Reinventing or Borrowing Hot Water? Early Latin and Tuscan Algebraic Operations with Two Unknowns

JENS HØYRUP

To the friends at IHNS, Jiaotong and Tsinghua
And to Enrico Giusti, whose editions made this possible

ABSTRACT

In mature symbolic algebra, from Viète onward, the handling of several algebraic unknowns was routine. Before Luca Pacioli, on the other hand, the simultaneous manipulation of three algebraic unknowns was absent from European algebra and the use of two unknowns so infrequent that it has rarely been observed and never analyzed.

The present paper analyzes the five occurrences of two algebraic unknowns in Fibonacci's writings; the gradual unfolding of the idea in Antonio de' Mazzinghi's *Fioretti;* the distorted use in an anonymous Florentine algebra from *ca* 1400; the regular appearance in the treatises of Benedetto da Firenze; and finally what little we find in Pacioli's Perugia manuscript and in his *Summa*. It asks which of these appearances of the technique can be counted as independent rediscoveries of an idea present since long in Sanskrit and Arabic mathematics – metaphorically, to which extent they represent reinvention of the hot water already available on the cooker in the neighbour's kitchen; and it raises the question why the technique once it had been discovered was not cultivated – pointing to the line diagrams used by Fibonacci as a technique that was as efficient as rhetorical algebra handling two unknowns and much less cumbersome, at least until symbolic algebra developed, and as long as the most demanding problems with which algebra was confronted remained the traditional recreational challenges.

AMS classification (MSC 2010): 01A32, 01A35, 11-03, 11A05, 51-03.

Gaṇita Bhāratī Vol. 41, No. 1-2 (2019) pages 23-67 (published 2021)
Corresponding Author E-mail: jensh@ruc.dk
DOI-https://doi.org/10.32381/GB.2019.41.1-2.2

Keywords: Indian Mathematics, Brahmagupta, Linear congruences, *kuṭṭaka*, Derivations, Algebraic identities, Algebra in several variables, Discourse analysis, Arabic algebra, Abbacus algebra, Fibonacci, Antonio de' Mazzinghi, Benedetto da Firenze, Luca Pacioli.

1. Introduction

In India, algebraic operations with several unknowns are earlier than anything similar to be found in the Islamic or medieval Latin world. Since this is not my subject, and since the technique is unrelated to what I am going to speak about, a reference to the section of Brahmagupta's *Brāhmasphumasiddhānta* where the topic is dealt with will suffice [ed., trans. Colebrooke 1817: 348–360].

Algebraic operations with several unknowns were also made in Islamic mathematics well before anybody in the Latin world practised or merely had heard about algebra. For this, a reference to Abū Kāmil's *Algebra* [ed., trans. Rashed 2012: 370, 396, 400–408] and to his *Kitāb al-Ṭayr*, his small treatise on the problem of the "hundred fowls" [ed., trans. Rashed 2012: 731–761] will do.

So, the present paper does not deal with priorities but with the borrowing or reinvention of hot water, about how it happened, and about the lack of short-term consequences.

Fibonacci

Before we address the textual evidence, a conceptual clarification is needed. Many traditional recreational problems speak about several unknown abstract or concrete numbers. As an example we may look at a "give-and-take" problem from Fibonacci's *Liber abbaci* – presented to him, he says, by a Constantinopolitan master [ed. Boncompagni 1857: 190; ed. Giusti 2020: 324]: One man (A) asks from another one (B) 7 δ (*denari*), saying that then he shall have five times as much as the second has. The second asks for 5 δ, and then he shall have seven times as much as the first. Fibonacci first uses a line diagram to reduce the problem to one where a single false position can be applied (in the last section of the paper we shall return to this diagram and how it serves). Expressed in words, the reduction runs like this: When B has given 7 δ, A shall have 5 times as much as B – that is, B shall be left with $^1/_6$ of their total possession. Therefore, B originally possesses $^1/_6$ of the total, plus 7 δ. For similar reasons, A originally has $^1/_8$ of the total, plus 5 δ. That is, removing $^1/_6$ and

$^1/_8$ of the total leaves 12δ. If the total had been 24 (a convenient false position), removal of $^1/_6$ and $^1/_8$ would instead have left 17. Etc.

The possessions of each of the two are unknown and asked for; that is what the problem is about. But they are not *algebraic unknowns*, not submitted to any kind of algebraic manipulations. A number of segments in the accompanying line diagram in the margin, or corresponding numbers in the verbal paraphrase, enter on an equal footing.

Afterwards, however, Fibonacci gives an alternative solution by means of *regula recta*, the "direct rule". We shall return to this technique but for the moment merely observe that this is first-degree rhetorical equation algebra with algebraic unknown *res* ("thing"): B is *posited* to possess *a thing* and 7δ.[1] After having received 7δ, A therefore has 5 *things*, originally thus 5 *things* less 7δ. If instead B gets 5 δ from A, he shall have a *thing* and 12δ, while A shall have 5 *things* less 12 δ. In consequence, a *thing* and 12δ equals 7 times 5 *things* less 12δ. Once this equation is established, algebraic transformations can start:[2]

$$35\ \textit{things} - 84\delta = 1\ \textit{thing} + 12\delta$$

and then, "since when equals are added to equals, the totals will be equal":[3]

$$35\ \textit{things} = 1\ \textit{thing} + 96\delta$$

1 In a false position, some unknown quantity is *posited* to have a particular (convenient but probably false) numerical value; the true value then follows from a consideration of proportionality.

Whether the *regula recta* is identified by name or not, this rule and its appurtenant application of algebraic reasoning are announced by the present different kind of positing, some entity being posited to be a *thing* (or whatever name be given to it), which leads to the construction of a (rhetorical) equation.

2 The calculations actually make use of the equality 12 δ = 1 ß (1 *soldo*), but in the end Fibonacci returns everything to *denari*.

3 My translation, as all translations from Latin and from Tuscan vernacular in the following. I strive to keep as close to the original grammar (indicative/subjunctive, singular/plural) as possible, since this grammar provides the conditions under which the rhetorical argument functions. Even in texts where these differentiations had probably lost their original meaning, I also conserve the distinctions between multiplication respectively division *in* and *by* – cf. [Høyrup 2007: 16 n. 35, 161 n. 12]. Italics used to indicate what functions as algebraic unknowns are my addition.

and further, “as when from equals equals are removed, the remainders will be equal”

$$34\ \textit{things} = 96\delta$$

and hence each *thing* equals $2^{14}/_{17}\ \delta$. From here it follows that the original possession of B is $1\textit{thing}+7\delta = 9\ ^{14}/_{17}\ \delta$, etc.

From our point of view, this is basic first-degree equation algebra. From that of Fibonacci, it is not what he speaks about as *algebra et almuchabala* – coming from Arabic *al-jabr wa'l-muqābalah*, which fundamentally is a second-degree technique; that topic he deals with much later in the *Liber abbaci*, in chapter 15 section 3. At the present point, the *regula recta* is introduced instead as “much used by the Arabs” and “immensely praise-worthy”. The difference is further made clear by the more or less Euclidean explanations of the operations as adding or subtracting equals to/from equals.[4] In order to keep clearly apart our generic idea of what is algebraic (*regula recta* as well as what Fibonacci designates *algebra et almuchabala*) from *algebra et almuchabala* alone, I shall henceforth speak about the latter as *aliabra* (a form regularly used in abbacus writings), the former as *algebra* (understood as equation algebra, not theory, which belongs to a much later epoch).

At all events, from *our* point of view the *regula-recta* operations are algebraic, whereas the first solution by false position is not.

In [2010: 61], Albrecht Heeffer formulated a list of criteria for a problem solution to be algebraic *and* solved by several unknowns, which extends the preceding reflections:

1. The reasoning process should involve more than one rhetorical unknown which is named or symbolized consistently throughout the text. One of the unknowns is usually the traditional *cosa.* The other can be named *quantità*, but can also be a name of an abstract entity representing a share or value of the problem.

2. The named entities should be used as unknowns in the sense that they are operated upon algebraically by arithmetical operators, by squaring or root extraction. [...].

4 Fibonacci obviously understands the affinity between *regular recta* and *al-jabr*. While keeping things straight at the present point where the former is introduced, at times [ed. Boncompagni 1857: 260, 265; ed. Giusti 2020: 421, 427] he uses the “restoration” terminology which had given *al-jabr* its name, *al-jabr* meaning precisely “restoration”. However, “restoration” can also be used as a non-algebraic term [ed. Boncompagni 1857: 276; ed. Giusti 2020: 443].

3. The determination of the value of the unknowns should lead to the solution or partial solution of the problem. [...].

4. The entities should be used together at some point of the reasoning process and connected by operators or by a substitution step.

Heeffer discusses instances of two or more unknowns from Antonio de' Mazzinghi (*ca* 1380; actually just mentioned, not discussed) to Stevin (1585) and of "the way it shaped the emergence of symbolic algebra" [Heeffer 2010:58]. What I shall do here is to supplement with some other instances, from Fibonacci to Benedetto da Firenze and Luca Pacioli, with a proper analysis of Antonio's text; this will illustrate that even what *a posteriori* looks as important steps forward (forwards *toward us*) may not have been considered significant in their own time – not even by their authors.

Abū Kāmil [ed., trans. Rashed 2012: 370, 396, 400–408, 736–755] gave to the second, third and fourth algebraic unknown the names of coins, *dinars*, *fals* and *khātam*; nothing in his words suggests that this was a new idea, so we may presume it to have been already an established routine.[5] One Latin source knows about this: The *Liber mahameleth* [ed. Vlasschaert 2010: 209*f*; ed. Sesiano 2014: 258–260] uses *res* and *dragma* a couple of times. This treatise – a less extensive counterpart of the *Liber abbaci* – was probably written in al-Andalus before the mid-12th-century and more or less freely translated into Latin by Gundisalvi or in his environment around 1260 (for this, see [Høyrup 2015b: 13–15]). It refers to this as a standard technique of "algebra", probably meaning that it is described in the chapter presenting this field – a chapter that is missing in all Latin manuscripts, perhaps already omitted from the original Latin translation. As we shall see, a coin is also used as second unknown once in the *Liber abbaci*, but in a way that makes it more than doubtful whether Fibonacci understood his source or just copied.

It is not totally excluded that Fibonacci knew the *Liber mahameleth*, but nothing in his text suggests so, and the details speak against it. Several parts of the *Liber abbaci* certainly seem to draw on the same environment [Høyrup 2015b], but the similarities never go beyond resemblances of mathematical style. When we turn to

5 Totally improvised is apparently the use – in the *Algebra*, in a problem about the division of 10 into two parts (say, a and b, $a>b$) – of "large thing" for $^{a}/_{b}$ and "small thing" for $^{b}/_{a}$ [ed., trans. Rashed 2012: 410; ed., trans. Sesiano 1993: 388]. This falls outside the coin-routine and suggests that Abū Kāmil in general saw the use of two algebraic unknowns as in need of no particular explanation.

the problem solutions involving two algebraic unknowns, even such resemblances are lacking.

Four problem solutions in the *Liber abbaci* make use of two algebraic unknowns[6]. Three belong to the category of *regula recta* solutions, coming long before the final *aliabra* section, where the fourth is to be found. The first [ed. Boncompagni 1857: 212; ed. Giusti 2020: 355] solves a problem of type "finding a purse",

> Two men, who have *denari*, find a purse containing *denari*. When they have found it, the first says to the second, "if I get the *denari* in the purse together with the *denari* I have, then I shall have three times as much as you". Against which the other answers, "and if I get the *denari* of the purse together with my *denari*, I shall have four times as much as you".

If A stands for the possession of the first man, B for that of the second, and p for the contents of the purse, the first solution proposed can be summarized as follows:

$$A+p = 3B$$

whence

$$A+B+p = 4B$$

and thus

$$A+p = {}^{3}/_{4}\,(A+B+p)\ .$$

A similar argument leads to

$$B+p = {}^{4}/_{5}\,(A+B+p)\ .$$

Now a false position is made, namely that $A+B+p$ is a number of which ${}^{3}/_{4}$ and ${}^{4}/_{5}$ can be found, for which 20 is chosen. Then $A+p = 15$, $B+p = 16$, and therefore

[6] What is said here about the former two of these problems could be claimed to repeat in part observations made in [Lüneburg 1993]. However, reading Fibonacci through the spectacles of modern computer science (see his p. 125) and school algebra, Heinz Lüneburg demonstrates not to have grasped the difference between algebraic and merely arithmetical reasoning, as also reflected in his cheap polemics against Johannes Tropfke – actually [Tropfke/Vogel et al., 1980], which was written by Kurt Vogel et al., not by Tropfke, as Lüneburg seems to believe. The only crime of Vogel et al. is a misprinted reference, [1;1, 236] instead of [1;2, 236].

$(A+p)+(B+p) = (A+B+p)+p = 31$, whence $p = 11$, $A = 4$, $B = 5$. Alternatively, with the same position, $B = {}^{1}/_{4}\,(A+B+p) = 5$, $A = {}^{1}/_{5}\,(A+B+p) = 4$, $p = 20–4–5 = 11$. Since the problem is indeterminate, this is a valid solution.

This may look algebraic, but it is *our* algebra; if anything beyond the words, Fibonacci's reader would probably be supposed to think of a representation by a line diagram, similar to the one serving the above-mentioned "give-and-take" problem.[7] Then, however, comes an alternative solution by *regula recta* (not identified by name here). *A* is posited to be a *thing*, and then Fibonacci operates with the *thing* and the *purse* (*bursa*) on an equal footing. Since *thing*+*purse* is thrice *B*, *B* must be ${}^{1}/_{3}$ (*thing*+*purse*). Therefore, if the second man gets the purse, he will have *purse*+${}^{1}/_{3}$ *purse*+${}^{1}/_{3}$ *thing*, which will be 4*thing*. Therefore 4*purse* = 11*thing*. In consequence, $p{:}A = 11{:}4$.

The non-algebraic part finds a single solution, and says nothing about the existence of others. By finding a ratio, Fibonacci shows implicitly that there are as many solutions as one may wish, but in agreement with prevailing norms for this kind of mathematic he needs no more than one. Thus, as he says, "*if* there are 11 δ in the purse, then the first man has 4", etc.

Since the purse conserves its name while changing its role, one should read attentively in order to discover that two *algebraic* unknowns are in play.

The second instance turns up within a sequence of problems about composite travels. The first of these [ed. Boncompagni 1857: 258; ed. Giusti 2020: 417] runs like this:

> Somebody proceeding to Lucca made double there, and disbursed 12 δ. Going out from there he went on to Florence; and made double there, and disbursed 12 δ. As he got back to Pisa, and doubled there, and disbursed 12 δ, nothing is said to remain for him. It is asked how much he had in the beginning.

This could be solved step by step backwards: Before disbursing 12 δ in Pisa, he had 12 δ, that is, coming to Pisa he must have had 6 δ, which have been left over in Florence after he disbursed 12 δ there. Before disbursing 12 δ in Florence he therefore had 18 δ, and coming to Florence hence 9 δ. Etc.

7 Most easily, a line segment consisting of three parts – to the left the possession of the first man, to the right that of the second man, in the middle the contents of the purse.

Fibonacci instead makes the tacit false position that the initial capital is 1 δ. He prescribes a sequence of unexplained numerical steps, whose underlying explanation is this: Without disbursements, the initial 1 δ would grow to a "Pisa value" of 8 δ. However, it *should* grow to equal the Pisa value of *the disbursements*, which by a similar argument is $(2\cdot2+2+1)\cdot12$ δ = 84 δ. We may say that the basic ideas of composite interest calculations and discounting are drawn upon.

The following problems are more complex: the rate of gain or the disbursements may vary; instead of the initial capital, the disbursement may be unknown though constant; etc. Sometimes solutions by *regula recta* are given. The basic idea underlying the solutions remains the same.

However, for this problem [ed. Boncompagni 1857: 264; ed. Giusti 2020: 426] that will not do:

> Again, in a first travel somebody made double; in the second, of two, three; in the third, of three, 4; in the fourth, of 4, 5. And in the first travel he expended I do not known how much; in the second, he expended 3 more than in the first; in the third, 2 more than in the second; in the fourth, 2 more than in the third; and it is said that in the end nothing remained for him. And let the expenditures and his capital be given in integers. We therefore posit by *regula recta* that his capital was an *amount* [*summa*], and the first expenditure a *thing*.

If we were to apply the technique used in the preceding problems, we would have to reduce the initial capital as well as the expenditures to final value, which inasfar as expenditures are concerned becomes somewhat arduous and at any rate involves the first unknown expenditure. Fibonacci instead makes the calculation stepwise, positing explicitly *amount* and *thing* as algebraic unknowns.[8] Moreover

[8] In a note to this problem, Laurence Sigler [2002: 626] observes that

> In this algebraic solution there are found two unknowns named the sum and the thing. Of course Leonardo has been solving all along problems with many variables, but this is the first instance where he uses two variables with the algebraic or direct method. [...] The remark [VEg: p265] that the first occurrence of two unknowns appears in the second half of the fourteenth century is therefore incorrect. This chapter and this book [the *Liber abbaci* and its chapter 12/JH] are full of problems with more than one unknown solved with the algebraic or direct method as well as elchataym [the double false position].

("[VEg: p265]" refers to [Van Egmond 1976: 265]). The first part of the quotation might make us believe that Sigler refers to the restricted notion of "two algebraic unknowns" as understood here and by Heeffer. The closing sentence shows that this is not the case – three or four instances do not amount to "full of", so Sigler must include many others.

we observe that Fibonacci knows the problem to be indeterminate, and asks for a solution in integers.

After the first travel, our merchant is seen to possess 2 *amount–thing;* after the second, he has 3 *amount*–$2^1/_2$ *thing*–3δ; after the third, 4 *amount*–$4^1/_3$*thing*–9δ; and after the fourth, 5 *amount*–$6^5/_{12}$*thing*–$18^1/_4$δ. In this way we end up with the indeterminate equation

$$5\ \textit{amount}-6^5/_{12}\textit{thing}-18^1/_4\delta = 0$$

or, "if all-over $6^5/_{12}$*thing* and $18^1/_4$δ are added",

$$5\ \textit{amount} = 6^5/_{12}\textit{thing}+18^1/_4\delta$$

with the request that *amount* and *thing* have to be integers. With a clever stepwise procedure Fibonacci finds as possible solution the *amount* to be 46, and the *thing* to be 33. In the end (since the equation can be transformed into 60 *amount* = 77 *thing*+219δ), other solutions are found by adding

> as many times as you will 60 to the first expenditure, that is, to 33, and as many times 77 to the capital that was found, that is to 46, and you will have what was asked for in ways without end.

In a variant of the present problem the traveller is left in the end with a net profit of 12 δ, in total thus with the initial capital and 12 δ. Here Fibonacci applies the *regula versa*, starting the construction of the equation from the final instead of the initial situation. Being left in the end with 1 *amount*+12δ, after disbursing 1*thing*+5, before disbursing he must have had 1 *amount*+1 *thing*+19δ; therefore he must have arrived with $^4/_5$ of this, that is, $^4/_5$ *amount*+$^4/_5$ *thing*+$15^1/_5$ δ, etc. Out of this comes the equation

$$1\textit{amount} = {}^1/_5\ \textit{amount}+{}^{77}/_{60}\ \textit{thing}+6^1/_{20}\delta$$

(both express the initial capital). If we count this as a separate instance, five and not four problems in the *Liber abbaci* make use of two algebraic unknowns.

The third of the four instances is an alternative solution "according to the investigation of proportions" of a problem about three men having *denarii* [ed. Boncompagni 1857: 338; ed. Giusti 2020: 529] (the first solution is based on a double false position),

> the first asks the last two for $^1/_3$ [of what they have], and states that then he shall have 14; the second asks the third for $^1/_4$ of his *denarii*, and says

> he shall then have 17 *denarii;* the third, indeed, asks the first for $^1/_5$ of his *denarii*, and says he shall have 19 *denarii*.

The alternative solution [ed. Boncompagni 1857: 339] asks to

> posit that the second and the third man have a *thing*. Therefore the first has 14, less a third of a *thing*. Then posit that the third has a *part* of a thing. Therefore the second has a *thing*, less a *part*.

This gives the equations

$$^{11}/_{12}\,thing - {}^{3}/_{4}\,part = 13\ {}^{1}/_{2}\ ,\quad {}^{4}/_{5}\,part + {}^{2}/_{15}\,thing = 16\ {}^{1}/_{5}\ ,$$

of which the latter after multiplication by $^5/_6$ becomes

$$^{2}/_{3}\,part + {}^{1}/_{9}\,thing = 13^{1}/_{2}\ .$$

The right-hand side being equal, the ratio $r : p$ can be determined, whence also the ratio $r - p : p$. This leads to the solution.

The fourth instance in found in chapter 15 part 3, the presentation of *aliabra*. Once again, it is found within an alternative solution. The problem [ed. Boncompagni 1857: 434; ed. Giusti 2020: 658] is the following:

> I divided 10 in two parts, and divided the larger by the smaller, and the smaller by the larger; and aggregated that which resulted from the division, and they were 5 *denarii*.

Here it is to be observed that Fibonacci often provides the pure numbers by the unit *denarius*, following the habit of Arabic algebra, where *dirham* (*dragma* in the Latin translations) has the same role – not very different from Diophantos's *monas* though with a different origin.

The alternative solution [ed. Boncompagni 1857: 435; ed. Giusti 2020: 660] starts that

> you posit one of the two parts a *thing*, and the other certainly 10 less a *thing*. And let from the division of 10 less a *thing* in a thing a *denarius* result.

Obviously, this draws of the Arabic use of coin names as second (and when needed third and fourth) unknown, cf. above, text before note 5. Since the calculation is complicated, and includes a number of errors, it is better to go on with a symbolic transcription. Here, we shall use *d* to render the algebraic unknown *denarius*, and δ to render the unit for pure numbers (Fibonacci mixes up the two[9]); I shall insert δ where Fibonacci does. For brevity, *r* shall stand for the *thing* (*res*), *C* for *census* (= *r*·*r*),[10] *CC* for *census census* (*C*·*C*). So,

$$\frac{10-r}{r} = d$$

and since

$$\frac{10-r}{r} + \frac{r}{10-r} = \sqrt{5}$$

we have

$$\frac{r}{10-r} = \sqrt{5}-d \;, \quad d\cdot r = 10-r.$$

Therefore,

$$r = (10-r)\cdot(\sqrt{5}-d) = \sqrt{500}-\sqrt{(5C)}-10d+10\delta-r,$$

which can be rearranged as

$$10d = \sqrt{500}-\sqrt{(5C)}+10\delta-2r.$$

Fibonacci's text [11] writes the left-hand side "10 et denario"; the outcome of the division by 10, however, is correct, and leaves out the δ:

$$d = 1+\sqrt{5}-\sqrt{(^{1}/_{20}C)}-{}^{1}/_{5}r \;.$$

9 Abū Kāmil (for instance, but our best source for the Arabic usage), does not mix the two things. His unit for numbers is *dirham*, but the coins that serve as unknowns are *dinar*, *fals* and *khātam*.

10 In al-Khwārizmī's *al-jabr*, second-degree problems are presented as dealing with a *māl*, "possession", becoming *census* in Toledo Latin and soon *censo* (with plural *censi*) in Italian, and its (square) root; but in problem solutions, al-Khwārizmī identifies the *thing* with the *root*, and its square therefore with the *census*.

11 All manuscripts, see [Giusti 2020: 660 apparatus]. In the reconstructed text, Giusti, presupposing Fibonacci's calculation to have been correctly intended, rectifies this as well as the ensuing $^{1}/_{2}$ that should have been $^{1}/_{20}$.

Multiplication by r and substitution of $d{\cdot}r$ by $10–r$ should give

$$10-r = \sqrt{(5C)}+r-{}^{1}/_{5}C-\sqrt{({}^{1}/_{20}CC)}$$

whereby d is eliminated. However, from this point onward all manuscripts change ${}^{1}/_{20}$ into ${}^{1}/_{2}$ (I shall instead continue with the correct calculations). This is rearranged (with δ reappearing) as :

$$\sqrt{({}^{1}/_{20}CC)}+{}^{1}/_{5}C+10\delta = \sqrt{(5C)}+2r\ ,$$

and then multiplied by $\sqrt{500}–20\delta$, which gives

$$C+10\cdot(\sqrt{500}-20) = 10r\ .$$

This equation is identical with one obtained in the first solution to the problem, so Fibonacci stops here.

While the calculation of $(10–r)\cdot(\sqrt{5}–d)$ had been explained in great detail, the normalization leading to the final equation is left unexplained. For us, it is not too difficult to verify this, since

$$\sqrt{500}-20 = 10\times(\sqrt{5}-2)\ ,\ \sqrt{\tfrac{1}{20}CC}+\tfrac{1}{5}C = \tfrac{1}{10}(\sqrt{5}+2)C,\ \sqrt{5C}+2r = (\sqrt{5}+2)r.$$

For Fibonacci, however (even if we should suppose the change of ${}^{1}/_{20}$ into ${}^{1}/_{2}$ to be a secondary mistake), the calculation would require several supplementary steps, among which

$$\sqrt{\tfrac{500}{20}CC} = 5C\ ,\ 20\sqrt{\tfrac{1}{20}CC} = \tfrac{\sqrt{500}}{5}C\cdot$$

Moreover, in order to get the idea, Fibonacci would need to realize that

$$\sqrt{\tfrac{1}{20}CC}+\tfrac{1}{5}C = \tfrac{1}{100}(20+\sqrt{500})\cdot$$

It is not quite excluded that Fibonacci simply found it too difficult to explain this. In view of the other errors (the use of *denarius* in two different functions, possibly the substitution of ${}^{1}/_{20}$ by ${}^{1}/_{2}$ (used in subsequent steps), and also possibly that of 10 *d* by "10 et denario") and of his normal pedagogical inclinations it seems more likely that he copied from a source without fully understanding it.[12] So, whether this is really an instance of *Fibonacci* making use of a second algebraic unknown is more than disputable, even though his text obviously does so.

There are, if I am not mistaken, no more instances of problems solved by means of two algebraic unknowns in the *Liber abbaci*.[13] But there is one in Fibonacci's *Flos* ("The Flower") [ed. Boncompagni 1862: 236], observed already by Vogel [1971: 610] - a pure-number version of an unusual variant of the "purchase of a horse", presented as "about finding five numbers from given proportions", and asking (emphasis added in order to facilitate reading) for

> five numbers, of which the *first with the half* of the second and third and fourth makes as much as the *second with the third part* of the third and fourth and fifth numbers, and as much as the *third with the fourth part* of the fourth and the fifth and the first numbers, and also as much as the *fourth with the fifth part* of the fifth and the first and the second numbers, and besides as much as the *fifth number with the sixth part* of the first and the second and the third numbers.

[12] Sigler [2002: 579, 632] sees that the *denarius* is introduced as a second unknown, but being unaware of the Arabic technique he obviously does not make the link. Moreover, he mercifully corrects the errors of the text.

[13] Another problem from chapter 15 section 3 [ed. Boncompagni 1857: 448; ed. Giusti 2020: 676] might give the impression that it makes use of two algebraic unknowns. It deals with three numbers in geometric proportion (say, *a*, *b*, and *c*, $a<b<c$). It posits *b* to be a *thing*, while in a first solution *a*, in a second solution *c* is posited to be a *dragma*. However, as shown by the calculation, the *dragma* is simply the numerical unit, the number 1. So, what we have here is a combination of algebra with one unknown and a simple false position. In the first solution, use of algebra leads to the simple equation CC = C+1, told correctly to be as the case *census* equal to *thing* and number, and by means of a geometric argument similar to *Elements* II.6 the solution is found to be

$$a=1,\quad b=\sqrt{\frac{1+\sqrt{5}}{2}},\quad c=\frac{1+\sqrt{5}}{2}.$$

This gives a wrong sum of the squares, and therefore has to be adjusted by multiplication with a common factor called again a *thing*. A second solution instead posits *c* to be a *dragma*, and runs in the same way. In a third solution, *b* is posited to be *2 dragmas*, and *a* to be a *thing*. The general principles of this solution are the same.

In symbolic abbreviation:

$$A+{}^{1}/_{2}(B+C+D) = B+{}^{1}/_{3}(C+D+E) = C+{}^{1}/_{4}(D+E+A) = D+{}^{1}/_{5}(E+A+B) = E+{}^{1}/_{6}(A+B+C),$$

an indeterminate problem which it would not be easy to solve without some kind of algebraic reflection or calculation. Accordingly, Fibonacci goes on:

> In order to find this, I thus posited[14] for the first number *causa*,[15] for the fifth *thing*, and for the number to which they are equal under the given conditions, I randomly posited 17.

After protracted arguments and reduction (almost 700 words), this yields two equations:

$$\mathit{thing} = (3-{}^{1}/_{33})\ \mathit{causa}+3^{20}/_{33}$$

and

$$\mathit{thing}+{}^{8}/_{15}\ \mathit{causa} = 15^{13}/_{15}.$$

Inserting the former into the latter and multiplying by 165 Fibonacci finds that

$$578\ \mathit{causa} = 2023$$

whence

$$\mathit{causa} = 3^{1}/_{2}\ .$$

Preferring integers, and knowing that the problem is indeterminate (though not saying that it is), Fibonacci instead chooses *causa* = A = 7, and derives with further intricate and somewhat elliptic arguments that B will then be 10, C will be 19, D will be 25, and E will be 29.

---

14 The *Flos* reports how Fibonacci solved problems with which he had been confronted, whence this first-person singular perfect (*posui*).

15 In medieval Latin, *causa*, originally "cause" or "legal case", had come to sometimes mean an "object" or "movable thing", whence Italian *cosa* and French *chose* for "thing". Fibonacci is likely to have taken the term from medieval Catalan or Castilian, cf. [Costa & Terrés 2001: 41] and [Corominas & Paqual 1980: I, 928]. Provenḍal is also a possibility, cf. [Raynouard 1838: I, 358].

Why not directly from Latin? We should remember that "medieval Latin" was not a language, in any case not *one* language. Many words and values found in medieval-Latin dictionaries were never part of some long-lasting Latin general discourse but borrowings from one or the other vernacular of the time, made when the facts and habits of social everyday had to be spoken of in official or scholarly documents.

Antonio de' Mazzinghi

After Fibonacci, we have to wait until 1380–1390 and until Antonio de' Mazzinghi's *Fioretti* ("Small Flowers") [ed. Arrighi 1967a] before known sources make use of two algebraic unknowns.[16] The *Fioretti* contain the first instance mentioned by Heeffer, and are indeed what Van Egmond refers to (above, note 8).

We only know the *Fioretti* as a whole from the copy which Benedetto da Firenze inserted as book XV, chapter 3 in his *Trattato de praticha d'arismetrica* (autograph Siena, Biblioteca degl'Intronati L.IV.21) from 1463.[17] Occasionally Benedetto's text refers to Antonio in the third person [ed. Arrighi 1967a: 28, 38, 47, 72]; yet on the whole it can be judged faithful in the respects that concern us here.[18]

What Benedetto copied can be seen to be a working version, or at least a text where Antonio does not hide the traces of his progress. At one point [ed. Arrighi 1967a: 63] Antonio attacks a problem that translated into symbols becomes

$$10 = a+b,\ a^2+b^2+\sqrt{a}+\sqrt{b} = 86;$$

he makes a position (t stands for *thing*) $a = 5-t$; $b = 5+t$, which leads to

$$\sqrt{5-t}+\sqrt{5+t} = 36-t^2.$$

At this point, Antonio says ("exclaims" might be the right word") "I do not like it, and therefore I do not complete it" – and goes on with a problem about three numbers in continued proportion.

This character of the work should be kept in mind when we look at what Antonio does with two algebraic unknowns.

16 The dates of Antonio have to be derived from discordant information; it seems plausible that he was born between 1350 and 1355, started teaching very young (perhaps at the age of 15), and died somewhere between 1385 and 1391 [Ulivi 1996: 110-114].

17 The presence of select problems out of order in other manuscripts [Franci 1988: 244] is of no help for the present analysis.

18 More than that, indeed. On one point [ed. Arrighi 1967a: 47] Benedetto points out how something *could* be expressed, but "since we speak like Master Antonio, we shall say" – and then follows a formal fraction involving algebraic polynomials. It thus seems certain that Benedetto tries to render notation as well as mathematical procedures faithfully, and that the third-person references can be regarded as separable external commentary.

In problem 9[19] the beginning of the procedure suggests the use of two unknowns. It deals with two numbers, which for brevity we may designate A and B, fulfilling the conditions that

$$AB = 8\,,\quad A^2+B^2 = 27.$$

A first solution [ed. Arrighi 1967a: 28], "though the case does not come in discrete quantity", makes use of *Elements* II.4, according to which (when it is read as dealing with "quantities" and not line segments)

$$A^2 + B^2 + 2AB = (A + B)^2.$$

This leads to

$$A = \sqrt{10\tfrac{3}{4}} + \sqrt{2\tfrac{3}{4}}\;,\quad B = \sqrt{10\tfrac{3}{4}} - \sqrt{2\tfrac{3}{4}}\;.$$

and at the same time tells us that Antonio's use of "quantity" has nothing to do with that of Aristotelian or scholastic philosophy (where it would refer to lengths, weights and other continuous magnitudes, and be opposed to numbers). A "quantity", for Antonio, is a number or, when needed (as here) an expression involving radicals.

Next Antonio teaches (underlining of "root" renders an encircing in the manuscript and means that the root is to be taken of an ensuing binomial) that

> we can also make it by the equations [*aguagliamenti*] of algebra; and that is that we posit that the first quantity[20] is a thing less the root of some quantity, and the other is a thing plus[21] the root of some quantity. Now you will multiply the first quantity [A] by itself and

[19] This numbering is found in Benedetto's manuscript; it is too similar to what is done elsewhere in the *Trattato* to be safely ascribed to Antonio.

[20] We observe that the two numbers of the statement have now become "quantities". There is nothing unusual in this, Antonio often replaces one word by the other. As we see in the following lines, that creates some confusion, only to be kept under control by keen unspoken awareness of what the various "quantities" refer to. As we shall discover further on, however, Antonio is aware of the difficulty and knows how to eliminate it.

[21] "Plus" translates *più*, literally "more" – but the expression "una chosa più la radice d'alchuna quantità" is ungrammatical if *più* is understood in this literal way. The word instead functions as a quasi-preposition, just like our "plus". Fortunately the English word "less" can serve as a quasi-preposition as well as in adjective function.

> the second quantity [*B*] by itself, and you will join together, and you will have 2 *censi* and an unknown quantity, which unknown quantity is that which there is from 2 *censi* until 27, which is 27 less 2 *censi*, where the multiplication[22] of these quantities [those of which the square root was taken] is $13^1/_2$ less a *censo*. The smaller part is thus a *thing* minus the root of $13^1/_2$ less a *censo*, and the other is a thing plus the root of $13^1/_2$ less 1 *censo*. [...].

If Antonio had worked with two algebraic unknowns, taking the "some quantity" as second unknown (say, q), he would have started with these steps (C stands for *censo*):

$$A = t+\sqrt{q}, \quad B = t-\sqrt{q}$$

$$A^2+B^2 = 2C+2(\sqrt{q})^2 = 2C+2q$$

whence

$$q = 13^1/_2-C,$$

which corresponds to the numerical steps in Antonio's argument, and obviously to his understanding. But what he does can instead be expressed

$$a = t+\sqrt{?}\ , \quad b = t-\sqrt{?}$$

$$a^2+b^2 = 2C+??,$$

and the fact that "??" equals two times "?" stays in his mind.

From this point onward, the method is algebraic, but with only one unknown (and the procedure is impeccable).

In the following problem 10 [ed. Arrighi 1967a: 30] we read:

> Find two numbers whose squares are 100, and the multiplication of one by the other is 5 less than the squared difference. Posit that the first number be a thing plus the root of some quantity, and the second be a thing less the root of some quantity, and multiply each number by itself

[22] Antonio, as other abbacus writers as well as Fibonacci, uses the same term for the *process* of multiplying and the *outcome*. We may add that our term *product* strictly speaking means nothing but "outcome" of any process, even though we have become accustomed to restrict it within the context of arithmetic to the outcome of a multiplication.

> and join the squares, they make two *censi* and something not known. And these squares should make up 100. Whence this unknown something is the difference there is from 100 to 2 *censi*, which is 100 less 2 *censi*. [...].

As we see, Antonio here gets even closer but still does not fully implement the possibility of working *algebraically* with two unknowns. But he can be seen to be preparing mentally, and in problem 18 [ed. Arrighi 1967a: 41] the idea comes to fruition:

> Find two numbers which, one multiplied with the other, make as much as the difference squared, and then, when one is divided by the other and the other by the one and these are joined together make as much as these numbers joined together. Posit the first number to be a *quantity* less a *thing*, and posit that the second be the same *quantity* plus a *thing*. Now it is up to us to find what this *quantity* may be, which we will do in this way. We say that one part in the other make as much as to multiply the difference there is from one part to the other in itself. And to multiply the difference there is from one part to the other in itself makes 4 *censi* because the difference there is from a *quantity* plus a *thing* to a *quantity* less a *thing* is 2 *things*, and 2 *things* multiplied in itself make 4 *censi*. Now if you multiply a *quantity* less a *thing* by a *quantity* plus a *thing* they make the square of this *quantity* less a *censo;* so the square of this *quantity* is 5 *censi*. And if the square of this *quantity* is 5 *censi*, then the *quantity* is the root of 5 *censi;* whence we have made clear that this *quantity* is the root of 5 *censi*. And therefore the first number was the root of 5 *censi* less a *thing* and the second number was the root of 5 *censi* plus a thing. We have thus found 2 numbers which, one multiplied in the other, make as much as to multiply the difference of the said numbers in itself; and one is the root of 5 *censi* less a *thing*, the other is the root of 5 *censi* plus a *thing*. Now remains for us to see whether one divided by the other and the other by the one and these two results joined together make as much as the said numbers. Where you will divide the root of 5 *censi* less a *thing* by the root of 5 *censi* plus a thing, this results, that is, $\frac{r.\text{ of } 5\ C\text{ less } 1\rho}{r.\text{ of } 5\ C\text{ plus } 1\rho}$ And then you will divide the root of 5 *censi* plus 1 *thing* by the root of 5 *censi* less a *thing*,

[23] Benedetto and, almost certainly Antonio, uses ρ (evidently not the Greek letter but something fairly similar) as a symbol for the *thing*. Since it is used within formal calculations involving formal fractions like these, it is justified to speak of them as *symbols* and not mere abbreviations, cf. [Høyrup 2010: 30–35].

$\frac{r.\text{ of } 5\ C \text{ plus } 1\rho}{r.\text{ of } 5\ C \text{ less } 1\rho}$ results.[23] And these two results should be joined together; where you will multiply the root of 5 *censi* plus a *thing* across,[24] that is, by the root of 5 *censi* plus *a thing*, they make *censi* plus the root of 20 *censi of censo*; and further multiply root of 5 *censi* less a *thing* across, that is, by root of 5 *censi* less a thing, they make 6 *censi* less root of 20 *censi of censo*.[25] Which, joined with 6 *censi* and root of 20 *censi of censo*, make 12 *censi*. And this quantity we should divide in the multiplication of the root of 5 *censi* less a *thing* in root of 5 *censi* plus a *thing*, which multiplication is 4 *censi* because root of 5 *censi* in root of 5 *censi* make 5 *censi*, and a *thing* plus multiplied in a *thing* less[26] make a *censo* less, and when it is detracted from 5 *censi*, 4 *censi* remain, and multiplying 1 *thing* plus by root of 5 *censi* and 1 *thing* less by root of 5 *censi*, their joining makes 0. So the said multiplication, as I have said, is 4 *censi*, so these two results are 12 *censi* divided in 4 *censi*, from which comes 3. And we want they should make as much as the sum of the said numbers, whence it is needed to join the root of 5 *censi* less a *thing* with the root of 5 *censi* plus a thing, they make 2 times the root of 5 *censi*, which is the root of 20 *censi*. Whence the joining of the said numbers is the root of 20 *censi*, and we say that is should be 3; so 3 is equal to the root of 20 *censi*. Now multiply each part in itself, and you will have 9 to be equal to 20 *censi;* so that, when it is brought to one *censo*, you will have that the *censo* will be equal to $^9/_{20}$. So the *thing* is equal to the root of $^9/_{20}$, and if the *thing* is equal to the root of $^9/_{20}$, the *censo* will be worth its square, that is, $^9/_{20}$. So the first number, which was the root of 5 *censi* plus a *thing*, was $1^1/_2$ plus the root of $^9/_{20}$; and the second number, which was the root of 5 *censi* less a *thing*, was $1^1/_2$ less the root of $^9/_{20}$. And so are found the said two numbers [...].

This probably goes beyond what Antonio was able to do by mental implicit use of a second unknown, or at least beyond what he found it possible to convey to a "model reader" in this way. This seems the likely reason that he now makes the use of two unknowns explicit, and also chooses a more stringent language, pointing

[24] The cross-multiplication is shown in a symbolic operation on the two formal fractions in the margin in the manuscript (fol. 458v) – Benedetto's autograph, but certainly copied from Antonio, as argued in [Høyrup 2010: 31–33].

[25] *Censo of censo* is the fourth power of the thing. At its second occurrence, Arrighi has 20 *censi* only, but the manuscript (fol. 458v) is correct.

[26] We observe a distinction between additive and subtractive (not yet negative) numbers.

out that the *same* quantity is meant in the two positions. Awareness that something new and unfamiliar is presented to the reader is reflected in the explanation that now "it is up to us to find what this *quantity* may be" – it is never stated that the *thing* has to be found, neither here nor elsewhere in problems with a single algebraic unknown.

It is also noteworthy that from this point onward, *quantity* in general use (cf. note 20) disappears from all problem solutions where that term is used to designate one of two algebraic unknowns (but not from other problems – in these *quantity* is still used profusely as a synonym for "number".[27]

The procedure can be translated into more familiar symbols as follows:

$$AB = (A-B)^2 \ , \ \ {}^{A}/_{B}+{}^{B}/_{A} = A+B$$

with the algebraic positions

$$A = q-t \ , \ \ B = q+t \ .$$

Then

$$(A-B)^2 = 4C \ , \ \ \text{while} \ \ AB = q^2-C \ ,$$

whence

$$q^2 = 5C \ ,$$

that is,

$$q = \sqrt{5C} \ .$$

In consequence we have the preliminary result

$$A = \sqrt{5C} - t \ , \ \ B = \sqrt{5C} + t \ .$$

Inserting this in the other condition we get

$$\frac{A}{B}+\frac{B}{A} \ = \ \frac{\sqrt{5C}-t}{\sqrt{5C}+t}+\frac{\sqrt{5C}+t}{\sqrt{5C}-t}$$

27 There are two apparent exceptions, one in the present problem ("this quantity we should divide in the multiplication of the root of 5 *censi* less a *thing* in root of 5 *censi* plus a thing"), one in problem 28 [ed. Arrighi 1967a: 61*f*]. Both, however, turn up after the algebraic *quantity* has been eliminated, and the problem thus reduced to one with a single unknown *thing*.

which, after cross-multiplication, becomes

$$\frac{A}{B}+\frac{B}{A} = \frac{(\sqrt{5C}-t)^2+(\sqrt{5C}+t)^2}{5C-C} = \frac{6C+6C}{4C} = \frac{12C}{4C} = 3$$

Therefore, since

$$A+B = 2q = 2\sqrt{(5C)},$$

$$2\sqrt{5C} = \sqrt{20\,C} = 3,$$

whence

$$20C = 9\ .$$

Tacitly interchanging "first" and "second" number, Antonio thereby obtains that

$$B = 1^1/_2+\sqrt{\,^9/_{20}}\,,\ A = 1^1/_2-\sqrt{\,^9/_{20}}.$$

This would probably have been very difficult even for a mathematician of Antonio's calibre to do without the explicit use of two unknowns. Once Antonio had decided to make the step, things were easy. As we can see in the marginal calculations, Antonio routinely performed formal calculations involving ρ (standing for the *thing*, we remember) and *c* or c^o (standing for *censo*) – his "multiplication across" refers to that.

Now, once the method has been invented and introduced, Antonio makes use of it even in problem 19 [ed. Arrighi 1967a: 43], which *could* have been solved according to the pattern we know from problems 9 and 10:

> Find two numbers so that the root of one multiplied by the root of the other be 20 less than the numbers joined together, and their squares joined together be 700. It is asked, which are the said numbers? You will make position that the first number be a *thing* less some quantity, and posit that the other number be a *thing* plus some quantity. And then you take the square of the first, which we said was one *thing* less one *quantity*, and its square is one *censo* and the square of this *quantity* less the multiplication of this *quantity* in a *thing*. And the square of the second number, which we say is a thing and some quantity, is a *censo*

and the square of this quantity plus the multiplication of this quantity in a thing.[28] Which, joined together, make 2 *censi* and 2 squares of 2 *quantities*.[29] And we say that they should make 700, whence one of these squares is 350 less one *censo*. This quantity is thus the root of 350 less once *censo*. And we posited that the first number was one *thing* less one *quantity*, that is was hence one thing less the root of 350 less one *censo*. And the second number, which was posited to be a *thing* and a *quantity*, was one *thing* and root of 350 less one *censo*. And thus we have solved a part of our question, that is, to find two numbers whose squares joined together make 700. Now it remains for us to see what it makes to multiply the root of one by the root of the other. Therefore you thus have to multiply the general root of one *thing* less root of 350 less one *censo* by the general root of one *thing* plus root of 350 less one *censo*,[30] they make root of 2 *censi* less 350; and this is their multiplication. For these matters one has to keep the eye keen, I mean of the mind and the intellect, because even though they seem rather easy, none the less, who is not accustomed will err. Therefore we have thus found that this multiplication is the root of 2 *censi* less 350, and this we say is 20 less than the numbers joined together. And the said numbers joined together are 2 *things*, that is joining a *thing* less root of 350 less a *censo* with a *thing* plus root of 350 less a *censo*, which indeed make 2 things. Whence we have that 2 *things* less 20 are equal to the root of 2 *censi* less 350; whence, in order not to have the names[31] of roots, multiply each part in itself, and you will have that root of 2 *censi* less 350 multiplied in itself make 2 *censi* less 350, and 2 *things* less 20 multiplied in itself make 4 *censi* and 400 less 80 *things*. So 2 *censi* less 350 are equal to 4

28 Obviously, the product of *quantity* and *thing* should be taken twice here as well as in the square of the first number. Antonio knew perfectly well how to multiply two binomials. Since the "error" is repeated in subsequent problems, we may be sure that Antonio abbreviates, knowing that the two elliptical expressions cancel each other.

29 *2 quadrati di 2 quantità* is also in the manuscript (fol. 459r), Benedetto's autograph. Perhaps Antonio (or Benedetto) makes a mistake, perhaps and more likely Antonio thinks of "the two squares coming from the two distinct quantities".

30 A "general root" is taken of a composite of which one component is the root of a binomial . The "general root of one *thing* less root of 350 less one *censo*" is thus to be understood as $\sqrt{(t-\sqrt{[350-C]})}$.

31 *Nomi*. Normally, the algebraic powers (*cosa*, *censo*, *cubo*, etc.) are spoken of as "names"; as we see, Antonio sees the root as belonging to the same category.

censi and 400 less 80 *things*. Where you should make equal the parts giving to each part 80 *things* and removing 2 *censi;* and we shall have that 2 *censi* and 740 are equal to 80 *things*, which is the fifth rule.[32] Where you bring to one *censo*, and you will have one *censo* and 375 equal to 40 *things*. Where you will halve the *things*, and let the half be 20, multiply in itself, they make 400, detract the number, they will make 25, that is, detracting 375 from 400, of which 25 take the root, which is 5, and detract it from 25, 15 remain. And you will say that the thing is worth 15, and the *censo* will be worth its square, which is 225. Whence the first number, which we posited that it was a *thing* less root of 350 less a *censo*, detract 225, which is worth the *censo*, from 350, 125 remain. And you will say, one part was 15 less root of 125, and the second number was 15 plus root of 125. [...].

In our usual translation:

$$\sqrt{A} \cdot \sqrt{B} = A+B-20 \ , \quad A^2+B^2 = 700 \ ,$$

with the position

$$A = t-q \ , \quad B = t+q \ ,$$

where Antonio no longer feels the need to point out that the two "some quantity" (*alchuna quantità*) refer to *the same* quantity. He does not quite return to the formulation of problems 9 and 10, $A = t-\sqrt{q}$, $B = t+\sqrt{q}$, since with the explicit position of q he can now operate freely with its square. Antonio calculates

$$A^2 = C+q^2-[2]qt \ , \quad B^2 = C+q^2+[2]qt \ ,$$

whence

$$2C+2q^2 = 700 \ , \quad q^2 = 350-C \ , \quad q = \sqrt{(350-C)}.$$

Therefore

32 That is, the fifth standard "case" (equation type) of abbacus *aliabra* (and al-Khwārizmī's *al-jabr*), "*censi* and number are equal to *things*" (the case with a double solution, which Antonio neglects here – the alternative solution leads indeed to complex and thus impossible values for *a* and *b*). In what follows, Antonio makes use of the standard algorithm for this case, which explains the unusually awkward choice of verbal forms (slightly more awkward in the original than I am able to render in understandable English).

$$A = t-\sqrt{(350-C)}\ ,\quad B = t+\sqrt{(350-C)}\ ,$$

which is seen as a partial answer, and is inserted in the other condition:

$$AB = \sqrt{t-\sqrt{350-C}} \times \sqrt{t+\sqrt{350-C}} = \sqrt{C-(350-C)} = \sqrt{2C-350},$$

a calculation which seems straightforward but where, according to Antonio, the untrained will none the less err.[33] At all events, with the correct calculation we now have

$$\sqrt{2C-350} = A+B-20 = 2t-20.$$

whence after squaring

$$2C-350 = 4C+400-80t,$$

which can be reduced to

$$2C+750 = 80t.$$

Solving this equation by means of the standard rule or algorithm for the fifth algebraic case Antonio finds $t = 15$ – silently discarding the other solution $t = 25$, cf. note 32.

There are more problems in the *Fioretti* which are solved by means of two algebraic unknowns: number 20, number 21, number 22 (twice during the procedure), number 24, number 25 and number 28. All seven make the position

$$a = t-q\ ,\quad b = t+q,$$

and all seven *could* have been solved in the same way as number 9 and number 10, if only the position had been

$$a = t-\sqrt{?}\ ,\quad b = t+\sqrt{?},$$

that is, with an implicit second unknown. Apart from one detail, they tell nothing new about the use of two unknowns, and there is no reason to go in depth with them – except, that is, for this detail. Number 20 [ed. Arrighi 1967a: 44] begins

Find two numbers so that their roots joined together make 6 and their

[33] Those who doubt Antonio's words should be aware that near-contemporary algebraic writings might presume that $\sqrt{a+\sqrt{b}} = \sqrt{a}+\sqrt{\sqrt{b}}$ – thus Parma, Biblioteca Palatina, ms. Pal. 312, ed. [Gregori & Grugnetti 1998: 116].

> squares be 60, that is, the joining of the squares be 60. Posit the first number to be a thing less the root of some quantity, that is less some quantity; the other posit to be a thing plus the said quantity. [...].

Firstly, this confirms that Antonio as copied by Benedetto presents us with a work in progress – if the *Fioretti* had been polished, there would have no reason to leave a formulation "root of some quantity" then to be corrected. Secondly, the slip shows that Antonio at first had in mind the method of problems 9 and 10; it is a plausible guess, and it can be no more, that he used an earlier solution of the problem – probably his own, nobody else in Italy between Fibonacci and Antonio is known to have possessed adequate mathematical capabilities except perhaps Dardi of Pisa, who however worked on different problem types.

Borrowed or reinvented?

As said initially, operations with two algebraic unknowns precede Fibonacci. Did he reinvent, or did he borrow his technique from elsewhere? In [Høyrup 2009: 82 n. 104], knowing only the problem from the *Flos*, I took it for granted (and so obvious that it did not deserve explicit statement) that Fibonacci had made an independent reinvention. With the three or four problems from the *Liber abbaci* (four if we believe Fibonacci to have understood what he took over for chapter 15 part 3), the evidence suggests otherwise.

All known manuscripts of the *Liber abbaci* go back to the second edition, dedicated to Michael Scot and dated in some of them to 1228 – with one exception: In [2017], Enrico Giusti showed that chapter 12 in the manuscript Florence, Biblioteca Medicea Laurenziana, Ms. Gaddi 36 (henceforth **L**) is rather different from what is found in [Boncompagni 1857]. Strong internal evidence shows it to be older. As argued by Giusti, it is likely to represent the original 1202 version; at the very least it precedes what is found in the other manuscripts.

The first two problems from the *Liber abbaci* that were discussed above are precisely from chapter 12. Both *problems* are also in **L**. However, for the problem from [Boncompagni 1857: 212; Giusti 2020: 355], only the first two solutions by means of false positions are offered, there is no trace of the algebraic solution with its two unknowns. As regards the problem from [Boncompagni 1857: 264; ed. Giusti 2020: 426], on the other hand – the one where the only solution given is the one by *regula recta* identified by name – the algebraic solution with its posited *amount* and *thing* is also in **L** [ed. Giusti 2017: 134*f*].

Fibonacci's introduction of the *regula recta* follows a similar pattern. The alternative solution offered to the "give-and-take" problem with which Fibonacci had been confronted by a master from Constantinople [ed. Boncompagni 1857: 191; ed. Giusti 2020: 324] is not in **L**; in consequence, the pedagogical introduction to the rule ("much used by the Arabs", and "immensely praiseworthy") is also absent. That does not mean, however, that the *regula recta* is not used, not even that it is not spoken about, in **L**. It is referred to and used repeatedly [ed. Giusti 2017: 70, 78, 125], just without any explanation; and then, of course, in the problem about repeated travels, where it is used with two unknowns.

It appears – and no other explanation seems at hand – that Fibonacci used the *regula recta* as something with which he was familiar in the first version of the *Liber abbaci*, or at least in the early version of chapter 12 that is contained in **L**. Then, when adding "certain necessary things" [ed. Boncompagni 1857: 1; ed. Giusti 2020: 3] in the revised version dedicated to Michael Scot, he quite appropriately *explained* it. Since one of the places where the rule is used without explanation in **L** [ed. Giusti 2017: 134*f*] involves two unknowns, it goes almost by itself that the use of two unknowns within the *regula recta* was also something "just known".

That fits the appearance of two unknowns in the *Flos*. We have no certain knowledge of the date of the *Flos* – a reference [ed. Boncompagni 1862: 234] to the *Liber abbaci* in a passage addressed to the Emperor as "your book" suggests a date later than 1228 (or whatever the precise date of the second edition of that work), but the single problems with their solution are told by Fibonacci himself [ed. Boncompagni 1862: 227] to antedate the treatise in which they were put together.

In consequence, Fibonacci seems to have used two algebraic unknowns for the first time in 1202, in a problem that was too complex for his normal methods; then, to have had recourse to it in a similarly tangled situation in the *Flos*, using however a different set of names (*causa* and *res* instead of *summa* and *res*); and finally, when making the revised version of the *Liber abbaci* in 1228, to have employed it (now with unknowns *borsa* and *res*) in a situation where it was not strictly necessary but brought in as an alternative, perhaps for pedagogical reasons – in parallel to the explanation of the *regula recta* which was regarded as one of the necessary things that had to be inserted. No reinvention, merely recourse to a known but rarely needed technique – similarly to what is suggested by Abū Kāmil's "large thing" and "small thing" (above, note 5).

Pedagogical or not, Fibonacci's use of two unknowns did not inspire Antonio's use of two unknowns in the *Fioretti* (however much he is reported to have appreciated Fibonacci's work in general). That is obvious if we recapitulate the steps in which he approached the idea: at first two problems (9 and 10) where an intuition of a second unknown is operated mentally; then a more intricate situation (problem 18) which does not allow quick elimination of the second unknown, and therefore goes beyond what can be mastered by intuition; then another bunch of problems (numbers 20, 21, 22, 24, 25 and 28) where the intuitive method of problems 9 and 10 *would* have worked, but where Antonio now sticks to the explicitation developed in problem 18, and where the slip in number 20 points to the existence of an earlier version or earlier idea based on the intuitive approach.

Antonio *may* have been aware of Fibonacci's use of two unknowns. However, what he develops here is something different. With exception of the problematic instance from chapter 15 section 3 of the *Liber abbaci*, the questions where Fibonacci uses two unknowns are all linear, as the *regula recta* in general. Those of Antonio are not. Moreover, Antonio understands his problems to belong within the area of *aliabra* – his *thing* multiplied by itself becomes a *censo*. Whether this is the reason that a (quite hypothetical) awareness of Fibonacci's expanded *regula recta* method is left aside is hardly to be decided. What is clear is that the actual method developed by Antonio is an independent creation. No absolute first in the history of mathematics – already Brahmagupta [ed., trans. Colebrooke 1817: 361*f*] had given rules for certain problems involving products of different unknowns; but clearly no borrowing but something Antonio had laboured to find by himself.

Who's next?

Enough abbacus books have survived to allow a generic portrait of abbacus mathematics, and even to delineate broad developments from one century to the next; but too many manuscripts have gone lost or have never been read in detail to trace the emergence and maturation of particular ideas. With this important proviso we may claim that Antonio's invention had no immediate consequences – except perhaps for one strange and partial exception to which we shall return below (hardly inspired by Antonio, however; text around note 39).

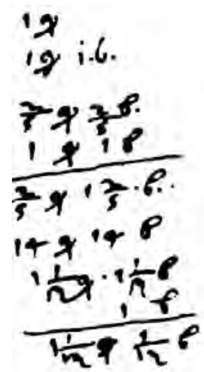
[34] Siena, Biblioteca degl'Intronati L.IV.21 (Benedetto's autograph).

Two algebraic unknowns proper only again rise over the horizon in 1463, in Benedetto's *Trattato de praticha d'arismetrica*[34] – that is, in the parts composed independently by Benedetto.

Beyond Antonio's *Fioretti*, Benedetto's *Trattato* contains several other extensive borrowings, always identified as such with reference to the original author (Fibonacci as well as Antonio and other abbacus writers). But precisely the conscientious identification of borrowings allows us to distinguish Benedetto's own mathematics – certainly no fresh invention but firmly rooted in abbacus tradition though on a much higher mathematical level than average abbacus books.[35]

On fol. 262^{r-v} we find two algebraic unknowns in a problem about five men finding a purse:

> Five men have *denari*, and going on a road they find a purse with *denari*. The first says to the others, if I got the *denari* of the purse, then I would have $2^1/_2$ times as much as you. The second says, if I got the *denari* of the purse, then I would have $3^1/_3$ times as much as you. The third man says to the other 4, if I got the *denari* of the purse, I would have $4^1/_4$ as much as you. The fourth man says to the other 4, if I got the *denari* of the purse, I would have $5^1/_5$ as much as you. The fifth man says to the other 4, if I got the *denari* of the purse, I should have $6^1/_6$ as much as you. It is asked how much each one had, and how many *denari* there were in the purse. You will make the position that the first had a *quantity*, and having got the purse he had a *quantity* and a *purse*, and he says to have $2^1/_2$ of the others. [...].



The purse is not explicitly posited, we observe. But after having written this introduction, the last part of which takes up the first two lines of fol. 263^v, Benedetto starts calculating in the margin, using *q* for the *quantity* and *b* for the *purse* (*borsa*) – the diagram to the right (redrawn for clarity) shows the first steps of

[35] Benedetto's independent work is also often characterized by being accompanied by extensive marginal calculations – better, by accompanying marginal calculations that were made before the text proper in the autograph, see [Høyrup 2010: 32*f*]. Such parts of the text evidently cannot be copied from an already finished model or source.

[36] The organization of the page shows beyond doubt that first these two lines were written, then the marginal calculations made, and finally the rest of the text written in whatever space was left over – see the depiction of its structure in [Høyrup 2010: 32]).

the very complex calculation.[36] So, not only is Benedetto operating with two unknowns, he also performs symbolic operations in which the unknowns are represented by one-letter abbreviations.

Did Benedetto learn this from Antonio, whose *Fioretti* he was to insert in the *Trattato* at a later point? Is such a borrowing supported by his use of *quantità* as one of the algebraic unknowns?

Not necessarily, and hardly. In his shorter, more elementary *Tractato d'abbacho* [ed. Arrighi 1974: 168, 181][37] Benedetto introduces the *regula recta* under the name *modo recto* (or *recpto* or *repto*, the orthography of the manuscript varies), suggesting that he took it from the school tradition and not from the *Liber abbaci*.

As a matter of fact, the abbacus school tradition may well have had direct access to the Arabic rule, and need not have learned about it from Fibonacci. In the *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304–371], translated into Latin in the 12th century, it is used abundantly as an alternative to the double false position under the unqualified name *regula*. If two Latin authors had encountered it independently, why not also some other early abbacus writer? In particular since an abacus treatise from c. 1300 (Siena, Biblioteca degl'Intronati L.VI.47_2) has adopted a term for prime numbers from spoken Maghreb Arabic independently of Fibonacci, see [Høyrup 2018: 4].

Further evidence that Fibonacci is not Benedetto's source for the method is the name for Benedetto's (primary) unknown: *quantità*, not "thing". The *Liber augmenti et diminutionis* uses *census* in the same function: as we remember, this was the Toledo standard translation of Arabic *māl*, meaning precisely *quantity* (of money).

"Primary unknown", indeed, since all but one of the examples of the use of the rule in Benedetto's *Tractato d'abbacho* make use of two algebraic unknowns. Initially [ed. Arrighi 1974: 168] there are three problems of type "purchase of a horse" (cf. above, on the problem from the *Flos*). The first of them, involving only two buyers, is solved by a means of a single unknown called *quantità*, the other two make use of *quantità* and *cavallo* ("horse", standing for its price). Then [ed. Arrighi 1974: 181–183] come three about men having *denari*, going on a road and finding there a purse. Here, as in the purse problem from the *Trattato de praticha*

[37] Misled by a wrong identification of the author in his manuscript, Arrighi ascribed the text to Pier Maria Calandri. Van Egmond [1980: 356] ascertained its identity with Benedetto's abbacus book.

d'arismetrica, the algebraic unknowns are *quantità* and *borsa*. In two of them, the *quantità* is the original possession of the first man, in the third it is the collective possession of the three men together with the contents of the purse. With great probability we may assume that Benedetto took the idea of using *quantity* as a basic unknown not from Antonio but from the same school tradition which gave him the *modo recto*, and that this school tradition used *modo recto* algebra with *quantità* as primary unknown regularly in Benedetto's Florentine mid-15th century.[38] What was concluded above concerning Fibonacci suggests, together with the similar naming styles ("purse", "amount", "horse"), that Benedetto's use of two algebraic unknowns may have been no 15th-century innovation but already a characteristic of the Arabic *regula recta* as Fibonacci and early abbacus masters encountered it.

An anonymous Florentine from *ca* 1390

Antonio does not seem to have treated of first-degree problems by means of two unknowns. At least, there are none in his *Fioretti* and, more important, there are none in the collection of 21 "miraculous" algebra problems of his student and successor Giovanni di Bartolo [ed. Arrighi 1967b] as copied in another "abbacus encyclopedia" (Florence, Bibl. Naz., Palat. 573). In this collection, difficult versions of such types as the "give-and-take" are constructed not by increasing the number of participants but by introducing square roots in the conditions – for instance [ed. Arrighi 1967b: 19]:

> Two have *denari*. The first says to the second, give me the root of your *denari*, I shall have as much as you have. The second says to the first, give

[38] The two other approximately contemporary Florentine "abbacus encyclopediae" (Vatican, Ottobon. lat. 3307 und Florence, Bibl. Naz., Palat. 573) both use *quantità* (abbreviated *q* in marginal calculations) in *regula recta* calculations (as far as I have noticed in my two fairly illegible scans never two algebraic unknowns).

Added in proof: Actually, there are at least two rudimentary occurrences of *q* together with *b* in the margins of the Ottoboniano manuscript (fols181^{v}, 185^{r}). More interesting is the introduction of the *modo recto* on fol. 28^{v} of the same manuscript, with reference to "Leonardo [Fibonacci] and all the others who understand". Obviously, the writer knows the method from the *Liber abbaci*, as also confirmed by the reference to the Arabic origin of the method. But the reference to "all the others who understand" (*tucti gl'altri intendenti*) shows that he also knows if from a general abbacus tradition, within which, as he says, "some say it is one of the exemplary modes of algebra" (*alchuno lo dice uno d'esemplari modi dell'algebra*). Every time the technique is used, marginal notes use *q* for the unknown *quantità*. This terminology leaves no doubt that the main reference of the writer is the living abbacus tradition, not Fibonacci.

me such part of your *denari* as I gave to you, and I shall have 10 more than you.

For this, the *thing*, its square (the *censo*) as well as its reciprocal have to be manipulated; but there is no need for a second unknown.

Support for the hypothesis that a *regula recta* tradition involving the use of two unknowns may none the less have inspired Benedetto is offered by a Florentine manuscript written around 1390, *Tratato sopra l'arte della arismetricha*[39] – sufficiently different from what we know from Antonio's hand to exclude more than possible (and, given temporal, geographical and professional proximity, probable) acquaintance.[40]

The author (assuming that we are confronted with an original composition) is a brilliant algebraist – see [Høyrup 2019: 331*f*] for his transformation of cubic equations (unfortunately he is less brilliant when it comes to grammar and style). As one of the illustrations of the algebraic case "cubes and *censi* equal to things" we also find a "give-and-take" problem involving the square of one of the possessions [ed. Franci & Pancanti 1988: 68][41] – not the same as what we find in the contemporary Giovanni di Bartolo, but clearly belonging to the same family.

This general acknowledgment of the author's competence is not our present concern, but it illuminates the last four of a final collection of problems falling outside what is solved by the 22 standard rules. They constitute the "strange and partial exception" referred to above.

Two of these problems are of type "finding a purse", two "purchase of a horse". All four make use of two algebraic unknowns (partial use, as we shall see), but none of

[39] Florence, Bibl. Naz. Centr., fondo princ. II.V.152. [Franci & Pancanti 1988] is an edition of its extensive algebra section.

[40] See, for example, [Høyrup 2015a: 18]. The *Tratato* in question introduces a naming of algebraic powers identifying these as "roots". The second power is "*censo* or *radice*", the third power "*cubo* or *radice cubica*", ..., the fifth power "*cubo di censi* or *a root that is engendered by a square quantity against a cubed quantity*, or some say *radice relata*", ... These root names for powers return, for example, in Luca Pacioli's *Summa* [1494: fol. 143r], and even in Jacques Peletier's *L'Algebre* [1554: 5]; but Antonio does not know them, and uses the simple sequence *cosa*, *censo*, *cubo*, *censo di censo*, *cubo relato*, *chubo di chubo* (according to what is reported in the above-mentioned ms. Palatino 573, fol. 399r).

If we assume the "some" who say *radice relata* to refer to Antonio, we see that the familiarity is not close enough to exclude misunderstanding.

[41] Similarly pp. 59, 65, 73, 75, 78, 82, 84

them take note of that, in spite of being provided with a metamathematical commentary (here emphasized). At first we have a purchase, not of a horse but of a goose:

> Three have *denari* and they want to buy a goose, and none of them has so many *denari* that he is able to buy it on his own. Now the first says to the other two, if each of you would give me $^1/_3$ of his *denari*, I shall buy the goose. The second says to the other two, if you give me $^1/_4$ plus 4 of your *denari* I shall buy the goose. The third says to the other two, if you give me $^1/_4$ less 5 of your *denari* I shall buy the goose. Then they joined together the *denari* all three had together and put on top the worth of the goose, and the sum will make 176, it is asked how much each one had for himself, and how much the goose was worth. *Actually I believe to have stated similar questions about men in the treatise,*[42]*but wanting to solve certain questions in a new way I have found new cases which I do not believe to have (already) treated. [...]. Therefore I have made it in such way that in this one and those that follow it will have to be shown that the question examined by the thing will lead to new questions that cannot be decided without false position.* [...]. I shall make this beginning, let us make the position that the first man alone had a *thing*, whence, made the position, you shall say thus, if the first who has a *thing* asks the other two so many of their *denari* that he says to be able to buy the goose, these two must give to the first that which a *goose* is worth less what a *thing* is worth, which the first has on his own. So that the first can say to ask from the other two a *goose* less a *thing*, and you know that the first when he asks for the help of the others asks for $^1/_3$ of their *denari*. So the two without the first must have so much that $^1/_3$ of their *denari* be a goose less a *thing*, and in this way you see clearly that the second and the third together have 3 *geese* less 3 *things*. Now it is to be seen what all the three have, and it is clear that the first by himself has a *thing* and the other two have 3 *geese* less 3 *things*, so that all three have 3 *geese* less 2 *things*. Now we must come to the second, who asks from the other two $^1/_4$ plus 4 of their *denari* and says to buy a *goose*. I say that when the second has had as help of the other two the part asked for, he shall find to have a *thing* (*sic*[43]).

After longwinded arguments it is concluded that *B* is $^1/_3$ *goose* plus $^2/_3$ *things*

[42] Namely in the sense that fols 97^v–110^r contain a large number of "give and take", "purchase of a horse" and "finding a purse" problems.

[43] The manuscript, correctly, has *ocha*, "goose".

less $5^1/_3$ in number (*A*, *B* and *C* being the three original possessions). Since *B*+*C* has been seen to be 3 *geese* less 3 *things*, *C* is $2^2/_3$ *geese* and $5^1/_3$ in number less $3^1/_2$ *things*. Using then that $C+{}^1/_4(A+B)-5$ is a *goose*, it is found (again I skip intermediate steps) that $1^3/_4$ *geese* equals $3^1/_4$ *things* and 1 in number or, multiplying "in order to eliminate fractions",

7 *geese* = 13 *things*+4.

Moreover, since *A*+*B*+*C* was seen to equal 3 *geese* less 2 *things*, and these together with the *goose* equalled 176

4 *geese*–2 *things* = 176.

Now, for instance, the *thing* might be found from the latter equation (namely, to be 2 *geese* less 88); inserting that in the former equation would easily lead to the goal. Instead the author goes on,

> So, you have two equations (*aguagliamenti*), which are solved one by means of the other in this way: You have on one side (*parte*) that 7 *geese* must be worth as much as 13 *things* and 4 in number, on the other side you will have that 4 *geese* must be worth as much as two things and 176 in number, put the sides together, now I shall make the position that the *goose* is worth 40, and take the first side, that is that 7 *geese* are worth as much as 13 *things* and 4, if the *goose* is worth 40, the 7 will be worth 280, thus 13 things and 4 are worth 280, and the *thing*, dividing the 276 by 13, the *thing* will be worth $21^3/_{13}$. With this go to the other side, and you will say, if the *goose* is worth 40 and the thing is worth $21^3/_{13}$ we shall see that 4 *geese* is worth as much as 2 *things* and 176, where we know that so much should be worth one as the other, from where it is manifest that the 4 *geese* are worth 160, and this is on one side, on the other side the 2 *things* and 176 in numbers will be worth $218^6/_{13}$, and we indeed said that they should be worth 160, there comes $58\ {}^6/_{13}$ more for us [than there should]. Thus save in this first position for 40 that you posited the *goose* to be worth there comes $58\ {}^6/_{13}$ more for us. Now make the other position and posit that the *goose* is worth 80 [...], so you shall say in the second position for 80 that you posited the *goose* to be worth, $58^6/_{13}$ are missing for me. Now take the two positions made and follow the way to be made for positions that become plus and less, and you shall find

> that the price of the *goose* was 60. When the price of the *goose* is known you shall say, if the *goose* is worth 60, then 3 (*sic*[44]) *geese* are worth 420, and 13 *things* and 4 in number are worth 420, the *thing* is thus worth 32 [...].

As we see, not only does the author not speak about using two algebraic unknowns; he evidently does not really see these as such, and therefore does not eliminate one by means of the two equations, as Fibonacci had done in the *Flos*, and as Antonio did repeatedly. This would have been very easy, but instead the author makes use of the non-algebraic double false position, a familiar but opaque technique – more opaque in the present context than normally.[45] The next three problems are quite similar. The style – taking the *goose* as an unknown that can be added, subtracted and multiplied by a coefficient – is too similar to what we find earlier in the Fibonacci problems and later in Benedetto's two treatises to be an independent invention. Instead, the author must have borrowed an idea in circulation – so rarefied circulation, however, that he only grasps half of it, so to say; and then he has completed it in his own way, drawing on a familiar technique.

Luca Pacioli

Explicit evidence for rarefied circulation is offered by Luca Pacioli in his *Summa*. Before considering that, however, we shall look at his Perugia manuscript from 1477–78. Here [ed. Calzoni & Gavazzoni 1996: 311–312], two "horse"-problems make use of the algebraic unknowns *thing* and *horse*, explaining that *horse* is nothing but the price of the horse, and positing in both problems that the first man has a *thing* and the other two together 2 *horses* less 2 *things* (given that in both problems the first, having received half of what they have together, will have 1*horse*).

So far, this seems close to what Fibonacci and Benedetto had done in similar problems. However, the rest of the calculation is not straightforwardly algebraic;

[44] The manuscript correctly has 7.

[45] In principle, the solution by means of a double false position follows the alligation principle: If the first position gives an excess of p and the second a deficit of q, then we make a weighted average, taking the first position q times and the second p times, dividing by the total number $p+q$ of times we have taken a position. However, I have never seen that explained in the texts making use of the technique.

An analysis of the present problem in modern symbolism is given by Raffaella Franci and Marisa Pancanti [1988: xxiii–xxiv].

actually, in order to discover the ratio between the possessions of the second and the third, twice a new *thing* is introduced (with no distinction of name). As the anonymous, Pacioli has borrowed part of the idea of two unknowns but either does not fully grasp it, or he prefers a different method.

Let us now turn to the *Summa*, where we see that at least in 1494 he understood. Here [Pacioli 1494: fol. 191^v] we find this:

> Three have *denari*. The first says to the other 2, if you give me half of yours, I shall have 90. The second says to the others, if you give me $^1/_3$ of yours I shall have 84. The third says to the others, if you give me $^1/_2$ of yours plus 6 I shall have 87. I ask what everyone has on his own. I ask this merely to show you how one operates with a deaf [*sorda*] quantity, which the ancients called second thing to differentiate it from the first positions. Posit that the first has 1 *thing*, remove it from 90, remains 90 less 1 *thing*, and this ought to be $^1/_2$ of the other two. These will thus have 180 less 2 *things*, and all three will have 180 less 1 *thing*. Now for the 2nd, posit that he has a *quantity*, which I depict in this way φ̃', and for the two will remain 180 less 1 *thing* less 1 *quantity*. Take $^1/_{33}$, from it results 60 less $^1/_3$ *thing* less $^1/_3$ *quantity*.

If *A*, *B* and *C* designate the three possessions, the conditions are thus

$$A+{}^1/_2\,(B+C) = 90\ ,\quad B+{}^1/_3\,(A+C) = 84\ ,\quad C+{}^1/_4\,(A+B)+6 = 87,$$

and with *A* posited to be a *thing*, *B* to be a *quantity*, Pacioli has found that

$${}^1/_3\ (A+C) = 60 - {}^1/_3\,\textit{thing} - {}^1/_3\,\textit{quantity}.$$

Inserting this in $B+{}^1/_3\,(A+C) = 84$ and using that $B = 1$*quantity* Pacioli derives the equation

$$1\ \textit{quantity} = 36+{}^1/_2\,\textit{thing},$$

which is the second possession.

Now comes something new:

> Now for the 3d do similarly: Posit that he has a *quantity*, remove it from 180 less 1 *thing*, that is, still from the amount of all three. [...].

That is, Pacioli operates with three algebraic unknowns, though only with two at a time, which allows him to recycle the name *quantity*. This second position allows Pacioli to derive the equation

$$1\ \textit{quantity} = 48+{}^{1}/_{3}\ \textit{thing}\ ,$$

which is the third possession. That brings him back to a single unknown,

$$A+B+C = 1\ \textit{thing}+36+{}^{1}/_{2}\ \textit{thing}+48+{}^{1}/_{3}\ \textit{thing}\ .$$

But it was shown in the beginning that $A+B+C = 180–1$ *thing*. This solves the problem. In the end Pacioli specifies that one shall always with this method isolate the *quantity*, and explains that

> And by means of these deaf quantities which the ancients called second things a great many strong problems can be solved by the one who handles the equations well.

Even though we cannot identify Pacioli's "ancients"[46] he leaves no doubt that he knew some kind of tradition making algebraic use of a "second *thing*". With the proviso that a chapter explaining the basics of algebra and most of another one containing "chapters of algebra" have been lost (fols 325–349),[47] however, the notions of "deaf quantities"[48] and "second things" as well as the pair *thing*–*quantity* are absent from the Perugia manuscript. It therefore seems doubtful whether the traditions was as sharply defined as the *Summa* might make us believe – but since neither Fibonacci nor Antonio nor Benedetto (nor for that matter the anonymous) are direct precursors for what Pacioli does in the Perugia manuscript, nor in essential details (terminology combined with subject-matter) for what we

[46] Obviously they cannot be the habitual "ancients" of the Humanists of the time, ancient Greeks and Romans – unless Pacioli misidentifies, which would indeed not be unseen in the epoch. A few decades earlier, Leon Battista Alberti had classified Savasorda together with Columella and the agrimensors as *antichi* and Fibonacci as "modern" [ed. Grayson 1973: 151].

But the reference to the source for the technique as "ancients" may also be deliberate invention. As not unusual in the world of learning past and present, Pacioli's use of references and citations of those who have inspired him is first of all strategic. The unpaginated initial *summario* thus claims that the whole first part of the work (the second part treats of geometry) is "according to [...] Euclid and Severinus Boethius, and our moderns Fibonacci, Jordanus, Blasius of Parma, Sacrobosco and Prosdocimo de' Beldomandi" and primarily taken from them.

[47] Well before this (fols 229–264, ed. [Calzoni & Gavazzoni 1996: 377–436]), however, there are two long chapters containing algebraically solved "divided 10" and other classical problem types, even discussing such higher-degree problems for which no general solving rule exists.

[48] Here, *sordo* is used only about irrational roots and quantities, corresponding to our *surd*.

find in the *Summa*, the tradition must have been fairly widely diffused though rarefied.

In any case, Pacioli shows (here as elsewhere) to know about strains in abbacus culture that are unknown to us. Our ignorance by far exceeds our knowledge.

Why no takeoff?

In spite of abundant anti-Whiggish proclamations, the historiography of mathematics often presupposes some kind of Galilean dynamics: once an insight has been reached, it is expected to unfold by it own impetus, at most disturbed by adverse external conditions. Why then was the use of several unknowns not adopted widely and its carrying capacity not explored to the full after the technique had been presented by Fibonacci (explored not even by Fibonacci himself)? Why not at least after Antonio's presentation in the *Fioretti?*

Galilean motion – already Galileo knew – is valid only in a vacuum. Mathematics, however, develops not in a vacuum but in an environment of mathematical practice – on its part embedded in a larger socio-cultural environment, but that needs not to be considered for our present question. So, what was the practice where Fibonacci, Antonio, the anonymous Florentine, Benedetto and Pacioli made use of several unknowns?

Like the practice of Viète and Descartes – those who were to really unfold the use of several unknowns – it was a practice of *problem solving;* and even, like this Early Modern practice, of *competitive* problem solving. The problems it considered were of a different type, however. Not Archimedean and similar geometric problems but intricate variations, either of classical recreational problems of types "give and take", "purchase of a horse", "finding a purse", "hundred fowls", etc., or (thus Abū Kāmil and Antonio) of *al-jabr/aliabra* classics like the "divided 10". The former are mostly problems of which Diophantos had solved somewhat simpler variants (in pure-number version) in book I of his *Arithmetic* by means of a single algebraic unknown *arithmós;* the latter are more intricate (much more intricate) variations on a problem type already used by al-Khwārizmī to illustrate the power of the *al-jabr* technique which he sets forth.

Moreover, the public for whom the virtuosity of problem solvers was displayed was different. In the epoch where Mersenne took care of organized information exchange, the circle that judged the virtuosity of, say, Descartes, Fermat, Mydorge, Pascal and Roberval, encompassed Descartes, Fermat, Mydorge, Pascal and

Roberval: The 17th-century competition for prestige was a competition between peers. Not between peers only, of course, the Republic of Letters just as later Enlightenment *philosophie* had its periphery; but the presence of competent judges was decisive. Fibonacci may perhaps have found a similarly competent public in the circle of philosophers around Frederick II (the *Flos* suggests so). The judges of abbacus masters competing for jobs, on the other hand, were municipal authorities or fathers of prospective students, rarely possessing more mathematical expertise than what survived from their 1½–2 years passed in an abbacus school before the start of commercial apprenticeship. Encyclopedic treatises like that of Benedetto were written for friends or patrons – Benedetto speaks of a friend. They may have been copied and received a somewhat wider circulation (that of Benedetto is an example), but those who were at the level of a Benedetto, Antonio or the anonymous Florentine were too scattered to be likely to get into effective communication. That only changed when mathematics went into print, and after 1494 (the year of Pacioli's *Summa*) we do indeed encounter cumulative emulation as well as criticism from intellectual peers.[49]

Until then, there was no push to go beyond the two types of traditional abbacus problems just discussed when abbacus masters wanted to exhibit their algebraic prowess. And within both types, two algebraic unknowns are only brought into play in exceptionally complicated questions. That explains that even Fibonacci, Antonio and Benedetto only use the technique in a few cases – most systematically Antonio, whose *Fioretti* however did not invite emulation by others (Benedetto copied the whole treatise for his encyclopedia, but that does not amount to emulation and further development). That the Florentine anonymous uses a famous traditional problem type when he introduces his idiosyncratic and only halfway algebraic use of two unknowns can come as no surprise, this is and was a common way among mathematicians to illustrate the potency of a tool they introduce.

The mathematical practice in which abbacus mathematicians were engaged thus gave them no reason to generalize the use of two algebraic unknowns and to explore more widely the carrying capacity of the technique. But to this comes a factor to which we are blinded by our own prejudice. Leaving out of consideration

[49] For instance, the final chapter of Cardano's *Practica arithmetice, et mensurandi singularis* is a fierce though posthumous attack on "Friar Luca's errors" [Cardano 1939: QQ v$^{\text{r–viii}}$r].

the heated but fuzzy debate about "geometric algebra" (see [Høyrup 2017]) we are accustomed to recognize two types of pre-Abel algebra – with some disagreement about where to trace the line separating them: "rhetorical algebra" (sometimes more or less "syncopated") and symbolic algebra. But for the kind of problems here dealt with, except those of Antonio, a third technique was at hand. Let us go back to Fibonacci's "give-and-take" problem [ed. Boncompagni 1857: 190; ed. Giusti 2020: 323]. One man (A) asks from another one 7 δ, saying that then he shall have five times as much as the second (B) has. The second asks for 5 δ, and then he shall have seven times as much as the first.

As told above, Fibonacci's first solution builds on a line diagram:

a e g d b

ab represents the sum of the two possessions, *ag* the possession of A. *gb* is therefore the possession of B. *gd* is 7, that is, the amount which A asks for; similarly, *eg* is 5, that which B asks for. If A receives 7 = *gd* from B, he shall have *ad*, while *B* keeps *bd*. So, *ad* is 5 times *db*, whence *db* is $^{1}/_{6}$ *ab*. Similarly, if B receives 5 = *eg* from A, he shall have *eb*, A retaining *ae*, whence *eb* = 7 times *ea*, and *ea* = $^{1}/_{8}$ *ab*. Therefore *bd*+*ea* = $^{1}/_{8}+^{1}/_{6}$ of *ab*, while *ed* = 5+7 = 12. Now a false position can be made, namely that *ab* = 24. Then *bd*+*ea* would be 3+4 = 7, and *ed* would be 24–7 = 17. But *ed* should be 12, whence (by the rule of three) *ab* must be $^{12\cdot24}/_{17}$, while *bd* = $^{4\cdot12}/_{17}$, and *ae* = $^{3\cdot12}/_{17}$.

As we see, this is very similar to an algebraic calculation with several unknowns. In a way it is superior by allowing freer play with the various unknown quantities represented by the segments. Line diagrams allow addition, subtraction and ratio taking – all that is needed for first-degree problems. Like algebra it is analytic, taking the existence of a solution for granted and representing it by a symbol – not by a word nor by a letter but by a stroke on paper.

Benedetto does not make use of such line diagrams in book X of his *Trattato de praticha d'arismetrica* (the book where his purse problem is found), although they play an important role in book XI, "dealing with certain proportions and demonstrations that serve as principles for continued proportions" (fol. 300^{r}). They are absent from the anonymous *Tratato sopra l'arte della arismetricha*, and also from Benedetto's copy of Antonio's *Fioretti*. The availability of this tool thus does not explain much about why the two algebraic unknowns did not take root in the abbacus environment. But it shows us that Fibonacci, in spite of having a mathematically competent public at Frederick's court, was not urged to make systematic use of them. Line diagrams, instead, are used in great quantity in the *Liber abbaci* – sometimes borrowed (as indicated by use of the letter sequence *a–b–g–d–...*, sometimes (when the letter sequence is *a–b–c–d–...*) almost certainly produced by Fibonacci himself.

They are also used for problems of the second degree, in particular in chapter 15, sections 1 and 2, where they draw on the line versions of *Elements* II.5 and II.6. As an example we may look at the first problem from 15.1 [ed. Boncompagni 1857: 387; ed. Giusti 2020: 595],[50] dealing with three numbers in continued proportion, represented by *ab*, *bc* and *cd*, *ab* : *bc* = *bc* : *cd*,

a b c e d

where *ab*+*bc* = 10 and *cd* = 9. At first proportion transformations are used, *ab*+*bc* : *bc* = *bc*+*cd* : *cd*, that is, 10 : *bc* = *bd* : 9, and therefore (these are numbers represented by segments but not segments) *bc*·*bd* = 90.

> Therefore, if the number *cd* is divided at the point *e*, namely into two equals, and the number *bc* is joined to it, then the multiplication of the adjoined *bc* in the whole *bd* with the square of the number *ce* will be equal to the square of the number *be*. And the multiplication from *bc* in *bd* is 90; and the square of the number *ce* is $20^{1}/_{4}$, which joined together make $110^{1}/_{4}$ for the square of the number *be*. Whose root, that is, $10^{1}/_{2}$, is the number *be;* from which is removed the number *ce*, that is, $4^{1}/_{2}$, remains 6 for the number *bc*. When it is detracted from the number *ac*, that is, from 10, remains 4 for the number *ab*.

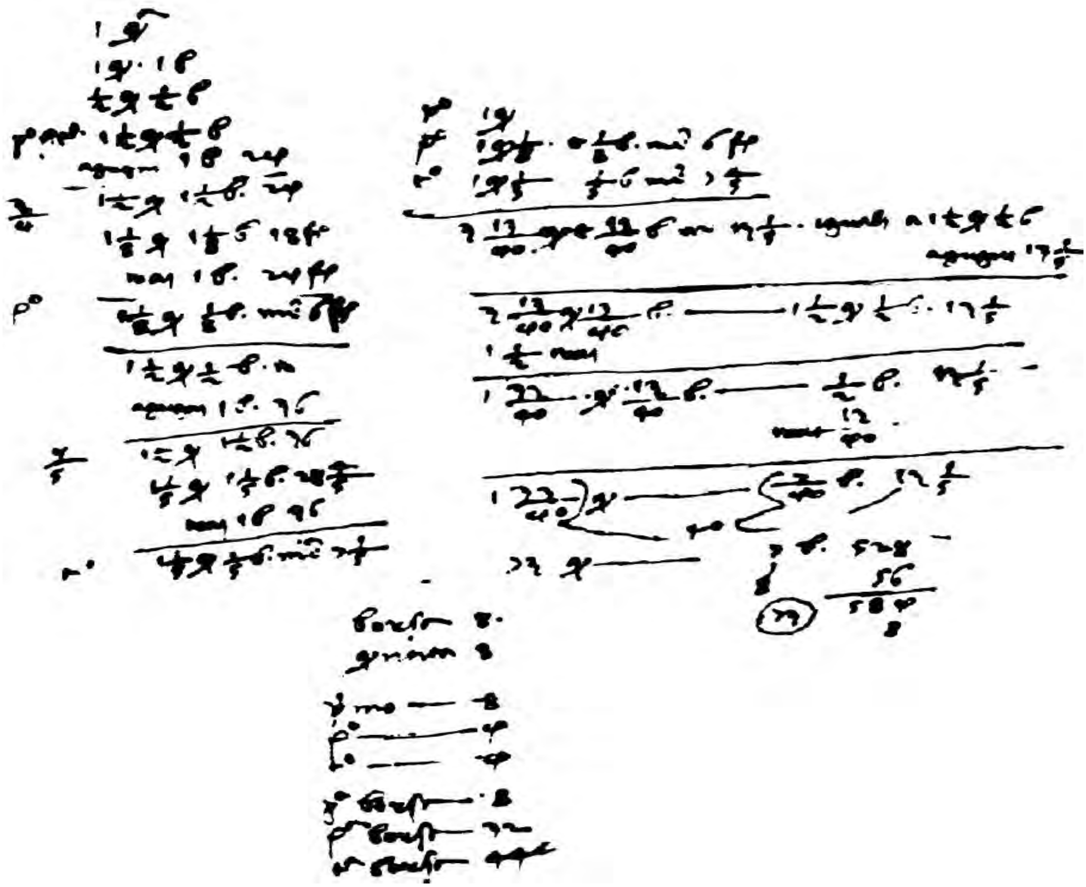
Euclid's proof for *Elements* II.6 is evidently geometric. But what is used here is a statement about numbers and does not take its proof into consideration. Though elsewhere fond of citing Euclid, Fibonacci also refers to neither Euclid nor the

[50] A complete overview of section 15.1 is in [Høyrup 2011: 97–100].

[51] The margin in Boncompagni's edition contains "per 7ma secundi euclidis". This (misguided) marginal commentary is not in the early Vatican manuscript (see note 11), nor in Benedetto's translation of the section (*Trattato de praticha d'arismetrica* fol. 304v); we may safely assume that the mistake has been added by the copyist of Boncompagni's manuscript or in an intermediate copy, and cannot be ascribed to Fibonacci.

Elements here.[51] The argument is wholly in the style of those making use of line diagrams for purse-problems and their kin earlier in the *Liber abbaci*. Together they show that Fibonacci possessed a technique for solving first- and second-degree problems that made application of several algebraic unknowns within a rhetorical algebra dispensable – and even makes it appear cumbersome if we consider the specimens we have looked at. Whether this was *another kind of algebra* or a possible *substitute for algebra* is a question of taste and definition.

We may ask why Benedetto, in spite of knowing the line technique from Fibonacci, did not adopt it. The margins of his *Trattato de praticha d'arismetrica* tell



Redrawn marginal calculation from Benedetto's *Trattato de praticha d'arismetrica*, fol. 266v.
The long horizontal strokes between algebraic expressions stand for equality.

us why. His text itself solves the intricate problems about purses etc. by means of rhetorical algebra: but *first* he has solved them in the margin by means of incipient *symbolic* algebra (an example solving a purse problem involving a *quantity* and a *purse* is shown on the previous page redrawn from fol. 266^{v}) – rudimentary, but already even easier to handle that the line diagrams.

Once the idea of symbolic writing carrying the mathematical argument (and not just abbreviating the rhetorical exposition) was maturing over the next century, and once different, more demanding problem types came to the fore, *then* – and *only then* – was there a reason to explore the possibilities of two, three or more algebraic unknowns. The hot water being already at hand, the midwives of the new mathematics did not need to reinvent, it only needed some extra heating.

References

[1] Arrighi, Gino (ed.), 1967a. Antonio de' Mazzinghi, *Trattato di Fioretti* nella trascelta a cura di M° Benedetto secondo la lezione del Codice L.IV.21 (sec. XV) della Biblioteca degl'Intronati di Siena. Siena: Domus Galilaeana.

[2] Arrighi, Gino, 1967b. "Una trascelta delle «miracholose ragioni» di M° Giovanni di Bartolo (secc. XIV–XV)". *Periodico di Matematiche*, 4. Ser. 45, 11–24.

[3] Arrighi, Gino (ed.), 1974. Pier Maria Calandri, *Tractato d'abbacho*. Dal codice Acq. e doni 154 (sec. XV) della Biblioteca Medicea Laurenziana di Firenze. Pisa: Domus Galilaeana.

[4] Boncompagni, Baldassare, 1854. *Intorno ad alcune opere di Leonardo Pisano*. Roma: Tipografia delle Belle Arti.

[5] Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

[6] Boncompagni, Baldassare (ed.), 1862. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. II. *Practica geometriae* ed *Opusculi*. Roma: Tipografia delle Scienze Matematiche e Fisiche.

[7] Calzoni, Giuseppe, & Gianfranco Cavazzoni (eds), 1996. Luca Pacioli, *"tractatus mathematicus ad discipulos perusinos"*. Città di Castello: Delta Grafica.

[8] Colebrooke, H. T. (ed., trans.), 1817. *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*. London: John Murray.

[9] Corominas, Joan, & José A. Pascual, 1980. *Diccionario Crítico etimológico castellano e hispánico*. 6 vols. Madrid: Gredos, 1980–1983.

[10] Costa, Mercè, & Maribel Tarrés, 2001. *Diccionari del català antic*. Barcelona: Edicions 62.

[11] Franci, Raffaella, 1988. "Antonio de' Mazzinghi: An Algebraist of the 14th Century". *Historia Mathematica* 15, 240–249.

[12] Franci, Raffaella, & Marisa Pancanti (eds), 1988. Anonimo (sec. XIV), *Il trattato d'algibra* dal manoscritto Fond. Prin. II. V. 152 della Biblioteca Nazionale di Firenze. Siena: Servizio Editoriale dell'Università di Siena.

[13] Giusti, Enrico (ed.), 2017. "The Twelfth Chapter of Fibonacci's *Liber abbaci* in Its 1202 Version". *Bollettino di Storia delle Scienze Matematiche* 37, 9–216.

[14] Giusti, Enrico (ed.), 2020. Leonardi Bigolli Pisano *vulgo* Fibonacci *Liber Abbaci*. Firenze: Leo S. Olschki.

[15] Gregori, Silvano, & Lucia Grugnetti (eds), 1998. Anonimo (sec. XV), *Libro di conti e mercatanzie*. Parma: Università degli Studi di Parma, Facoltà di Scienze Matematiche, Fisiche e Naturali, Dipartimento di Matematica.

[16] Heeffer, Albrecht, 2010. "From the Second Unknown to the Symbolic Equation", pp. 57-101 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*. London: College Publications.

[17] Høyrup, Jens, 2007. *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. Basel etc.: Birkhäuser.

[18] Høyrup, Jens, 2009. "'Proportions' in and around the Italian Abbacus Tradition". *Physis* 46 (2009; published 2011), 55-110.

[19] Høyrup, Jens, 2010. "Hesitating Progress – the Slow Development toward Algebraic Symbolization in Abbacus and Related Manuscripts, c. 1300 to c. 1550", pp. 3-56 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*. London: College Publications.

[20] Høyrup, Jens, 2011. "«Proportions» in the *Liber abbaci*", pp. 83–100 *in* Sabine Rommevaux, Philippe Vendrix & Vasco Zara (eds), *Proportions: Science – Musique – Peinture & Architecture*. Actes du LI^e Colloque International d' Études Humanistes, 30 juin – 4 juillet 2008. Turnhout: Brepols.

[21] Høyrup, Jens, 2015a. "Embedding: Another Case of stumbling progress". *Physis* 50, 1-38.

[22] Høyrup, Jens, 2015b. "Advanced Arithmetic from Twelfth Century Al-Andalus, Surviving Only (and Anonymously) in Latin Translation? A Narrative That Was Never Told". Contribution to the Conference "Narratives on Translations", Max-Planck Institut für Wissenschaftsgeschichte, Berlin, 16–20 November 2015. *Preprint*, Roskilde University. Forthcoming in Sonia Brentjes & Alexander Fidora (eds), *Premodern Translation Comparative Approaches to Cross-Cultural Transformations.* Turnhout: Brepols, 2021.

[23] Høyrup, Jens, 2017. "What Is 'Geometric Algebra', and What Has It Been in Historiography?". *AIMS Mathematics* 2, 128-160.

[24] Høyrup, Jens, 2018. "Abbacus School", *in* M. Sgarbi (ed.), *Encyclopedia of Renaissance Philosophy*. Online, https://doi.org/10.1007/978-3-319-02848-4_1135-1.

[25] Høyrup, Jens, 2019. *Selected Essays on Pre- and Early Modern Mathematical Practice*. Cham etc.: Springer.

[26] Libri, Guillaume, 1838. *Histoire des mathématiques en Italie*. 4 vols. Paris: Jules Renouard, 1838–1841.

[27] Lüneburg, Heinz, 1993. *Leonardi Pisani liber abbaci oder Lesevergnügen eines Mathematikers*. 2. überarb. und erw. Aufl. Mannheim: B. I. Wissenschaftsverlag.

[28] Pacioli, Luca, 1494. *Summa de Arithmetica Geometria Proportioni et Proportionalita.* Venezia: Paganino de Paganini.

[29] Peletier, Jacques, 1554. *L'algebre*. Lyon: Ian de Tournes.

[30] Rashed, Roshdi (ed., trans.), 2012. Abū Kāmil, *Algèbre et analyse Diophantienne*. Berlin & Boston: de Gruyter.

[31] Raynouard, François-Just-Marie, 1838. *Lexique roman, ou, Dictionnaire de la langue des troubadours*. 6 vols. Paris: Silvestre, 1838-1844.

[32] Sesiano, Jacques (ed.), 1993. "La version latine médiévale de l'Algèbre d'Abū Kāmil", pp. 315–452 *in* M. Folkerts & J. P. Hogendijk (eds), *Vestigia Mathematica*. Studies in Medieval and Early Modern Mathematics in Honour of H. L. L. Busard. Amsterdam & Atlanta: Rodopi.

[33] Sesiano, Jacques, 2014. *The* Liber mahameleth: *A Twelfth-Century Mathematical Treatise*. Heidelberg etc.: Springer, 2014.

[34] Sigler, Laurence (ed., trans.), 2002. Fibonacci's *Liber Abaci*. A Translation into Modern English of Leonardo Pisano's Book on Calculation. New York: Springer.

[35] Tropfke, Johannes / Vogel, Kurt, et al., 1980. *Geschichte der Elementarmathematik.* 4. Auflage. Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.

[36] Ulivi, Elisabetta, 1996. "Per una biografia di Antonio Mazzinghi, maestro d'abaco del XIV secolo". *Bollettino di Storia delle Scienze Matematiche* 16, 101-150.

[37] Van Egmond, Warren, 1976. *The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence*. Dissertation, Indiana University.

[38] Van Egmond, Warren, 1980. *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600*. Firenze: Istituto e Museo di Storia della Scienza.

[39] Vlasschaert, Anne-Marie (ed.), 2010. *Liber Mahameleth*. Wiesbaden: Franz Steiner (page numbers refer to the edition part).

[40] Vogel, Kurt, 1971. "Fibonacci, Leonardo", pp. 604–613 *in Dictionary of Scientific Biography*, vol. IV. New York: Scribner.

Contact details:

Jens Høyrup
Roskilde University,
Section for Philosophy and Science Studies
Max-Planck-Institut für Wissenschaftsgeschichte,
Berlin, Germany.

Email : jensh@ruc.dk