History of Mathematics Education in the European Middle Ages

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Periodization and institutional types

By lucky but adequate accident, "the Middle Ages" have become plural in English. The millennium that separates the definitive demise of the Western Roman Empire from the discovery of the New World and the Reformation can hardly be understood as one homogeneous period under any point of view – and certainly not if we look at mathematics education.

The earliest phase, lasting until c. 750 CE, is known as "the Dark Ages" – both because surviving sources for this period are rare and because this rarity reflects a very low intensity of literate culture. During the ensuing "Central Middle Ages" (c. 750 to c. 1050, the epoch of Charlemagne and the Ottonian emperors), attempts at statal centralization led not only to creation of the cathedral school system but also to corresponding developments of monastic learning. The High (c. 1050 to c. 1300) and Late (c. 1300 to c. 1500) Middle Ages are characterized by the rise of city culture, which led to the emergence of the university system as well as to the appearance of institutionalized lay education – connected to but not identical with the Renaissance current.

In principle we should also distinguish between partially or fully separate types of education – that of the Latin school and university tradition, and those of various kinds of practitioners. Among the latter, however, only the education of Late Medieval merchant youth is well documented in sources – for that of craftsmen we have very little evidence.

The Dark Ages

From the point of view of mathematics education, the Dark Ages are even "darker" than other aspects of literate culture. During the centuries after the final collapse of the Western Roman empire, some members of the social elite of the new Barbarian states in Italy, Visigothic Spain and Gaul were still taught Latin letters; notarial and legal services were still needed in royal and (what remained of) municipal administration in the same areas; and monks who entered monasteries in not too late age would learn to read at least the psalter-book – in some places and periods considerably more, and not only sacred literature.[1]

However, if ever the seven Liberal Arts had been a serious curriculum, already in Augustine's youth (later fourth century) no more than grammar and

[1] A detailed investigation (which also makes clear the absence of every kind of mathematical studies) is [Riché 1976].

rhetoric remained – the *quadrivium* (arithmetic, geometry, mathematical astronomy+geography and mathematical musical theory)[2] had neither teachers nor students. Augustine himself had profounder mathematical interests (and had read Euclid on his own), but his *De doctrina christiana* [ed., trans. Robertson 1958] – which set a higher aim than ecclesiastical teaching was able to attain for many centuries – mentions only the need to understand certain numbers in the scriptures (II.25, III. 51). He points out that mathematical truths are of divine origin – but that is not seen as a reason to pursue them, Augustine only warns against being too interested in them (II.56–57).

Isidore, the learned Visigothic bishop of Seville (c. 560 to 636) certainly praised mathematics (more precisely "the science of number") in his monumental *Etymologies*, which was to become one of the most-quoted authorities of the Middle Ages, but his own knowledge of the quadrivial disciplines does not go beyond a few ill-digested definitions and a few concepts borrowed from late Latin encyclopediae; no wonder that for more than a century there is no trace of anybody being *taught* according to his modest programme.

In this as in other domains, however, the Dark Ages are made darker by the absence of sources. Administration, taxation and the household accounting of monasteries were not possible without some calculation and land measurement. We have no traces of how this non-quadrivial mathematics was taught, but we may safely presume that the necessary skills were trained in apprenticeship and "on the job", as they had already been in Antiquity. Merchants – a class that had been reduced but had not disappeared – must also have known how to calculate; even here we have no direct information, but we shall return to a possible indirect trace.

We might have expected some teaching of *computus* (Easter and other sacred calendar reckoning) within monasteries; since the early fourth century it had been considered a problem that various regions celebrated Easter at different times, and tables as well as (discordant) calculation methods had been developed

[2] This group of disciplines and the collective name used about it since Boethius are the closest we can get to a *unified concept* of mathematics in the Medieval Latin school tradition at least until the thirteenth century; only Aristotelian philosophy brought in the notion of "more physical" mathematical disciplines (which beyond astronomy included optics and the science of weights). However, when asking about mathematics education – a modern concept – we shall need to include also mathematical activities falling outside the quadrivial framework, such as practical computation. Unfortunately, the sources are mostly mute on this account.

that should allow the prediction of the right day.[3] It appears, however, that the matter did not enter any monastic teaching programme except in Ireland from the seventh century onward. From here it entered Anglo-Saxon Britain in the earlier eighth century – or at least the monastery where the Venerable Bede was teaching and writing.

Carolingian to Ottonian Times

During the second half of the eighth century, Charlemagne first took over the Frankish kingdom and next subdued much of Western and Central Europe. Probably as part of an effort to create a stable power structure (but apparently also because of sincere personal concern), he initiated an ecclesiastical, liturgical and educational reform after having attached to his court the best scholars he could find in Italy and Britain. A "general admonition" was issued in 789, which called for the creation of schools in all monasteries and bishoprics where (select) boys could be taught hymns, notes, singing, computus and grammar (in this order, which may reflect Charlemagne's concerns.[4] As we see, mathematics only appears in the programme through its service for computus (for which Charlemagne had a passion); we are still left in the dark when asking how the underlying calculational skills were to be taught. However, another circular letter to the clergy exhorted those who had the ability (presumably the scholars at his court) to teach "the liberal arts" to others, and we do indeed have small treatises (mostly primers) written by Alcuin of York, second-generation scholarly descendant of Bede and the central figure of the "palace school" (a "school" about whose character we know nothing precise). They deal with elementary grammar, rhetoric, dialectic and computus; in the treatise concerned with the latter subject it is again clear that the reader is supposed to know simple calculational operations (including divisions).[5]

One more possible trace of mathematical instruction in the environment exists: a collection of mixed "recreational problems", *Propositions to sharpen the*

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[3] The ways the problem was confronted from the beginning until the early eighth century is accounted for in [Jones 1943: 1–114].

[4] The relevant passage of the *Admonition* appears in [Riché 1979: 352*f*]. [Brown 1994] is a fine presentation of the whole reform effort.

[5] This picture is confirmed by the *Manual for My Son* written by the mid-ninth-century noblewoman Dhouda. When speaking about sacred numerology, in Augustinian style, she tacitly assumes the son to understand basic computation (and shows that she does so herself) [ed. Riché 1975: 326–334].

minds of youngsters ascribed in some manuscripts to Alcuin.[6] Once again, their solution presupposes familiarity with elementary calculation. Some of the problems refer to the monastic environment, many others to the world of trade. The collection may well have been put together by Alcuin or a contemporary, but most of the problems are likely to have circulated since late Antiquity and thus to reflect the teaching of basic arithmetic of young monks and merchant youth.

In the long run, the obligation of bishops to take care of teaching developed into the cathedral school system – but at first the breakdown of Charlemagne's realm after his death did not allow schools to flourish. In some monasteries, however, the attempt to fill out the full gamut of liberal arts gave rise to a hunt for manuscripts. Already in Charlemagne's time, Martianus Capella's *Marriage of Philology and Mercury* turned up [Stahl 1971: 61–64]) – a late ancient encyclopedic presentation of the liberal arts (not teaching much mathematical substance, yet more than other encyclopedic works that started to circulate at the same time). In the early ninth century Boethius's *Arithmetica* and *De musica* (free translations from c. 500 of Greek originals written by Nicomachos around 100) were rediscovered, and the surviving agrimensor writings (Latin treatises on practical surveying) were collected and combined with surviving fragments of a translation of Euclid's *Elements* (or, plausibly, of a digest omitting most proofs) and reclassified as quadrivial geometry (to which surveying geometry had never been reckoned since the ancient invention of the "liberal arts") [Ullman 1964].

Until the early twelfth century, Boethius' *Arithmetic* and the agrimensorial tradition (with the same fragments of Euclid) were the fundament of all teaching of arithmetic and geometry (to the extension such teaching existed). The former offered a philosophical discussion of number; the concepts of odd and even; prime and composite number; figurate numbers and their properties; an extensive classification of ratios;[7] and various means (arithmetical, geometric, harmonic, and seven more) (no practical computation). The "sub-Euclidean" geometry derived from the agrimensors and from the Euclidean fragments contained

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[6] Edition and German translation in [Folkerts & Gericke 1993].

[7] A ratio was not understood as the number resulting from a division but as a *relation* between two numbers; it might be *multiplex* (of type $m : 1$), *superparticular* (of type $m+1 : n$), *multiplex superparticular* (of type $mn+1 : n$), superpartient (of type $n+p : n$), *multiplex superpartient* (of type $mn+p : n$); or inverses of any of these. Depending on the numbers involved, ratios had specific names – 5 : 2 (and 10 : 4, etc.), for instance, were "duplex sesquialter".

Euclidean definitions, postulates, the proposition statements from books I–III, and the proofs for the first three propositions – and from the agrimensor side mainly rules for area calculation (not always correct). Its most mature expression is the eleventh-century so-called *"Boethius" Geometry II* [ed. Folkerts 1970].

When this compilation was made, however, the tradition had already developed, primarily in the cathedral schools of Lotharingia.[8] Most important was the introduction of a new type of abacus (plausibly a transformation of a type that was already around), using counters marked by Hindu-Arabic numerals. It may have been designed by Gerbert of Aurillac after his stay in Catalonia (*not* in Muslim Iberia) in the late 960s and was at least taught by him while he was the head of the cathedral school in Rheims (972–982, 984–996).[9] It seems not to have spread outside the monastic and school environment, and it is likely to have served more in teaching than for practical calculation. The topic was pigeonholed under geometry, not arithmetic – perhaps because of its use of a plane surface, perhaps because it would serve the area calculations of sub-Euclidean geometry, perhaps because ancient geometry was known from Martianus Capella to have made its drawings on a sand board similarly designated *abacus*. In any case, a categorization under quadrivial arithmetic would have been no more adequate.

Linked to this abacus by using at times its board but also to the teaching of arithmetic was a newly invented game board *rithmomachia*.[10] The players had to know the Boethian theory of ratios as well as the whole gamut of figurate

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[8] See [Bergmann 1985] (to be used with some care).

[9] Since the slave trade route to Muslim Spain passed through Lotharingia, Gerbert's stay in Catalonia was not the only numerate cultural contact at hand. [Thompson 1929] lists a number of further contact on the courtly and literate levels. Since the names given to the nine figures seem to be of mixed Magyar-Arabic-Latin-German origin [Köppen 1892: 45], the slave traders could be the most likely inspiration.

[10] Arno Borst claims in his fundamental study [1986] that the game can be traced back to c. 1030 and no further. However, Walther von Speyer's *Libellus scolasticus* [ed., trans. Vossen 1962: 41, 52*f*] clearly speaks of a very similar game played around 970 (without indicating the name, which may indeed be later) on the abacus board and using its counters. Borst dismisses this testimony, asserting that Walther does not understand what he is speaking about, and Vossen because he does not know that the abacus board belonged with geometry.

The didactical use of the game was discussed by Gillian R. Evans [1976]. A recent discussion of the game and its survival is [Moyer 2001]. A short presentation of the way the game was played is in [Beaujouan 1972: 644–650].

numbers, and there is little doubt that the game contributed to keeping alive the interest in Boethian arithmetic until the sixteenth century (and vice versa).

Computus was still taught, but astronomy was now more than computus. This *may* have depended on the incipient interest in astrology – the first compilations using Arabic material are from the late tenth century [Van de Vyver 1936; Burnett 1987: 141*f*]. However, Gerbert's teaching of the topic as described by his former student Richer [ed. Bubnov 1899: 379*f*] points to Martianus Capella and shows no hint of astrological preoccupations – it deals with the horizon, tropics, ecliptic and other circles of the heavenly sphere. On the other hand, the first treatises on the astrolabe turn up around the same time; one has been ascribed to Gerbert. This instrument came from an area where astrology was a central motivation for work on astronomy; whether its complicated use was taught in any organized way at the moment is dubious.

Even music had changed. In Charlemagne's time, as we saw, it was no mathematical topic at all; with the discovery of Boethius's *De musica*, it once more became a mathematical discipline (albeit hesitatingly), and singing was reclassified as *cantus*. But even theoretical music split in the early eleventh century. Guido di Arezzo, known as the inventor of (the earliest form of) the modern musical notation, used musical theory in the teaching of singing and developed it for that purpose (*musica practica*, in a later term) [Wason 2002]; predecessors in the tenth century had started this process. Gerbert, teaching in Rheims, taught *musica theorica*, Boethian theory, which was to remain the music of quadrivial teaching throughout the Middle Ages (whereas a number of outstanding university scholars, some – like Jean de Murs – known as mathematicians, developed theory far beyond Boethius in the thirteenth and fourteenth centuries).

When Richer speaks of Gerbert's geometry teaching, he only mentions the abacus. Letters written to Gerbert by a former student and correspondences between ex-students (or students' students) who themselves had become school heads show that any further teaching of agrimensorial geometry, if existing, had been in vain.[11] As Paul Tannery [1922: 79] says about these correspondences, they belong not to the history of science but to that of ignorance. One correspondent does not understand why the determination of a triangular area from triangular numbers does not coincide with that following from base and height and asks Gerbert (now Pope Sylvester II) for an explanation; two others

[11] The existence of a *Geometria Gerberti* decides nothing, since it may well be a compilation from the later eleventh century.

discuss the meaning of the notion of "exterior angles" of a triangle (which they have found in Boethius) without coming to the correct result; one of the latter also supposes that the Archimedean formula for the circular area[12] had been found by cutting and reassembling a parchment circle. Other writings of theirs show them to have been both well educated and intelligent; their failure thus reveals the absence of any adequate teaching (and the unavailability of relevant manuscripts); simultaneously, the letter exchanges testify to a vivid interest in the topic at least among schoolmasters and former school heads who had risen to the rank of bishops.

Gerbert's fame allows us to conclude that nobody else at the time reached his level. What he did *could* be done in his time; but we should not believe that others did as much at the moment.[13]

The "twelfth Century Renaissance"

The translation of medical writings from the Arabic began in the later eleventh century, but the golden age of philosophical and scientific translation arrived with the twelfth. However, the same factors as caused this new beginning at first produced a culmination of the autochthonous Latin scholarly tradition. One factor was the growth of towns, of artisanal industry and of urban wealth; another (largely dependent on the first one) was the growth of schools, in absolute number as well as number of students at each. Already because of the latter increase, a single scholast could no longer take care of the whole school. In consequence, masters became free scholars, teaching with permission of the local see but living from the fees of the students. Most famous of these at the time – and one of the most famous philosophers of the Middle Ages, and not only because of his love affair with Héloïse – was Abelard, whose reputation contributed to make Paris and Ile de France the school region par excellence [Haskins 1927: 377–379].

However, Paris – one of the most important cities of Europe – had been a school city before Abelard had any influence, and the one where we get information about mathematics *education*. Wealthy burghers wanted their sons to be educated, and the only institutions where education was disbursed were those of the church. In particular, the Saint Victor monastery in Paris had an

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[12] Known by him from one of Boethius's treatises on Aristotelian logic, even though it is amply used by the agrimensors.

[13] Uta Lindgren [1976: 48–59] discusses some of them in detail and comes substantially to the same result.

external school, whose head Hugue wrote a study guide to the arts (liberal as well as mechanical) and to sacred scriptures, the *Didascalicon*. He also produced a *Practica geometriae* [ed. Baron 1956], which we must presume was connected to his teaching. Since the former work was famous enough to be plagiarized around 1500 and the second influenced the terminology of all Latin practical geometry already a few decades after it was written (probably, like the *Didascalicon*, in the 1120s), we must presume the work to have been at the forefront of what was possible at the time in the most advanced region of Western Europe, and not typical – but at least it was sufficiently close to the level of other scholars to be understood. The success shows that subsequent generations soon reached Hugue's level – presumably also in teaching.

When presenting sacred history in the study guide [trans. Taylor 1961 : 135–137], Hugue asks (as his students would perhaps do) whether knowledge of this topic is really necessary. He argues that seeming trifles are useful, and gives as an example his own boyhood experience: not at all dealing with sacred history, but with experiments on numbers, on area measurement, on the sound of strings, and his observations of the stars – that is, the full quadrivium, which he must somehow have been taught as a boy, if we are to believe his words.

The presentation of the quadrivium in the *Didascalicon* is *metatheoretical*, in part arithmological (presenting the meaning of numbers in a perspective derived from ancient Platonizing writers, including Augustine), in part metamathematical (presenting the distinction between continuous and discrete quantity, the view of mathematical objects as abstractions), in part concerned with the division into subdisciplines – and in part simply etymological, explaining the names of the four disciplines. As befits a study guide, it does not enter into the mathematical subject matter, but here the geometry treatise shows what Hugue might teach.

The title presupposes a distinction between theoretical and practical geometry; however, what is said about geometry in the *Didascalicon* corresponds exactly to the practical branch, which we may thus assume to be what was taught (at what age and to whom remains an open question). The subject matter is to a large extent derived from the best of the sub-Euclidean writings, but everything is thought through – Hugue thus begins with explanations of concepts which are *not* repetitions of the familiar Euclidean fragments. Hugue's own contributions are also conspicuous, in particular concerning the section on "cosmimetry", measurement of the (spherical) world.[14]

[14] An analysis of this part of the treatise and its inspiration from ancient philosophical sources is in [Tannery 1922: 208–210]. Tannery rejects Hugue's authorship as probably

Abelard and Hugue may stand for the culmination of autochthonous Latin knowledge, but the thirst for more among the brighter scholars of the epoch is already symbolized by the name Héloïse gave to the son she had with Abelard some time around 1120: Astralabius.[15] The astrolabe, though already known in the eleventh century and prescribed by Hugue for the measurement of angles, was first of all the central tool for that "medico-astrological naturalism" which was a main motive for the translations from the Arabic and the Greek – the other motive being the desire to get hold of those famous works which were known by name and fame from Martianus Capella and other Latin authors but not in body.

From the perspective of mathematics education, the first important acquisitions were Euclid's *Elements* and the Hindu-Arabic numerals. The first translation of the *Elements* (from the Arabic, known as "Adelard I") was presumably made by Adelard of Bath (probably assisted by somebody who knew Arabic better). Adelard's general orientation was toward naturalism, astrology and magic, and his own mathematical upbringing as reflected in his juvenile *De eodem et diverso* and *Regule abaci* had not gone beyond the traditional quadrivium; he may have worked on the *Elements* because this work was known by Arabic astronomers to be the fundament for the mathematics of the *Almagest* (two other twelfth-century translators of the *Elements* also translated the latter work). In the wake of Adelard I, a family of derived versions emerged (collectively known as "Version II", ed. [Busard & Folkerts 1992]), seemingly produced by an informal network of Adelard's former students [Burnett 1996: 229–234].

Version II is clearly marked by didactical concerns. Instead of giving full proofs, it often just gives hints of how a proof should be made; at this point it is clear that the matter presented in the work had become the primary aim, while further utility for astronomy (and, still further, for astrology) had retreated into the background.

Hindu-Arabic numerals, however, were introduced and studied (at first outside every formal framework) as a tool for astronomical calculation and for understanding astronomical tables; initially, some writers experimented with alternatives, such as use of the Roman numerals I through IX within a place value system, or of Latin letters as numerals (as known also from the Greek) [Burnett

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a thirteenth-century reconstruction (pp. 319–321), but better editions of the texts on which his arguments are based turns the conclusion upside-down – cf. also [Baron 1955].

[15] Abelard, *Historia calamitatum*, ed. [Muckle 1950].

2010, articles III and X]. However, well before the end of the twelfth century, the Hindu-Arabic numerals had forced out these possibilities.

We know nothing about the way these innovations made their way into the schools during the twelfth century; the situation is no different if we think of the *Almagest*. But somehow they must have reached a fair number of students. Indeed, toward the end of the century the conservative theologian (and head of the school of the St Geneviève monastery in Paris) Étienne de Tournais (translated from [Grabmann 1941: 61]) complained that many Christians (and even monks and canons) endangered their salvation by studying

> poetical figments, [Aristotle's] philosophical opinions, the [grammatical] rules of Priscian, the Laws of Justinian [Roman Law], the doctrine of Galen, the speeches of the rhetors, the [logical ambiguities of Aristotle, the theorems of Euclid, and the conjectures of Ptolemy. Indeed, the so-called liberal arts are valuable for sharpening the genius and for understanding the Scriptures; but together with the Philosopher [i.e., Aristotle] they are to be saluted only from the doorstep.

As we see, the "new learning" of the twelfth century, encompassed a new level of literary, grammatical and rhetorical studies; Roman Law; Galenic rational medicine; Aristotelian (natural) philosophy and advanced logic; and finally the planetary hypotheses of Ptolemy, and the *Elements*. Hindu-Arabic numerals go unmentioned – they were probably seen only as a tedious tool by those who used them, hardly something that could call forth undue enthusiasm. Unmentioned are also other mathematical topics to which the translations had given access (geometrical optics, spherics and algebra), as well as such that were mere continuations of the previous age – neither computus nor the abacus had been raised to a new level as had the study of Latin poetry, nor had they been linked to "the Philosopher". However, the *Elements* are there, and Étienne may even have meant them as *pars pro toto*, as a stand-in for mathematical studies in unspecific general.

The era of universities

Étienne's complaint is located at a watershed. As he was writing, the number of teachers and students had reached the level in some towns where the mutual protection provided by a guild could serve. Since neither teachers nor students were normally citizens of the town where they stayed, the need for juridical protection was obvious. Such guilds – in Latin *universitates* – are first attested around 1200 in Paris, Oxford and Bologna (in the latter town, the guild was for students only, the masters of Roman Law being ordinary citizens of the town

and possessing their own organizations).[16]

One of the weapons possessed by such a guild was emigration – students often brought money with them from home, and if they left a town, its commercial life might suffer severely. Even when an agreement was reached, some masters might stay together with their students. In this way, an emigration from Oxford produced that of Cambridge in 1209, while that of Padua resulted from an emigration from Bologna in 1222.

The northern universities grew out of the cathedral school system, and thus had as their original core the liberal arts as these had been shaped from the Carolingian age onward; they can thus be expected to be relevant for discussions of mathematics teaching. Those of Bologna and Padua were initially schools of law, later also of medicine; in this context, mathematics was an auxiliary discipline for astronomy, itself an auxiliary discipline for astrology, which served in medicine.

Since Paris eventually came to serve as a general model, we may look at what we know about its mathematics. At least when the structure crystallized during the earlier decades of the thirteenth century, the university was divided into *faculties*. A student (always a boy) first entered the Arts Faculty around the age of 14 or so, studying there for at least 6 years (unless part of the corresponding studies had been achieved elsewhere, as was gradually becoming possible); during the final two years, when he had acquired the degree of a *baccalaureus*, he was allowed to make his own "cursory" lectures under supervision. The name of the faculty refers to the hypothesis that it taught the liberal arts (whereas the later alias "faculty of philosophy" refers to the sway which Aristotelian philosophy possessed from the mid-thirteenth century onward). Most students left after having finished the arts study, if not on the way (even less than the full curriculum might serve to obtain a post in the Church or in secular administration); some of those who graduated and got the *licentia docendi* stayed as masters at the faculty while normally pursuing studies at the "lucrative faculties" – the faculties of Medicine and Canon Law. Having graduated from one of these, they had the possibility to teach there, perhaps pursuing studies at the Theological Faculty. Mathematics was taught at the Arts Faculty.

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[16] The medical schools of Salerno and Montpellier were older but only came to be characterized as "universities" at a moment when this term had acquired new meanings.

The whole process by which the universities emerged is much too complex to be treated justly in the present context. A recent fairly detailed description is [Pedersen 1998: 138–188].

The first approach to the definition of a curriculum is found in a Papal decree (issued on the Pope's behalf by Robert de Courçon, a local theologian) from 1215.[17] It rules in a few lines what should be taught in Aristotelian logic and grammar, and what must not be presented in cursory lectures. All that is said about mathematics is that the masters

> shall not lecture on feast days except on philosophers and rhetoric and the quadrivium and *Barbarismus* [a section of Priscian's grammar dealing with stylistic and rhetorical topics] and ethics, if it please them, and the fourth book of the *Topics*.

That mathematics was not compulsory seems to be confirmed in a decree from 1252 [trans. Thorndike 1944: 53–56]: an arts student presenting himself for the disputation leading to the bachelor's degree should at least be in his twentieth year, he shall have followed lectures on advanced grammatical and logical subjects (including Aristotle's *Prior* and *Posterior Analytics* – not easy stuff) and on Aristotle's *On the Soul* – about things mathematical not a word. However, since all of this belongs on the advanced level, the student *may* have been supposed to have pursued quadrivial studies along with elementary grammar and logic. A new decree from 1255 [Thorndike 1944: 64–66], famous as the demarcation of the complete Aristotelization of the faculty, leads to the same conclusions.

However, other kinds of evidence are at hand. One is a satirical poem "The Battle of the Seven Arts" [ed., trans. Paetow 1914], describing the fight between Orléans, a representative of twelfth-century learning at its literary best, and the university of Paris, where "the arts students, they care for naught except to read the books of nature" (that is, Aristotle's natural philosophy), but which none the less starts by loading "the trivium and the quadrivium in a tub on a large cart" as its arms. Among the warriors are necromancy, coming from Toledo and Naples (where translations from the Arabic had been made), together with her accomplice "the daughter of Madam Astronomy" (that is, astrology). Further, we encounter among the Parisian warriors Arithmetic, who counts and calculates (and thus appears to have has nothing to do with the Boethian tradition), Geometry drawing a circle, and Madam music, presented in a way which suggests *musica practica* rather than Boethius. Astronomy herself also turns up repeatedly on the Parisian side.

This actual presence of astronomy and of what it presupposed is confirmed by some famous pedagogical treatises. One was written by Alexandre de Villedieu (c. 1175–1240), who directed a (presumably pre-university) school in

[17] Trans. [Thorndike 1944: 27–30].

Paris in 1209: the *Carmen de algorismo*, a versified introduction to the Hindu-Arabic numerals and their use. It became very popular, but was soon followed by Sacrobosco's *Algorismus vulgaris*, a prose work which may have been meant as an explanatory commentary to the *Carmen* but soon became the foundation on which most subsequent expositions of the Hindu-Arabic system built (Sacrobosco may have been taught on Oxford, but he was a Paris master from 1221 until his death in 1244 or 1256). Sacrobosco also wrote an introductory treatise *De sphaera*, whose use in certain universities lasted until the seventeenth century. There is thus no doubt that a fair number of students were interested in these two subjects, both supports for astronomy. Alexandre also wrote a versified introduction to computus, and Sacrobosco an advanced treatise on the same topic; in particular the former was widely used for a long time in universities.

As to the *Elements*, two apparently contradictory statements confirm that they were read at the university in the 1240s. One was made by the ever-polemical Roger Bacon [ed. Brewer 1859], according to whom the *philosophantes* of his time – probably those whom he had met when in Paris in the 1240s – ran away after the fifth proposition of book I. The other is a collection of *quaestiones* [Grabmann 1934] – a specific university genre emulating the university disputation, raising a question, giving arguments in favour of one answer, formulating the counter-arguments, refuting these, etc. The collection was made in Paris in the 1240s and deals (so it says) with matters that can be discussed at examinations (thus reflecting the advanced level of the whole curriculum). Concerning mathematics, the contents of all fifteen books of the *Elements* is analyzed. Since nothing promises that students were supposed to know them in detail, we may perhaps conclude that a commentary possibly written by Albert the Great [Tummers 1980] reflects better what a teacher would go through – namely the first four books.

Jordanus de Nemore, competing with Fibonacci for the honour of being the best thirteenth-century Latin mathematician, probably taught in Paris somewhere between 1215 and 1240. According to its style and contents, an anonymous *Liber de triangulis Jordani* is a student *reportatio* of a lecture series held over one of Jordanus's works while it was still in process, and thus plausibly by Jordanus himself [Høyrup 1988: 343–351]; however, his teaching appears to have influenced the happy few only.

One of these few – and one who certainly learned from Jordanus, in person or from his writings, was Campanus of Novara. He wrote a *Theorica planetarum*, which (together with a namesake) served university teaching for a couple of

centuries, and which was certainly much more accessible than the *Almagest* (even though the namesake, wrongly attributed to Gerard of Cremona, became the favourite scapegoat of the famous Vienna astronomers Peurbach and Regiomontanus in the fifteenth century). More influential, however, was his version of the *Elements*,[18] written around 1259, which replaced the preceding versions and was only itself replaced as the standard version by that of Clavius in the later sixteenth century. Like the Clavius version in later times, it owed its success to its accomodation to the pedagogical contexts in which it served. As the equally didactic Version II, Campanus thus points out the parallels between geometric and arithmetical propositions – but in agreement with the philosophical mood of the thirteenth-century university, Campanus discusses *why* apparently identical matters are treated twice.[19]

The situation of mathematics at the Paris Arts Faculty seems not to have changed much during the later thirteenth or the fourteenth century. Some scholars connected to the university were certainly interested in mathematics – some, like Nicole Oresme (c. 1320–1382) even made impressive contributions to the field. None the less, the statutes of 1366 [Denifle & Châtelain 1889: III, 143] only require that students admitted to the license should have "heard some mathematical works" along with a specified list of Aristotelian books on natural philosophy; it is not excluded, given the language of the time, that some of works thought of would actually have dealt with the astrological "daughter of Madam Astronomy". A document antedating 1350 explains that the minimal requirement was that bachelors had "heard" *De sphaera* and were following lectures on another work with intention to finish them [Denifle & Châtelain 1889: II, 678].

In any case, astrological chairs were established at the same time at the Paris Faculty of Medicine [Lemay 1976: 200–204], inaugurating a local alliance between medicine and astrology which was to last until the 1530s. Ideally, according to a fourteenth-century list [Lemay 1976: 210*f*], the fundament for astrology included algorism (Sacrobosco's, or a later work on the topic), *De sphaera*, computus, Boethius's *Arithmetica* and *De musica*, Euclid's geometry, Ptolemy's book on the astrolabe and *Almagest*, Theodosios's and Menelaos's treatises on spherical geometry, Jābir ibn Aflaḥ's and al-Bitrujī's works on planetary astronomy, and finally a number of works on the principles of judicial astrology – all considered as "mathematics". How much of this was really taught to the medical students in Paris remains a guess.

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[18] Now available in critical edition [Busard 2005].

[19] Cf. the discussion in [Murdoch 1968].

Fourteenth-century Oxford is somewhat more explicit than the Arts Faculty of Paris. In the statutes from 1340 [Gibson 1931: 33], students passing the baccalaureate were requested to have heard six books of Euclid, Boethius's *Arithmetica*, computus with algorism, and *De sphaera*. It is even stated that geometry was to be heard for five whole weeks, Boethius for three whole weeks, and algorism, sphere and computus each during 8 days (not counting feasts). Since Oxford was the home of the mathematically innovative "Merton College group" (Thomas Bradwardine, Richard Swineshead, etc.), we may safely assume that lectures were held on more advanced topics (proportion theory and its new links to natural philosophy and theology) without being part of the compulsory curriculum – cf. also [Weisheipl 1964: 149]. In later statutes, Euclid may be replaced by Witelo's *Perspectiva*, book I of which is indeed an introduction to geometrical theory.

As we have seen, mathematics belonged with medicine in Bologna and other Italian universities (for Padua, cf. [Siraisi 1973: 67*f*, 77]); so did natural philosophy. From Bologna we have a list of the compulsory mathematical readings for the medical students [Rashdall 1936: I, 248, cf. Thorndike 1944: 281*f*] (undated, but almost certainly fourteenth century): an algorism for integers and fractions (namely the sexagesimal fractions used in astronomical calculation); the astronomical tables of Alfonso X (the "Alfonsine tables"), with rules for using them; the Campanus version of *Elements* I–III; treatises on the use of the astrolabe and the quadrant (another instrument for measuring angles); a *Theorica planetarum*; and book III (the theory of the sun) of the *Almagest*. Boethian quadrivial works are absent, in good agreement with the frequent employment of qualified abbacus masters (see below) as mathematics teachers.

On the whole, the northern universities that were established during the fourteenth and fifteenth centuries emulated Paris. However, Vienna at least was more explicit than Paris about mathematics in its regulations from 1389. Before the baccalaureate, the student should have followed lectures about "the sphere, algorism, the first book of Euclid, or other equivalent books" [Kink 1854: II, 180]; for the *licentia*, they should have followed "*Theorica planetarum*, five books of Euclid, [Pecham's] *Perspectiva communis*, some treatise about proportions, and one on the latitude of forms [the innovations of the fourteenth century, in which Bradwardine and Oresme had been involved], some book on music and some on arithmetic" [Kink 1854: II, 199]. A roughly contemporary document from the newly founded Heidelberg University (closer to the Paris model) requires a student who is examined for the *licentia* to have followed lectures on "several mathematical books in their entirety", and further *De sphaera*; another one fixes

the fees for lectures on a variety of books, including *De perspectiva*, *Elements* I–IV, *De sphaera*, algorism, computus and *Theorica planetarum*, which must thus have been lectured on regularly [Winkelmann 1886: 38, 42]. The list of books that were printed time and again in university towns between 1450 and 1500 [Klebs 1938] indicates the Vienna and Heidelberg documents reflect widespread interests in the late medieval university environment.

We should take note, on the other hand, that even in Vienna mathematics was considered more a pastime than a really serious matter by teaching authorities. In the statutes from 1389 we read [Kink 1854: 196] that since it is better that students "visit the schools than the taverns on feast days, fighting with the tongue rather than with the sword", after noon at such days the bachelors of the arts faculty "should dispute and read gratuitously and for the sake of God on computus and other *mathematicalia*".[20]

We may perhaps wonder why the medieval university, with all its success in the domains of logic and natural philosophy, and in spite of the activity of several noteworthy mathematicians, never brought it far in the domain of mathematics education. At least a partial answer can be derived from its favourite teaching methods. Lectures alone, of course, do not give much, neither in philosophy nor in mathematics (in particular not when students do not have the textbook allowing them to reflect on their own – and we are pre-Gutenberg). However, combined with intensive discussion, they are an ideal means for furthering philosophical perspicacity. As far as mathematics is concerned, lectures combined with discussion favour the development of *metamathematics* – that is, also philosophy. But in order to become creative in mathematics itself, and possibly to enjoy it, one has to *do* mathematics, not only to speak about it. Inside the curriculum of the learned schools and the universities, the areas where one could do mathematics were few. Computus was one such area – but its mathematics did not go beyond simple arithmetical computation. *Rithmomachia* was another one, and the game indeed remained popular until the sixteenth century. The third was computation with Hindu-Arabic numerals in the use of astronomical tables – perhaps not too inspiring either, but none the less a domain that was practised assiduously well into the Renaissance, whether for its own sake or (rather) because it was a *sine qua non* for simple astrological prediction.

[20] More information about mathematics teaching at medieval universities (despite various imprecisions) in the first chapter of [Schöner 1994].

Lay schooling

We know very little about the education of burghers' children after the twelfth-century revival of city life. A few institutions like the Saint-Victor school in Paris admitted them, but what they offered seems to have been badly adapted to a future in commercial life (future artisans were in any case taught as apprentices); Pirenne [1929: 20] relates that a Flemish merchant's son was put into a monastic school around 1200 in order to learn what was needed in trade – but then became a monk. Some clerks served as house teachers in wealthy families [Pirenne 1929: 21*f*], some probably held private schools. That Italian merchants had been taught by Latin-writing clerics is illustrated by Boncompagno da Signa's description (1215) of their letters as written in a mixture of corrupt Latin and vernacular.[21] Computation was presumably learned on the job, during apprenticeship – but even this is nothing but an educated guess built on what we know from later times.

The region which provides us with the earliest detailed information is northern to central Italy. In his *Cronica*, the former Florentine banker Giovanni Villani [1823: VI, 184*f*] states about Florence in 1336–1338 that the children that were baptized

> numbered every year by then 5,500 to 6,000, the boys exceeding the girls by 300 to 500 per year. We find that the boys and girls that were learning to read numbered from 8,000 to 10,000. The boys that were learning the abbacus[22] and the algorism in 6 schools, from 1,000 to 1,200. And those who were learning grammar and logic in 4 higher schools, from 550 to 600.

Allowing for a pre-school mortality of c. 50% (which seems reasonable from what we know about wealthy families[23]), we see that the majority of all children (within the city, *not* the surrounding countryside) learned to read and write (for the reliability of this information, cf. [Goldthwaite 2009: 354]). At least one third of the boys went to the two-year abbacus school learning practical arithmetic, and perhaps one out of ten went to a grammar school (which lasted longer).

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[21] *Rhetorica antiqua*, ed. [Rockinger 1864: 173]. In any case, since notarial documents were written in Latin, merchants needed to understand the rudiments of that language.

[22] The "abbacus" is not, as one might believe, the calculation board; the word (mostly in this spelling) had come to designate practical mathematics – thus already in Leonardo Fibonacci's *Liber abbaci*.

[23] In Fiesole outside Florence, in the relatively benign years 1621-1626, 20% died with the first year of life; later in the century, this rate doubled, with peaks above 50% [Cipolla 1993: 221].

What interests us here is the abbacus school. From around 1260 onward, such schools were created in the commercial towns between Genova, Milan and Venice to the north and Umbria to the south.[24] It was attended in particular by merchants' and artisans' sons, but patricians like Machiavelli and even Medici sons also visited it.

Two documents inform us about the curriculum, one from the 1420s [ed. Arrighi 1967], the other from 1519 [ed. Goldthwaite 1972: 421–425]. Scattered remarks in some of the texts written by abbacus masters confirm their general validity.

At first, the boys learned how to write numbers with Hindu-Arabic numerals. Then they were taught the multiplication tables and their application; the sources do not speak about addition and subtraction, perhaps because these techniques were implicit in the learning of the number system. Division came next, first with divisors known from the multiplication tables, then by multi-digit divisors. Then came calculation with fractions.

After this followed commercial mathematics (in varying order): the rule of three; monetary and metrological conversions; simple and composite interest, and reduction to interest per day; partnership; simple and composite discounting; alloying; the technique of a "single false position"; and area measurement. All teaching from the multiplication tables onward was accompanied by problems to be solved as homework. More complex matters, like the use of a double false position and algebra, are amply treated in many abbacus books but seem not to have been part of the curriculum. They may have been part of the training of assistant-apprentices, but this is another speculation with no support in the sources; what we do know is that proficiency in such difficult matters played a role in the competition for employment (smaller towns often employed abbacus teachers) or for pupils.

Strikingly, the accounting techniques of the great commercial firms were *not* taught in abbacus school, and are not described in the abbacus books. These do not even mention the abacus boards used in the counting houses [*Cambridge Economic History of Europe* III, 90], nor double-entry book-keeping before Luca

[24] Recent discussions of the social history of this institution are [Ulivi 2002a] and (dealing particularly with Florence) [Ulivi 2002b].

Contrary to what is often claimed (also repeatedly by Ulivi), the abbacus school does *not* descend from Fibonacci's *Liber abbaci* – cf. [Høyrup 2005]. There is some (mostly indirect) evidence that the Italian tradition (as already Fibonacci) was inspired from what went on in the Iberian region, but we have no information of how teaching was organized there before the fifteenth century.

Pacioli borrowed a whole *Libro di mercatantie et usanze de' paesi* (already printed in 1481) and inserted it into his *Summa* [1494] [Travaini 2003: 164]. These techniques were assimilated on the job by apprentices who had already visited an abbacus school [Goldthwaite 2009: 83*f*, 91*f*, 354].

Flanders, also home to a wealthy merchant class already in the twelfth century, offers information about the effort of burghers to create their own schools at least from c. 1150 onward – Pirenne [1929: 24–28] shows how the effort was mostly successful, even though church and feudal princes often did their best to keep control and sometimes monopoly. It appears that the schools, like those in Renaissance Germany, taught reading, writing and calculation together (from the thirteenth century onward basic Latin as well as vernacular literacy).

A mercantile arithmetic inserted in the thirteenth-century Picardian *Pratike de geometrie* [ed., trans. Victor 1979: 550–601] probably reflects the kind of computation the Flanders merchants made use of when visiting the fairs of Champagne. Here they met the merchants from Italy, and the arithmetic in question also meets what we know from Italy in some of the problem types it deals with. However, the contact is no more intimate in one than in the other case. The abbacus books generally offer methods that can be justified theoretically and do not excel in unexplained shortcuts, as could be expected from books written by professional teachers of (elementary) mathematics; the Picardian treatise is much closer to what appears to have been tricks developed and used by practical traders, and it was hardly based in a school tradition. We may surmise that there was no mathematics teaching in Flanders similar to that of the Italian abbacus school. That this was indeed so seems to be confirmed by the purely Italian inspiration of the German *Rechenmeister* tradition and the German *Schreib- und Rechenschulen* that emerge in the sixteenth century: they appear to have found nothing of interest in Flanders.

Apart from Iberia and Provence (similar in this respect to Italy), other European regions probably had even less lay teaching of mathematics than Flanders. The Norwegian *Speculum regale* (written perhaps c. 1195) may illustrate this common situation. It contains a long section where a father advises his son, a merchant *in spe*. All it says about mathematics is "practise [*gerðu* / "do"] number skill [*tölvisan*] well, that is much needed by merchants" [ed. Keyser et al 1848: 7]. No school is certainly implied.

Master builders and other artisans

It cannot be excluded that some abbot or bishop asked a master builder to put some sacred number into a sacred building to be constructed – for instance, *three* for Trinity. However, this does not mean that the geometrical knowledge needed for the actual construction had any scholarly or sacred origin. Many artisans may certainly have been taught at the workshops connected to ecclesiastic building activities, but they were taught by more experienced artisans working there, not by monks or priests [*Cambridge Economic History of Europe* II, 772].

The best evidence we have for the actual type of geometric training received by master builders comes from the writings of Mathes Roriczer, himself an experienced master mason – in particular from his *Geometria deutsch* (between 1486 and 1490) [ed., trans. Shelby 1977]. Roriczer is not ignorant of the way more scholarly geometry was written – he uses lettered diagrams, and only uses each letter once; his way to explain the diagrams, however, is not in the scholarly tradition. Some of his constructions are exact and might for that matter come from the scholarly tradition. In their totality, however, they belong to a tradition that had been handed down within the craft since Antiquity and even longer. This tradition, moreover, was wholly separate from that of "practical geometry", which dealt with geometric calculation (and thus with scribal/administrative practice) and not with construction – cf. [Høyrup 2009].[25] Like the geometry of shipbuilders it was never taught in any school of the abbacus type, but only in apprenticeship – until, with the emergence of the engineering profession, the profession-specific and largely oral tradition was crowded out by scholarly mathematics more or less adapted to practitioners needs. That, however, was long after the end of the Middle Ages.

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[25] Shelby [1970] reaches similar conclusions concerning late medieval English masons.

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