [Boncompagni 1851: 32], the ninth chapter does not simply begin with the words *Incipit capitulum nonum de baractis mercium atque earum similium* as found in [Boncompagni 1857: 118][15] but *Hic incipit magister castellanus. Incipit capitulum nonum de baractis mercium atque earum similium*. It is difficult to see why a copyist should insert a claim that a certain chapter was copied from a Castilian master if the claim was not in his original; if he did, it would at least show that he knew about such a Castilian treatise and believed to recognize its contents in Fibonacci's text. The passage thus offers evidence of Castilian writing on barter, probably before 1228 (or even 1202, the date of the first version of the *Liber abbaci*, now lost), and in any case before the end of the thirteenth century.

[14] Vatican, Palatino 1343. This manuscript is from the late thirteenth century and thus one of the two oldest manuscripts (and older than the one used by Boncompagni for his edition.

[15] Henceforth, I shall refer to the *Liber abbaci* by simple page number, always referring to [Boncompagni 1857].

Direct evidence for Iberian algebra integrated with commercial arithmetic goes back to the twelfth century – but to Arabic practice. I refer to the *Liber mahamaleth*,[16] a twelfth-century Latin treatise whose title points to Arabic commercial mathematics (*muʿāmalāt*). A systematic presentation of algebra is lost in all manuscripts, but repeatedly referred to; it may (but need not) be drawn from al-Khwārizmī. There are further repeated references to Abū Kāmil. However, there are also algebraic problems of a kind which we do not find in al-Khwārizmī nor in Abū Kāmil: problems involving the square root of profit and of capital [Sesiano 1988: 80] and the square root of a wage [Sesiano 1988: 83]; such problems, though not very common, also turn up in abbacus algebra, starting with Jacopo (assuming that the Vatican algebra is really his).

There is some evidence for further influence from the Maghreb on fourteenth-century developments in abbacus algebra.[17] Firstly, apparent setoffs of the incipient symbolism developed in the Maghreb in the later twelfth century turn up in various Italian manuscripts in the course of the fourteenth century, (whereas Jacopo's algebra is totally deprived of symbolism) – see [Høyrup 2009a: 16–25]. The scattered character of these setoffs suggest interaction within an open space during the first half of the century, interaction about which we are however unable to say any more. After the mid-century, Italian abbacus algebra appears to develop largely on its own premises, within its own closed space[18] (developing quite slowly, it must be said).

A *Tratato sopra l'arte della arismetricha*, written in Florence in c. 1390 (Bibl. Naz. Centr., fondo princ. II.V.152) contains an extensive algebraic section [ed. Franci & Pancanti 1988], which suggests a last case of (direct or indirect) inspiration from Arabic algebra. Firstly, in a wage problem, an unknown amount of money is posited to be a *censo*; Biagio *il vecchio* as quoted by Benedetto da Firenze [ed. Pieraccini 1983: 89*f*] had already presented the same problem before

[16] As long as Anne-Marie Vlasschaert's dissertation remains unpublished, the best description is [Sesiano 1988]; the most complete manuscript is Paris, BNF Lat. 7377A. Manuscript references for the points made here are in [Høyrup 2009a: 9].

[17] Since key figures like ibn al-Yāsamin were active on both sides of the Gibraltar strait, here and elsewhere I use "Maghreb" in the original sense, indicating the whole Islamic West including al-Andalus.

[18] Or even within a plurality of fairly closed spaces: schemes for calculation with polynomials, though present in some manuscripts before the mid-fourteenth century, are not even mentioned in the Florentine school tradition culminating in Benedetto da Firenze's *Trattato de praticha d'arismetrica* from 1463 and referring back to Biagio il Vecchio, Paolo dell'abbaco and Antonio de' Mazzinghi.

c. 1340, though positing the money to amount to a *cosa*. However, the present author does not understand that a *censo* can be a simple amount of money, and therefore feels obliged to find its square root – and then finds the solution as the square on this square root. The author hence cannot himself be familiar with the Arabic meaning of *māl*, nor can he however have taken it from Biagio. He thus uses the terminology without understanding it, and therefore cannot have invented it himself. On the other hand, this rather characteristic problem could not be shared with Biagio if the author's inspiration did not come from the same area – ultimately from the Maghreb, immediately from somewhere in the Romance Ibero-Provençal region.

Another highly plausible borrowing from Maghreb algebra in the same treatise is a scheme for the multiplication of three-term polynomials[19] which emulates the algorithm for multiplying multidigit numbers; the text itself justly refers to the multiplication *a chasella* [ed. Franci & Pancanti 1988: 11]. The "Jerba manuscript" of ibn al-Hāʾim's *Šarḥ al-Urjūzah al-Yasmīya* does something very similar [Abdeljaouad 2002: 33].

Since these two borrowings occur in the same manuscript and nothing else from the period which I know of suggests any recent contact, interaction through a single gate seems more likely that exchanges in an open space.

I shall say little about an episode in the reception of Arabic algebra which goes back to the earlier thirteenth century but had negligible impact. Benedetto refers in his *Trattato* [ed. Salomone 1982: 1] to a translation made by Guglielmo de Lunis (otherwise known as a translator of Aristotle); Raffaello Canacci [ed. Procissi 1954: 302] is more explicit, and speaks of a translation of "La regola dell'algibra" by Guglielmo "d'arabicho a nostra linghua". In 1521, Francesco Ghaligai copies Canacci [Karpinski 1910: 209], but with reference also to Benedetto; other features of his text confirm that he is familiar with both versions of the story [Høyrup 2008: 38]; finally, one manuscript of Gherardo's translation of al-Khwārizmī [Hughes 1986: 223] claims to represent Guglielmo's translation, the existence of which is thus confirmed, even though the ascription itself is obviously wrong.

It has been proposed that translation into "our language" should be understood as "into Latin", and in particular that Guglielmo's translation be identical with the version found in the manuscript Oxford, Bodleian, Ms. Lyell

[19] Earlier manuscripts only present schemes (wholly different in character) for the multiplication of binomials.

52.[20] This idea can be safely discarded, since all our evidence (apart from the erroneous ascription) lists a number of Arabic terms together with Italian explanations; of these terms and explanations there are no traces in the Latin manuscript, which furthermore translates al-Khwārizmī's technical terms differently.

One of the Arabic terms is *elchal*, which according to the explanation must stand for *al-qabila*. As observed by Ulrich Rebstock (personal communication), the disappearance of the *b* indicates an Ibero-Arabic pronunciation. Apart from this very unspecific confirmation of the role of the Iberian (but probably Islamic-Iberian) environment, nothing is known about this lost translation – apart from a vague possibility that Fibonacci's occasional use of *avere* instead of *census* in the algebra section of the *Liber abbaci* could be borrowed from Guglielmo.

The "Columbia algorism"

The "Columbia algorism" (Columbia X 511 A13) is a fourteenth-century copy of a late thirteenth-century original (cf. note 9). It is interesting in the present context for several reasons.

Firstly, it makes (sparse) use of a notation for ascending continued fractions, known in Christian Europe primarily from Fibonacci's writings. For instance (p. 155), Fibonacci would write $\frac{9\ \ 5}{25\ 12}16$ where our notation would be $16+\frac{5}{12}+\frac{9}{12\cdot 25}$ (Fibonacci's fractions may be much longer).[21] The Columbia algorism does not write mixed numbers with the fraction to the left, nor does it follow the corrupted usage of the *Livero*.[22] However, it does use the notation for continued fractions, sometimes written from right to left (the Maghreb way), sometimes from left

[20] Without adopting the thesis, Wolfgang Kaunzner [1985: 10–14] gives an adequate survey.

[21] This is the notation which the *Livero* (above, p. 3) mixes up with the normal notation for mixed concrete numbers, writing for instance [ed. Arrighi 1989: 18] "d. $\frac{17}{49}$7 de denaio", "*denari* $\frac{17}{49}$7 of *denaro*" where his source must have had "7 denari, $\frac{17}{49}$ de denaro" or "denari 7, $^{17}/_{47}$ de denaro").

Both the notation for ascending continued fractions and the habit to write the fractional part of a mixed number to the left are borrowings from the Maghreb, where they were created in the twelfth century.

[22] For instance, in #4 we find "9 e $\frac{1}{2}$" and "8$\frac{3}{4}$", and in #23 "d 11 e $\frac{22}{25}$ di d" [Vogel 1977: 33, 51].

to right (an adaptation to the European writing direction).[23] Since nothing else in the treatise points toward Fibonacci (and since Fibonacci's continuous fraction line is broken here into two), we may safely assume that he is not the source.

Two other features of the treatise suggest an Iberian connection. Firstly, one of its problems is an atypical use of the dress of a purse. Usually, the purse is found by several persons, which gives rise to a complicated set of linear conditions; what we find in the Columbia algorism [ed. Vogel 1977: 122] is much simpler (and analogous to what Fibonacci (p. 173) calls "tree problems", in accordance with the usual dress for this problem type): "Somebody had *denari* in the purse, and we do not know how many. He lost $^1/_3$ and $^1/_5$, and 10 *denari* remained for him". The same problem, only with the unlucky owner of the purse being "I" and the remaining *dineros* being only 5, is found in the *Libro de arismética que es dicho alguarismo*, an undated Castilian treatise known from a copy from 1393 (ed. Caunedo del Potro, in [Caunedo del Potro & Córdoba de la Llave 2000: 167]). Both treatises, moreover, solve the problem by way of a counterfactual question, "If 7 were 10 [respectively 5], what would 15 be?". This is the standard approach of the Columbia Algorism as well as the Castilian treatise, but not of other Italian treatises; since the Columbia Algorism appears not to have been widely known, the problem type is most likely to have circulated in the Ibero-Provençal area and to have been borrowed from there into the Columbia algorism, even though the opposite passage cannot be excluded.

The rule of three

The second characteristic feature of the Columbia algorism, on the other hand, can be quite safely attributed to Iberian (or at least Ibero-Provençal) influence: the way the rule of three is dealt with.[24]

The initial pages of the Columbia algorism are missing; if the rule of three was presented here, we cannot known in which terms this was done. However,

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[23] In #39, $\frac{1}{4}\frac{1}{2}$ stands for $^5/_8$, and $\frac{3}{4}\frac{1}{2}$ for $^7/_8$ – but in #60, $\frac{1}{4}\frac{1}{2}$ stands for $^3/_8$. In #39, moreover, $\frac{1}{gran}\frac{1}{2}$ stands for $1\,^1/_2$ gran [ed. Vogel 1977: 64, 81].

[24] The "rule of three" is *a rule*, and to be kept apart from the sort of *problems* (problems of proportionality, "to *a* corresponds *b*, to *c* corresponds what) to which it is applied. The rule can be identified through the order of operations to be performed: "first multiply *b* and *c*, then divide by *a*". The intermediate result *bc* has no concrete meaning, whereas the intermediate results of the alternatives (division first) have a concrete interpretation; either "to 1 corresponds $^b/_a$" or "to *c* corresponds $^c/_a$ times as much as to *a*".

all references to the rule within problems are of the kind also encountered in the problem just quoted, through counterfactual questions, “if *a* were *b*, what would *c* be?”.

Such questions are not absent from other Italian treatises. However, they always occur as secondary examples of the rule of three, after problems confronting two different species of coin, or coin and commodity – or they are found wholly outside the presentation of the rule of three. The *Livero* [ed. Arrighi 1989: 14], for instance, introduces them separately and at a distance from the rule of three (its very first topic) as a “rule without a name”. In all Ibero-Provençal treatises I have inspected,[25] however, such counterfactual questions (or related abstract number questions like “if $4^1/_2$ are worth $7^2/_3$, what are $13^3/_4$ worth?”) always provide the first and basic exemplification of the rule of three.

This observation leaves little doubt about the dependence of the Columbia algorism on an Iberian (or Ibero-Provençal) model, since standard Arabic treatises introduce the rule in a wholly different way. However, the rule of three has much more to say about our topic.

The earliest statement of the rule is found in the *Vedāṅgajyotiṣa* [Sarma 2002: 135], cautiously to be dated to c. 400 BCE [Pingree 1978: 536]. In Kuppanna Sastry’s translation as quoted by Sarma, this version of the rule runs

> The known result is to be multiplied by the quantity for which the result is wanted, and divided by the quantity for which the known result is given.

The reference to “the result that is wanted” has some similarity to what we find in the abbacus books – for instance, in Jacopo’s *Tractatus*,[26]

> If some computation should be given to us in which three things were proposed, then we should always multiply the thing that we want to know against that which is not similar, and divide in the third thing, that is, in the other that remains.

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[25] That is, beyond the just-mentioned *Libro ... dicho alguarismo*, in chronological order: the “Pamiers algorism” from c. 1430 [Sesiano 1984a]; the anonymous mid-fifteenth franco-Provençal *Traicté de la pratique* [ed. Lamassé 2007]; Barthélemy de Romans’ slightly later, equally Franco-Provençal *Compendy de la praticque des nombres* [ed. Spiesser 2003]; Francesc Santcliment’s Catalan *Summa de l’art d’Aritmètica* from 1482 [ed. Malet 1998]; and Francés Pellos’ *Compendion de l’abaco* from 1492 [ed. Lafont & Tournerie 1967: 103–107]. The Pamiers algorism, the *Traicté* and the *Compendy* are connected, but the others are independent of each other and of this group.

[26] From [Høyrup 2007: 236*f*], with correction of an error (“in the third thing” instead of “in the other thing”).

It is not clear from Sarma's quotation (but unlikely from the context of his discussion) whether already the *Vedāṅgajyotiṣa* refers to a "[rule of] three things", but so do at least Āryabhaṭa, Brahmagupta, Mahāvīra and Bhāskara II.[27] All of them also refer to that which is wanted (*iccha*). Āryabhaṭa's formulation (translated from Kurt Elfering's German) is

> in the (rule of) three magnitudes, after one has multiplied the magnitude *phala* ["fruit" / "outcome"] with the magnitude *icchā*, the intermediate outcome is divided by the *pramāṇa* ["measure"].

Here, there is no reference to what is similar / not similar. However, this turns up as secondary information in the formulations of Brahmagupta, Mahāvīra and Bhāskara II[28] – but in ways so different that direct descent can be excluded.

The earliest reference to the rule in an extant Arabic work is in al-Khwārizmī's algebra. Al-Khwārizmī speaks of four quantities, not three. For the rest, interpreters differ on the meaning of his words. For four quantities in proportion $\frac{a}{b} = \frac{c}{d}$, Rosen [1831: 68] takes al-Khwārizmī to claim that *a* is "inversely proportionate" to *d*, and *b* to *d*, while Rashed [2007: 196] states that *a* is "not proportional" to *d* (etc.). A slightly later passage states according to Rosen that among the three known quantities, two "must necessarily be inversely proportionate the one to the other", according to Rashed that there are two numbers, each of which is not proportionate to its associate; in both cases, these two numbers have to be multiplied. None of this makes much sense mathematically,[29] and the Latin translations of Gherardo da Cremona [ed. Hughes 1986: 255] and Robert of Chester [ed. Hughes 1989: 64] are indeed different (while agreeing with each other). Both interpret the essential adjective as "opposite";[30] as long as we restrict ourselves to the first statement, this "opposition" could refer to a graphical scheme (*our* scheme, and the scheme used in twelfth-century Toledo, cf. note 31; al-Khwārizmī has nothing of the kind); the second passage, however, leaves only one possibility; that the term *mubāyn*,

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[27] See [Elfering 1975: 140] (Āryabhaṭa), [Colebrooke 1817: 33, 283] (Bhāskara II, Brahmagupta), and [Raṅgācārya 1912: 86] (Mahāvīra).

[28] In Bhāskara I's commentary to Āryabhaṭa [ed. Keller 2006: I, 107*f*], on the other hand, no such reference is present.

[29] We may presume that both translators have drawn from their familiarity with Euclidean proportion theory, without asking themselves whether al-Khwārizmī would be likely to use the same resource.

[30] Boris Rozenfeld [1983: 45] also agrees, and translates *protiv*.

translated “différent” by Mohamed Souissi [1968: 96] with reference to exactly this passage, means *dissimilar* – in exact agreement with the secondary explanations of the Sanskrit mathematicians from Brahmagupta onwards.

Most Arabic treatments of the rule have as their primary examples problems confronting commodity and price, and designate the four terms accordingly. They also often present the rule after a short introduction of the proportion concept and the rule of cross multiplication. Sometimes proportions and rule of three are linked, sometimes they are not – and often a formulation including the similar / non similar is involved.

Al-Karajī’s *Kāfī fi’l ḥisāb* does not link the rule with the preceding presentation of the proportion. His rule (translated from [Hochheim 1878: II, 16*f*]) runs as follows:

> You find the unknown magnitude by multiplying one of the known magnitudes, for instance the sum or the quantity, by that which is not similar to it, namely the measure or the price, and dividing the outcome by the magnitude which is of the same kind.

Ibn al-Bannā [ed., trans. Souissi 1969: 88] integrates proportions and the rule of three, and gives the rule in this shape:

> You multiply the isolated given number, (that is, the one which is) dissimilar from the two others, by the one whose counterpart one does not know, and divide by the third known number.

Ibn Thabāt [ed., trans. Rebstock 1993: 43–45] also integrates proportions and rule of three, and first gives rules based on the former. Then comes this rule, almost identical with the Italian abbacus formulation:

> The fundament for all *muᶜāmalāt*-computation is that you multiply a given magnitude by one which is not of the same kind, and divide the outcome by the one which is of the same kind.

Ibn Thabāt was active in Baghdad in the earlier thirteenth century, and primarily a legal scholar rather than a “mathematician” or “astronomer-mathematician”. That precisely *his* words should have been taken over by the abbacus school is not credible. We must rather assume that they reflect the formulation used by merchants in a wide area (apart of course from the passage “fundament for all *muᶜāmalāt*-computation”, which in the commercial milieu went by itself). If Fibonacci had been taught for more than “some days” in Bejaïa he might even have encountered it there; in any case, the Italian formulation cannot have been adopted from Fibonacci[31] nor probably from any other specific "gate", but by way of participation in a shared open space. The reference of Italian abbacus authors as well as Sanskrit mathematicians to a "rule of three" suggests that this open space encompassed not only the shores of the Mediterranean but also those of the Arabian Sea.

The origin of the Iberian recourse to counterfactual questions is more enigmatic. It could of course represent a local development; the abstract number question is not difficult to produce by simple abstraction, al-Khwārizmī's example "ten for eight, how much for four" is not very different; nor would the step from the merely abstract to the explicitly counterfactual be more difficult to make in the Iberian world than elsewhere.

However, there is some reason to believe that at least the abstract formulation circulated in the Arabic commercial world. As it turns out, al-Khwārizmī's "ten for eight ..." is found in Rosen's, Rashed's and Robert of Chester's translations – but Gherardo has concrete numbers, "ten *cafficii* for six dragmas ...".[32] The abstract formulation may thus very well have crept into the manuscript tradition after al-Khwārizmī's time. Moreover, ibn al-Khidr al-Qurašī, a little-known mid-eleventh-century author from Damascus, explains [ed., trans. Rebstock 2001: 64] that the foundation for "sale and purchase" is the seventh book of Euclid, and then goes on that "this corresponds to your formulation, 'So much, which is known, for so much, which is known; how much is the price for so much, which is also known?'". Finally, a hint of Persian (pre-Islamic?) properly counterfactual usage may exist: according to A. S. Saidan (Mahdi Abdeljaouad, personal communication), al-Baghdādī refers to the way profit and loss are calculated

[31] Fibonacci, when introducing the rule (p. 83*f*) does not speak of a "rule of three things", as done by the Sanskrit as well as Italian authors but (as common among Arabic mathematicians) of "four proportional numbers, of which three are known but the last unknown"; his rule prescribes the inscription of the numbers on a rectangular *tabula* (represented in the treatise by a rectangular frame). This method was also known to the compiler of the *Liber mahamaleth*, and thus in twelfth-century Toledo. It is likely to have inspired Robert's and Gherardo's understanding of *mubāyn* as "opposite".

[32] This informs us about three manuscripts: the main Arabic manuscripts Oxford, Bodleian, Hunt 214, and the two lost manuscripts used by Gherardo and Robert; since Rashed's critical apparatus is incomplete (he has some references to Gherardo's edition, but omits some of its variants on this point; what else he may omit is a guess), it is not possible to know how the other Arabic manuscripts look.

by the Persian expressions *dah yazidah*, “ten (is) eleven”, and *dah diyazidah*, “ten (is) twelve”.[33]

Because of the possibility to identify specific markers in the formulations of the rule of three, scrutiny of a larger number of Sanskrit, Arabic and Christian-European presentations of the rule would probably yield more information about points of contact, transmission roads and communities.

Set phrases, abbacus culture, and Fibonacci

Whoever has read (in) a few abbacus books will be familiar with phrases like these:

– “make this computation for me”;
– “this is its rule”
– “now say thus”;
– “and it is done, and thus one makes the similar computations”;
– “make similar computations thus”;

They all point to a teaching concentrated on the solution of problems serving as paradigmatic examples, and they will only have made sense within an institution similar in that respect to the abbacus school. In the *Livero*, they are particularly copious in those problems that are not taken from Fibonacci, but some of them are glued onto Fibonacci problems without being present in the original.

Fibonacci himself mostly avoids these characteristic locutions; in general, he tries to emulate the style of “philosophical” mathematics (just as he often tries to reformulate the mathematical substance *magistraliter*, “in the way of [school] masters” – this word is found on pp. 163, 215, 364). However, an occasional “make similar computations thus” *can* be found in the *Liber abbaci*.

The appearances of the set phrases in Fibonacci’s works are by far too few to have inspired their ubiquitous presence in abbacus writings. We may conclude that Fibonacci was so immersed in the style that later unfolds in the abbacus books that he did not manage to eliminate it completely.

In some cases, he distinguishes his own style from what “we are used to do” or from what is done *vulgariter*. An example of the former distinction is in his exposition of the simple false position (p. 173*f*), taught by means of a tree, of which $^1/_4+{}^1/_3$ are below the ground, which is said to correspond to 21 palms. He searches for a number in which the fractions can be found (taking 12 as the

[33] Neither Abdeljaouad nor I have so far been able to get hold of Saidan’s edition of al-Baghdādī; verification of the precise context of the formula is thus impossible.

obvious choice), and next argues that the tree has to be divided in 12 parts, 7 of which must amount to 21 palms, etc. But then he explains that there is another method "which we use" (*quo utimur*), namely to posit that the tree be 12. He concludes that

> therefore it is customary to say, for 12, which I posit, 7 result; what shall I posit so that 21 result?

and finds the solution by the rule of three. This corresponds exactly to what can be found in abbacus books – for instance, in the Columbia algorism [ed. Vogel 1977: 79]:

> The $^1/_3$ and the $^1/_5$ of a tree is below the ground, and above 12 cubits appear. [...] If you want to know how long the whole tree is, then we should find a number in which $^1/_3$ $^1/_5$ is found, which is found in 3 times 5, that is, in 15. Calculate that the whole tree is 15 cubits long. And remove $^1/_3$ and $^1/_5$ of 15, and 7 remain, and say thus: 7 should be 12, what would 15 be?

Vulgariter, per modum vulgarem (etc.) are used (at least) four times (pp. 115, 127, 204, 364) to characterize simple stepwise calculation as opposed to a single combined operation (by means, e.g., of composite proportions); this would probably be what a practical reckoner preferred. On p. 63, addition of $^1/_3$ and $^1/_4$ *secundum vulgi modum* is made by taking both fractions of a convenient number (*in casu* 12), similarly a method easily understood by reckoners without theoretical training. More informative is what we find on p. 170. After having found the fourth proportional to 3–5–6 as (5·6)/3, Fibonacci says that the same question is proposed "in our vernacular" (*ex usu nostri vulgaris*) as "if 3 were 5, what would then 6 be?". Next, he asks for the number to which 11 has the same ratio as 5 to 9, and restates the question *secundum modum vulgarem* as "if 5 were 9, what would 11 be?". This tells us that the vernacular practice in which Fibonacci participates (*vide* his repeated first person plural, "we use", "our vernacular") is of the Ibero-Provençal kind, not similar in its approach to the rule of three to what is later found in Italy. Actually, Santcliment [ed. Malet 1998: 163] introduces the presentation of the rule of three by saying "and this species begins in our vernacular, 'if so much is worth so much, how much is so much worth'".[34]

Fibonacci is certainly no abbacus author, his scope as well as his ambition goes much beyond that. But as we see, he knew that mathematical culture of which the Italian abbacus school became the most famous representative. His

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[34] "E comença la dita specia en nostre vulgar si tant val tant: que valra tant".

book, furthermore, informs us about how this culture looked at a moment it had not yet reached Italy[35] – though not very specifically, it is up to us to try to sort out what comes from which place.

Byzantium

As an example of what may perhaps be dug out by careful analysis I shall mention the question of Byzantium. As quoted, Fibonacci tells us in the introduction to the *Liber abbaci* that one of the places where he encountered study of the art of "the nine digits of the Hindus [...] with their varying methods" was Greece, that is, in Byzantium. On several occasions, moreover, he states that a particular problem was presented to him by a Byzantine master (pp. 188, 190, 249); finally, a number of problems tell stories taking place in Constantinople (pp. 161, 203, 274, 296). Of the former group, all examples but one state prices in *bizantii* (the one on p. 190 deals with unspecific "money", *denarii*), and all the latter deal with the same Byzantine currency. We may infer that the metrologies occurring in the book, even in wholly artificial problems, were as a rule not chosen at random but thought of in connection to the location where they were in use. Since most problems do not specify where they are supposed to take place,[36] nor where Fibonacci was confronted with them,[37] the metrologies and

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[35] That it had not yet reached Italy is illustrated by the yet another reference in the book to vernacular methods (the last one, if I am not mistaken), namely on p. 114. Here the Pisa method to find the profit corresponding to each *libra* invested in a commercial partnership (certainly a real-life method, since it starts by removing as some kind of tax or as payment for the shipping one fourth of the profit) is confronted with calculation *secundum vulgarem modum*, which turns out to be the usual partnership rule.

A number of apparent Italianisms in the text (*baracta*; *viadium* / *viagium*; *pesones* [Latinized plural of *peso*; *avere*; and various names of goods] might be taken to suggest an Italian background to the *Liber abbaci*. However, apart from the possibility that Fibonacci – a Tuscan speaker – might introduce such loanwords on his own, it should be noticed that all may just as well come from the Catalan of his epoch.

[36] *Bizantii*, for example, occur on these pages: 21, 83, 84, 93-96, 102, 103, 107-109, 113, 115, 119, 120, 121, 126, 131, 137, 138, 159-163, 170, 178, 180, 181, 203-207, 223-225, 228-258, 266, 273-277, 279, 281, 283, 310, 313-318, 323, 327-329, 334, 335, 347, 348, 349, 396, 401. Not all of these passages are of course relevant for Byzantium, *bizantii* were also minted in Arabic and crusader countries [Travaini 2003: 245].

[37] Actually, I am fairly sure that there are no specified references to locations for such confrontations other than Byzantium, in spite of the open-ended reference to "the give-and-take of disputation" of the introduction.

currencies are likely to carry otherwise hidden information of one or the other kind – or both.

The *Liber abbaci* shares with later books in the abbacus tradition another kind of likely indirect information about the role of Byzantium. Regularly, they start alloying problems with a phrase "I have silver / gold" of this and this fineness.

Fibonacci uses the structure a few times. On p. 143 it stands in a reference to "our" (vernacular) way of expressing ourselves.[38] Then on p. 156 it stands as what "you" should say when stating a problem about alloying of silver. Finally, the locution is used to indicate that an alligation problem dealing with grain is equivalent to one about the alloying of silver (p. 163); obviously Fibonacci sees the "I have"-structure as characteristic for such problems.

In Jacopo's *Tractatus, all* alloying problems start with "I have"; the locution is also used in one problem about exchange of money, and in one about money in two purses [Høyrup 2007: 125]. All alloying problems in the non-Fibonacci part of the *Livero* do as much. Later, we find the same opening for instance in Paolo Gherardi's *Libro di ragioni* from 1328 [ed. Arrighi 1987a: 29–31, 89];[39] in a *Libro de molte ragioni d'abacho* from c. 1330 [ed. Arrighi 1973: 95–106];[40] in Giovanni de' Danti's *Tractato de l'algorismo* from c. 1370 [ed. Arrighi 1987b: 50–52]; in a *Libro di conti e mercatanzie* probably from c. 1390 [ed. Gregori & Grugnetti 1998: 72–74];[41] in Francesco Bartoli's *Memoriale* [ed. Sesiano 1984b: 134*f*], a private notebook written in Avignon before 1425 and containing excerpts from earlier abbacus works[42]; in Piero della Francesca's *Trattato d'abaco* [ed. Arrighi 1970: 56–59]; and (with the slight variation, also known by Piero della Francesca [ed. Arrighi 1970: 74], "Io mi trovo ...") in Pacioli's *Summa* [1494: 184^r–185^v]. It is also found in a Castilian merchant handbook *De arismetica* (Real Academia Española, Ms. 155, ed. [Caunedo del Potro 2004: 45]), and it survives in Christoff Rudolff's *Coss* from 1525 [ed. Kaunzner & Röttel 2006: 201, 202, 215*f*].

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[38] "When we say, I have bullion at some ounces, say at 2, we understand that one pound of it contains 2 ounces of silver".

[39] In the first of these passages, the first person only initiates problems about gold, whereas a silver problem starts "There is somebody who has ...".

[40] Alternating with the formula "A man has ...".

[41] Gold problems only. Problems about silver are neutral or start "Somebody has ...".

[42] Ten instances of "I have" regarding gold as well as silver, and a single of "Somebody has" (about gold).

This distribution of the opening "I have .." seems to point to an origin in a particular environment, distinct from that where abbacus problems in general were circulating (a money-dealers' environment, it would seem).

In one Byzantine treatise of abbacus-type (Ψηφηφορικὰ ζητήματα καὶ προβλήματα, "Calculation Questions and Problems") from the early fourteenth century [ed. Vogel 1968: 21–27], the first person singular serves not only for alloying problems but also for other problem types (mostly but far from always dealing with possession of or payment in gold coin). If this characterized Byzantine practical mathematics in broader general, it would be tempting to believe that the Italian and Iberian way to formulate alloying problems had its roots in a *Byzantine* money-dealers environment.[43]

Absence of Hebrew influence

A similar argument can be used to *rule out* another possible line of influence. In Roman Law, it was customary to represent participants in paradigmatic cases by the names Gaius and Titius[44] (and less often Maevius). The habit became so familiar in Medieval Italy that "some guy" is spoken of in modern Italian as *un tizio*. Similarly, the Babylonian Talmud sometimes (less pervasively) uses Jacob's oldest sons Reuben, Simeon, Levi and Judah for this purpose.[45] Medieval Hebrew practical mathematics[46] took over this usage and applied it much more

[43] Admittedly, a Byzantine treatise from the next century [ed. Hunger & Vogel 1963] shows no trace of the style. On the other hand, the older treatise is local Byzantine in its choice of coins referred to, whereas the younger one is heavily influenced by Italian treatises in this respect [Scholz 2001: 102]; it may therefore not say much about Byzantine habits in the twelfth and thirteenth centuries.

[44] A search for "Titius" in the electronic version of [Scott 1932] finds more than 1860 appearances. Often, of course, the name occurs repeatedly within the discussion of a single case – this is exactly why it is useful to have names to refer to. None the less, the omnipresence of this fictive person is impressing.

[45] For instance, In *Yebamoth* [ed., trans. Slotki 1964: 28b], Reuben and Simeon have married two sisters, and Levi and Judah two strangers; in *Baba Kamma* [ed., trans. Kirzner 1964: 8b], Reuben sells all his lands to Simeon, who then sells one of the fields to Levi; none of this has anything to do with Genesis.

[46] Represented by ibn Ezra's twelfth-century *Sefer ha-mispar* [ed., trans. Silberberg 1895], written in Lucca or Rome in c. 1146 [Sela 2001: 96]; the problem collection accompanying Levi ben Geršom's *Sefer maaseh hoshev* [ed. Simonson 2000]; and Elia Misrachi's *Sefer ha-mispar* [ed., trans. Wertheim 1896] (early sixteenth-century).

systematically.[47]

However, with a single exception abbacus and related Ibero-Provençal writings I know of never adopted the stylistic scheme; the parallel of the "I have"-opening of alloying problems shows that they would certainly have done so if they had borrowed from the Hebrew tradition.

The single exception is Muscharello's *Algorismus* [ed. Chiarini et al 1972: 154–158, 193], written in Nola in 1478;[48] in three problems dealing with the settling of accounts, the protagonists are, respectively, Piero+Martino, Rinaldi+Simoni and Roberto+Martino, and in one about four gamblers, these are Piero, Martino, Antonio and Francischo. Whether this exception is really a borrowing or an independent creation cannot be decided – in particular because German cossic writings begin at the same time to use capital letters for the same purpose – first in Magister Wolack's Latin university lecture about abbacus mathematics from 1467/68 [ed. Wappler 1900: 53f], later occasionally in Rudolff's *Coss* from 1525 [ed. Kaunzner & Röttel 2006: 211], and probably by others in between (in this case, inspiration from university teaching of Aristotelian logic is possible). There was thus a need for a way to identify the actors of a problem beyond the traditional "the first", "the second", etc., and Muscharello's use of names may have been a self-invented way to meet this need.

A pessimistic conclusion

In my initial disclaimer I promised that "the following offers elements that have to go into a synthetic answer, but too few and too disparate to allow the construction of this synthesis". I am afraid that in particular the negative second part of this pessimistic pledge has been respected. I am also afraid that further research may dig out more elements that have to go into the answer, while making it even more difficult to produce a convincing synthesis – too much of the process has taken place in oral interaction and left no permanent traces. The only reason the Italian situation in itself is *somewhat* better documented is that the Italian merchant class was the effective ruling class of its cities, and eventually even nobility; for these merchant-patricians, mathematics books were objects of prestige – *sacred* objects, almost as the sword was a sacred object for other nobilities. For the ruling classes or culturally hegemonic strata of other areas

[47] Some of ben Geršom's examples, however, deal with anonymous "travellers", "merchants" etc., as usual in *muᶜāmalāt*- and abbacus texts.

[48] Nola is located in Campania, and thus well outside the core region for the abbacus tradition (which reaches from northern Italy to Umbria).

of importance for our process they were not. The Italian books were therefore conserved with much higher probability than similar books elsewhere; and even in Italy, as one discovers at any attempt to trace *development* – for instance, the development of incipient symbolism – the holes predominate, and the cheese turns out to be all too scarce to satisfy our appetite.

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