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Research article

What is "geometric algebra", and what has it been in historiography?[1]

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Denn eben, wo Begriffe fehlen,
Da stellt ein Wort zur rechten Zeit sich ein.
Mit Worten lässt sich trefflich streiten
Goethe, *Faust* I, 1995-1997

Abstract: Much ink has been spilled these last 50 years over the notion (or whatever it is) of "geometric algebra" – sometimes in disputes so hot that one would believe it to be blood.

However, nobody has seemed too interested in analyzing whether others have used the words in the same way as he has himself (*he*, indeed – as a feminist might declare, "all males, of course"). So, let us analyze what concepts or notions have been referred to by the two words in combination – if any.

Keywords: Ancient Greek mathematics; Babylonian mathematics; Geometric algebra; P. Tannery; H. G. Zeuthen; O. Neugebauer; Á. Szabó; M. Mahoney; S. Unguru: B. L. van der Waerden; H. Freudenthal

Contents

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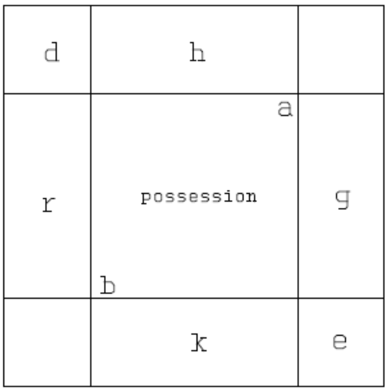
Before “geometric algebra”

Since the possibly earliest written treatment of algebra carrying that name, algebra and the geometry of rectangles and line segments have been linked. When al-Khwārizmī was asked by the caliph al-Ma’mūn to write a brief presentation of the art of *al-jabr wa'l-muqābalah*, he decided not only to let it contain “what was most subtle in this calculation and what is most noble, and what people need” in various commercial and mensurational practices.[2] Knowing from his familiarity with those who were engaged in translation of Greek mathematics (so we may reasonably surmise) that mathematics ought to be based on argument, he also provided geometric proofs for the algorithmic prescriptions for the solution of the mixed second-degree equation types – not by appealing to the propositions of *Elements* II, which his target group was perhaps not likely to know, but by borrowing from the surveyors' geometric riddle tradition.[3] This is most obvious in the case of the algorithm for the equation type “possession and square roots equals number”, normally translated $x^2+\alpha x = \beta$,[4] whose algorithm corresponds to the formula

$$x = \sqrt{\beta + (\frac{\alpha}{2})^2} - \frac{\alpha}{2} \; .$$



The diagram shows the justification as it appears in Gherardo's version [ed. Hughes 1986: 237] (the Arabic version only differs by using Arabic letters). It is adapted to the equation “a possession and 10 roots equal 39 dirham”. The central square represents the possession, and the 10 roots are represented by four rectangles, each of breadth 2½ and length equal to the side of the square, that is, the square root of the possession. The four corners, with area 4×2½×2½ = 25, are filled out, and the resulting larger square thus has an area equal to 39+25 = 64; etc. This is fairly different in style from what we find in Euclid, not strictly deductive but an appeal to what can be “seen immediately”.[5]

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[2] My translation (as all translations in the following if no translator is identified), here from the French in [Rashed 2007: 94]. When the two aims have been in conflict (as in most of the German quotations), my priority has been literal faithfulness (not always obtainable, however) rather than elegance.

[3] On this, see for instance [Høyrup 2001].

[4] A pedantic note: Literally (and probably in an original riddle appearance), the translation should be $\mathrm{y} + \alpha\sqrt{\mathrm{y}} = \beta$. However, by presenting the type “possession equals number” in normalized and the type “roots equal number” in non-normalized form al-Khwārizmī shows that *his* real unknown is the root. If not, a statement “possession equals number” would already be its own solution, while the normalized “root equals number” would be a problem.

In order to see that, one needs to look at Gerard of Cremona's Latin version [ed. Hughes 1986: 233], which represents an earlier stage of the work than the surviving Arabic manuscripts (see [Høyrup 1998], and cf. [Rashed 2007: 86]). In these, all equation types appear in non-normalized from, even though the primary accompanying illustration examples are invariably normalized.

[5] This procedure corresponds to the formula

$$x = \sqrt{\beta + 4 \cdot (\frac{\alpha}{4})^2} - 2 \cdot \frac{\alpha}{4} \; ,$$

not to the algorithm that is used. Al-Khwārizmī must have chosen it, either because it was the first to come to his own mind, or because he supposes his readers to be familiar with this configuration (which corresponds to

According to the rules of grammar, “geometric algebra”, whatever it means, must refer to some kind of algebra, only modified or restricted by the adjective. Accordingly, this is *not* geometric algebra but merely a justification of a certain algebraic procedure by means of a borrowing from a different field.

Half a century later, Thābit ibn Qurrah offered new proofs [ed., trans. Luckey 1941]. He does not mention al-Khwārizmī at all but only refers to the procedures of the *ahl al-jabr*, the “*al-jabr* people”– presumably those reckoners whose technique al-Ma’mūn had asked al-Khwārizmī to write about. Most likely, Thābit did not regard al-Khwārizmī 's justifications as proofs proper; his, indeed, are in strict Euclidean style, with explicit reduction to *Elements* II.5–6.

Abū Kāmil does refer to al-Khwārizmī in his algebra, and he writes a full treatise on the topic; but his proofs are equally and explicitly Euclidean [ed., trans. Rashed 2013: 354 and *passim*]. Roshdi Rashed (p. 37) speaks of Thābit 's and Abū Kāmil's proofs as “geometric algebra” (Rashed's quotes).[6] However, what we see is once again *not* (“geometric”) algebra but merely a justification by means of a borrowing – this time from rigorous Euclidean and not from intuitively obvious geometry.

Later Arabic algebrists are no different, and there is no reason to discuss them separately. The same can be said about Fibonacci's *Liber abbaci*, and about the use of geometric justifications from Pacioli to Cardano and his contemporaries.

Slightly different is the case of Jordanus de Nemore's *De numeris datis*. In his attempt to create a theoretically coherent stand-in for Arabic algebra based on axiomatic arithmetic,[7] Jordanus created the arithmetical (and quasi-algebraic) analogues of a number of theorems from *Elements* II (and much more). If the phrase had not already been occupied by a different signification, it would not be totally misleading to speak of this reversely as “algebraic geometry”, that is, geometry translated into something like algebra. In any case, it is no more “geometric algebra” than what we have already discussed. Nor is, of course, Nuñez' or Descartes' use of algebra as a tool for solving geometric problems (on very different levels, to be sure).

When did it start? Tannery or Zeuthen?

In conclusion, we find no “geometric algebra” in the mathematics that was produced since algebra got its name if not necessarily its essence (whether we identify this essence with equation techniques or with Noether-Artin theory). If it makes sense to speak about “geometric algebra”, then it must be as a description of techniques that antedate al-Khwārizmī. And that is indeed what those who speak about it have done, with the exception of Rashed (and a few lesser figures). Since none of those who produced mathematics at that time can have had the idea that they worked on some kind of algebra, “geometric algebra” can only be an interpretive tool, and thus a tool wielded by historians of mathematics.

When discussing historiography, some writers (mostly but not only French) point to Paul

the riddle “the four sides and the area of a square equal *A*”, in circulation since 2500 years in the area and staying alive until Luca Pacioli).

It is easy to make a proof that corresponds directly to the algorithm that is used, and al-Khwārizmī does present it as an alternative. Here, the style is closer to the Euclidean norm, closer also than the proofs corresponding to the two other mixed equation types. The assumption is near at hand that this alternative was added in a second instance, perhaps after discussion with mathematicians more engaged in Euclid.

[6] Rashed identifies it with what “certain historians since Zeuthen have erroneously believed to find in Euclid and Apollonios, among others” − mistakenly, as we shall see.

[7] For this interpretation of Jordanus's intentions, see [Høyrup 1988: 335*f*].

Tannery as the one who introduced the idea of “geometric algebra”. This is misleading. The onslaught on the idea was launched by Árpád Szabó in [1969: 457*f*], and we may therefore look at what is said there. Szabó refers to an article from Tannery's hand, originally from 1882 but republished in [Tannery 1912: 254–280], when stating that

‹1› Those propositions in Euclid which are habitually – since a work by P. Tannery – regarded as “algebraic propositions in geometrical dress” – have in reality only this much to do with algebra that *we* can indeed quite easily point to *our* algebraic equivalents of these propositions.

He quotes only Tannery's title “De la solution géométrique des problèmes du second degree”, and would indeed have been unable to find the idea of “algebraic propositions in geometric dress” expressed in the article. Instead, Tannery [1912: 254] points out exactly what Szabó parades as his own objection:

‹2› When we speak about a second-degree problem, by our educational habits we are immediately brought to think of the general equation:

$$x^2 + px + q = 0 .$$

Maybe Szabó has been entangled in his polemical intention – p. 456 n. 3 he speaks scornfully of Thomas L. Heath's translation of the *Elements* as “his compilation”; alternatively, he is a prisoner of his own educational habits and does not know that the mathematical notion of “second degree” is not restricted to algebraic equations, and was not so in Tannery's times.[8] Slightly later in Tannery's text it is made clear that the more abstract *problem* can be expressed *either* in terms of an algebraic equation *or* as a geometric problem.

An article from 1880, “L'arithmétique des Grecs dans Pappus” (reprinted in [Tannery 1912: 80–105]), makes it clear that the idea of translation between mathematical disciplines is not totally foreign to Tannery. *But it goes the other way.* In a discussion of Pappus's “means” he points out that the calculation of the sub-contraries to the harmonic and the geometric mean involves second-degree equations. He goes on [Tannery 1912: 93]:

‹3› We are led to the conclusion that the inventor knew how to resolve these equations; it is hardly doubtful, after what we know about works made during this epoch (that of Plato) that is was relatively easy for him to find the geometric solution; but since the theory of *means* was, after all, a speculation about numbers, there are strong reasons to believe that he already knew to translate into an arithmetical rule the construction to be made geometrically.[9]

In one passage – but only one, as far as I can find out[10] – in the three volumes of Tannery's *Mémoires scientifiques* that are dedicated to the “exact sciences in Antiquity” does Tannery speak of

[8] One needs only take a look at Michel Chasles' *Rapport sur les progrès de la géométrie* from [1870], where second-degree curves and surfaces turn up repeatedly. These can certainly be described in analytic geometry by second-degree equations, but their characteristic geometric properties do not depend on the choice of that tool − in projective geometry, second-degree curves are those that are equivalent to circles.

It can be added that even an “equation” is not in itself algebraic − that depends on the way it is solved. The use of nomograms is one non-algebraic alternative. Cf. below, quotation ‹69›.

[9] Tannery, we notice, speaks of arithmetic, not algebra. The rules in question (the counterparts of *Elements* II.5−6 and *Data* 84−85) are indeed exactly those which I argued (*pace* Rashed) should not be seen as algebra, at least not when used as mere justifications of an algorithm.

[10] It is also the only passage to which I have been able to find a reference − see below, quotation ‹59›.

"geometrical algebra", namely in an article from 1903. The context is a discussion of Greek mathematical *analysis* and *synthesis*, and here Tannery [1915: 167*f*] states that the geometrical language developed in the fourth century, combining diagrams and words,

> ‹4› presented at the same time all the advantages of the use of letters in Viète's analysis, at least for powers 2 and 3. They had thereby been able to form, probably already at the time of the first Pythagoreans, a veritable geometrical algebra for the first degrees, with very clear awareness that it corresponded precisely to numerical operations.
>
> Even though they did not, on the other hand, reach the general concept of coordinates, their way to examine the conics was fully analogous to our analytical geometry [...].

Here, Tannery takes over the phrase Zeuthen had coined in 1886 (see imminently), but with a slight reserve ("a veritable ..."). In *La géométrie grecque* [Tannery 1887], the phrase does not turn up at all, it seems.

In conclusion: Tannery did not originate the idea of a "geometric algebra"; he used the phrase in a single case only; and he did not use other terms for what Szabó took it to cover.

Since Szabó (and various followers of his) claim that Hans Georg Zeuthen borrowed the idea of a "geometric algebra" from Tannery, and since Zeuthen does use the expression in *Die Lehre von den Kegelschnitten im Altertum* from [1886] (first Danish edition in 1884), Zeuthen is the likely originator.[11] Then what did he mean by it?

Not what Szabó and many others believe or at least claim. Zeuthen was a mathematician engaged in advanced geometry. His starting point [Zeuthen 1886: 6] was the *theory of proportions* as provided ("as generally assumed", thus Zeuthen) by Eudoxos with a new and generalized foundation, eventually adopted by Euclid as a way to handle the similarity of figures. The agreement in terminology and propositions makes it perfectly clear – Zeuthen again – that the Greek mathematicians were fully aware of the link between the arithmetical proportion theory of *Elements* VII–IX and the general theory of *Elements* V:

> ‹5› From this follows, however, that also when using the propositional instruments of the theory of proportions, the ancients – just as we, when we express our algebraic operations in proportions – were able to use the thought of the calculational operations underlying the proportions as personal inspiration.
>
> According to present-day conceptions, however, a use of proportions that can somehow be mastered is inseparable from the employment of a symbolic language that makes manifest their connections and the transformations that are possible according to familiar theorems, and allows one to impress them firmly in memory. Truly, Antiquity had no such symbolic language, but a tool for *visualization of these as well as other operations in form of the geometrical representation and handling of general magnitudes*[12] *and the operations to be undertaken with them.*

[11] That is also the opinion of Bernard Vitrac [1990: I, 366], even though he sees in Tannery's reference to a "geometric algorithm" in the 1882 article the origin of the "algebraic interpretation" of *Elements* II (with an incomplete bibliographic reference to Tannery's article).

[12] A "general magnitude" (*allgemeine Größe*) is explained thus on p. 3: Through the Eudoxean definition of a proportion,

> these definitions of the magnitude of a ratio in its relation to others were the same as those that characterize the general magnitudes which underlie the algebra of today, and which continuously go through all values, not only such as stand in a rational ratio to a certain unit.

Symbolic algebra and geometry are thus seen as parallel in this respect, none of them expresses the other. If anything, Zeuthen claims that *we* make use of an "algebraic arithmetic of proportions" and *the Ancients* of a "geometric arithmetic of proportions".

A description follows of how a line in a diagram, even though its magnitude is actually determined, can function as a representative of a completely general magnitude, restricted only by the explicitly assumed presuppositions. Further (p. 7),

‹6› The application of this tool allowed one to continue irrespectively of the discovery of irrational magnitudes. This discovery, which hampered the use of arithmetical tools, would for that very reason be particularly favourable for the development of that geometrical tool.

This leads to the introduction of the concept of a "geometric algebra" (which, in the way it is explained by Zeuthen, can legitimately be considered a concept):

‹7› In this way, a *geometrical algebra* developed; one may call it thus, since, on one hand, like algebra, it dealt with general magnitudes, irrational as well as rational, on the other because it used other tools than common language in order to visualize its procedures and impress them in memory. In Euclid's time, this geometric algebra had developed so far that it could handle the same tasks as our algebra as long as these did not go beyond the treatment of expression of the second degree.

That is, Zeuthen uses the term not because ancient Greek geometrical theory (or a part of it) "translated" algebraic propositions or procedures but because it *fulfilled analogous functions*.[13] That is also stated in the *Vorrede* (pp. IX–X), which promises to show

‹8› that the geometry with the ancients was developed not only for its own sake, but that it served at the same time as an instrument for the theory of general magnitudes, just as algebra today, and that in this respect the doctrine of conics went beyond elementary investigations.

This must be kept in mind when Zeuthen further on in the book formulates himself in a way that suggests a more directly algebraic reading of ancient geometry – for instance on p. 12, when it is stated that

‹9› The first 10 propositions in the Second Book of Euclid can be written in the following way:

$$1.\ a\,(b+c+d+...) = ab+ac+ad+...\,,$$
$$2.\ (a+b)^2 = (a+b)\,a+(a+b)\,b\,,$$
$$[...]$$

That this is a shorthand and no interpretation of "what really goes on" is made clear on p. 13, which explains that

‹10› Our first equation merely expresses that a rectangle is cut by parallels to one of the sides (the height) in new rectangles, whose bases together make up that of the given rectangle.

When Zeuthen uses symbolic algebra, it is explicitly "our [...] algebraic representation" (p. 18).[14]

[13] Analogous, not fully identical functions. Zeuthen points out repeatedly when one of the two representations (not always the same) is more flexible than the other.

[14] And, as Ivo Schneider [2016: vii] points out:

In order to judge the adequacy or inadequacy of such a request [that of Sabetai Unguru to rewrite

It should be kept in mind, however, that Zeuthen sees certain *problem types* as being *above* the distinction between algebra and geometry (as discussed above in connection with Tannery and the “second degré”). They can be expressed and solved by both, but that does not effect the problem itself – even though Zeuthen allows himself to speak of them as equations he does not believe (as do many later historians of mathematics, nobody mentioned, nobody forgotten) that a problem is in itself algebraic just because *we* are tempted to solve is by means of algebraic manipulations. That his “equations” are not meant to be algebra expressed through geometry but as descriptions of genuinely geometric procedures can also be seen in the reference on p. 21 to Apollonios's “use of the application of areas or quadratic equations”.

Further on, first phase: Thomas Heath and Moritz Cantor

In [1896], Heath published a translation of Apollonios's *Conics* “edited in modern translation” with a long introduction; this introduction is what is of interest here. He is close to Zeuthen (even though he allows himself to disagree on certain points, for instance on p. lxxii), and takes over Zeuthen's concept of a “geometric algebra” identified with proportion theory combined with the application of areas (pp. ci–cv). But he is adamant, already in the first lines of the preface (p. vii), that Apollonios reaches his results “by purely geometrical means”.[15] On p. cxi he also points to the contrast between Apollonios's geometric method in *Conics* III.26 and Pappos's treatment of the same matter (*Collection* III, lemma 4), which proceeds “semi-algebraically”.

So, Heath had understood Zeuthen perfectly, and uses the concept with the same care but perhaps with greater sharpness. When stating (p. cii) that the theory of proportions (one of the two constituents of his geometric algebra) “is capable of being used as a substitute for algebraical operations” this is meant transhistorically, not (as Szabó would have it) as a claim that the Greeks were in possession of algebra and then created a substitute.

In the preface to his similar *The Works of Archimedes* [1897: xl] as well as in his *History of Greek Mathematics* [1921: 150–153], we find the same explanation though abbreviated. In the latter work we also find the idea that the geometrical algebra was used to solve numerical problems; this is concluded from the assumed invention of *Elements* II.9–10 for the purpose of “finding successive integral solutions of the indeterminate equations $2x^2 - y^2 = \pm 1$”. This was a dubious assumption already at the time (it has to do with the “side-and-diagonal-number algorithm”), but in any case this shows us that Heath did not believe in the existence of an arithmetical algebra that was translated into geometry.

Finally, in his translation of the *Elements* [1926: I, 372–374], Heath returns to the matter, still emphasizing that “geometrical algebra” builds on application of areas and proportion theory combined, and still in terms that echo those of Zeuthen. On p. 373 it is further stressed how important it is

‹11› to bear in mind that the whole procedure of Book II is geometrical; rectangles and squares are shown in the figures, and the equality of certain combinations to other combinations is proved by those figures.

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the history of Greek mathematics completely, see below] it is, for example, necessary to distinguish whether an author represents the contents of a Greek mathematical text in algebraic dress while referring to the underlying geometrical argumentation of the original, or he claims the algebraic representation to correspond to the proper thought of the Greeks.

[15] Actually this is said about “what amounts to the complete determination of the evolute of any conic”, but it sets the tone.

So, the semi-algebraic way to prove the propositions from II.2 onward presumably introduced by Heron is argued to be later; once again, that does not suggest the hypothesis that *Elements* II.1-10 should be a translation of a pre-existent algebra.

Slightly later (p. 383) comes the line which provoked Szabó's disdain for Heath's "compilation", namely "Geometrical solution of a quadratic equation". It turns out to be a headline, covering a discussion of *how* proposition II.5 and 6 can be used to solve "problems corresponding to the quadratic equations which are directly obtainable from them". So, this is no claim about what these propositions *are*, but about what they can be used for (and *were* used for from Thābit onward); and as everywhere, Heath speaks about correspondence, not of identity or underlying/preceding algebra.

The third edition of Moritz Cantor's *Vorlesungen* I contains a reference to Zeuthen's idea, namely the passage [Cantor 1907: 285]

‹12› The geometrical shape, in which those problems [*Data 84–85* and related matters in the *Elements*] appear, and which not improperly have been designated geometrical algebra [a footnote refers to Zeuthen], would not suffice to refute all algebraic awareness [...]. Euclid must have had to do with numerical quadratic equations, as this is the only way to explain the creation of Book X of his *Elements* [another footnote referring to Zeuthen].

This seems a very reduced variant of Zeuthen's point of view, and almost a rejection: Cantor sees Zeuthen's "geometric algebra" not as an expression of "algebraic awareness" but as possible objection to the presence of such an awareness (an objection which he then rejects). A comparison with the corresponding passages in the first and second editions [Cantor 1880: 245; 1894: 270] is illuminating. In the former, antedating Zeuthen's book, we find this:

‹13› The geometrical shape, in which those problems appear, would in any case not suffice to refute all algebraic awareness.

The latter has

‹14› The geometrical shape, in which those problems appear, would not suffice to refute all algebraic awareness [...]. Euclid must have had to do with numerical quadratic equations, as this is the only way to explain the creation of Book X of his *Elements* [a footnote referring to Zeuthen].

As it becomes obvious, Cantor is not illuminated by Zeuthen's discussion, he simply glues Zeuthen's phrase to a formulation he had made himself years before, as a commentary to the "geometrical shape" which should not mislead. His own idea is indeed rather different from that of Zeuthen – there is nothing about analogous function, nothing about a technique involving application of areas *and* theory of proportions. Instead, Cantor appears to see *Elements* II.5–6 and *Data* 84–85 as instances of a geometric form impressed upon an underlying algebraic awareness. The second part of the formulations from 1894 and 1907, on the other hand, is a genuine borrowing from Zeuthen, but once again it serves as support for a thesis that was already present in 1880.

Interestingly, all three editions contain a Chapter 35 "Number theoreticians, calculators, geometrical algebraists from c. 950 to c. 1100". This was thus written for the first time well before Zeuthen. Cantor spoke of certain mathematicians as "geometrical algebraists". They seem to be such as used theoretical geometry as a tool for genuine *al-jabr* algebra, for example al-Karajī and al-Khayyāmī.

Further on, second phase: Otto Neugebauer and Thureau-Dangin

Neither Heath nor Cantor had any noticeable influence on later references to the "geometric algebra”. That the notion did not die a quiet death was due to Otto Neugebauer and his interpretation of *Elements* II in the light of the newly discovered Babylonian “algebra”.[16]

Neugebauer's designation of this Babylonian technique as an “algebra” hinged upon his somewhat idiosyncratic delimitation of this latter term as given in the first of three articles “Studien zur Geschichte der antiken Algebra”. Here he says [Neugebauer 1932a: 1] to understand

‹15› the word “algebra” as substantially broad as possible, that is, I include also problems with a strong “geometric” emphasis, if only they seem to me to be on the way toward a formal operation with magnitudes that is ultimately “algebraic”.

That is, what Neugebauer sees as *a developmental step* toward the mature algebra of the seventeenth century (CE) is *eo ipso* covered by the algebraic heading. “Formal operations” as Neugebauer finds them in the Babylonian context are operations that seem ontologically meaningless, acting on *measuring numbers* but not allowing any corresponding operation on the entities that are measured (adding for instance linear and planar extensions).

The second article, dedicated to Apollonios, starts by arguing why Apollonios is pertinent for a history of ancient algebra [Neugebauer 1932b: 215*f*]: This has, firstly,

‹16› a purely external reason: certain cuneiform texts, whose appurtenance to the area of algebraic problems is not to be doubted [...], call for a precise insight in the Greek theory for second-degree expressions, in particular the “application of areas”, as a precondition for a profounder historical interpretation. In this way the ancient theory of conics came automatically into the centre of the investigation.

This must have seemed enigmatic at the moment – the explanation of why Neugebauer sees a connection between the application of areas and Babylonian “algebra” (which he understood as a purely numerical technique) was only to be given in the third article. In 1932, readers will have had to concentrate on the second part of the argument: that

‹17› in the apparently purely geometric theory of conics much is hidden that can provide us with keys to the so to say latent algebraic components of classical Greek mathematics. Here I do not refer to the familiar fact of the “geometric algebra”, which we encounter everywhere; but I think of a wholly different facet of the “algebraic” (I am quite aware that such a conceptual delimitation can be challenged): the existence of certain “algorithms”, according to which analogous cases can be dealt with quite schematically. Once such an “algorithm” exists, this may have direct immediate consequences that lead directly into the purely algebraical: Renunciation of homogeneity of the dimensions of the magnitudes that appear, emergence of a conventional symbolism, which then on its part leads to a widening of all conceptualizations, etc. That precisely such things do not occur in Greek geometry is part of our basic arsenal of historical insight. The first impression certainly confirms this claim. In the following I shall, however, try to produce the proof for specific examples that the external construction may differ strongly from the inner motivation of the demonstrations, and that precisely in this substructure very much hides which in a certain sense can be characterized as “algorithm”.

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[16] As anybody familiar with the history of Mesopotamian mathematics knows, a distinction between periods is mandatory − in the present case at least between the Old Babylonian and the Seleucid epochs, from which “algebra” texts are known. But since this distinction is rarely made and never emphasized in the texts I discuss, I shall ignore the wringing of my bowels and speak as they do.

Whether or in which sense Old Babylonian or Seleucid “algebra” are “algebras” is unimportant in the present connection − after all, it depends on definitions.

As we see, Neugebauer does not claim to continue Zeuthen's mode of analysis – it is simply dismissed as uninteresting old stuff. Instead, what he says here is in line with his earlier idiosyncratic notion of algebra as concerned with "formal operations"; the reference to "algorithms" merely introduces a new facet. Neugebauer's underlying algebra is not defined from method nor from analogous use, as is Zeuthen's geometric algebra. It seems rather to be determined through opposition to the surface appearance of Greek geometry, which Neugebauer then argues *is* sometimes only surface.[17] Below the surface, he sees it as much closer to the Babylonian numerical technique that one would expect; the difference is mainly one of style (p. 217):

‹18› So, when I claim that there may be a deep difference between outward construction and inner method in Apollonios' *Conics*, then I thereby emphasize the necessity of questions that can almost be characterized as dealing with the "history of style".

This appears to have had even less impact than Zeuthen's thinking. In principle, it may be justified to launch new ideas in the context of an analysis of the *Conics* if the *Conics* is the text which illustrates their carrying power best. Strategically, however, it is a mistake – the unavoidable technicalities of the topic reduce the audience to a very restricted circle, as Neugebauer knew – on p. 218 he observes that

‹19› From Zeuthen's fundamental work "Die Lehre von den Kegelschnitten im Altertum" (Copenhagen 1886), astonishingly little has entered the literature. This may in part be due to the not always convenient and clear exposition, but in part also on the fact that Apollonios himself is anything but easy to understand.

What gained influence (duly transformed) was the third article, entitled "Zur geometrischen Algebra" [Neugebauer 1936]. At first, Neugebauer speaks about Babylonian mathematics (p. 246):

‹20› The most important outcome of the interpretation of Babylonian mathematics is the revelation of its algebraic character. I have often pointed out that this character mainly relies on the existence of a symbolic writing system, which in itself allows a kind of formal writing, and analyzed how the emergence of this technique is related to the general history of the Babylonian culture.

That we really have to do with essentially algebraic matters follows, in spite of frequent (yet not at all exclusive) geometric dressing, from the repeated appearance of non-homogeneous expressions (addition of "segments" to "surfaces" and "volumes". Similarly, "days" and "people" are added without reserve).

The first paragraph of the quotation refers to Neugebauer's belief that the use of ideographic writing in the Babylonian texts should be understood as an algebraic symbolism. This had been more fully explained in Neugebauer's *Vorgriechische Mathematik* [Neugebauer 1934: 68]:

‹21› Any algebraic working presupposes that one possesses certain fixed symbols for the mathematical operations as well as the magnitudes. Only the existence of a conceptual notation of this kind makes it possible to combine magnitudes that are not numerically identified with each other and to derive new combinations from them.

But a symbolic notation of this kind was automatically at hand in the writing of Akkadian texts. As we have seen, two different ways to express oneself were indeed at

[17] Actually, certain formulations indicate that Neugebauer also thought in terms of analogous function − but analogous to recent algebraic theory, not (as Zeuthen) to analytical geometry. Thus, p. 219 n. 5:

> The addition of the conjugated hyperbola is a genuinely new idea (which methodically corresponds precisely to the introduction of "ideal elements" enlarging the area of validity of a formal system).

hand here: either to make use of the syllabic way of writing, or to write with ideograms. Most Akkadian texts shift continually and quite arbitrarily between the two ways of writing. Now this outcome of a purely historical process was of fundamental importance for the mathematical terminology. There, indeed, it became the fixed habit to write mathematical concepts ideographically, operations as well as magnitudes. That means, then, that in a text written in Akkadian precisely the decisive concepts were written by means of conventional single symbols. Thereby one disposed from the very beginning of the most important basis for an algebraic development, namely an adequate symbolism.

For the present purpose it is immaterial that Neugebauer is demonstrably mistaken on this account.[18] Even if he had been right, however, we would once again be confronted with an idiosyncratic understanding – there is no request that *operations* be performable at the level of "symbols". In Nesselmann's classical terms [1842: 302], this would be a case of *syncopated*, not symbolic algebra.

Returning to the article on "geometrical algebra", we see that the other argument for the algebraic character of Babylonian mathematics is the use of "formal" operations, that is, of problem statements in terms of the measuring numbers of magnitudes.

The next section then deals with Greek geometry. It begins thus [Neugebauer 1936: 249]:

‹22› Zeuthen we may thank for an insight that is fundamental for the understanding of the whole of Greek mathematics, namely that in particular Book II and VI of Euclid's *Elements* express geometrically problems that are properly algebraic. In particular he has pointed out in many passages that the problems about "application of areas" in Book VI and the appurtenant propositions of the *Data* contain a full discussion of the equations of the second degree. He has further shown how this "geometric algebra" forms the basis for the "analytical geometry" of Apollonios's *Conics*, whose designations "ellipse", "hyperbola" and "parabola" still point back to the fundamental cases of the "application of areas" today.

This distorts what Zeuthen had actually said. Instead of analogous use, we get a statement about an algebraic essence of geometrically formulated problems; proportion theory seems to have left the scene completely (but see note 19). Moreover, as we have already seen, Neugebauer's idiosyncratic understanding of "algebra" differs strongly from what Zeuthen had meant by that word.

Neugebauer continues (pp. 249–250):

‹23› The central problem that remains after Zeuthen's investigations is the question: *How does one come to so peculiar questions as asked by the "application of areas"* To "apply" a given area on a given line in such a way that a rectangle of given shape is missing ("elliptical" case) respectively in excess ("hyperbolic" case).

The answer to this question, that is, to the question about the historical cause of the fundamental problem of the whole of the geometric algebra, can now be given in full: It lies, on one hand, in the request of the Greeks (following from the discovery of irrational magnitudes) to ensure the general validity of mathematics through a transition from the domain of rational numbers to that of general ratios between magnitudes;[19] on the other in the ensuing need *to translate also the results of*

[18] Firstly, only a few "magnitudes" and one operation (not really an operation, but functioning as such) is almost always written with a single-symbol ideogram. All genuine operations are sometimes written in one, sometimes in the other way even within the same text. Secondly, the ideographic writing (and for that matter, but less often, syllabic text) is sometimes ambiguous and only to be understood from the global context.

[19] Though it is not mentioned earlier, Neugebauer is thus aware that it is not so much geometry as the theory of proportions that has to take the place of arithmetic.

pre-Greek "algebraic" algebra into a "geometric" algebra.

Once one has formulated the problem in this way, then everything else is completely trivial and provides *the smooth junction of the Babylonian algebra to Euclid's formulations*. Still, one must then start from the state of Babylonian algebra [...]: the "normal cases" of the Babylonian equations of the second degree are the problems, to determine two magnitudes x and y from

$$(1)\quad \begin{matrix} xy = a \\ x + y = b \end{matrix} \qquad \text{respectively} \qquad (1^*)\quad \begin{matrix} xy = a \\ x - y = b \end{matrix}$$

Indeed, the immediate translation into geometry obviously runs: Given a segment b and an area c^2 [...]. On shall divide b into two partial segments x and y in such a way that $x + y = b$ (the discussion of case (1) is sufficient) and that $x \cdot y = c^2$.

After a short exposition of how the application of an area with deficit works Neugebauer can conclude (p. 251):

‹24› Thereby is has been shown that the whole application of areas is nothing but the mathematically evident geometric formulation of the Babylonian normal form of quadratic equations. It is equally trivial to show that even the Greek solving method is nothing but the literal translation of the Babylonian formula (2):

$$(2)\qquad \left.\begin{matrix} x \\ y \end{matrix}\right\} = \frac{b}{2} \pm \sqrt{(\frac{b}{2})^2 - c^2}$$

This is the foundation for subsequent references to the "geometric algebra" (again with further reinterpretations). The riddle why somebody should ask the odd questions of applying an area along a line as a rectangle with deficit or excess *before* its usefulness in the theory of conics had become manifest, together with the structural similarity of both question and method leads him to the following scenario: *The discovery of incommensurability*[20] *made a direct continuation of the Babylonian arithmetical algebra impossible, at least within theoretical mathematics. In order to save its results, the Greeks therefore undertook to translate these into the language of geometry.* Basically and below the surface, however, the translations remained algebraic – thus Neugebauer.

That this article set the scene for the future understanding of "geometric algebra" is illustrated by Len Berggren's introduction to the topic in his "History of Greek Mathematics: A Survey of Recent Research" [1984: 397]:

‹25› The scholarly point at issue here is whether it is historically justified to interpret parts of Greek mathematics, typified by Book II of Euclid's *Elements*, as translations of Babylonian algebraic identities and procedures into geometric language.

Soon after the appearance of the article "Zur geometrischen Algebra", Neugebauer switched his main interest to astronomy [Høyrup 2016: 185*f*]; after the publication of the third volume of the *Mathematische Keilschrifttexte* in 1937 his only major publication on Babylonian mathematics was the volume *Mathematical Cuneiform Texts*, coedited with Abraham Sachs in 1945. So, he had little occasion to return to the matter, except in the popularization *The Exact Sciences in Antiquity* (first published in 1951, second edition in 1957 reprinted in [1969]), where he presents the same scenario rather briefly, concluding the topic with these words [Neugebauer 1969: 150]:

‹26› Attempts have been made to motivate the problem of "application of areas" independently of this [Babylonian] algebraic background. There is no doubt, however,

[20] Already taken into account by Zeuthen but brought to the fore by the presumed identification of a "foundational crisis" to which this discovery should have given rise [Hasse & Scholz 1928].

that the above assumption of a direct geometrical interpretation of the normal form of quadratic equations is by far the most simple and direct explanation. I realize that simplicity is by no means equivalent with historical proof. Nevertheless the least one must admit is the possibility of the above explanation.

In the article “The Survival of Babylonian Methods in the Exact Sciences of Antiquity and Middle Ages”, Neugebauer still argues in favour of the link [Neugebauer 1963: 530]:

‹27› For Greek mathematics the picture now becomes quite clear. It hardly needs emphasis that one can forget about Pythagoras and his carefully kept secret discoveries. It is also clear that a large part of the basic geometrical, algebraic, and arithmetical knowledge collected in Euclid's *Elements* had been known for a millennium and more. But a fundamentally new aspect was added to this material, namely the idea of general mathematical proof.

The notion of a “geometric algebra”, however, does not occur.

Not only Neugebauer virtually stopped his active work on Babylonian mathematics in 1937. So did François Thureau-Dangin. However, his tense but polite race with Neugebauer during the 1930s is a likely background to an article “L'origine de l'algèbre” which he published in [1940],[21] even though the direct occasion (p. 292) was a recent claim by Gino Loria, who,

‹28› in order to draw a limit between arithmetic and algebra, has proposed to characterize algebra by the methodical use of symbols for known and unknown quantities as well as for operations.

As Thureau-Dangin further reports Loria, algebra would thereby become a wholly modern method, not known by the ancients.

Thureau-Dangin instead (p. 293) wants to understand algebra as

‹29› an application of the analytical method in the resolution of numerical problems. The essence of the procedure is that one thinks of the number as a known number and formulates the problem, considered as already solved, in the shape of an equation, then transforming this equation step by step, which leads to a final shape in which the unknown number appears alone on one side and a known quantity on the other. If more than one unknown appears in the problem, it is reduced to a single unknown through elimination of the others by appropriate procedures.

Thureau-Dangin therefore sees a long prehistory, discussing at first the Arabic technique that provided algebra with its name (mentioning also Diophantos's explicitly analytic rules for reducing an equation). He then goes back in time (p. 295):

‹30› As long as only rational numbers were recognized as numbers and one did not know how to calculate with irrational radicals, the field of application of the algebraic method remained strictly limited. This is the likely reason that the Greeks were brought to develop a method which Zeuthen has called geometric algebra. Instead of operating on numbers, as does algebra proper, the geometric algebra operated on geometric magnitudes (line segments and rectilinear figures), that is, on continuous magnitudes, commensurate as well as non-commensurate.

[21] Here, Thureau-Dangin is also in dialogue with an article by Solomon Gandz from [1937], “The Origin and Development of the Quadratic Equations in Babylonian, Greek, and Early Arabic Algebra”. In that article, Gandz speaks as a matter of course about Euclid's “geometric algebra (II,1−10)”. Since he says nothing more, it is impossible to know whether he was inspired by Zeuthen, by Heath, by Cantor, and/or by Neugebauer.

A philologically informed discussion of the application of areas follows (simple, or with excess or deficiency). In this connection there is also a reference on p. 296 to Tannery's above-mentioned article from 1882.

Thureau-Dangin does not mention the use of proportion techniques; not does he take up the use of "geometric algebra" in the *Conics*. On their part, neither Tannery nor Zeuthen speak of the analytic character of algebra (both probably take it for granted, after all it is inherent in the name "analytic geometry"). All in all, however, Thureau-Dangin is a more faithful reader of the two than Neugebauer, who had used Zeuthen freely for his own purpose (and not mentioned Tannery).

After a reference to Tannery's suggestion of the primacy of arithmetic, Thureau-Dangin goes on with a presentation of Babylonian "algebra". He considers it an algebra in his sense, even though, as he says, the texts (as he read them) are purely synthetical, being convinced that the synthesis must have been based on some kind of analysis (p. 300):

‹31› In general, the Babylonian algebraic problems are resolved in a purely synthetic fashion: the analysis that has guided the scribe to the operations which he performs, the very shape of the equations which he thinks of, can only be reconstructed by conjecture.

Thureau-Dangin does not think of the geometric algebra of *Elements* II etc. as a translation of Babylonian knowledge. Truly, one might believe this if reading a formulation on p. 309 superficially:

‹32› Propositions 5, 6, 9 and 10 from the second book of Euclid translate into geometrical terms the equations by which the Babylonian method expresses algebraically the product of two unknowns or the sum of their squares.

But "translate" (*traduisent*) is in the present, not the past tense; it expresses correspondence or perhaps (much less likely) *our* translation. And on p. 300 we read that

‹33› Let us say it immediately: there is no trace in Babylonia of geometrical algebra. In this way the question of the relative age of the two kinds of algebra is decided. In its origin, the geometrical algebra is a purely Greek method and the numerical algebra, known by the Greeks, very probably has a Babylonian origin, as we shall see.

All in all, Thureau-Dangin's thinking *around* the idea of a "geometric algebra" is at variance with that of Neugebauer. Even their respective ideas about what "geometrical algebra" *is* seem discordant

In the wake

As often happens, others climbed on the shoulders of the giants (not exactly dwarfs, but historians of more normal stature). Did they see longer?

Let us first look at B. L. van der Waerden, who probably exerted the largest influence, in particular through his *Science Awakening*. This work first appeared in Dutch as *Ontwakende Wetenschap* in 1950, which was translated into English in 1954 (I have not seen either of these). A corrected German edition (*Erwachende Wissenschaft*) appeared in [1956], and a second, further corrected English edition in [1961].

The second English edition was the main channel for spreading a transformed interpretation of Neugebauer's idea, so I shall refer to that. On pp. 118–124 we find a chapter " 'Geometric Algebra' ", while the immediately following pages 125–126 discuss the question "Why the Geometric Formulation?"[22] The former begins

[22] In the German edition [van der Waerden 1956: 193] the corresponding headings are "Die geometrische Algebra" (without quotation marks) and "Wozu die geometrische Einkleidung?".

‹34› When one opens Book II of the Elements, one finds a sequence of propositions, which are nothing but geometric formulations of algebraic rules. So, e.g., II 1: *If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments*, corresponds to the formula

$$a(b+c+...) = ab+ac+...$$

II 2 and 3 are special cases of this proposition. II 4 corresponds to the formula

$$(a+b)^2 = a^2+b^2 + 2ab\ .$$

[. . .]

Quite properly, Zeuthen speaks in this connection of a "geometric algebra". Throughout Greek mathematics, one finds numerous applications of this "algebra". The line of thought is always algebraic, the formulation geometric. The greater part of the theory of polygons and polyhedra is based on this method; the entire theory of conic sections depends on it. Theaetetus in the 4th century, Archimedes and Apollonius in the 3rd are perfect virtuosos on this instrument.

This is accompanied by diagrams, close to those of the *Elements* though not identical.

As we see, there is no explanation here of what is meant by algebra; nor is there any earlier explicit discussion. Possibly, the author of *Moderne Algebra* thought the answer to be obvious. In any case, his implicit understanding owes nothing to that abstract algebra whose prophet he had been - see also quotation ‹63› for van der Waerden's explanation of his usage in 1976. He seems to have endorsed Neugebauer's argument from formal operations: on p. 72 it is argued that the Babylonians may well have found some of their rules from geometric diagrams, as suggested among other things by their geometric terminology.

‹35› But we must guard against being led astray by the geometric terminology. The thought processes of the Babylonians were chiefly algebraic. It is true that they illustrated unknown numbers by means of lines and areas, but they always remained numbers. This is shown at once in the first example, in which the area *xy* and the segment *x–y* are calmly added, geometrically nonsensical.

Elsewhere it seems that the possibility of direct expression as a symbolic equation and possibly the use of operations that emulate those made on equations (in other words, homomorphic structure) are taken as symptoms of algebraic thought.

As we see, van der Waerden follows Neugebauer, Heath and/or Zeuthen, speaking about the use of geometric algebra in the conics. However, even though van der Waerden must have read Zeuthen's *Lehre von den Kegelschnitten*,[23] he did not understand what Zeuthen had meant by "geometric algebra". On p. 4 he states that

‹36› Neugebauer, following in the tracks of Zeuthen, succeeded in discovering the hidden algebraic element in Greek mathematics and in demonstrating its connection with Babylonian algebra.

The idea of a hidden algebraic element had been foreign to Zeuthen, as we have seen; moreover, whereas Zeuthen's "geometric algebra" was a synthesis of application of areas and proportion theory

[23] Mostly, his fairly copious references to the book are not fully specific, but on p. 259 there is a correct page reference.

(the condition that he could speak of an analogous function), van der Warden sees the two techniques as clearly distinct and essentially unconnected (pp. 264, 266).

Since van der Waerden had seen Neugebauer's “Apollonius Studien” [Neugebauer 1932b] (p. 259 n. 1 contains a reference), and almost certainly Neugebauer's article on geometric algebra from [1936],[24] we may assume that van der Waerden's inspiration when he speaks about “geometric algebra” is Neugebauer.

Neugebauer is also the indubitable source for van der Waerden's view of Greek “geometric algebra” as a translation of Babylonian algebraic knowledge into geometric language necessitated by the discovery of incommensurability.

But van der Waerden is no mere copyist or epigone. He develops Neugebauer's idea into a general interpretive tool for Greek mathematics, used, for instance, to analyze the side-and-diagonal numbers (pp. 126*f*),[25] the Delic problem (p. 161) and Theaitetos (p. 119 and *passim*).

On one point, van der Waerden also disagrees rather openly with Neugebauer. While Neugebauer (admittedly only in [1963: 118] in these words, though with a similar attitude in [1933: 316]) would state that it “hardly needs emphasis that one can forget about Pythagoras and his carefully kept secret discoveries”, van der Waerden takes from Proclos/Eudemos that the technique of application of areas (and thus the assumed translation of Babylonian knowledge) should be ascribed to the Pythagoreans (p. 118 and *passim*).

Van der Waerden's book covers precisely the two mathematical cultures that are central to the notion of a geometric algebra as developed by Neugebauer. We may also have a brief look at a few general histories of mathematics and see how they deal with the topic.

First Dirk Struik's *Concise History of Mathematics*, published in [1948] (and often later, with revisions). About the mathematics of the Hammurabi epoch Struik states (vol. I p. 26) that here “we find arithmetic evolved into a well established algebra”. There is no explanation of what is meant by that, but since Struik also characterizes the Egyptian *pws*- or *pesu*-problems as “primitive algebra” (p. 22), he is likely simply to see problems which *we* would solve by means of algebraic equations as algebra. In any case, the passage and what follows immediately after it is inspired from Neugebauer's survey article “Exact Science in Antiquity” from [1941: 28*f*]:

‹37› Those texts are pure mathematical texts, treating elementary geometrical problems in a very algebraic form, which corresponds very much to algebraic methods known from late Greek, Arabian, and Renaissance times.

In Struik's presentation of Greek mathematics, only three passages are pertinent. On p. 58 it is said that

‹38› Among these other texts [by Euclid] are the “Data,” containing what we would call applications of algebra to geometry but presented in strictly geometrical language,

and on p. 60*f* that

‹39› Algebraic reasoning in Euclid is cast entirely into geometrical form. An expression $\sqrt{a}$ is introduced as the side of a square of area A, a product *ab* as the area of a rectangle with sides *a* and *b*. This mode of expression was primarily due to Eudoxos' theory of proportions, which consciously rejected numerical expressions for line segments and in

[24] Apart from the obvious though not specific reference “Neugebauer, following in the tracks of Zeuthen”, we may observe that [van der Waerden 1938] was published in *Quellen und Studien* B. Van der Waerden can be presumed to have followed the journal at the time.

[25] This could be a borrowing from Heath, cf. above.

this way dealt with incommensurables in a purely geometrical way.

As we see, the problem arising from incommensurability is mentioned, and underlying algebraic thought is taken for granted. Nothing is said, however, about translating and saving Babylonian insights. In the third revised edition [Struik 1967: 52] the sentence "Linear and quadratic equations are solved by geometrical constructions leading to the so-called 'application of areas'" is inserted before "This mode".

Nothing more is said elsewhere about "translation" of Babylonian results (nor is it indeed told in [Neugebauer 1941]). Babylonian algebra is supposed only to have inspired Diophantos (p. 74):

‹40› The Oriental touch is even stronger in the "Arithmetica" of Diophantos (c. 250 A.D.). Only six of the original books survive; their total number is a matter of conjecture. Their skilful treatment of indeterminate equations shows that the ancient algebra of Babylon or perhaps India not only survived under the veneer of Greek civilization but also was improved by a few active men.

Joseph Ehrenfried Hofmann knows and refers to the Dutch first edition of *Science Awakening* in his equally concise *Geschichte der Mathematik* from [1953]–1957. Even he, however, avoids the idea of translation. In the discussion of the Babylonians in vol. I he does not refer to algebra at all, and only twice (pp. 13, 14) to "equations".

Even when dealing with the Greeks, Hofmann does not endorse the idea of a "geometric algebra", even though he does mention algebra a few times.

About the *Elements* we find on p. 32 that

‹41› The second book contains algebraic transformations such as the calculations of $a(b+c)$ oder $(a+b)^2$ in geometrical dress. This serves the resolution of the general quadratic equations, which is presented through the example $x^2 = a(a–x)$

(a debatable claim, to be sure). On p. 33, now about *Elements* VI,

‹42› Of particular importance is the handling of quadratic equations through application of areas (in continuation of II, 4/6), that was taken up by Apollonios and reinterpreted.

Two other references speak of algebra proper. So, on p. 42, about Hipparchos:

‹43› The Muslims also ascribe to him an algebraic work, in which quadratic equations were perhaps dealt with.

This seems to be a reference to what is found in al-Nadīm's *Fihrist* [ed. trans. Suter 1892: 22, 39] – that Hipparchos wrote a book about algebra, and that Abū'l Wafa' made a commentary to it provided with geometric demonstrations.[26]

Finally, on p. 45, Diophantos is spoken about:

‹44› Completely different is the most significant arithmetical work of Diophantos of Alexandria (c. 250 ad), who contrary to Greek manner took up and continued Egyptian-Babylonian traditions.

So, like Struik, Hofmann distances himself indirectly from the idea that the techniques of *Elements* II and VI should be geometric translations of Babylonian results (such translation being "contrary to Greek manner"/*in ungriechischer Weise*). While Struik may not have known about the thesis in 1948 (but in 1967 he certainly knew), Hofmann is familar with van der Waerden's *Ontwakende Wetenschap*. He refers to [Zeuthen 1886] in his bibliographic notes, but like everybody else since

[26] Woepcke, who was the first to point to this passage in [1851: xi], abstained from having an opinion whether this suggests something like al-Khwārizmī's demonstrations. Cantor [1880: 313] dropped the doubts.

Neugebauer (Thureau-Dangin excepted) he ignores Zeuthen explanation of what *he* meant by a "geometric algebra". At least neither he nor Struik claim to follow Zeuthen.

Carl Boyer's *A History of Mathematics* from [1968] looks much more (on this account) like a diluted version of [van der Waerden 1961]. The Babylonians, we understand, had an algebra that is adequately expressed in symbolic equations, and the "geometric algebra" of the *Elements* was a translation of Babylonian knowledge. On p. 33 we read:

‹45› One table for which the Babylonians found considerable use is a tabulation of the values of n^3+n^2 for integral values of n, a table essential in Babylonian algebra; this subject reached a considerably higher level in Mesopotamia than in Egypt. Many problem texts from the Old Babylonian period show that the solution of the complete three-term quadratic equation afforded the Babylonians no serious difficulty, for flexible algebraic operations had been developed. They could transpose terms in an equation by adding equals to equals, and they could multiply both sides by like quantities to remove fractions or to eliminate factors.

The beginning of the quotation shows that Boyer knows the Babylonian material only from uncritically absorbed hearsay – there is exactly *one* extant Babylonian problem (a cubic problem) where the table in question is used (referred to by the name "equal, one added" – it was actually understood as a tabulation of $n \cdot n \cdot (n+1)$). Any mathematical reflection would have shown that it can play no role whatsoever in the solution of second-degree problems, whether understood as algebra or not.

On p. 34 this follows:

‹46› The solution of a three-term quadratic equation seems to have exceeded by far the algebraic capabilities of the Egyptians, but Otto Neugebauer in 1930 disclosed that such equations had been handled effectively by the Babylonians in some of the oldest problem texts.

finally, about the Greeks, pp. 85*f* offers this:

‹47› The dichotomy between number and continuous magnitude required a new approach to the Babylonian algebra that the Pythagoreans had inherited. The old problems in which, given the sum and the product of the sides of a rectangle, the dimensions were required[,] had to be dealt with differently from the numerical algorithms of the Babylonians. A "geometric algebra" had to take the place of the older "arithmetic algebra," and in this new algebra there could be no adding of lines to areas or adding of areas to volumes. From now on, there had to be a strict homogeneity of terms in equations, and the Mesopotamian normal forms, $xy = A$, $x \pm y = b$, were to be interpreted geometrically. [...] In this way, the Greeks built up the solution of quadratic equations by their process known as "the application of areas," a portion of geometric algebra that is fully covered by Euclid's *Elements*. Moreover, the uneasiness resulting from incommensurable magnitudes led to an avoidance of ratios, insofar as possible, in elementary mathematics.

This, and in particular the reference to the Pythagoreans as the translators of "Babylonian algebra," shows that Boyer has learned from [and watered down] van der Waerden rather than Neugebauer.

Apollonios gets a chapter of his own. There are references to equations, and also a remark on p. 172 on the impossibility to consider negative magnitudes in "Greek geometrical algebra". In the end there is the rather perspicacious remark that

‹48› Of Greek geometry, we may say that equations are determined by curves, but not that curves were defined by equations, which can be read as a suggestion that the “algebraic” reading (whether à la Zeuthen or more modernizing) has its limitations.

Morris Kline's *Mathematical Thought from Ancient to Modern Times* appeared in [1972], after the attacks against the notion of a “geometrical algebra” has set in, but it was relatively unaffected by them. It deserves a look.

“Babylonian algebra” gets a section of its own (pp. 8–11). It is, even on the conditions of the interpretations of the times, rather badly informed. It seems that Kline has taken his information about the topic from Neugebauer's mathematical explanations of *why things work*, which were never meant to interpret the thinking of the Babylonians; sometimes Kline even misses the numerical procedure – cf. [Høyrup 2010: 12].[27]

With the Greeks, Kline shows himself to be a mathematician who understands the sources[28] and sometimes forms his own opinions. On p. 49 he introduces geometric algebra in this way:

‹49› The Eudoxian solution to the problem of treating incommensurable lengths or the irrational number actually reversed the emphasis of previous Greek mathematics. The early Pythagoreans had certainly emphasized number as the fundamental concept, and Archytas of Tarentum, Eudoxus' teacher, stated that arithmetic alone, not geometry, could supply satisfactory proofs. However, in turning to geometry to handle irrational numbers, the classical Greeks abandoned algebra and irrational numbers as such. What did they do about solving quadratic equations, where the solutions can indeed be irrational numbers? And what did they do about the simple problem of finding the area of a rectangle whose sides are incommensurable? The answer is that they converted most of algebra to geometry.

As we see, here is no hint of a translation of Babylonian knowledge, nor is there much of a role for the Pythagoreans. To the contrary, Kline supposes it to be Eudoxos who made the invention, in opposition to the views of his (Pythagorean) teacher Archytas.

About *Elements* II, Kline has this to say (p. 64; and much more, indeed):

‹50› The outstanding material in Book II is the contribution to geometrical algebra. [...] In Book II all quantities are represented geometrically, and thereby the problem of assigning numerical values is avoided. Thus numbers are replaced by line segments. The product of two numbers becomes the area of a rectangle with sides whose lengths are the two numbers. The product of three numbers is a volume. Addition of two numbers is translated into extending one line by an amount equal to the length of the other and subtraction into cutting off from one line the length of a second. Division of two numbers, which are treated as lengths, is merely indicated by a statement that expresses a ratio of the two lines; this is in accord with the principles introduced later in Books V and VI.

About these principles and book V, this is said on p. 70:

‹51› We know that we can operate with irrationals by the laws of algebra. Euclid cannot and does not. The Greeks had not thus far justified operations with ratios of incommen-

[27] According to the bibliography for the chapter, Kline's sole sources for information about Babylonian mathematics were Neugebauer's *Vorgriechische Mathematik* and *Exact Sciences in Antiquity*, van der Waerden's *Science Awakening* and Boyer's *A History of Mathematics*.

[28] Even here, however, Kline mostly relies on the secondary literature such as Heath's *History of Greek Mathematic*, as he tells on p. xi. As a matter of fact, however, he has also looked at Heath's translations of the *Elements* and of Archimedes, Apollonios and Diophantos, and further translations of a few other works.

surable magnitudes; hence Euclid proves this theorem by using the definitions he has given and, in particular, Definition 5. In effect, he is laying the basis for an algebra of magnitudes.

As Heath's “geometrical algebra” (and Zeuthen's, but Kline does not refer to him), that of Kline encompasses the application of areas and other operations from *Elements* II.1–10 as well as the theory of proportions. Kline does not explain what he means by “algebra”, but the phrase “an algebra of magnitudes” shows that his understanding is broader than what is suggested by the reference on p. 176 to “heuristic, empirical arithmetic and its extension to algebra”.

And then, on p. 80, we find (Kline’s square brackets)

‹52› Book X of the *Elements* undertakes to classify types of irrationals, that is, magnitudes incommensurable with given magnitudes. Augustus De Morgan describes the general contents of this book by saying, “Euclid investigates every possible variety of line which can be represented [in modern algebra] by $\sqrt{\sqrt{a}+\sqrt{b}}$, a and b representing two commensurable lines.” Of course not all irrationals are so representable, and Euclid covers only those that arise in his geometrical algebra.

This last observation is likely to be inspired by [Heath 1926: 4*f*], where a 13 lines long quotation from [Zeuthen 1896: 56] to this effect (but in very different words) can be found.

On p. 88, a strange claim turns up:

‹53› Euclid's *Data* was included by Pappus in his *Treasury of Analysis.* Pappus describes it as consisting of supplementary geometrical material concerned with “algebraic problems.”

What Pappos actually says is that the first 23 propositions (of a total of 90) deal with *magnitudes* [Jones 1986: 84]. The only plausible source for Kline's invention in the literature listed in the bibliography for the chapter is van der Warden's statement [1961: 198] that “The ‘Data’ is a book of great importance for the history of algebra” (but Pappos is not mentioned here, only on p. 200, in connection with Euclid's *Porisms*). Since Kline obviously did not inspect the Euclidean text, it is doubtful whether anything follows as to what he means by “algebra” and “geometric algebra”.

The use of the techniques of “geometric algebra” by Archimedes and Apollonios is mentioned on pp. 108 and 92, respectively. In Archimedes' case, solution by means of conic sections is counted as “geometric algebra”, which is definitely a broadening of the meaning.[29] There is no reason to go into details.

Inspiration from the Babylonians *is* mentioned, but in connection with Heron (p. 136 – mainly, as a matter of fact, the pseudo-Heronic compilers of the *Geometrica* collection; Kline is certainly not alone in this confusion) and Diophantos (p. 143).

All in all we see that the various authors using “geometric algebra” as an interpretive tool share the phrase but do not agree upon what it means (with the exception of Zeuthen/Heath and the partial exception of Neugebauer/van der Waerden). Even their notions of “algebra” point in many directions. So, the words of Goethe's Mephisto, originally referring to theology, are adequate also in the present case: “where concepts are absent, there, at the due moment, arrives a word”.

[29] Kline may have got his idea from van der Waerden [1961: 222], who however does not mention “geometric algebra” in this connection. [Heath 1897: cxxv−cxxvii], apparently van der Waerden's source, explicitly says that the method in question is *no* generalization of the application of areas (and thus, according to Heath's understanding, not a case of “geometric algebra”).

The decade of objections

The continuation is no less adequate: “with words, quarrel is easily made”. *Quarrel* was initiated by Szabó in [1969]. We already discussed him. At present we shall restrict ourselves to summing up that he evidently did not bother to find out what Tannery and Zeuthen had actually thought; he took Neugebauer's pretended borrowing from Zeuthen at face value without control and believed from Tannery's title that he could ascribe priority to him.

Michael Mahoney's essay review from [1971a] of the reprint of [Neugebauer 1934] is more thoughtful, but still plagued by statements adopted from hear-say and embellished by free invention. On p. 371 he writes:

‹54› The group of theorems that embody the Babylonian importation had already attracted the attention of historians of mathematics long before anyone even suspected the existence of Babylonian mathematics. The term “geometrical algebra” was coined by Tannery and adopted by Zeuthen as both men, in the late nineteenth century, tried to make some sense of theorems that seemed to them incongruous to the *Elements* that contained them. No wonder, ran their answer, these theorems seem out of place. They are not geometrical theorems, but algebraic theorems in geometrical guise, as the theorems of Books VII-IX are geometrically-clad arithmetic. What neither man did, however, was to analyse the reasons for his initial discomfiture. Why should the theorems seem incongruous in the first place? What is it about Greek geometry that would make algebraic theorems stick out like the proverbial sore thumb? Van der Waerden and Neugebauer provide the elements of an answer.

Mahoney has obviously not read neither Tannery nor Zeuthen on this account (this is confirmed by the absence of any precise reference); whether he ever inspected *Elements* VIII–IX can be questioned. He borrows from Szabó (who is cited slightly earlier), apparently convicted that nobody would dare write with as strong emphasis and conviction as Szabó if hanging in thin air.

On the whole, however, Mahoney accepts Neugebauer's and van der Waerden's supposed facts: That the Babylonians had created a mathematical discipline that is easily explained by means of algebra; and that this was taken over by Greek mathematicians, more precisely by the Pythagoreans (p. 373), and reshaped. His main objection is directed against the interpretation of either discipline as “algebra”.

For this, Mahoney builds on *his* conception of what algebra should be (p. 372):

‹55› What are the characteristics of the algebraic approach? In its most developed form, that is, in modern algebra, it appears to have three: first, algebra employs a symbolism for the purpose of abstracting the structure of a mathematical problem from its non-essential content; second, algebra seeks the relationships (usually combinatory operations) that characterize or define that structure or link it to other structures; third, algebra, as a mathematics of formal structures, is totally abstract and free of any ontological commitments. The first and second characteristics mark algebra as an operational approach to mathematics; the relations symbolized are combinatory operations and the symbolism expresses the constituents and the results of the operations. Moreover, the second characteristic points to the foundations of algebra in relational logic. The third characteristic emphasizes the purely formal nature of mathematical existence, that is, consistent definition within a given axiom system.

In contrast,

‹56› And what of Greek geometry? What are its characteristics? It employs no symbols, for it is concerned not with structures formed by relations between mathematical objects,

> but with the objects themselves and their essential properties. It is not operational, but contemplative; its logic is the predicate logic of Aristotle's Organon. When Plato castigates mathematicians for speaking as if their task were to do something, he places the Greek seal on geometry.

In quotation ‹55›, firstly, the distinction between a merely “algebraic approach” and its “most developed form” is important; the latter, “modern algebra”, is indeed not only “a creation of the seventeenth century – AD!”, as formulated on p. 375, but at best the teleological interpretation of seventeenth-century algebra in the light of that “moderne Algebra” which van der Waerden had written about – seventeenth - to eighteenth-century algebra was not exactly axiomatic.[30] Secondly, there is a certain (unrecognized) convergence with Zeuthen's distinction between “Geometry ... for its own sake” and geometry “as an instrument for the theory of general magnitudes” (quotation ‹8›), the latter being also, in Zeuthen's opinion, “totally abstract and free of any ontological commitments”.[31] Neugebauer's inclusion of geometrically oriented problems under algebra if only they seem to be based on “formal operation with magnitudes” (quotation ‹15›) also comes to mind. Precisely the addition of magnitudes of different dimensions (which was Neugebauer's main criterion for that) seemed to him as well as van der Waerden to suggest thinking free of ontological commitments.

To quotation ‹56› it should be observed that preaching against sin is evidence of the existence of sin, not of pervasive virtue; so, if anything, Plato's reproach is proof that Greek geometry was not pure contemplation in his times (and, after all, four of the five postulates as well as the first three propositions of the *Elements* deal with doing something[32]). Whether even the proportion theory of *Elements* V and the classification of irrationals (and investigation of the relations between these classes[33]) in *Elements* X deal “not with structures formed by relations between mathematical objects, but with the objects themselves and their essential properties” is at least disputable (but perhaps these are not included in Mahoney's concept of “Greek geometry”?); the former was a constitutive part of

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[30] The “purely formal nature of mathematical existence, that is, consistent definition within a given axiom system” is not even adumbrated in any seventeenth-century author; it belongs to Hilbert's times.

[31] Remember that the Euclidean plane is just as categorically distinct from the paper where diagrams are drawn as is the number 3 from a collection of 3 pebbles!

[32] The famous “parallel postulate” 5 may seem not to do so. However, operating only with the potentially infinite, that is, with the *possibility to produce a line segment ad libitum,* everything in Greek geometry which *we* would spontaneously and naively explain in terms of infinite lines is inherently concerned with “doing”.

Mahoney's note 16 also shows his actual knowledge of the matter to be in conflict with his apodictic appeal to Plato's testimony:

> I do not want to overstate the non-operational nature of Greek geometry. Greek geometrical analysis in particular maintained something of the operational approach and shows, in fact, many “algebraic” traits; see my “Another Look at Greek Geometrical Analysis”, *Archive for History of the Exact Sciences*, **5** (1968), 318−48.

[33] Evidently, Euclid does not speak of “classes” but about unspecified magnitudes possessing a specified character − for instance (X.61, trans. [Heath 1926: III, 135], cf. [Heiberg 1883: III, 186])

> The square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial.

Euclid, indeed, does not possess a fully developed and flexible language for second-order logic. For the same reason, his definition of equal ratio in *Elements* V.5 becomes ambiguous − cf. [Heath 1926: II, 120]. Attributing “classes” to Euclid is as much an imposition of our language as is correcting his definition of equal ratio into “any ... any” − but no more.

Zeuthen's “geometric algebra”, as we remember.[34]

Instead of exploring such questions in more depth, Mahoney has this preliminary conclusion (p. 373):

‹57› In characterizing Babylonian mathematics as algebraic, I do not want to confuse the algebra of the seventeenth century AD with that of the seventeenth century BC. A shared typology need not imply shared content or shared concepts. For if it did, then algebra would constitute a bond that links modern mathematics more closely to the Babylonians than the Greeks. If it did, then, at least in the realm of mathematical thought, the mythopoeic mind and the rational mind would not be as far apart as one has good reason to believe they are.

This is dressed as an indirect proof -- but since the presumed *reductio ad absurdum* has to be argued unspecifically from what “one has good reason to believe” and from the ethnocentric belief in the “mythopoeic mind” of the others, then it must rather be characterized as teleological logic, as a *petitio principii*. If Edward Said had cared about the historiography of mathematics, Mahoney might well have figured prominently in his *Orientalism* [1978].

Sabetai Unguru's “On the Need to Rewrite the History of Greek Mathematics” from [1975] is very different (although it cites Mahoney as well as Szabó with approval and uses rhetorical emphasis and scare quotes no less than Szabó to convince). An example (p. 68*f*), among others dealing with Tannery and Zeuthen (or so it must be presumed − the formulations are conveniently hazy):

‹58› As to the goal of these so-called “historical” studies, it can easily be stated in one sentence: to show how past mathematicians hid their modern ideas and procedures under the ungainly, *gauche*, and embarrassing cloak of antiquated and out-of-fashion ways of expression; in other words, the purpose of the historian of mathematics is to unravel and disentangle past mathematical texts and transcribe them into the modern language of mathematics, making them thus easily available to all those interested.

Neither Tannery nor Zeuthen have evidently said or done anything like this; I doubt Unguru would be able to find anybody who has. At least, however, he has read in the writings of Tannery and Zeuthen, which nobody else since Heath appears to have done – unfortunately without seriously trying to understand them. So, p. 70, n. 7 he quotes some lines from Tannery's article from 1903 (cf. quotation ‹4› – square brackets are Unguru's):

‹59› Indeed, while their algebraic symbolism [sic !] developed painfully, they had already in the fourth century BCE created one for geometry, ... That language presented at the same time all the advantages of the use of letters in Viète's analysis [!], at least for powers 2 and 3.

The “sic” can safely be taken as evidence that Unguru has overlooked that Tannery, when speaking about an algebraic symbolism that develops “painfully”, speaks about Diophantos and his predecessors,[35] not about geometry. It is unclear to me whether “[!]” means that it should be illegitimate to speak about “Viète's analysis”, (which is after all the term Viète uses in order to avoid

[34] From a different angle: *If* a ratio is really *an object* and not a relation (as *Elements* V, def. 3 says that it is), then the ratios dealt with in *Elements* VII−IX are really “disguised” or “translated” fractional numbers (as can actually be argued to be the case from the naming of such ratios, cf. [Vogel 1936]); and the ratios of *Elements* V are nothing but disguised positive real numbers, or at least their equivalents!

[35] Tannery, indeed, is quite aware that there are predecessors − see [Tannery 1887: 51].

the filthy Arabic word “algebra”), or it is deemed illegitimate to compare *the efficiency* of one tool to another one when both investigate similar matters. None of the possibilities makes historiographic sense.

The same note contains quotation ‹7› from Zeuthen, “In this way ...”. There is no attempt to find out what Zeuthen means by his words – Unguru knows what algebra *is*, namely from a shorter version of quotation ‹55› taken from [Mahoney 1971b: 16]. Unguru misses that this is from Mahoney's hand a characterization of that algebraic mode of thought which emerges in the seventeenth century and reaches maturity with abstract group theory or perhaps with category theory. None of those who have spoken about “Babylonian algebra” or Greek “Geometric algebra” would claim that any of these could be algebra *in that sense*. Unguru also *knows* that equations means algebra (but see note 8, above).

This style goes on. Heath [1926: I, 372] states that

> ‹60› Besides enabling us to solve geometrically these particular quadratic equations, Book II gives the geometrical proofs of a number of algebraical formulae.

Then, after listing the algebraic *equivalents* of II.1-10 and a commentary, Heath goes on that

> ‹61› It is important however to bear in mind that the whole procedure of Book II is geometrical; rectangles and squares are shown in the figures, and the equality of certain combinations to other combinations is proved by these figures.

Unguru overlooks that the sequel shows that “gives the geometrical proofs of a number of algebraical formulae” does not claim that Euclid intends this, only that the propositions *can* serve in this way.[36] Instead he accuses Heath of apparently “not grasping the inconsistency involved”.

These few excerpts are characteristic of the whole paper (48 pages). Unguru makes his task easy (for example, pp. 79*f*) by inventing a representative of the stance he tries to refute, having him “grant ... (though reluctantly!)” an objection proving only Unguru's ignorance of matters Babylonian, and which a van der Waerden would never admit; and then, after having this imaginary opponent defend himself, asks “what does one answer to such an interlocutor?” This is, on the whole, a case of allergy, not argumentation.

Four years later, Unguru published a sequel. I shall not go through it in detail – its eventual influence was quite modest, as was also that of [Unguru & Rowe 1981] (thanks to David Rowe more cautiously formulated). But the beginning of the former of the two [Unguru 1979: 555] is striking:

> ‹62› The history of mathematics typically has been written as if to illustrate the adage “anachronism is no vice.” Most contemporary historians of mathematics, being mathematicians by training, assume tacitly or explicitly that mathematical entities reside in the world of Platonic ideas where they wait patiently to be discovered by the genius of the working mathematician. Mathematical concepts, constructive as well as computational, are seen as eternal, unchanging, unaffected by the idiosyncratic features of the culture in which they appear, each one clearly identifiable in its various historical occurrences, since these occurrences represent different clothings of the same Platonic hypostasis.

This confirms the allergic interpretation: As soon as Unguru sees the word “algebra”, he stops reading the explanations of the writer. *He*, if anybody, is the Platonist who knows that algebra *is*,

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[36] That the diagram of II.1 *can* indeed serve to justify that

$$a\,(b + c + d + ...) = ab + ac + ad + ...$$

will be familiar to many modern mathematics teachers.

eternal and unchanging.

For Unguru, however, this was not the end of the road (we have all been young and sometimes perhaps overly eager). Firstly, he told me around 1995 (thus years before the appearance of [Fried & Unguru 2001]), that so far (by then) the only consistent interpretation of the *Conics* was unfortunately that of Zeuthen. Even later (well after 2001, I believe in 2011) he told that even he had to start with symbolic algebra in order to grasp Apollonios. Similarly, Szabó was very interested when I told him (in 2002) how the geometric reading of the Babylonian material led to an interpretation of "Babylonian algebra" similar to what the slave boy does (in Plato's *Menon*) when asked by Socrates to double a square – a story that plays a major role in Szabó's own scenario.

Objections to the objections – and what followed

Already on the final page of [Unguru 1975] there was an "Editorial Note: A defense of his views will be published by Professor van der Waerden in a succeeding issue". It appeared as "Defence of a 'Shocking' Point of View" [van der Waerden 1976], the tone of which is generally as calm as that of Unguru had been violent.

Commenting upon Unguru's use of the shorter version of quotation ‹55›, van der Waerden points out (p. 199) that

‹63› If this definition of "algebraic thinking" is accepted, then indeed UNGURU is right in concluding that "there has never been an algebra in the pre-Christian era", and that Babylonian algebra never existed, and that all assertions of TANNERY, ZEUTHEN, NEUGEBAUER and myself concerning "Geometric algebra" are complete nonsense.

> Of course, this was not our definition of algebraic thinking. When I speak of Babylonian or Greek or Arab algebra, I mean algebra in the sense of AL-KHWĀRIZMĪ, or in the sense of CARDANO'S "Ars magna", or in the sense of our school algebra. Algebra, then, is:

> the art of handling algebraic expressions like $(a + b)^2$ and of solving equations like $x^2 + ax = b$.

We may remember Thureau-Dangin's reaction to quotation ‹28› (Loria).

Slightly later, we find the only sharp formulation, namely in the beginning of a section presenting "Babylonian algebra":

‹64› UNGURU denies the existence of Babylonian algebra. Instead he speaks, quoting ABEL REY, of an arithmetical stage (Egyptian and Babylonian mathematics), in which the reasoning is largely that of elementary arithmetic or based on empirically paradigmatic rules derived from successful trials taken as a prototype.

> I have no idea on what kind of texts this statement is based. For me, this is history-writing in its worst form: quoting opinions of other authors and treating them as if they were established facts, without quoting texts.

> Let us stick to facts and quote a cuneiform text BM 13901 dealing with the solution of quadratic equations. Problem 2 of this text reads:

> I have subtracted the (side) of the square from the area, and 14,30 is it.

> The statement of the problem is completely clear: It is not necessary to translate it into modern symbolism. If we do translate it, we obtain the equation

$$x^2 - x = 870$$

Actually Unguru does not even *quote* Rey, whom he mostly treats as a fool because they mostly disagree. He gives a supporting reference to scattered pages for a summary which, as it stands, is adopted as Unguru's own point of view. The very ideas of a "Babylonian algebra" or a "Greek geometrical algebra” are a priori [Unguru 1975: 78]

‹65› historically inadmissible. There is (broadly speaking) in the historical development of mathematics an *arithmetical* stage (Egyptian and Babylonian mathematics) in which the reasoning is largely that of elementary arithmetic or based on empirically paradigmatic rules derived from successful trials taken as a prototype [first reference to two distinct pages in Rey], a *geometrical* stage, exemplified by and culminating in classical Greek mathematics, characterized by rigorous deductive reasoning presented in the form of the postulatory-deductive method, and an *algebraic* stage, the first traces of which could be found in DIOPHANTOS' *Arithmetic* and in AL-KHWĀRIZMĪ'S *Hisab al-jabr w'al muqàbalah*, but which did not reach the beginning of its full potentiality of development before the sixteenth century in Western Europe [a second reference to three places in Rey],

which in view of what was well known in 1975 about Egyptian and Babylonian mathematics could indeed have deserved sharper commentaries than those of van der Waerden. One is tempted to quote Tom Lehrer’s *Be Prepared*, “Don’t write naughty words on walls if you can’t spell”.

Returning to quotation ‹64›, van der Waerden's “facts” are of course based on the interpretation of Babylonian mathematics and its terminology that was current at the time; but none of the critics had ever challenged *that;* Mahoney and Szabó accepted it, perhaps with some reticence, Unguru rejected it *a priori* with his reference to Rey, whose ignorance can perhaps be excused by his date (but shouldn't a philosopher stay quiet when he does not know?).

Van der Waerden goes on with further arguments supporting his position, some of them quite interesting and innovative. One might ask why these things were not explained in *Science Awakening*, but without being asked van der Waerden gives the answer on pp. 203*f:*

‹66› We (ZEUTHEN and his followers) feel that the Greeks started with algebraic problems and translated them into geometric language. UNGURU thinks that we argued like this: We found that the theorems of EUCLID II can be translated into modem algebraic formalism, and that they are easier to understand if thus translated, and this we took as “the proof that this is what the ancient mathematician had in mind”. Of course, this is nonsense. We are not so weak in logical thinking! The fact that a theorem can be translated into another notation does not prove a thing about what the author of the theorem had in mind.

No, our line of thought was quite different. We studied the wording of the theorems and tried to reconstruct the original ideas of the author. We found it evident that these theorems did not arise out of geometrical problems. We were not able to find any interesting geometrical problem that would give rise to theorems like II 1–4. On the other hand, we found that the explanation of these theorems as arising from algebra worked well. Therefore we adopted the latter explanation.

Now it turns out, to my great surprise, that what we, working mathematicians, found evident, is not evident to UNGURU.

A key phrase here is “interesting geometrical problem”. Indeed, van der Waerden, in almost Wittgensteinian manner, asks not “for the meaning” but “for the use”. *In this respect* he is thus almost faithful to Zeuthen when enrolling him (the first to be so since Heath). As we remember, Zeuthen spoke exactly about the Euclidean propositions being *used* in the same manner as algebra is used in latter-day analytical geometry. Both knew from their experience as creative mathematicians that “mathematical entities” do *not* reside ready-made and immutable “in the world of Platonic ideas where they wait patiently to be discovered by the genius of the working mathematician” (cf.

quotation ‹62›).[37] Unguru instead, trained as a philosopher and a classical philologist, is an essentialist, convinced that, incorruptibly, algebra is algebra is algebra.

In [1977], Hans Freudenthal published another commentary. He starts with a double motto, Juliet Capulet's “What's in a name”, and Mephisto's “Mit Worten lässt sich trefflich streiten”. Accordingly, Freudenthal is much sharper than van der Waerden, starting thus (p. 189)

‹67› Whoever starts reading Greek mathematics is struck by large parts that are overtly algebraic as well as other parts where algebra seems to hide under a geometrical cover. [...].

S. UNGURU has recently challenged this view. All who have written about Greek mathematics have been wrong, he claims. On what grounds? Has he discovered sensational new facts? No, nothing! He has not even interpreted old facts in a new way. He simply says they are wrong, and does so with resounding rhetorical emphasis. If the rhetoric is disregarded, the remainder consists of large extracts from the work of others, decorated with numerous exclamation and question marks, and a few, more concise statements, which can properly be submitted to analysis.

As van der Waerden, Freudenthal asks (among other things) for a reading of the Greek mathematical texts which asks for the *use* of theorems like *Elements* II.5 and VI.28 (pp. 189*f*). Geometrically seen, they are “badly motivated” and “unattractive” in themselves. He goes on, “it appears that these propositions were used as algebraic tools within Greek geometry”; that is, without at all mentioning Zeuthen in the article, he too is more or less back at his position. He also asks for discrimination – some renderings of a verbal text with “algebraic” symbols are faithful to it, others are misleading:

‹68› *Unguru* suggests a quite different origin for the common interpretation of II5 and VI28: people discovered that you can note down these propositions in modern algebraic language, and then concluded that they were algebra, geometrically disguised. This brings us to the question of how Greek mathematics should be edited. In fact, there are various levels on which this can be done. In a plain translation, such as by T. L. HEATH the Greek text (V11) may appear in the version

as A is to B, so let C be to D,
and, as C is to D, so let E be to F.
I say, that, as A is to B, so is E to F.

whereas in a comment or in a summary you might find

Algebraically, if $a : b = c : d$
and c : d = $e : f$
then $a : b = e : f$.

No doubt this is allowable but it would be absolutely inadmissible in the same context to replace propositions like $a : b = c : d$ by their more modern analogues $ad = bc$. It would not only spoil the context but even make nonsense of it. [...] Some delicacy is needed to know in any particular case which language is most suitable. [...] Anyhow, it is unwarranted to quote modern style algebraical formulas from historians of mathematics without identifying the level of presentation to which they belong and to insinuate that conclusions are drawn from wrong translations.

Freudenthal goes on with discussion of specific points. Of major interest is what he says about Unguru's claim that the Greek texts contain no equations:

[37] Long before “mathematical practice” became a concern for philosophers, working mathematicians of course knew it from the inside even if they did not verbalize it as such.

‹69› “Equation” has three meanings
formal identity,
conditional equality involving unknowns to be made known,
conditional equality involving variables.
Which meanings are absent in Greek mathematics? Is

$$(a+b)^2 = a^2 + 2ab + b^2$$

not an equation, if it is formulated in words? Are the “symptoms” for circle, parabola, ellipse, hyperbola not equations, just because they are written in the language of rectangles and squares? And finally what about linear and quadratic equations? Of course II5 is not the solution of a quadratic equation, but nobody ever claimed it was. VI28, however, is explicitly formulated as problem-solving, and the problem is a quadratic equation not for a number but for a magnitude,

which gives further substance to the point made above in note 8.

As he is indeed obliged to after attacking Unguru for lack of facts, Freudenthal then goes through a number of textual examples, on which he can illustrate his objections to Unguru's claims.

A final reply was formulated by André Weil in [1978]. It castigates “Z” (Weil's alias for Unguru) for various misunderstandings and contains a number of interesting observations (and it is certainly worth reading); but it does not add anything very significant to the topic of “geometric algebra”, so I shall not discuss it.

Then, “what followed”?

Unguru's attack had provoked van der Waerden and Freudenthal to make explicit many things that had been taken for granted by those who spoke about one or the other kind of “geometric algebra”. Without their recognizing it, they had even been induced to return to arguments similar to those of Zeuthen, which nobody had cared about since half a century or more.

Nobody listened, however. Just as Zeuthen's phrase was taken over by others who did not care to read precisely what Zeuthen had really said, so Unguru's attack was broadly accepted as a standard reference by a generation of historians who argued in principle for precise reading of *sources* but did not bother to read their own standard references in a similarly careful manner.[38]

As formulated by David Rowe [2012: 37]:

‹70› Today it would appear that most historians of mathematics have come to accept this central tenet [of Unguru]. Indeed, at the recent symposium honoring Neugebauer at New York University's Institute for Studies of the Ancient World, Alexander Jones told me that Unguru's position could now be regarded as the accepted orthodoxy. Sabetai Unguru, however, begs to differ; he quickly alerted me to recent work by experts on Babylonian mathematics who, in his view, continue to commit the same kinds of sins he has railed about for so long.[39]

[38] Less kindly, Schneider [2016: viii] suggests that

> a group of present-day historians has appropriated Unguru's request to reinterpret Greek mathematics and thereby tried to get an alibi to ignore, against all scientific probity, results that have been reached earlier.

[39] I suspect that Unguru refers to my intervention at a workshop on the “history of algebra” in which we both participated in 2011. Here I discussed (as asked for by the workshop theme) what the Old Babylonian technique had in common with later algebras (in the plural) and what not, leaving explicitly open whether this would qualify it as an “algebra”. For a Platonist who has spent a third of a century using abusive language in the matter (the explanation the *Oxford English Dictionary* gives of “to rail”) it is obviously a sin to leave

In a similar vein but explicitly endorsing the "accepted orthodoxy" (which neither Rowe nor Jones does), Nathan Sidoli [2013: 43] sums up a survey introduction in this way

‹71› The new historiographic approach that was so hotly debated in the 1970s has become mainstream. There are now almost no serious scholars of the subject trying to determine how Greek mathematics must have originated based on what seems likely from some mathematical or logical perspective, or trying to understand the motivation for methods found in Apollonius or Diophantus using mathematical theories and concepts developed many centuries after these mathematicians lived.

Obviously, no "serious scholars of the subject" ever tried to do so; if anybody has argued *a priori* from a "mathematical or logical" perspective, it will have been an Abel Rey – no "serious scholars of the subject" but a philosopher speaking from second-hand knowledge.[40] It has nothing to do with Zeuthen, Neugebauer, or van der Waerden (with whom one may agree or disagree) – but is seems that Sidoli continues the tradition from Neugebauer, inventing the opinions of those whom he cites, or worse, speaking of things he never read or read carefully.

So, even by otherwise good scholars, Unguru's report of the opinion of those whom he attacked have been broadly accepted on faith. Would it not be time to start reading not only the historical sources but also that part of the literature which everybody feels obliged to cite in order to demonstrate appurtenance to the "in-people" – the "superclassics" of Derek Price [1965: 149]?[41] Thus deciding whether it should stay on this privileged shelf or should be gently moved to the archives of the history of historiography?

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things open.

[40] Perhaps also Mahoney: as we have seen in quotation ‹55›, he explains the aims and motivations of seventeenth-century algebra from twentieth-century views.

[41] Actually, I am not the only one to be of this opinion. After I began writing this, Viktor Blåsjö published online "In Defence of Geometrical Algebra" − now published as [Blåsjö 2016], the aim of which is not "to argue *for* geometrical algebra, but rather to argue against the arguments against it" (p. 327). This "geometric algebra" refers (1) to Zeuthen's claim that "the Greeks possessed a mode of reasoning analogous to our algebra", and (2) the idea that the Greeks "were well aware of methods for solving quadratic problems (such as those exhibited in the Babylonian tradition), and that *Elements* II and VI "contain propositions intended as a formalisation of the theoretical foundations of such methods" (p. 326).

Blåsjö's conclusion (pp. 357*f*) is that

> The geometrical algebra hypothesis has, for the past few decades, been a kind of scapegoat in a war of historiography. As the hallmark of a currently unpopular mode of scholarship, this hypothesis has been condemned with zeal by a new generation of historians. Because of its unfashionable association, the geometrical algebra hypothesis has seen objections of all sorts hurled its way. And with no one to defend it, bystanders are likely to assume that it is justified. But the geometrical algebra hypothesis deserves a fair trial. In this paper I have attempted to address every substantial argument ever raised against the geometrical algebra hypothesis. I have argued that none of them are at all compelling. I urge, therefore, that it is time to take a step back from perfunctory opposition to geometrical algebra and to look at its case afresh with an open mind.

Blåsjö has read the texts, from Zeuthen until Reviel Netz [2004] (whom I do not deal with here), showing that Netz's apparent support for his former teacher Unguru (and for Jakob Klein) actually comes down to a total destruction of their claims about cognitive incompatibility.

As I discover only at last-minut re-reading, Blåsjö's title transforms that of [van der Waerden 1976], just as mine is inspired by that of [Freudenthal 1977].

Conflict of Interest

The author declares no conflicts of interest in this paper.

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September 2010. (The published version had to be so strongly shortened that the relevant passage disappeared.)

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