The World of the *Abbaco*

Abbacus mathematics analyzed and situated historically between Fibonacci and Stifel

Jens Høyrup

Manuscript in progress, 1 March 2022

If referring or quoting, please indicate this status of the text

Preface

Five years ago, Vinzenzo de Risi approached me (7 November 2016, to be precise), inviting me to write a volume for a new series at Birkhäuser which he edited. It was supposed to be based on some as yet unpublished paper of mine and be between 70 and 150 pages.

That should be quite easy – so thought Vincenzo and so thought I. We agreed on a volume dealing with the Italian abbacus tradition. That should still be easy, and the material I had at hand still seemed to fit the planned length.

How naive I was! One might as well have sent an alcoholic well provided with money to the supermarket expecting him to buy nothing but bread. Like Oscar Wilde I can resist anything but temptations, and in my case *sources* took the place of alcoholic beverages. Having no predefined deadline I dived into Fibonacci's *Liber abbaci*, Benedetto da Firenze's *Praticha d'arismetricha*, Luca Pacioli's *Summa*, and many other works, all of which I had worked on at earlier occasions but always from some particular perspectice. Obviously, there was more to say about all of them, but saying it presupposed reading and analyzing, and writing the outcome asked for the many pages that follow – and a cobweb of close to cross-references.

Nobody lives eternally; so, I did have an implicit though not sharply defined deadline. Somehow suspecting that I decided from an early moment to include normal abbacus geometry but to disregard the large vernacular translations of Fibonacci's *Pratica geometrie*. Figures at the periphery of abbacus culture – Nicolas Chuquet, for one, in spite of his impressive work – were left at the periphery, or beyond the horizon.

I started in earnest in early October 2019. My thanks to Vincenzo for having kept me busy during the Covid-19 pandemic!

Technically: All translations into English from original sources or secondary literature are mine where nothing else is stated. When translating, I try to keep as close to the original text as possible, often at the cost of stylistic elegance (with the exception, due to typesetting convenience, that fractions, which in the manuscripts are invariably written with a horizontal fraction line, appear with slashes); terms and phrases in the original language may inserted in square brackets. Illustrations taken from manuscripts are redrawn for clarity, not reproduced directly.

When an edition of a manuscript exists, my references will be to this edition; however, if I have had access to the manuscript, I have controlled critical points.

References are made according to the author/editor-date system, in the format [NN year], or alternatively "NN [year]"; for works that cannot be ascribed to an author or an editor, [*Title*] is used.

My thanks to Fabio Acerbi, Ahmed Djebbar, Enrico Giusti and Ulrich Rebstock for interaction along the road.

Contents

Introduction

As I was around fourteen years of age, my Danish school arithmetic taught topics like these:

- Applied proportionality (no longer called the "rule of three", it is true, but a few decades before it had been *reguladetri*);
- alligation and fineness of bullion;
- the partnership rule (proportional sharing);
- simple and composite interest;
- bills of exchange;
- discounting;
- and bonds and stocks.

At the time I probably understood vaguely that this was the mathematics of the financial infrastructure of capitalism (not of capitalism *tout court*, it goes by itself) – around the same time I read the *Communist Manifesto*. What I did not understand, and what my teacher probably did not know, was that all of this, bonds and stocks excepted, belonged within a tradition reaching back to the Italian 13th–14th century – that is, to the beginnings of Italian commercial capitalism, and to the Italian *abbacus school* and its mathematics.

The preceding line invites two misunderstandings, which have to be cleared away. Firstly, the abbacus school did not thrive in the whole of Italy but between the Genova-Milan-Venice arc to the north and Umbria to the south. Secondly, much more important, "abbacus" has only the etymology in common with "abacus". The abbacus school taught calculation with Hindu-Arabic numerals (what we mostly speak of as "Arabic numerals" today) *on paper*, and never made use of a reckoning board. Those of its students who later as bank employees had to make use of a calculating board for accounting purposes were trained in that during their apprenticeship, following after their frequentation of the abbacus school. "Abbacus" (*abbaco*) can be understood approximately as "practical calculation" – but of the particular kind which was taught in the school.

In any case, in Western, Central and Northern Europe, from around 1300 and until around 1960, those who learned mathematics of the abbacus kind constituted the majority of those who were at all subjected to systematic mathematics teaching. In this sense, we may say that abbacus mathematics and its direct descendants were highly visible for two thirds of a millennium – all who were professionally engaged in commercial activities knew at least its basic level.

Yet in the historiography of mathematics, abbacus mathematics as a specific undertaking went completely unnoticed until the mid-20th century; at most it was subsumed under the general heading "practical arithmetic" (and its geometry under that of "practical geometry”).

Change was announced – but hardly more than announced – by Amintore Fanfani, an economic historian and leading Christian Democratic politician (six times Italian prime minister), in a lecture held at Université de Liège in 1950 and published in [1951]. As already Henri Pirenne [1929] had done with emphasis on north-western Europe, Fanfani contradicted Werner Sombart’s opinion [1919: I.1, 296–298] that late medieval merchants were almost illiterate and hardly able to calculate (Sombart, it is true, admits that the Italian situation was not quite as gloomy). Accordingly, Fanfani’s main interest was the abbacus *school* system; but he also referred to specific “abbacus books”, which he had obviously inspected.

Louis Karpinski had published a couple of descriptions of abbacus treatises in [1910] and [1929], but without seeing them as representatives of a particular genre; editions of full texts only began in the 1960s with Gino Arrighi’s work, much of which will be drawn upon below. In the late 1970s, Warren Van Egmond undertook to produce a complete catalogue (published in [1980]) of all abbacus manuscripts he could trace in Italian libraries – still almost complete four decades later. By then, the existence of the particular abbacus tradition was finally established beyond doubt.

The continuation of this tradition (as we shall see, transforming adoption) by the German *Rechenmeister* and other teachers of practical arithmetic, as well as its impact on writers such as Luca Pacioli, Michael Stifel, Johann Scheubel, Nicolò Tartaglia and Rafael Bombelli was also recognized. Lacking, however, and on the whole lacking to this day, is understanding of how one particular aspect of the abbacus tradition – namely its algebra – through these contributed to the redefinition of higher-level mathematics from the 17th century onward.

Outside that restricted part of the scholarly community which understands Italian, knowledge about the abbacus tradition in general is also missing to this day. Only two abbacus texts have been translated into any language – the short *Larte de labbaco* or “Treviso Arithmetic”, originally printed in 1478, translated into English by David Eugene Smith (published in [Swetz 1987]), and the Vatican version of Jacopo da Firenze’s *Tractatus algorismi* from 1307 translated into English by myself [Høyrup 2007].

The purpose of the present volume is, firstly, to present a fairly detailed portrait of the abbacus tradition as it developed historically; secondly, in less depth, to show how the adoption of abbacus mathematics in German lands gave rise to the creation of a different tradition; thirdly, in the very end, to investigate that interplay of abbacus algebra with other intellectual currents which turned the whole mathematical endeavour upside-down in the 17th century.

I. The home of abbacus mathematics: the abbacus school

The abbacus school was a school type that thrived between Genova-Milan-Venice to the north and Umbria to the south. The earliest evidence for its existence is truly accidental: in 1265, a certain Pietro characterized as abbacus master appears as a witness in a contract in Bologna. Within the next decades, however, documents appear confirming the existence of abbacus schools financed by the city communes;[1] such schools remained in existence until well into the 16th century, after which they seem to have merged with the elementary schools that taught reading and writing [Grendler 1989: 22*f*]. Big towns like Florence and Venice also allowed a number of private abbacus schools to flourish.[2]

Abbacus teaching was a craft, and the trade was often handed down from father to son; for instance, the Bologna master serving as witness in 1265 had a son who wrote his testament in 1279 and also identified himself as an abbacus master. The students were mainly artisans' and merchants' sons, who frequented the school for 1½ to 2 years, as a rule around the age of 11 or 12. Even those of higher social standing, however, often frequented the school. So did Niccolò Machiavelli, a lawyer's son, in 1480 (he was born in 1469) [Black 2007: 379], and in 1479 a brassworker wrote to Lorenzo de' Medici *il magnifico*, the *de facto* ruler of Florence, that he had gone to the abbacus school together with Lorenzo's father Piero [Goldthwaite 2009: 552].

We have some numbers. Writing at a few years' distance about the Florentine situation of the years 1336–38, Giovanni Villani [1823: VI, 193*f*] states in his *Cronica* that some 5500 to 6000 children were born each year in the city, and that 8000 to 10000 went to school learning to read; taking child mortality in account, this means that at least half of all children, boys and girls together, learned to read and write (which explains why vernacular writing matured in Italy well before it did in the rest of Europe). Six abbacus schools taught 1000 to 1200 boys – which, if true, means that at least 20% of all boys went through an abbacus school. Grendler [1989: 72] argues that real numbers must have been considerable lower, from the premise that each school will have had a single teacher with no assistants. A Florentine contract (on which below, p. 4) shows that this premise was not always true. Moreover, a probably Florentine manuscript (Vatican, Vat. lat. 10488), written in 1424 by several hands (see below, note 275) looks as if it is the product of the collaboration of a master and several assistants, or between assistants alone: hands may change in the middle of a page, and occasionally those who write express their own

[1] Elisabetta Ulivi [2002a: 124–126] offers a convenient summary.

[2] For Florence, see [Ulivi 2004].

opinion about how matters should be treated, showing that they were not professional scribes. Data from 1480, finally, indicate that more than one third of Florentine boys aged between 6 and 14 went to school (and remember that only those who went to a grammar school continued beyond the age of 12). All in all, Villani's numbers seem reliable. We should not believe, however, that the situation in average cities was similar to that of Florence (that in the countryside even of Florence certainly was not).

We have two sources specifying the curriculum taught in the school. One [ed. Arrighi 1967c], from the first half of the 15th century, sets out how the abbacus is "taught in the Pisa way".

At first the boy is taught how to write the digits from 9 to 1 (an order that still reflects Arabic writing from right to left as interpreted in an ambience that wrote left-to-right), then the place-value system and the use of the multiplication table (learned by heart), and squares until 99×99 (to be calculated) as well as further multiplications of two-digit numbers. Next follow monetary and metrological conversions and shortcuts, and in many steps more advanced multiplications and divisions and the computation with fractions.

This is followed by commercial calculation: simple and composite interest and reduction to interest per day; the rule of three, with extension to partnership; area calculations; discounting, with simple and composite interest and per day; alloying; and finally a single false position.[3] From multiplications beyond the 10×10-table everything

---

[3] "Single" because there is also a "double false position". We may exemplify by the corresponding ways to solve the problem "a quantity, with $^1/_6$ of it added to it, gives 40".

We may try the convenient guess or "position" that the number is 6. Then the total would be 7, and not 40 as it should be. Therefore, the true value must be $^{40}/_7$ times as large, that is, $^{40}/_7 \cdot 6 = 34^2/_7$. In the abbacus books, the last step would instead be made by means of the rule of three, "to the false value 7 corresponds the true value 40; what corresponds to the false value 6?", which leads to the formula $^{40 \cdot 6}/_7$.

The double false position normally served for more complex (though still linear) problems, but it can also be used in the present case. One position may be that the quantity is 6; that still yields 7 for the total. Alternatively, we may try 60, which yields 70 for the total. The former value falls 33 short of what we need, the latter exceeds by 30. The texts never explain the basis of the method, but the trick is to "mix" the two positions in such a way that the errors cancel out. We might take the former guess 30 times, this would give a total deficiency of 30·33; and the second guess 33 times, that would give a total excess of 33·30. But we should only make 1 guess, not 30+33 = 63 guesses, and therefore we have to divide by 63. The total true value is therefore $^{30 \cdot 6+33 \cdot 60}/_{33+60} = {}^{2160}/_{63} = {}^{240}/_7$. If both guesses had been deficient or both in excess, we would have had to use subtraction.

The principle of the double false position is the same as the "alligation principle": if we are to mix gold of 15 carats and gold of 22 carats in such relative quantities that the mixture will be of 20 carats we have to take 2 measures of 15 carats and 5 measures of 22 carats – and in order to get only one measure, divide both by 2+5.

is trained as problems done as daily homework.

The other specification of the curriculum is a Florentine contract [ed. Goldthwaite 1972: 421–425], signed in 1519. It states what the assistant has to teach: multiplication, division and fractions, as in the Pisa document; finally, the rule of three and the (complicated) Florentine monetary system. It can be imagined that those more advanced matters that are listed in the Pisa curriculum were to be taught in Florence by the master Francesco Ghaligai himself; it is also possible that the syllabus of Ghaligai's school went no further than the rule of three and the monetary system, although Ghaligai's *Summa de arithmetica* from [1521] (depicting on its title page Ghaligai himself throning over four students, one of whom is engaged in weighing and another one busy with a compass) makes it unlikely. As pointed out by Goldthwaite, however, homework problems (*ragioni*) about the rule of three might also have been understood to involve the application of this rule to partnership, alloying, interest calculation, etc. Even the single false position is so closely connected to the rule of three that it may have been subsumed under this heading – after all, the contract primarily deals with the salary to be paid per student in each section and with the mutual obligations of master and assistant, the curriculum only comes in as description of the contents of the single sections.

The so-called *abbacus books* cannot be used uncritically as information about the school curriculum. Abbacus books were not textbooks for the students. The term refers to all kinds of manuscripts about practical mathematics written by authors who were or had once been connected – as masters or as students – to the abbacus school institution.

Abbacus books may be messy problem collections (*zibaldoni*), or more orderly, looking like "teachers' books" without being necessarily meant to serve in this function. The orderly books may indeed also have been written for friends or patrons, and some claim that they can serve self-instruction. Three (to which we shall return below, p. 249) are genuine mathematical encyclopedias. Some are author's autographs – but still, as so many mathematics textbooks from all epochs, drawing heavily on named or unnamed predecessors; some are booksellers' copies. Some are anonymous, some indicate the name of an author, some borrow the name of a famous author but alter or maltreat his material. Some authors display a thorough understanding of the mathematics they present, others make gross blunders as soon as they arrive at the inverse rule of three or volume determination. Some cheat, or naively plagiarize the fraud of predecessors (Piero della Francesca's abbacus collection famously falls in this category, as shown by Enrico Giusti [1991: 64]). Some stay within the limits of the Pisa curriculum, some go far beyond (not least taking up algebra). All types are represented in Van Egmond's above-mentioned catalogue.

Fraud (to be discussed in detail below) mostly concerns the algebra. Cheating at the level that was to be taught and used in commercial practice was of course excluded, it would readily drive the teacher out of business. But nobody would ever discover in a commercial dispute that a formula for solving a third-degree equation was fake. Such formulas might therefore serve to impress mathematically incompetent municipal authorities and fathers of prospective paying students; it may also have been meant to bewilder rivals in competitions for employment – rivals who might not understand the deceit, and who would in any case find it difficult to explain to the judging authorities that something was fishy.

II. An example: Jacopo da Firenze's *Tractatus algorismi*, the short version

In 1307, one otherwise unidentified Jacopo da Firenze, at the time living in Montpellier in Provence, wrote a *Tractatus algorismi*. Aftr the Latin title and an equally Latin incipit, the language of the work is Tuscan. Three manuscripts claim to represent the treatise:

– Vatican, Vat. Lat. 4826, datable by watermarks to *ca* 1450 (henceforth **V**[4]);
– Milan, Trivulziana 90, to be dated in the same way to *ca* 1410 (henceforth **M**);
– Florence, Riccardiana 2236, written on vellum and therefore undatable (henceforth **F**).[5]

F and **V** are very close to each other, **F** with somewhat more errors than **M**.

The present chapter discusses the version of the treatise represented by **M** and **F**. **V** is longer and probably closer to the original than **M**+**F**, as shall be argued later (and in much greater depth in [Høyrup 2007: 5–25]). **M**+**F** look like an adaptation of the original treatise to what was actually taught in the school, and are therefore a better introduction to the general enterprise of abbacus mathematics.

I shall follow the semi-critical edition of the two manuscripts given in [Høyrup 2007: 382–456] – "semi-critical" because I worked directly on **M** but used Annalisa Simi's transcription of **F** [1995] and not the manuscript itself (however, comparison of Simi's transcription with a facsimile of one page from the manuscript that is included in her publication shows the transcription to be precise). Page references in the following point to this edition. When the same matter appears in **V** and in **M+F** I shall refer to its location in both versions (that of **V** anchored to the edition in [Høyrup 2007: 193–376]); when translating I shall build on **M+F**.

4 I shall write sigla for manuscripts in boldface; they correspond to the list on p. 417.

5 Van Egmond's dating of **F** merely repeats the date given in the shared incipit and is thus no dating of the manuscript but only of the original from which the manuscript is derived – more or less faithfully.

The introduction

As mentioned, the treatise opens with a Latin incipit. It states where and when the treatise was written, and moreover (p. 383) that the

> art [of algorism] consists of nine species, namely, numeration, addition, subtraction, mediation, duplation, multiplication, division, progression, and root extraction.

The same list is found in **V** (p. 193), and it is equally misleading in both versions. It is copied verbatim from Sacrobosco's *Algorismus vulgaris* [ed. Pedersen 1983: 174*f*] and is indeed a precise description of the contents *of that work*.

This is followed immediately by an introduction, equally shared with **V**, and actually copied more often than any other introduction to abbacus writings during the following two centuries (from which we may conclude that it expresses a widely shared attitude):[6]

> Admittedly, all those things which the human race of this world know or are able to know, are obtained in two main ways, which ways are these. The first is discernment [*senno*], the second is science. And each of these two ways is accompanied by two gentle and noble partners. One is the grace of God. And the other is knowledge by reason. And of the partners of science, one is mastery of what has been written. And the other is understanding with good intelligence. And according to what the Holy Scripture says, discernment is the noblest treasure that there is in the world. And you shall know that Solomon, who was close to being the wisest man of all the world, asked the Lord in his youth to give him discernment. And our Lord said to him that his request was the highest request that he could have asked. Wherefore he gave him one third of the discernment of Adam, and this discernment was by grace of God. The Holy Scripture also says that no man until now asked God for any request more beautiful or higher than that, since all God's good and pure gifts descend from this request. It is true that one may call discernment and science, one natural discernment, the other accidental science. And you shall know that everything men do naturally and by accident, our Father has granted (them) to know in his most holy virtue and grace and compassion. And therefore we are all obliged to thank Him who is such a sweet Father and Lord, who has given us to know so much subtlety for our use.[7]
>
> And therefore in His most holy name and His most holy honour we begin our treatise, which is called algorism. And know that we call it algorism because this science was first made in Arabia, and those who found it were similarly Arabs. And art in Arabic is called

---

[6] Beyond **V**, **F** and **M**, it is recycled more or less completely in no less than 10 treatises from between *ca* 1370 and 1513 described in [Van Egmond 1980] – detailed list in [Høyrup 2007: 46 n.120].

[7] "for our use" is not in **V** – probably by omission, since it is in Giovanni de' Danti's copy of the text [ed. Arrighi 1985: 9].

> algo, and the number is called rismus, and so it is called algorism. Which algorism distinguishes five chapters, which we shall show you manifestly in our treatise ordered according to the said matter, as the said science asks for. And we begin in the honour and reverence of our Lord Jesus Christ and his most holy mother Virgin Mary and the whole celestial court, and with the assistance of our predecessors, and in honour of all masters and scholars of this science, and of every other honest person who might see and read this treatise with dedication and sense.
>
> Now we shall show the properties of the five chapters spoken of above according to what Boethius says in his *Arithmetic*.[8] The first chapter is to multiply. The second chapter is to divide. The third chapter is broken numbers. The fourth chapter are the rules. The fifth chapter is the general understanding which is drawn from the said four chapters. And you shall know that the said five chapters have many subdivisions and sections, such as multiplying by two or three or four or more figures [i.e., digits]. Division falls in whole numbers and fractions. The fractions are to multiply, to divide, to join, to subtract, and to say which fraction is greater that the other, or how much smaller, and which. And to recognize them, seeing them written by figures. The rules comprise many routines [*maniere*] and insights and subtleties, which you will hear in orderly manner according to their nature which is explained.
>
> As in this treatise the mind and good intelligence grants us to know the great subtlety of the prophecies and the philosophies and the celestial and temporal writings, it will grant us to know even more henceforth, since by mind and good and subtle intelligence men make many investigations and compose many treatises which were not made by other people, and know to make many artifices and written arguments which for us bring to greater perfection things that were made by the first men. Hence as we have said above, our treatise is called in Arabic algorism, and so we should write the ten figures of the said algorism according to the custom of the Arabs, since they were those who found this science. That is, we shall write backwards and read to the right according to (what is customary with) us, that is to say, we shall begin by writing from the smallest number and read from the greatest number.

A number of observations can be made on this introduction. First of all, the praise of knowledge and the belief in the continuous growth of knowledge; this goes further than Bernard of Chartres' oft-quoted point that *we* are like dwarfs perched on the shoulders of giants[9] – Jacopo is convinced that others shall still climb onto *his* shoulders. At the

---

8 Another case of deceptive namesdropping. This time, the description corresponds to what Jacopo deals with in his treatise, but there is no connection to Boethius's *Arithmetic*.

9 "Bernard of Chartres used to compare us to [puny] dwarfs perched on the shoulders of giants. He pointed out that we see more and farther than our predecessors, not because we have keener vision or greater height, but because we are lifted up and borne aloft on their gigantic stature" –

same time we notice the strong religious key, rather different from what we see in the works of even outspokenly strong believers among the university scholars of the time.[10] Admittedly, the distinction between discernment (*senno*) and science, of which the former is "natural" and the latter is "accidental" is a trace of the Aristotelian philosophy of the time (where "accidental" often replaces the Aristotelian notion of being "by art"); but as we see, Solomo, not Aristotle is called in as witness.[11]

That the art (pp. 194, 383) is

> called algorism [...] because this science was first made in Arabia, and those who found it were similarly Arabs. And art in Arabic is called algo, and the number is called rismus, and so it is called algorism

is a reflection of Sacrobosco's *Algorismus vulgaris*. It is hardly direct – Sacrobosco does not mention the Arabic origin and does not state that *algo* means "art" *in Arabic* (*rismus*, shared with Sacrobosco, obviously reflects Greek *arithmós*). However, *The Art of Nombryng*, an English version of Sacrobosco, states that *algos* means "art" *in Greek*. So does the *Craft of Nombrynge*, an amplified translation of a commentary to Alexandre de Villedieu's *Carmen de algorismo* [ed. Steele 1922: 33, 3]. Both are known from 15th-century manuscripts, but it seems likely that the ascription of the meaning to a prestigious language (whether originally Greek or Arabic is a guess) goes back to a common source.

---

thus John of Salisbury in the *Metalogicon* from 1159 [ed. trans. McGarry 1971: 167].

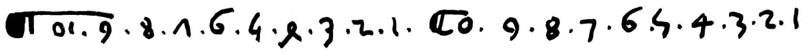
[10] For example, William of Ockham, Duns Scotus and Dietrich von Freiberg, all three as dry as Aristotle's *Second Analytic* whether writing about theology or philosophy.

[11] On the whole, the Solomo story is borrowed from 1. Kings 3:5–14 and 2 Chr. 1:7–12. But the borrowing is clearly indirect: none of the two Biblical versions refer to "one third of the discernment of Adam"; nor is this part of the story to be found anywhere in the Bible (including apocrypha), the Qur'ān, or the *ca* 170000 densely printed pages of the *Patrologia latina*, Latin Christian writings written before 1200. I presume it comes from the lay pious environment which is reflected in Jacopo's and other abbacus writings.

About the numerals and the place-value system

So, Sacrobosco, albeit indirectly, is one source for the presentation of the Hindu-Arabic numerals. But he is not the only one. Jacopo goes on,

> These are our abbaco figures, by means of which you may write whatever number you wish, or of whatever quantity it were. And these are the figures of the old art and the new.[12]



The idea of presenting two variants of the figures goes back to the Maghreb/al-Andalus mathematician Ibn al-Yāsamīn († *ca* 1204) [Burnett 2002: 240] (if not to some predecessor). Ibn al-Yāsamīn shows the *ghubār* ("western") as well as the "Eastern" shapes of the numerals. Jacopo, as we see, lists the shapes current in 13th- and 14th-century Latin Europe (both derived from the *ghubar* shapes). The idea of presenting both of these together recurs in a *Trattato di tutta l'arte dell'abacho* written in 1334, probably in Avignon,[13] and independent of Jacopo's *Tractatus;* we may surmise that Jacopo follows a more general Provençal habit, on its part ultimately inspired by Ibn al-Yāsamīn in ways we cannot trace.

Jacopo goes on (pp. 196, 385),

> Further we shall write here below how the said figures denote. And so that they may be understood better and more clearly we shall write them by figures, and similarly by letters, so that one may understand by himself without any master teaching him. And you shall know, and it is thus that the zero by itself does not signify anything, but it surely has the power to make signify when it is accompanied, but not always but according to where it is put, either before or behind. That is, if the zero is put before[14] another figure it does not have the power to make signify anything, but if placed behind the figure then it has the power to give to signify according to which figure it is. That is, if it were beside 1, it signifies 10, and if it were beside 2, it signifies 20. And if it were beside three, it signifies 30. And thus according to the figure which it makes signify.

This is followed by an extensive table containing numbers in Hindu-Arabic writing together with the corresponding writings by means of Roman numerals. Whereas *we* would explain that "ccxxxiiii means 234", subliminally understanding "234" as *being* the number and not being a mere writing, Jacopo obviously expects his reader to see things in the opposite

---

[12] Redrawn after the Trivulziana manuscript.

[13] Both in the compiler's draft autograph ($\mathbf{T_F}$, fol. 23^r) and in a copy of the final version ($\mathbf{T_R}$, fol. 3^v); the treatise and its dating is discussed in [Cassinet 2001] – cf. below, p. 201.

[14] "before", as we notice, is to the left – the local writing direction has taken over.

way.

That “zero by itself does not signify anything, but it surely has the power to make signify” is borrowed from Sacrobosco [ed. Pedersen 1983: 176]; that this power depends on whether it is written left or right, however, is not from Sacrobosco, and it may well be Jacopo’s own addition to the text.

The table is followed by a very pedagogical exposition of the place value principle – more fit for self-study, in agreement with what is promised (“without any master teaching him”), than as support for a teaching master who already knows.

Multiplication, division, fractions

After this follows (pp. 203, 389) another table containing first squares n^2, $2 \leq n \leq 10$, then products $m \times n$, $m < n \leq 10$ (called *librettine minori*, "minor booklets"), continued by other examples where one or both factors is multiplied by a power of 10 (for example, 300×600).[15] Then (pp. 206, 391) come *librettine maggiori* ("major booklets), products $m \times n$, $11 \leq n \leq 20$, $m < n \leq 20$, and then (p. 392 – in tables organized in the Arabic way, from right to left) all squares from 11×11 until 99×99 and select other products of two-place numbers, all controlled by casting out nines – but that this is the reason that, for instance, the field containing the numbers 840, 24 and 35 (meaning 840 = 24×35) also contains the number 3 is not explained. Another (equally unexplained) set of products (p. 403) involves mixed numbers (also written according to Arabic custom, with the fraction to the left); this time, the control is made by casting out sevens (after transforming the mixed numbers into pure fractions). Since the fractional and integer part of the mixed numbers are widely separated in **M** as well as **F**, the compiler of the archetype for **M+F** is likely not to have understood what was meant (which is also the likely reason that he did not normalize the writing of the mixed numbers).[16]

Two types of division follow (pp. 220, 408) – in general known as *a regolo* ("by ruler") and *a danda* ("by giving"), here unexplained and unnamed. The former are sequences of 10 to 12 short divisions, starting from a dividend of 6 to 8 digits.[17] The *danda* method was the outcome of the transfer of a division algorithm performed on a dustboard to paper, where deletions were no longer possible – a forerunner of our long division. *A regolo* division is shown for divisors from 2 to 12, *a danda* is used on 12 examples – for example, 71422330÷37.

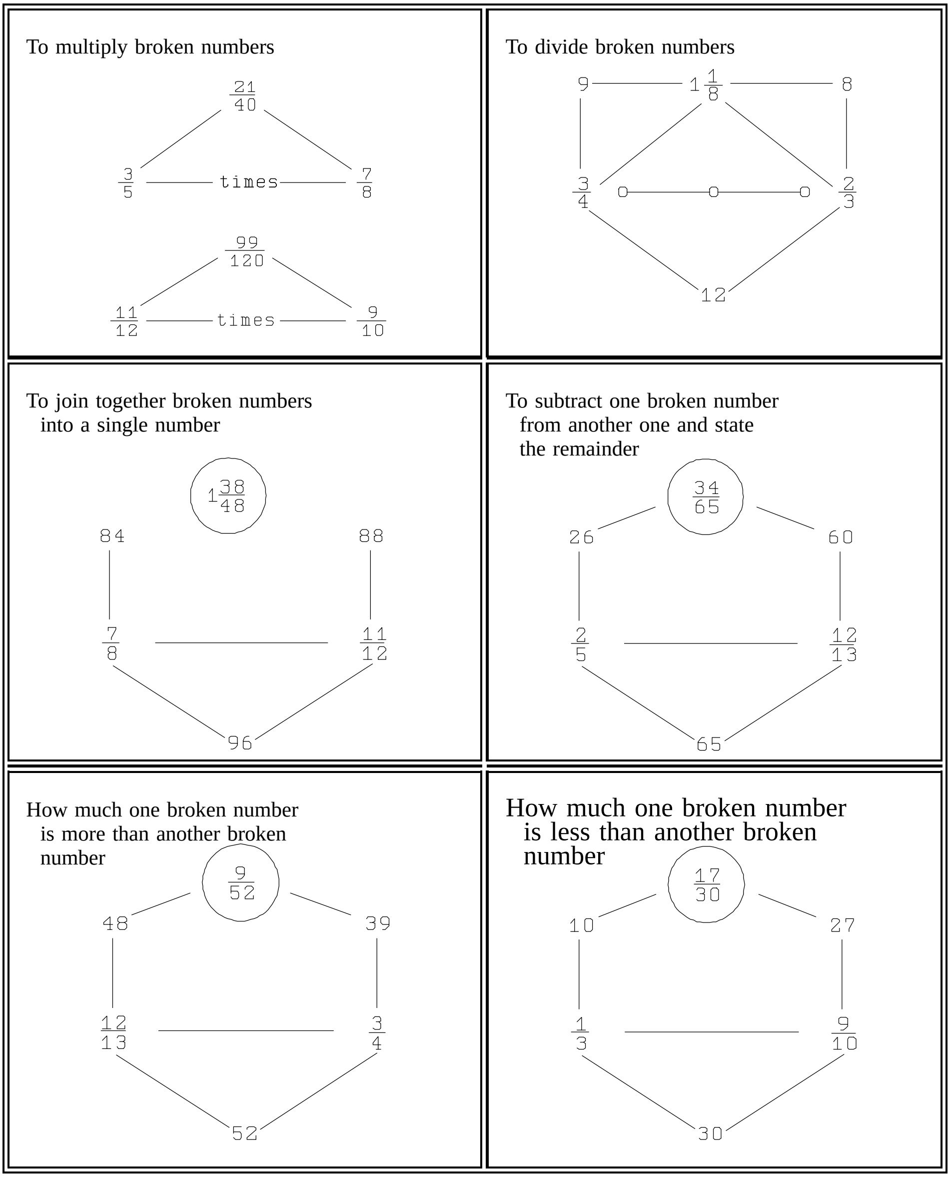
As in the Pisa programme, fractions close the section about pure arithmetic, and as explained in the introduction to the treatise it is taught here how "to multiply, to divide, to join, to subtract, and to say which fraction is greater that the other, or how much smaller, and which". At first (pp. 228, 415), the procedures are summarized in schematic examples:

---

15 We observe that addition and subtraction go unmentioned, just as in the Pisa and Florence programmes. In Sacrobosco's *Algorismus vulgaris*, on the other hand, they are the first operations to be explained [ed. Pedersen 1983: 177–181].

16 **V** contains the squares from 11×11 to 99×99 (organized left to right) with check by casting out nines, but nothing more.

17 With the successive remainders put in as a first digit in the results, so as not to make the students "run out of numbers, as they would soon do if remainders were not picked up", as Pacioli explains [1494: 32v].



After the schemes follows (pp. 230, 416) this introduction to the topic,

> We have spoken about the multiplications and the divisions and of all that is necessary concerning this. Now we leave this, and we shall speak in proper and legitimate rule about all routines about broken numbers, such as we proposed before in the prologue, since they give tools for the other computations, and without them this art cannot be subtly exercised

> nor learnt.

Then 19 examples are explained – first this one:

> Let us first begin in the name of the supreme God and say thus, say me, how much is, joined together, $^1/_2$ and $^1/_3$. Do thus, say, a half and a third are found in six because 2 times 3 makes 6. And take the half and the third of 6, which are 5, and divide 5 by 6, from which comes 5 sixths. And we shall say that $^1/_2$ and $^1/_3$ joined together are 5 sixths. And in this way you may join whatever broken number it be.

God as well as the product of denominators are absent from the other examples, but apart from that their general style is the same. The following 18 examples merely prescribe the finding of a common multiple when it is pertinent – actually, the least common multiple. Curiously, the writing of fractions with a fraction line (*verga*, literally "a cane") is only described after the third example.

The rule of three

In agreement with both curricula, fractions are followed by topics of specifically commercial relevance – at first, as in the Florence document, the rule of three. **V** (p. 236) has an introductory remark similar to the one that precedes the operations with fractions,

> We have said enough about fractions, because of the similar computations with fractions all are done in one and the same way and by one and the same rule. And therefore we shall say no more about them here. And we shall begin by doing and showing some computations according to what we shall say soon.

It is absent from **M** and **F**, but its stylistic agreement with the introduction to fractions suggests that it belonged to Jacopo's original and is no addition. It contains an important piece of information about abbacus meta-terminology: Addition and multiplication of fractions certainly do not follow the same steps. That they are "done [...] by one and the same rule" indicates that "rule" is not to be understood as a precise procedure or algorithm but refers instead to some general principle.[18]

In all three manuscripts (pp. 236, 419), the rule of three is introduced in this way:

> If some computation should be given to us in which three things were proposed, then we should always multiply the thing that we want to know against that which is not of the same (kind), and divide in the other, that is, in the third thing.

With no or minimal deviations, this was to remain the standard formulation of the rule of three for two centuries.[19]

---

[18] This, by the way, was to become the meaning of "algorithm" until the late 19th century – after having first, spelled *algorismus*, simply referred to the calculation with Hindu-Arabic numerals. The shift of meaning and to the hypercorrect spelling is marked by Christoph Rudolff's *Coss* [1525: 9v], who states that his second chapter *ist von gemeinem algorithmo der Pruch*, "is about the general algorithm for fractions" – precisely Jacopo's "one and the same rule".

[19] In **V**, the precise formulation is

> If some computation should be given to us in which three things were proposed, then we should always multiply the thing that we want to know against that which is not similar, and divide in the third thing, that is, in the other that remains.

The *Livero de l'abbecho* (discussed in detail below, p. 159), known from a fourteenth-century copy in the manuscript Florence, Ricc. 2404 but probably to be dated around or slightly after 1300 [Høyrup 2005: 27–28, 47], has almost the same formulation as **M** and **F** [ed. Arrighi 1989: 9], followed word for word by the anonymous *Liber habaci* [ed. Arrighi 1987: 111] from *ca* 1309:

> If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same (kind), and divide in the other.

Pacioli explains in the *Summa de arithmetica* [1494: 57r] that

> The rule of 3 says that the thing which one wants to know is multiplied by that which is not

A number of examples follow. The first one (pp. 237, 419) runs like this (*tornesi* are minted in Tours, *parigini* in Paris):

> I want to give you the example to the said rule, and I want to say thus, vii *tornesi* are worth viiii *parigini*. Say me, how much will 20 *tornesi* be worth. Do thus, the thing that you want to know is that which 20 *tornesi* will be worth. And the one which is not the same is that which vii *tornesi* are worth, that is, they are worth 9 *parigini.* And therefore we should multiply 9 *parigini* times 20, they make[20] 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $^5/_7$. And 25 *parigini* and $^5/_7$ will 20 *tornesi* be worth.

Then come three more examples with the same ratio between *tornesi* and *parigini*, the third however asking for the value of 150 £, 13 ß and 4 δ of *tornesi*.[21] With no intermediate calculations, 9 times 150 £, ß 13, δ 4[22] is stated (correctly) to be £ 1356,

---

> similar, and divided by the other which is similar, and that which results will be of the nature of that which is not similar, and the divisor will always be of the similitude of the thing which one wants to know

which merely adds the clarification "and that which results [...] wants to know".

20 This way of speaking characterizes the whole of **M**+**F** (but not **V**, abbacus writers had different views on the matter). The idea is that the number 20 (in itself seen as a singular) occurs 9 times, and these 9 constitute a plurality that together make 180.

21 £ stands for *lira*/*lire* (singular/plural), ß for *soldo*/*soldi*, δ for *denaro*/*denari*. The *lira* was a money of account (in Carolingian times the value of a pound of silver, but that was 500 years of monetary debasement ago). The *lira* was divided into 20 *soldi* (the *soldo* descending from the *solidus* introduced by the Emperor Constantine the Great in 312, by then $^1/_{72}$ of a Roman pound of gold); the *soldo* was divided into 12 *denari*. Those who remember the British monetary system from two decades ago will recognize it.

The main weight unit used in the treatise is the (light) pound, *libbra* [*sottile*], varying according to location but mostly *ca* 320 g. It was divided into 12 ounces. 8 ounces constituted a mark [*marcha*], and 2 marks in most places a "heavy pound" (which was thus close to a British pound). The ounce was subdivided into weight *denari* – with a few exceptions, 24 *denari*, and the *denaro* into 24 *grani*. See [Zupko 1981: 106, 129–135, 139*f*, 174–177]. Henceforth, Zupko's work will be the basis for all metrological information unless a different source is indicated.

Apart from the *denaro* δ being a specific monetary unit, the plural *denari* also had the generic meaning "money" (as *soldi* in present-day Italian and *pennies* in obsolete English). Even though the abbacus authors probably did not think of the difference (and used the same abbreviation in both cases) I shall try to reserve the abbreviation for the cases where the monetary (or, occasionally, metrological) unit is intended.

22 In the first example, Jacopo follows general spoken language, stating the "quantity" (7) before

and the outcome of the division is similarly announced directly.

The next examples astonish a modern ear. First (pp. 238, 420), "if 5 times 5 would make 26, what would 7 times 7 make at the same ratio?", next, "if 3 times 4 would make 13, what would 7 times 9 make?". As we shall see, these "counterfactual calculations" are informative about the historical process; for the moment we shall simply take note of their presence though as secondary examples.

Returning to monies, Jacopo now teaches what to do when fractions are involved in the parameters; at first, "3 $^1/_3$ *tornesi* are worth 4 *parigini*" is transformed by multiplication into "10 *tornesi* are worth 12 *parigini*". Three more examples follow.

the "quality" (*tornesi*). Now, he gradually shifts to the commercial technical order quality-quantity (still with us today). In my translations I shall try to be locally faithful to the originals (inconsistent though they often are, as here), while complying with the habits of spoken language in paraphrases and commentaries.

Basic commercial techniques

The rule of three, with examples mostly speaking of money, looks commercial. Actually, it is a general – we may say functionally abstract – technique, and as we shall see on p. 392 it had an almost axiomatic status.

It is followed, in Jacopo's *Tractatus* (pp. 242, 422) as well as the Florence curriculum by a specifically commercial subject: shortcuts to be used in calculation of simple interest. Interest was habitually specified as *denari* per month and per *lira*, and the problems dealt with (all corresponding to previously stated general rules) are to find

- how much is earned by 100 £ in six months if the £ is lent at 3δ per month;
- how many £ will earn 1 δ in a day if 1 £ is lent at 3 δ per month (the month being calculated at 30 days);
- how many £ will earn 1 δ a day if 100 £ are lent at 12 £ per year;
- in how much time 100 £ will be doubled if lent (at simple interest) at 3 δ a month;
- in how much time is doubled 1000 £ if 100 £ are lent at 6 £ per year;[23]
- how much does 100 £ earn per day if they are lent at 12 £ per year.

All rules are stated without argument; for the third problem the unexplained rule is thus to divide 150 by the number of £ earned per year by 100 £.

It is sometimes believed that the abbacus books could not deal with interest calculations because interest-taking was considered usury and hence forbidden by the Church; the preceding shows that this was not the case.[24] As to interest-taking in Florentine commercial practice, one may consult [Sapori 1955: I, 236–240]. There we notice that 3δ per £ per month (15 % per year) was in the high end of the acceptable, but still within the limits.

Another sort of calculational shortcuts follows (pp. 246, 423). At first comes a general rule:

[23] In this case, the preceding rule which the question is supposed to illustrate speaks of a different problem type: "if 100 £ earn me so and so many £ per year, how many £ will earn me 1 δ per day?" The same confusion is found in **V**. Two possible explanations are at hand: either Jacopo copied from a source and skipped an example and a rule; or all three extant manuscripts descend from an archetype already copied from Jacopo's text with a similar omission. Without being able to offer strong arguments (beyond the absence of other shared demonstrable omissions) I favour the first possibility.

[24] A rare expression of doubts caused by the sinfulness of usury is found in the encyclopedic manuscript Florence, BNC, Palatino 573, fol. 258^{r} [ed. Arrighi 2004/1967: 183]. Since the soul of the one who practices usury ends up in hell and his body in prison, the author promises to deal with the topic with brevity – which he then does over 104 folio pages!

> If some computation was given to us in this way, and let us say that the load[25] of pepper, or any other thing, which is 300 pounds, is worth so many £, or so many ß, or so many δ, and we want to know what the pound will be worth. Then you should know that for each £ that the load is worth, the pound is worth $^{4}/_{5}$ of a *denaro*, and for each soldo that the load is worth, the pound is worth $^{1}/_{25}$ of a *denaro*, and for each *denaro* which the load is worth, the pound is worth one three-hundredth of a *denaro*.

Examples, inversion, variations for a different value of the load and for other weight units follow – all of it evidently very useful for quick calculation in practical trade, where the rule of three would be utterly cumbersome for the determination of the price of a pound if, for instance, the load was told to cost 13 £ 8 ß.

[25] *Carica*. The load of 300 pounds or 3 quintals was a Provençal and the usual French unit. The normal Florentine load was 400 pounds. The exact number of pounds to a quintal might vary (a Barcelona quintal consisted of 104 pounds), as might also the weight of the pound. See [Høyrup 2007: 68].

Mixed problems

These two quite orderly sections, both of evident commercial pertinence, are followed (pp. 251, 426) by a disorderly collection of 39 mixed problems, of which some are still commercial in substance as well as dress, some are recreational problems – mostly traditional, and making use of methods (such as the single false position, the rule of three and the inverse rule of three) that could also serve commercial calculation.

We may look at some examples. The very first problem (pp. 251, 426) deals with a partnership (*compagnia*), but it is *not* solved by means of the usual partnership rule. Instead it runs:

> There are three partners who make partnership together. And one partner puts into the principal of the partnership £ 150, and the second partner puts into the principal of the partnership £ 230, and the third partner puts into the body of the partnership £ 420. Now it occurs after a certain time that they have earned £ 100 and want to divide. Say me how much comes to each one as his share, remaining untouched the capital of each of these three partners. Do thus, first join together all that which they have put into the principal of the partnership, that is, the £ 150 and £ 230 and £ 420, which in all are £ 800. Now divide that which they have earned, that is, £ 100, by 800, from which results ß 2 δ 6, and as much comes per £, that is, ß 2 δ 6. Now multiply 150 times ß 2 δ 6, which make £ 18 ß 15, and so much shall the first partner have, who put into the principal of the partnership £ 150, that is, 3 18 ß 15. Now multiply 230 times ß 2 δ 6, which make £ 28 δ 15, and so much shall the second partner have, who put into the principal of the partnership £ 230, that is £ 28 ß 15. Now multiply 420 times ß 2 δ 6, which make £ 52 ß 10, and as much shall third partner have, who put into the principal of the partnership £ 420, that is, £ 52 ß 10. And it is done. Now join together all these parts, that is, £ 18 ß 15, and £ 28 ß 15, and £ 52 ß 10, which make in all £ 100. We have thus divided well, and in this way and by this rule do with whatever partnership it be and whatever each one has put into the principal of the partnership, and you see how much comes per £.

The "partnership rule" – the parallel application of the rule of three – would have prescribed a different procedure. The share of the first partner would have been found as $^{150\cdot100}/_{800}$ £, that of the second as $^{230\cdot100}/_{800}$ £, that of the third as $^{400\cdot100}/_{800}$ £. The advantage of this procedure (and in general of the rule of three) is that it avoids the multiplication of rounding errors (mostly, the multipliers involved would be larger than 1); the disadvantage is that the intermediate results (230·100, etc.) have no intuitively meaning (as revealed by their dimension £^{2}). The intermediate result of the present procedure, instead, is meaningful, and explained to be the gain falling to each invested £. We may assume that Jacopo choose the alternative way for pedagogical reasons, but since he does not explain we cannot know.

In contrast we may next consider a problem (pp. 259, 429*f*) where the partnership rule proper is not only applied but seen to be a standard structure onto which other questions can be mapped – a functionally abstract representation of proportional sharing.[26] With occasionally varied parameters, the problem occurs in many abbacus texts; as pointed out by Moritz Cantor [1875: 146–149], it is first found in an ancient Roman jurisprudential text (Dig.2.13) – but since it stands there as a pure hypothesis ("If it be written thus ...") and does not agree with the normal principles of Roman Law, one may guess that the second-century jurist Salvius Julianus draws upon an already circulating piece of recreational arithmetic.[27]

> A man is ill and wants to make his will. And he has a wife, who is pregnant. And the good man decides in this way, and says to the wife, if you get a male child, I leave to him two parts of what I have, and to you the third; and if it happens that you get a female child, then I leave to her the third of everything I have, an to you I leave the two parts. And the good man departed from this life, and after a certain time the wife gave birth and made a male child, and a female. Tell me in which way one shall divide this possession, since one cannot divide in the way the father left to the wife and the children. Do thus, and this is its rule, firstly make a position of one and say thus, when the girl should have one, the wife were to have two. And when the mother were to have two, the boy were to have four. Thus, of whatever possession one were to divide between them, of every 7 the male child should have 4 and the wife two and the female girl one. We have thus brought this computation to a partnership, and we say thus, there are 3 partners who have made partnership together. And one partner puts in 4, the other partner puts in 2, and the third puts in one. And they have earned as much as that which the bequest was. How much comes to each. And this is done after that way of the partnership which we have shown earlier. Now let us posit that this bequest were 1400 gold *fiorini*. Say me, how much shall the mother have of it, how much the male child, and how much the female. Do thus, join together 4 and 2 and 1, which makes 7, and this is the divisor. Now multiply 4 times 1400 gold *fiorini*, it makes 5600 gold *fiorini*, and divide in 7, from which results 800 gold *fiorini*. And so much shall the male child have, that is, 800 gold *fiorini*. Now multiply 2 times 1400, it makes 2800 gold *fiorini*, and divide in 7, from which comes 400, and so much shall the mother have, that is, 400 gold *fiorini*. Now multiply 1 times 1400, it makes 1400, divide by 7, from which comes 200. And so many gold *fiorini* shall the female child have, that is, 200. And it is done, and in this way and by this rule you may divide whatever bequest that he left.

---

[26] We shall encounter another instance of this use of the abstract model below, p. 53 (in **V** only, eliminated in **M**+**F**).

[27] Quite similar structures are indeed to be found in even older Chinese mathematical texts – both the *Suàn shù shū* from 186 BCE or earlier [ed. trans. Cullen 2004: 45] and the *Nine Chapters on Arithmetic* [ed. trans. Chemla & Guo 2004: 285–287], from no later than the first century CE.

This explicit use of the partnership as a functionally abstract structure is not widespread in abbacus mathematics; in the early period it was characteristic of Jacopo and a few other treatises linked to the Provençal environment [Høyrup 2007: 129*f*]. More widespread, as we shall see repeatedly, is the use of the rule of three as the basic representation of proportionality.

Other recreational problems are based on methods with no evident commercial bearing – thus problems of meeting and pursuit and of combined works. Both types are widespread, and both are represented in the present collection of mixed problems. We may look at one of the meeting problems (pp. 262, 431) – pursuit problems, in which the two parties leave in the same direction, are absent from Jacopo's treatise:

> A man is in Rome, and wants to go to Montpellier and would go there in 11 days, neither more nor less. And another man is at Montpellier, and wants to go to Rome, and would go there in 9 days, neither more nor less. Now they leave at the same hour one from Rome and the other from Montpellier. Say me know in how many days they will meet on the way. Do thus and say, because one comes in 9 days and the other goes for 11 days, then multiply 9 in 11, it makes 99, and divide 99 in 20, because 11 and 9 make 20, from which results 5 less a twentieth, and after so much time the said men meet, that is, in 5 days less a twentieth of a day.

A strict parallel follows "in order to show it more clearly". There the calculation is organized differently, agreeing with the way both problems are solved in **V**, where the procedure is organized as follows (p. 262):

> Do thus and say, because one comes to Rome in 9 days and the other goes to Montpellier in 11 days, join together 11 and 9, which make 20. And this is the divisor. Now multiply 9 times 11, it makes 99. Divide in 20, from which results 4 and $^{19}/_{20}$.

"The divisor", with definite article (*il partitore*), indicates that a specific, pre-existing method is followed (we already encountered it on p. 22 in the twin problem, and it is indeed recurrent when the partnership rule is used). In the present case the rule in question is that for "combined works"; the underlying reasoning (whether understood or not by Jacopo) could be that in 9·11 days the first man can cover the distance 9 times, and the second man can cover it 11 times – in total thus 9+10 = 20 times. The distance is thus covered a single time by the two together in 9·11/20 days.[28]

Another favourite dress for the "combined works" in abbacus books is a ship with

[28] Alternatively, the idea could be that one man covers $^{1}/_{11}$ of the distance in a day, the other $^{1}/_{9}$. Using the rule for adding fractions one finds that they cover $^{9+11}/_{9\cdot 11}$ of the distance in a day, and hence the whole distance in $^{9\cdot 11}/_{9+11}$ days. This argument is made explicit in an analogous problem in the *Liber habaci* [ed. Arrighi 1987: 144], on which below, p. 177.

several sails.[29] In Jacopo's collection of mixed problems we find this version (pp. 268, 433):

> A galley is in Genova and wants to go to Aigues-Mortes. And the said galley has two sails such that, with one sail it would go there in 7 days, and with the other sail it would go in 9 days. Now it occurs that I hoist up both sails at a time. Say me in how many days the galley will have made its voyage from Genova to Aigues-Mortes, operating each of these sails by its force. Do thus, say, because with one sail it would go there in 7 days, and with the other it would go in 9 days, then join together 9 and 7, they make 16. And similarly multiply 7 times 9, they make 63, and divide 63 by 16, from which comes 4 less $^1/_{16}$, and in so many days will the galley have reached Aigues-Mortes, that is, in 4 days less $^1/_{16}$.

A problem type with many representatives, in Jacopo's collection as well as the abbacus tradition in general, is a quantity divided into parts, of which one is given absolutely and the other or others relatively. They may deal with a tree partially above ground and partially underground; with the parts of a fish, with the components of a goblet or the components or contents of a purse; etc. We may look at the fish variant (pp. 261, 430):

> A fish, whose head weighs the third of the whole fish, and the tail weighs the $^1/_4$ of the whole fish. And the body in middle weighs ounces 8. Say me, how much weighs the head alone, how much weighs the tail, and how much the whole fish. Do thus, say, $^1/_3$ and $^1/_4$ one finds in 12. And seize the $^1/_3$ and the $^1/_4$ of 12, they are 7. And say, from 7 until 12 there are 5, and this is the divisor. Now because the body in middle weighs 8 ounces, then multiply 8 times 12 ounces, they make 96, and divide by 5, from which comes ounces 19 and $^1/_5$, and as much weighs the whole fish, that is, ounces 19$^1/_5$. If you want to know how much weighs the head alone, then take $^1/_3$ of 19 and $^1/_5$, which is 6 and $^2/_5$, and as much weighs the head, that is, ounces 6 and $^2/_5$. If you want to know how much weighs the tail, then take $^1/_4$ of 19 and $^1/_5$, which is 4 and $^4/_5$, and as much weighs the tail, that is, 4 ounces and $^4/_5$ of an ounce. And it is done. If you want to prove it, join together what the head weighs, that is ounces 6 and $^2/_5$, and what the tail weighs, that is, ounces 4 and $^4/_5$, and that which the body in middle weighs, that is, 8 ounces, which in total are ounces 19 and $^1/_5$. We have thus done well. Thus are made all the similar.

The underlying idea is a single false position, even though the trick is not named. Since $^1/_3$ and $^1/_4$ are both found in 12 (as integers), the total weight is posited to be 12. If so, the weight of the head would be 4, and that of the tail would be three, leaving 5 for the

[29] The classical representative of the problem type, a cask or other container filled or emptied through several channels, is also found in **M** and **F** [ed. Høyrup 2007: 433] but not in **V**. Since the channels are mostly more than two (thus also here), the formula is slightly different.

body in middle. Now to these 5 correspond 8 ounces; the correspondent of 12, the total weight, is then found by means of the rule of three, as $^{8\cdot12}/_5$. The weights of head and tail are found in the same way.

As pointed out by Leonardo Fibonacci in the *Liber abbaci* [ed. Boncompagni 1857: 173; ed. Giusti 2020: 296],[30] a different approach is possible. He explains a tree example thus:

> There is a tree, of which $^1/_3$ $^1/_4$ is underground. And they are 21 palms. It is asked how much is the length of this tree. Since $^1/_3$ $^1/_4$ can be found in 12, understand this tree to be divided into 12 equal parts; of which the third, and the fourth, that is 7 parts, are 21 palms; therefore, proportionally, as 7 is to 21, thus is 12 parts to the length of the tree. And therefore, when four numbers are proportional, the multiplication of the first in the fourth is equal to the multiplication of the second in the third. Therefore, if you multiply the second, 21, by the third, known to be 12, and similarly divide by the first, namely by 7, 36 results as the fourth, unknown number, that is, for the length of that tree; or because 21 is the triple of 7, take the triple of 12, and you will similarly get 36.

However, the early abbacus school was not familiar with even the most elementary proportion theory; Fibonacci explains the procedure in terms of scientific ("magisterial") mathematics. Afterwards he explains the procedure of practical commercial reckoners:

> There is another methods which we use, namely, that for the unknown thing you posit a freely chosen number, that can be divided in whole numbers in fractions.

And then he goes on with this single false position, and calculation according to the rule of three (not named, as Fibonacci *never* gives a name to this procedure). Since this is not the method just taught, this "we" cannot be an authorial plural, it must refer to a community of which he considers himself a member – the "proto-abbacus" community, as we shall call it. Jacopo, also omitting the name of the rule of three although he uses it elsewhere, must draw on the same ultimate source.

---

[30] In [1857], Baldassare Boncompagni made an edition of the *Liber abbaci* based on a single good but not perfect manuscript. In 2020, Enrico Giusti published a new, critical edition. Since the former is easily accessible on the web, for instance (11 November 2020) at the addresses

https://archive.org/details/bub_gb_CrdUBgtAZFoC/page/n3/mode/2up

https://archive.org/details/bub_gb_w86fLKi88pYC

https://archive.org/details/scrittidileonard00bonc/page/n181/mode/2up

and

https://archive.org/details/bub_gb_G4IL1D5PUsoC

while the new edition may not be easily accessible in libraries for quite some time, further references to passages in the *Liber abbaci* (they are copious) will for brevity have the form [B*m*;G*n*], standing for page *m* in [Boncompagni 1857] and page *n* in [Giusti 2020] (indicating the pages where the passage begins if it extends over several pages).

Other classical recreational problems are dressed in a way that seems to connect them to commercial practices of the day but do not correspond to anything that would happen in the world of real trade; their role is to train the mathematical mind and, at times, particular methods.

One of them is of the type known as "the lazy worker" (pp. 266, 430):

> A master undertakes to construct a building in 30 days. And the day where the master works he shall have from the gentleman ß 5. And the day where he is not working he shall give the gentleman ß 7 back. Now the master has worked so much and has been so much away from work that he shall have nothing from the gentleman and shall give nothing back. Say me, how much the master was not working, and how much he worked, that is, how many days. Do thus, first join together 7 and 5, which make 12 ß, and this is the divisor. Now multiply 30 times 5 days, they make 150 days, and divide in 12, from which comes $12\frac{1}{2}$ days, and so much he was not working, that is, $12\frac{1}{2}$ days. And similarly multiply 7 times 30 days, they make 210 days, and divide in 12, from which results days 17 and $\frac{1}{2}$, and so much he worked, that is, days $17\frac{1}{2}$. Now we shall say that the master worked days $17\frac{1}{2}$ and was not working days 12 and $\frac{1}{2}$, which in all are 30 days. He thus made the said building in 30 days. If you want to prove it, say thus, the master worked days $17\frac{1}{2}$ and took ß 5 per day, then he took in all £ 4 ß 7 δ 6. And so much he took from the gentleman, that is, £ 4 ß 7 δ 6. And say, the said master was not working days $12\frac{1}{2}$ and gave back to the gentleman ß 7 per day, which in all are £ 4 ß 7 δ 6. He thus took as much from the gentleman as he gave back to him. And it is well done.

The basic trick is obviously the same as the one used in the double false position and in alligation; the reference to "the divisor", and thereby to a standard method, suggests that the creator of the problem thought of one or the other (not necessarily Jacopo, who may have copied uncritically, like countless compilers of mathematics textbooks from Antiquity until present times).

Another problem where the double false position would have been used by other authors but is shunned by Jacopo deals with the packing of cloth in bales (pp. 268, 433). 400 pieces of cloth are to be packed in 38 bales, of which some should contain 10 and some 11 pieces. The prescription only states the numerical steps to be taken, but the idea is quite simple: that if all bales contained 10 pieces, only 380 pieces would be packed. Therefore 20 of the bales must contain an extra piece.

There are two more recreational problems. One of them (pp. 271, 434) is of the classical "Chinese box" type: Somebody picks oranges in a garden which he has to leave through three guarded doors; to each doorkeeper he has to hand over half of what he has and one more, and in the end 3 oranges should be left. The solution goes by stepwise backward computation. Other abbacus books offer problems with the same mathematical structure while speaking instead of commercial travels in several steps with costs or customs payment (see [Tropfke/Vogel et al 1980: 582–584], and below, p. 89); but even then the mathematical problem is clearly recreational – no merchant would ever need this backward calculation.

Another question (pp. 263, 431) does seem to speak of trade: two merchants are transported by ship, one with 20, the other with 24 sacks of wool. Since they cannot pay the freight in coin, each gives a sack to the master of the ship to sell, asking him to take what is owed and to give back the rest; from the amounts the two merchants receive back the price of a sack as well as the total freight is determined. The recreational character of the problem is clear, firstly from the mathematical problem itself, which would never present itself to real-life merchants; secondly by the solution, which presupposes that the merchants themselves travel for nothing.

Certain problems really prepare directly for commercial life. A number of these are based on the intricacies of the monetary system. The first of them runs as follows (pp. 253, 427):

> I have to make in Bologna a payment of £ 100 of *bolognini piccioli*. And in Bologna the *bolognino grosso* is worth δ 13 and $^{1}/_{3}$ of *bolognino* picciolo. And in Florence the said *bolognino* is worth δ 15$^{1}/_{4}$. And in Bologna the gold *fiorino* is worth ß 31, and δ 6 of *bolognini piccioli*. And in Florence the said *fiorino* is worth ß 39 δ 6 of the coin of Florence. Say me what is better for me to carry to Bologna, starting from Florence, in order to make the said payment, either gold *fiorini*, or *bolognini grossi*, and how much it will be better for me at the said *libre* 100. Do thus, know firstly how many *bolognini grossi* it suits him to carry in order to make the said payment. And multiply 100 times 15 and $^{1}/_{4}$, which makes 1525, and divide by 13$^{1}/_{3}$, from which comes £ 123 ß 12 δ 11$^{25}/_{37}$ of *bolognino* [error for £ 114 ß 7 δ 6].[31] And so much will it suit him to carry in *bolognini grossi*, that is, £ 123 ß 12 δ 11$^{25}/_{37}$. Now let us know how much it suits him to carry in gold *fiorini*, and multiply 100 times 39 and $^{1}/_{2}$, they make 3950, and divide 3950 by 31 and $^{1}/_{2}$, from which comes £ 123 ß 7 δ 9 [error for £ 125 ß 7 11$^{15}/_{63}$],[32] and so much will it suit him to carry in gold *fiorini*, that is, £ 123 ß 7 δ 9. And it will thus be better to carry gold *fiorini* than *bolognini*, as much as there is from 123 £ 7 ß δ 9 until 123 £ ß 12 δ 11 and $^{25}/_{37}$. And we shall thus say that it will be better for him to bring gold *fiorini* than *bolognini grossi*, and on the whole payment of the said 100 £, it will be ß 5 δ 2 and $^{25}/_{37}$ precisely better for him.

---

[31] The correct result is found in **V**; the wrong result shared by **M** and **F** corresponds to a division by 12$^{1}/_{3}$ instead of 13$^{1}/_{3}$.

[32] Apart from a writing error probably due to a copyist ($^{15}/_{36}$ instead of $^{15}/_{63}$), **V** has the correct result. The wrong result of **M** and **F** appears to be the outcome of at least two errors. Possibly, $^{3950}/_{31½}$ was transformed into $^{7900}/_{64}$ instead of $^{7900}/_{63}$, and the outcome £ 123 ß 8 δ 9 then miswritten as £ 123 ß 7 δ 9.

It seems that the compiler of **M+F** has recalculated, made a mistake, and then trusted his own calculations. Apart from that, the solution is blameless; in order to see that we should take into account that the relation between £, ß and δ is the same in the monetary system of the two cities (namely 240 : 12 : 1), and that the ratio between the Bologna and the Florence *soldo* is determined by the values of the gold *fiorino* and the *bolognino grosso* (both actual, physical coins) expressed in the two kinds of *soldi* respectively *denari*. Evidently, the reader is supposed to understand this immediately.

Another handful of problems also deal with the difficulties arising from the complex monetary system; among these, one (pp. 277*f*, 436*f*) speaks specifically about how payments are made in Sicily and Puglia, and another one (p. 437, absent from **V**) about the practice of the fairs of Champagne.

These fairs had been of supreme importance for European long-distance trade in the 12th and 13th centuries; the presence of the corresponding problem in the revised version of Jacopo's *Tractatus* may be taken as evidence that its compiler wanted not only to adjust it to the school curriculum but also to include material of direct mercantile interest. The contents of the problem confirms this:

> In the fairs of Champagne purchases and sales and all payments are made in *provisini forti*,[33] and *provisini* are sold the dozen. And of this we shall give an example. The dozen of *forti*, that is, 12 *libre*, is worth *libre* 37, *soldi* 10. Say me, how much will 1443 *libre* of *forti* be worth. Now know that you should do thus, and say thus, 1200 *libre* are one dozen of hundreds, hence the 1200 *libre* of *forti* will be worth *libre* 3750. Now *libre* 243 are saved for you, and the 240 *libre* are two dozens of tens, and each 120 *libre* of *provisini* are worth *libre* 375, hence two dozens are worth *libre* 750, and you gave in total *libre* 4500. And we have to make the 3 *libre*, which are worth the $^1/_4$ of *libre* 37, *soldi* 10, that is, *libre* 9, *soldi* 7, *denari* 6. And you have in all *libre* 4509 *soldi* 7 *denari* 6. And it is done, and we shall say that *libre* 1443 of *forti* are worth *libre* 4509, *soldi* 7, *denari* 6 of whatever money you posit at the rate of *libre* 37 and $^1/_2$ the dozen of *provisini*.

The method here introduced was to be known among German *Rechenmeister* as *Welsche Praktik*[34]. It asks for much less use of paper and paper algorithms than the

---

[33] That is, minted in the Champagne town Provins [Travaini 2003: 37].

[34] At the time, *welsch* might refer to the Italian and French regions, preponderantly perhaps the latter. The *Rechenmeister may* thus have thought of the commercial practices of Flanders and northern France. But it might also be a practice of Italian merchants which rarely made its way into the writings of abbacus masters, and never systematically, as was to happen in 16th-century German writings.

The former possibility could be suggested by the appearance of the method together with a reference to the Champagne fairs; the latter, on the other hand, would be a parallel to the appearance of *tollet* calculation among the *Rechenmeister* (below, p. 372), which indubitably is of Italian origin rule of three (whose answer is $(1443 \cdot 37\,^{10}\!/_{20})/12$). A trained merchant would probably be able to make the partition 1443 = 1200+240+3 mentally and keep it in mind. If not, the converted payment could be made piecemeal.

The *welsche Praktik* is used again in a later problem (p. 439), discussed pp. 87*f*. This problem about cloth bought in Florence and sold in Nîmes involves both the relation between the length metrologies of the two locations and the ratio between *fiorini* and *tornesi*. The modern reader may note with satisfaction that even the medieval calculator gets lost here, mixing up the rule of three and the inverse rule of three.[35]

In spite of its practical advantage, few abbacus books introduce the *welsche Praktik;* after all, the proper practice of the abbacus school teachers was not trade but teaching, and as teachers they may have preferred mathematical coherence (or elegance, or what else we may call it).

None the less, one problem shared by **V** and **M**+**F** (pp. 252, 427) deals with discounting in the way it would probably be dealt with in practical life and not in the mathematically most elegant way:

> A merchant has to give to another one £ 200 two and a half month from now. The one who shall receive the said £ 200 says, give them to me now, and reduce your money at a rate of δ 2 per *lira* per month. Say me, how much he should give him in advance for the said 200 £. Do thus, say, in two months and a half, at 2 δ per *lira* the *lira* is worth δ 5. Do thus, put yourself at the 195 £, and know how much the said 195 £ are worth in interest, and they are worth £ 4, ß 1, δ 3, and is in all £ 199, ß 1, δ 3. ß 18 δ 9 are lacking there, that in interest are worth δ 5. Now detract δ 5 from ß 18 δ 9, ß 18 δ 4 remain. Now join ß 18 δ 4 above £ 195, and you have in all £ 195 ß 18 δ 4, it is done. And we shall say he shall pay him £ 195 ß 18 δ 4 in advance for the sais £ 200. And in this way do all the similar.

We may tend to observe that 1 £ = 240 δ grow to 245 δ in five months. Therefore, the

---

(the name comes from Italian *tavoletta*) but was a technique used by merchants and shunned by the abbacus masters. *Welsche Praktik* as well as *tollet* may indeed have been considered mathematically undignified by professional teachers of mathematics, however much mathematicians of our times might think the same about much abbacus mathematics. Though a relatively late witness, Stifel translates the name as *praxis italica* in the *Arithmetica integra* [1544: 83^{v}].

[35] The rule of three solves problems of type $a : p = A : P$, where a and A have the same dimension (for instance volume or weight) and p and P a different dimension (for instance, price). The inverse rule serves for problems of type $q \cdot w = Q \cdot W$, where q and Q are of one kind (for instance, the quality of an alloy) and w and W of a different kind (for instance, weight), which corresponds to a proportion $q : Q = W : w$. In the rule of three, the intermediate product in the rule of three ($p \cdot A$) has no concrete meaning, in the inverse case it has (in the example, [ounces silver per pound of alloy]·[pounds of alloy], that is, ounces silver in total, cf. the example on p. 51.

true answer has to be $^{240}/_{245}\cdot 200$ £. The medieval calculator could formulate the same according to the rule of three, and get the equally unhandy $^{240\cdot 200}/_{245} = {}^{9600}/_{49}$. The present approximate iteration is much more likely to correspond to what was done in practice (and perhaps to what we would do if we were requested to perform the *complete* calculation on paper). In the first approximation, we may observe that 1 £ = 240 δ grows to 245 δ in $2^1/_2$ months. Even this is slightly unhandy, and since it is only a preliminary step the text instead supposes that 200 grow to 205, or that 195 grow to 200. Then it is calculated that the value of 195 £ after $2^1/_2$ months is 199 £ 1 ß 3 δ – 18 ß 9 δ below the requested 200 £. But 18 ß 9 δ is almost 1 £, and therefore carry an interest of 5 δ in $2^1/_2$ months. Detracting these 5 δ we may claim that 18 ß 4 δ also carry an interest of approximately 5 δ, and that an advance payment of 195 £ 18 ß 4 δ is adequate; the error is obviously a fraction only of 1 δ (actually 0.40... δ).

The fifteenth-century *Libro di conti e mercatanzie* [ed. Gregori & Grugnetti 1998: 95] (probably copying from an earlier treatise) contains an analogous calculation, showing the equivalence of the solution by means of the rule of three and by (here exact) iteration in 7 steps, stopped only when the correction vanishes. It is concluded that the solution by means of iteration is *più breve*, "shorter" (the calculation by means of the rule of three leads to a division by $22^7/_{20}$).

Two more topics of genuine commercial interest are dealt with (simplified so as to allow the application of simple arithmetic): Alloying and washing of wool.

One alloying problem (p. 429, no counterpart in **V**) runs as follows:

> The mark of silver, which is 8 ounces, costs me ß 66 of *tornesi*. Now it happens that I have the said silver melted and refined. And when I take it from the fire I weigh it and find that each mark decreases by $^3/_4$ of an ounce, that is, that each mark becomes ounces 7 and $^1/_4$. Say me how much it suits me to sell the mark in order to reconstitute my capital. Do thus, say, 8 ounces of silver are worth ß 66; what will ounces 7 and $^1/_4$ be worth? Multiply 8 times ß 66, they make £ 26 and ß 8. And divide by 7 and $^1/_4$ in this way. Say, 4 times 7 and $^1/_4$ make 29. And say, 4 times 26 £ and ß 8 make £ 105 and ß 12, and divide £ 105 and ß 12 by 29, from which results ß 72 and δ 9 and $^{27}/_{29}$ of a δ. And at so much will it suit him to sell the mark of silver in order to

> reconstitute his capital, that is, ß 72 and δ 9 and $^{27}/_{29}$ of a δ. And it is done. Thus do all the similar.

The problem is a simple case of inverse proportionality, and solved in a simple way, referring neither (as **V** mistakenly does in several similar cases though without erring mathematically) to the language of the rule of three, nor to the inverse rule of three (none of the two versions of the treatise ever do so with any name). *Q* being the price for ounce of the refined silver, it simply uses that the total value should not change, that is, $66{\cdot}8 = Q{\cdot}7\,^1/_4$ (the value of the copper in the alloy being disregarded).

The other alloying problem (pp. 256, 428) is strictly analogous: gold containing 2 ounces of copper per pound is paid back with gold containing 3 ounces of copper per pound, the copper considered worthless.

Even the wool-washing problem (pp. 279, 437) is similar: 100 pounds of wool, bought for 10 £, becomes wet, and when dried its weight is reduced to 95 pounds. In the end there is a reference to the rule of three which is *not* found in the counterpart in **V**.

We may wonder at the story, but other texts betray what really happens and why it is commercially relevant: raw wool is dirty, and has to be washed; in this process, it looses weight – namely the weight of the dirt.

Finally, four of the mixed problems are geometric in character – several of them also in the view of fellow abbacus writers, who would deal with such problems in a geometry section.

First we may look at one about cloth (pp. 270, 434):

> A man wants to dress [in woollen cloth] and finds cloth of cubits [*braccia*] 11, which is sufficient for a robe, and the said cloth is palms 3 and $^1/_2$ broad. And he finds another cloth which is palms 5 and $^1/_2$ broad. Say me, how much will be enough to make a robe of this which is palms $5\,^1/_2$ broad at the same rate. Do thus, multiply 11 times 3 and $^1/_2$, they make $38\,^1/_2$, and divide $38\,^1/_2$ by 5 and $^1/_2$, from which comes 7, and we shall say that 7 cubits of cloth will be enough to make the robe.

It looks at first as if the text finds the area of the cloth in question and then divides by the breadth of the second kind of cloth in order to find the corresponding length. However, the units in the two dimensions are not the same. Maybe the compiler did not bother about this difficulty – explicit dimension analysis, after all, was half a millennium in the future; but maybe he merely used the technique of the inverse rule of three.

In **V**, the inverse rule of three is indeed used – and as always in this manuscript, the terminology used (the similar/not similar) is that of the direct rule of three, which at several points has baffled the original compiler of **M**+**F**. Most likely, this time he did not fall into the trap.

The next geometric problem certainly deals with area calculation (pp. 276, 436)

> A church, or indeed a building [*palazzo*],[36] is 1120 long, and cubits 36 broad, neither more nor less. And I want to flag it with flags or slabs that are all of one and the same magnitude. And each slab is long half a cubit and broad a quarter of a cubit. Say me how many slabs are needed to flag the said church or palace, neither more nor less. Do thus, firstly bring to square cubits the church or palace, and multiply the length against the breadth, that is, 120 times 36, they make 4320, and so many square cubits is the whole floor of the palace, that is, 4320. And similarly bring to square cubits the slab, and multiply the length of the slab against the breadth, that is, a half times $^1/_4$, it makes $^1/_8$, and we shall say that 8 slabs enter in each square cubit. And we want to know how many slabs enter in cubits 4320. Multiply 8 times 4320, which make 34560, and we shall say that in the whole floor of the church or palace enter 34560 slabs, neither more no less. And it is done. Make thus the similar.
>
> If you want to prove it, say thus, in the length enter 240 slabs, and in the breadth 4 slabs for each cubit, multiply thus 4 times 240, they make 960, and so many slabs enter in breadth for each cubit, that is, 960. And you want to know how many enter in 36 cubits. Multiply 36 times 960, they make 34560. We thus find our computation again.

The method by which the problem is solved asks for no explanation, but it still invites a commentary. We observe a particular kind of proof, namely a calculation by a different method; such checks, not only of the result but, so to speak, also of the method, turn up repeatedly in abbacus texts.

Finally, two problems deal with volumes; first this one (pp. 269, 433):

> Somebody lends to a friend of his a chest full of feeding grain. And this chest is in all directions 4 cubits, that is, long and broad and high. And after a certain time had passed, the one who had lent the grain asked his friend, who said, I do not have a chest made as the one which you lent me, but I have two chests, each of them in all directions 2 cubits, that is, 2 cubits in height and two cubits in breadth and 2 in length. Say me if he is paid with these two small chests for his large chest, or how many times he shall give them full. Do thus, firstly bring to square cubits the large chest, and multiply for the length and the breadth 4 times 4, they make 16. And for the height multiply 4 times 16, they make 64, and so many square cubits is the large chest, that is, 64 cubits, Now we bring

---

[36] *Una chiesa overo palazzo*. **V** has *una sala overo piazza*, which fits that only a length and breadth are spoken of much better – when referring to a "hall" or a "square" it is indeed possible to think only of the floor that is to be paved; for a "building" with several rooms this is less near at hand. The compiler of **M+F** seems to have misread *piazza* as an abbreviated *palazzo* (*p'lazzo*), and to have replaced **V**'s "hall" by a "church" under the influence of problems he knows. The problem type is indeed borrowed from the medieval post-agrimensor tradition, found in the Carolingian *Propositiones ad acuendos iuvenes* [ed. Folkerts 1978: 62] as well as in the *Geometria incerti auctoris* IV.38 [ed. Bubnov 1899: 355]. Both of these speak of a *basilica*.

> to square cubits the large chest, that is that we say it is 64 square cubits. And similarly, we bring to square cubits the small chest. And we multiply for the breadth and the length 2 times 2, they make 4, and for the height multiply 2 times 4, they make 8, and so many square cubits is the same chest, that is, 8 cubits. Now divide 64 by 8, from which comes 8, and we shall say that he ought to give back 8 small chests full of grain for one of the large ones. And it is done.

This problem type is rather common in abbacus books, with varying numbers but mostly powers of 2 for the sides, and mostly cubic chests, and mostly more or less explicit hinting at an intended fraud. As confirmed by other writings, stereometry was at the limit of mathematical intuition. The failing distinction between square and cubic cubits – in principle explainable as a conceptualization of volumes as made up of “thick surfaces”, surfaces provided with a standard thickness, which however is never made explicit – will not have helped.

Another stereometric problem determines the number of ashlars of given dimensions that go into a wall of given dimensions. The solution follows the same pattern as the determination of the number of slabs in the church floor, omitting only the proof.

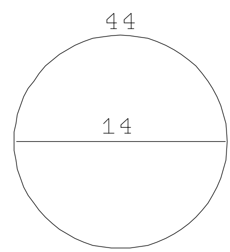
Practical geometry with approximate determination of square roots

The chapter on practical geometry is much more orderly, and is announced in these words (pp. 284, 440):

> In the name of God, amen. Here we shall begin to speak of all modes of measures, and firstly we shall speak of the compass-made round. And about this we shall show an example by proper rule.

We observe, firstly, that Jacopo (also in **V**) speaks of measures, not of geometry, as one might expect if the Latin post-agrimensor tradition had been in the background (or Fibonacci's "practice of geometry", for that matter, the *Pratica geometrie*).[37] Secondly, the use of non-technical terminology, "round" (*tondo*), not "circle" (*cerchio*); there are several similar examples, though sometimes the technical term is used first.

The promised first example runs like this:



> There is a terrain, which is all round by compass, and its circumference, that is, that which it goes around, 44 cubits. Say me how much is its diameter, that is, (how much it is) by the straight in middle. This is its proper and legitimate rule. Always, when you know the circumference of a round, and you want to know how much it is by the straight in middle, then divide its circumference by 3 and $^1/_7$, and that which results from it, so much will its diameter be, that is, the straight in middle. And similarly when you know the straight in middle of a circumference and you want to know in how much it goes around, then multiply the straight in middle by 3 and $^1/_7$, and as much as it makes, in so much does the said round go around. Thus, as our rule says, we should divide the circumference of the round, that is, 44, by 3 and $^1/_7$. And say, 7 times 3 and $^1/_7$ make 22, and say, 7 times 44 make 308. And it is as much to divide 308 by 22 as 44 by 3 and $^1/_7$, from which 14. And we shall say that the said circumference shall be 14 by the straight in middle. And it is done, and do in this way and by this rule with all the circumferences, when you want to know the diameter, such as I show you the form here.

As we see, the Archimedean approximation to the ratio between perimeter and diameter is taken to be plain exact truth; this is so in all abbacus writings. None know about arguments in the Archimedean style. Instead, **V** (p. 285) has its own kind of demonstration before the words "thus, as our rule says ...":

---

[37] Paolo Gherardi's *Libro di ragioni*, written in Montpellier in 1327, does distinguish between rules *di misure* and *di giomatria* [ed. Arrighi 1987], where the latter might indeed refer back to the Latin post-agrimensor tradition; but this seems to be unique, and therefore cannot count as firm evidence of anything.

> And if you should want to know for which cause you divide and multiply by 3 and $^1/_7$, then I say to you that the reason is that every round of whatever measure it might be is around 3 times and $^1/_7$ as much as is its diameter, that is, the straight in middle. And for this cause you have to multiply and divide as I have said to you above.

We may not find this demonstration very persuasive, but we should observe the presence of the *idea* of demonstration, an appeal to a general principle. We may indeed claim that the Archimedean approximation serves as *an axiom* (cf. below, p. 392), though obviously not a part of any axiomatic *system*.[38]

Next follows the reverse calculation, the determination of the circumference from the "straight in middle", taken to be 19 cubits.

As third comes the determination of the area (*terreno*) when the circumference is 22 cubits. At first the diameter is found to be 7 cubits, next the area is determined as the product of circumference and diameter divided by 4.

This primacy of the diameter does not characterize abbacus geometry broadly. Jacopo shares it with two other treatises also written in Provence, one (the *Liber habaci* – below, p. 177) around 1309, the other (the *Trattato di tutta l'arte dell'abacho*; above, p. 11) from *ca* 1334.[39] – clear evidence that Jacopo had gone to Montpellier in order to learn from the local environment, not only with the aim to disseminate Florentine knowledge (a point we shall return to).[40]

Two problems teach how to find the diagonal of a right triangle and the hypotenuse of a square (pp. 286, 441):

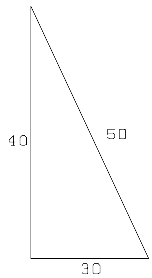
> A terrain with three edges, the two edges straight and the other edge skew, of that size that one side, that is, the straight side, is 30 cubits. And the other side is 40 cubits. Say

---

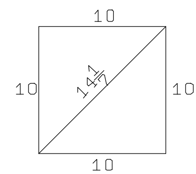
38 We are thus far away from the Euclidean system. Not necessarily very far from what Hippocrates of Chios had done in his analysis of lunes, however. Hippocrates appears in the same way to take as foundations in need of no further argument such things as the Pythagorean rule and the proportionality of areas to the square of a linear dimension – thus things which practical geometers had known and used for centuries; cf. [Høyrup 2019c: 163–177].

39 Apart from that, I have observed it in two 15th-century writings, not properly abbacus treatises: a *Praticha di geometria e tutte le misure di terra* [ed. Arrighi & Nanni 1982], written in the earlier 15th century by Tommaso della Gazzaia, a nobleman from Siena, "for his pleasure, taking delight in the science of geometry" [Van Egmond 1980: 187], and also in other respects close to Jacopo and thus probably (given the respective dates) borrowing from him; the other a *Praticha di gieometria* [ed. Arrighi 1970a] written by the military engineer Giorgio Martini around the mid-15th-century, sharing some particularities with Tommaso della Gazzaia.

40 His way to present the rule of three is Italian, however, not Provençal – cf. the analysis of Ibero-Provençal ways below, p. 180 onward.



me how much the skew side of the terrain will be, that is, from the tip of one side of the terrain to the other. Do thus, multiply 30 times 30, they make 900, and multiply 40 times 40, they make 1600. Now join together 900 and 1600, they make 2500. Now find the root of 2500, that is 50. And we shall say there are 50 cubits from one tip of the terrain to the other, as I show you its diagram [*forma*] here.



A square terrain, which is 10 cubits by each face. I want to know how much there will be from one corner[41] of the terrain to the other, measuring across. Do thus, multiply 10 times 10,[42] they make 100, and double 100, they are 200.[43] Now find the root of this number, that is, of 200, which is 14 and $^1/_7$. And it is done. And we shall say that this terrain is, measuring [*quadrare*] by the edge, 14 cubits and $^1/_7$. And in this way make all the similar.

V specifies that $14\,^1/_7$ is "the closest, because precisely it cannot be found". As we shall see, and as confirmed by other texts, this is a technical term for the first approximation obtained by the standard method,

$$\sqrt{n^2+r} \simeq n+\frac{r}{2n}$$

Later in the chapter there are other problems involving the Pythagorean rule, and a section teaching how to find "the closest" approximation to a square root. But first there is an intruder that has little to do with measure or geometry[44] (pp. 287, 441):

A serpent is at the foot of a tower, which tower is 30 cubits high, and the said serpent wants to climb to the top of the tower. And each day it climbs a third of a cubit, and in the night it descends a fourth of a cubit. Say me, in how many days the serpent will have

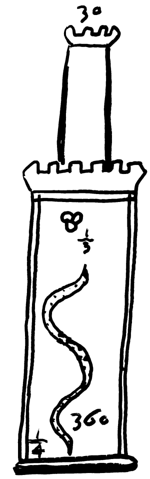
[41] *Canto*, the term here translated "corner", is used (alternatingly with the synonym *cantone*) in the preceding problem about the side, and there translated "edge". The terminology is obviously vacillating, not yet technical.

[42] Here, **V** has the misleading explanation "multiply one face of the terrain against the other, that is, 10 times 10".

[43] **V** insists on the misunderstanding, "further multiply the other two, 10 times 10".

[44] The problem type is known under the name *leo in puteo*, "the lion in the pit", after the first version offered in the *Liber abbaci* [B177;G302]. Afterwards Fibonacci offers another version speaking of *two* serpents, one coming from the top of a tower and the other from the ground, both with alternating motion.

> climbed to the crown of the tower. Do thus, say, $^{1}/_{3}$ and $^{1}/_{4}$ are found in 12, and multiply 12 times 30, they make 360. Now take $^{1}/_{3}$ of 12, which is 4, and take $^{1}/_{4}$ of 12, which is 3. Now detract 3 from 4, one remains, and divide 360 by 1, from which comes 360. And we shall say that in 360 days will the serpent climb the tower, as I show you in drawing. And you could also make the said computation in a different way and say, $^{1}/_{3}$ and $^{1}/_{4}$ are found in 12, a third is thus $^{4}/_{12}$ and a fourth is $^{3}/_{12}$. The $^{1}/_{3}$ is thus $^{1}/_{12}$ more than $^{1}/_{4}$. And because $^{1}/_{3}$ is $^{1}/_{12}$ more than a fourth, the serpent advances every day $^{1}/_{12}$ of a cubit. In 12 days it thus advances 1 cubit. And we want it to advance 30 cubits. Multiply 12 times thirty, they make 360. And it is done. It thus comes in one way as in the other.

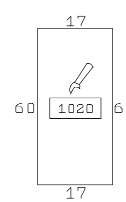


First of all we notice that the solution misses the recreational prank of the problem: after 357 days, the serpent has advanced $^{357}/_{12} = 29\,^{9}/_{12}$ cubits. The next day it reaches the top; it then probably does not slide down any longer, but in any case this is immaterial for the answer: after 358 days, the serpent has reached the top. The use of the dress simply as a pretext for subtracting one fraction from another and then dividing by the outcome is no particularity of the compiler of **M**+**F**, it is shared not only with **V** but also with abbacus books in general (we shall encounter an exception below, p. 174), and even with the *Liber abbaci*, which however formulates itself in terms of a single false position.

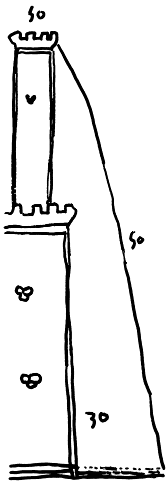
Noteworthy is also that we encounter a second instance of the check of the method, not merely of the result. The second way of the text is identical with that of **V**. The first, instead, is independent, or perhaps inspired by a different source.

After this arithmetical aside, the text returns to measures, first the measure of a rectangular area (pp. 288, 442):

> A terrain which by its two larger faces is 60 cubits, as you see drawn, and by the other two it is for each face 17 cubits, say me how much is this whole terrain in area [*quadro*]. Do thus, because it is by one face 60 cubits and by the other face 17 cubits, then multiply 17 times 60, which make 1020. And we shall say that this whole terrain is 1020 square cubits. And always, when you want to bring to area whatever terrain it may be with equal sides, as we have said, then multiply the length against the breadth.



This straightforward rule invites a linguistic observation: Jacopo has no specific term for a rectangle, technical or otherwise. Once a "terrain" is specified (in drawing and measures) to be quadrangular and to have equal opposing sides, then it is assumed by default to be rectangular. Such default understanding is much more common in mathematical thinking (even ours) that we are usually aware of: who, even among those who have learned in school about negative, broken and irrational numbers, would ever think of anything belonging to these categories when asked to “think of a number”? Even the mathematician will assume without reflecting that the one who asks means “a positive integer”, and will at most think of $-\sqrt{\pi}$ as a provocation.



After this, Jacopo returns to the tower, but now as a dress asking for application of the Pythagorean rule (pp. 289, 442):

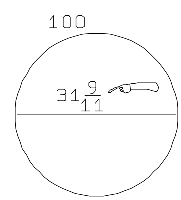
> A tower, which is 50 cubits high. And at the foot of this tower there is a moat, which is 30 cubits broad. Now I want to carry a rope or string which reaches from the border of the moat until the crown of the tower. Say me how long the said string will be. Do thus, say, since 50 cubits is the height of the tower, then multiply 50 times 50, they make 2500, and because the moat is 30 cubits broad, then multiply 30 times 30. they make 900. And join together 2500 and 900, they are 3400. Now find the root of 3400, which is 58 and $^9\!/_{29}$, and so long should the rope be that reaches from the border of the moat until the crown of the tower, that is, cubits 58 and $^9\!/_{29}$ And it is done. And here I show you the diagram in order to understand better.

The calculation itself asks for no explanation. Once again, nothing is said about the approximate character of the square root; in **V** (p. 289), even this time, we find “that is, the closest, and closer one cannot find”; but even in **V**, the explanation only comes later.

A similar problem follows, in which the height of the tower is told to be 40 cubits, and the length of the rope 50 cubits, the breadth of the moat being asked for.

Before explaining how to find “the closest” square root, Jacopo inserts another circle problem (pp. 290, 442):

> A compass-made round which goes around in 100 cubits. Do thus, and that is its proper rule, divide 100 by 3 and $^1\!/_7$ in this way, say, 7 times 3 and $^1\!/_7$ make 22, and say, 7 times 100 make 700. And as much is to divide 700 by 22 as 100 by 3 and $^1\!/_7$, from which comes 31 and $^9\!/_{11}$. And so much is this round by the straight in middle, that is, cubits 31 and $^9\!/_{11}$ of a cubit, such as I show you drawn in the diagram.



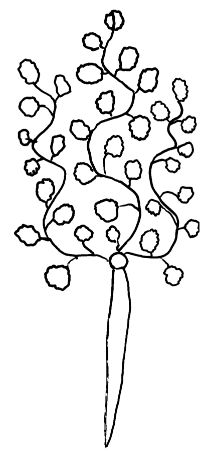
This, of course, is a repetition of what has been shown already (see above, p. 34). **V** (p. 290) is aware of that (“I have also said it to you above, (for) every round, if one wants to know how much is its diameter, one shall divide by 3 and $^1\!/_7$”), and also has a detailed explanation of the division (transforming $3\,^1\!/_7$ into 22 seventh, and 100 into 700 seventh). All these reasons for the repetition have been left out in **M**+**V**.

Then comes the explanation of what a square root is and how it can be found or approximated (pp. 291, 443):

This is a rule which shows us how to find the root of every number of which one can find the root, or indeed the closest root that one can find. And this we shall show by proper rule.

First we say thus, as example: The root of 4 is 2 because 2 times 2 make 4. And the root of 9 is 3 because 3 times 3 make 9. And the root of 16 is 4 because 4 times 4 make 16. And the root of 100 is 10 because 10 times 10 make 100. And the root of 169 is 13 because 13 times 13 make 169. And the root of 10000 is 100, because 100 times 100 make 10000. And thus happens with every other number which you multiply in itself, this same number is the root of its multiplication, as you have understood.

Now we shall say in which way the root can be found for every number for which it can be found, namely the closest root. Know that you shall do thus. You shall find a number which, when multiplied by itself, is closer to the number of which you want to find root than any other number. And then divide the remainder by the double of that number which you multiplied. And in this way one finds true or closest root.



And to this we shall say the example, and we shall say thus, find me the root of 10. Do thus, say, 3 times 3 make 9. And say, from 9 until 10 there is 1. Now divide 1 by the double of 3, that is, by 6, from which comes $^1/_6$. And join $^1/_6$ above 3, they are 3 and $^1/_6$. And we shall say that the root of 10 is 3 and $^1/_6$, that is, the closest root than can be found. And in this way and by this rule you can find root to every number, or indeed the closest root that can be found, by the rule stated above.

V is more detailed, both in the exposition of the rule for finding the “closest root” and the example. Neither **V** nor **M**+**F**, however, contains the least hint of an explanation why the rule works; this is characteristic of the abbacus tradition as a whole, as is the expressed belief that the “closest root” is indeed as close as one can get. *If* they had also presented the possibility to approach from above (say, approximating $\sqrt{15}$ as $4-{}^1/_{2\cdot4}$ = 3.875 instead of $3+{}^6/_{2\cdot3}$ = 4 – the true value is 3.87298...), they would have discovered that this is not true (not to speak of the possibility to iterate the process); but they very rarely do.[45] For geometrical use (or pretended use), what they offer was probably quite sufficient.

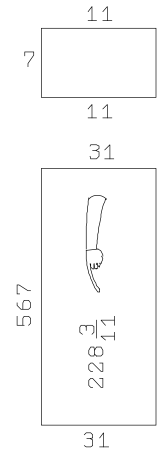
We may appreciate the drawing of a plant with root; in **V** it is much more beautiful.

Two more examples follow, the determination of the “closest roots” of 67 and 82.

[45] Below (p. 44) we shall encounter a case where Jacopo copies a calculation which starts with an approximation from above, and even goes on with a (mistaken) second approximation. But Jacopo seems not to understand what goes on.

Then follow applications of the formulas for finding areas and volumes. First rectangular areas (pp. 295, 444):

> A terrain which is 567 cubits long, and 31 cubits broad, as I show you drawn opposite by diagram. And I want to build on all of it. Say, I want to build on all of it with houses that are each 11 cubits long and 7 cubits broad, neither more nor less. Say me how many houses you can lodge there so that you fill the whole terrain. Do thus, first bring to square cubits the whole terrain, and multiply the length against its breadth, that is, 31 times 567, that make 17577. And so many square cubits is the whole terrain, that is, cubits 17577. And similarly bring the house to square cubits, and multiply the length against the width, that is 7 times 11, they make 77. And so many square cubits is the house, that is, 77 cubits. Now, if you want to know how many houses can be lodged there, then divide 17577 by 77, from which comes 228 and $^{3}/_{11}$. And it is done. And we shall say that in this whole terrain 228 houses and $^{3}/_{11}$ of a house can be made, neither more not less. And in this way make the similar.



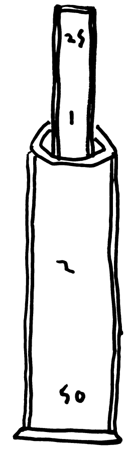
As in the serpent-problem, we observe that the dress is not taken seriously: 7 divides 567, but 11 divides neither 567 nor 31. It is therefore not possible to fill the terrain with houses of the requested dimension. Apart from that, the text is quite straightforward, and correct.

The problem type comes from the Latin post-agrimensor tradition. Three versions occur in the *Propositiones ad acuendos iuvenes* [ed. Folkerts 1978: 60*f*]. In one case, houses there have to be built within a trapezoid, in one within a triangle, and in one within a circle; the houses are rectangular, that is, unable to fit precisely. The problems in *Geometria incerti auctoris* IV.35–37 [ed. Bubnov 1899: 354*f*] are similar, and so are those of the *Artis cuiuslibet consummatio* I.34–36 [ed. Victor 1979: 212–218]. From the latter treatise the triangle- and the circle-version went into the late-13th-century vernacular (Picardian) *Pratike de geometrie* (I.34,36, ed. [Victor 1979: 504, 506]), which we shall encounter again in note 47.

When volumes are dealt with, Jacopo’s intuition fails (pp. 296, 444):

> A square well, which is 2 cubits by each face, and is 50 cubits deep, and it is quite full of water. Now it happens that a square column falls into it, which by each face is 1 cubit and which is 25 cubits long. Say me how much water flows out of the said well because of this column which falls into it. Do thus, first bring the well to square cubits, and multiply 2 by 2, they make 4. And for the depth multiply 4 by 50, they make 200. And so many square cubits is the whole well, that is, 200 cubits. Now, similarly bring the column to square cubits, and multiply 1 by 1, it makes 1. And for the length multiply by 25, it makes 25. And so many square cubits is the column, that is, 25 cubits. Now divide 200 by 25, from which comes 8. And we shall say that 8 square cubits of water

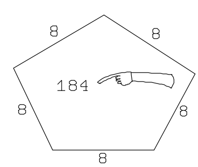
> flow out of the well because of this column which falls into it, such as I show you the diagram of the well and the column.



The fallacy is shared with **V**. Since the compiler of **M**+**F** has intervened actively in the text, this means that the elementary mistake committed by Jacopo has been accepted by somebody who was thinking through its subject-matter.[46] Stereometry was evidently not the strong point of abbacus mathematicians. We observe in this connection that there is no distinction between the units for area and volume, both are measured in "square cubits".

The origin of the error can be understood from similar problems in the *Liber mahameleth* (Latin version *ca* 1160, [ed. Vlasschaert 2010: 397; ed. Sesiano 2014: 536]) and the *Liber abbaci* [B403*f*;G618–620]. In these, together with the dimensions of the column and of the well (in both actually a cistern) the contents of the latter measured in barrels is given, and the quantity of outflowing water measured in barrels is to be determined. In both, as here, the ratio between the two volumes is found (here, 8), and this is then used to convert the volume of the immersed body into hollow measure measured in barrels. There can be little doubt that this problem type, known in Iberian area around 1160, reached the Provençal area and Jacopo from there, and was miscopied without understanding by Jacopo or some predecessor of his (and, in the latter case, re-copied without understanding by Jacopo and again by the compiler of **M**+**F**).

The next problem (pp. 297, 445) is also fallacious:



> A terrain with five equal faces, as you see it drawn here, which is called a pentagon, and by each face it is 8 cubits. Say me, how much is this whole terrain in area. Thus is its rule, multiply one of its faces by itself, that is, 8 times 8, they make 64, and multiply 3 times 64, they make 192. And from 192 detract one of the faces, that is, 8, 184 remain. And it is done, and we shall say that this whole terrain is 184 square cubits. and in this way and by this rule do whatever the terrain is by face, if the faces are equal and if there are 5 faces, multiply always one of the faces in itself and then make three times this multiplication, and from the total detract one of the faces, and the remaining will be this whole terrain, as we have said.

This strange formula – immediately understood to be impossible by anybody who knows about dimensional analysis (or its foundation in metrology, given that a change of unit will change the outcome) – comes from the Latin post-agrimensor tradition, and ultimately from the ancient theory of polygonal numbers. The nth pentagonal number is indeed $\frac{1}{2}(3n^2-n$

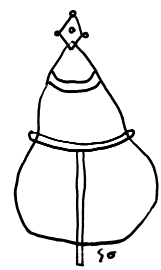
---

[46] **V**, on the other hand, may descend from Jacopo's original by professional scribal copying.

a formula taken over by Jacopo with omission of the factor $^1/_2$ – most likely by a predecessor of his[47]).

When dealing with a cone-shaped pavilion, Jacopo also runs into trouble (pp. 298, 445):

> A pavilion, whose mid-pole [**V**: the post that supports it] is 40 cubits, and the cloth from the peak of the pole until the lower border of the pavilion is cubits 50. Do thus, say, know how much is all this cloth, and how much ground the said pavilion occupies under itself. Do thus, say, because the cloth is 50 cubits long, then multiply 50 times 50, it makes 2500. And because the pole is 40 cubits long, then multiply 40 times 40, it makes 1600. Now detract 1600 from 2500, 900 remains, and find the root of 900, which is 30. And double 30, they make 60, and so much is broad the pavilion by the straight in middle, that is, 60 cubits. Now multiply 60 times $3\,^1/_7$, which makes $188\,^4/_7$, and so much is the whole circle of the pavilion around. Now, if you want to know how much ground it occupies under itself, then divide the straight in middle of the pavilion by half, that is, 60, from which comes 30. And similarly divide in half the circle of the pavilion, that is, $188\,^4/_7$, from which comes 94 and $^2/_7$. Now multiply 30 times 94 and $^2/_7$, which make 2828 and $^4/_7$. And the said pavilion occupies so much under itself, that is square cubits $2828\,^4/_7$. Now if you want to know how much is all the cloth, divide the diameter, that is, 60, by $^1/_2$, from which comes 30, and multiply 30 times 50, they make 1500, and so many square cubits is all the cloth of the said pavilion, that is, 1500 cubits. And it is done, as you see drawn in diagram.



The diagram in **M** is not very informative, as we see. The counterpart in **V** (next page)

---

[47] In **V** the factor 3 is indeed explained as the number of remaining sides, which presupposes that the product of one of the faces by itself is replaced by the product of two of the faces (cf. also note 42). Several variants of the formula must have been current in Provence – Paolo Gherardi, writing in Montpellier in 1327 [ed. Arrighi 1987: 61], has the "correct" formula $(3n^2–n)/2$. The *Trattato di tutta l'arte dell'Abacho* ($\mathbf{T_f}$, fol, $137^{r–v}$) gives two procedures, first *per l'arte di rismetricha*, then *per giometria*. The former prescribes to multiply 8 by 8, then by three, and then to subtract the square of a side, leaving in total 128. The latter prescribes to multiply half the side by the measured height (claimed to be $6\,^9/_{22}$ – it should be 5.5055...) and then by 5, which yields $128\,^2/_{11}$, almost the same. This can only come from a repair of "Jacopo's formula", made by somebody who had a better intuition of dimensional homogeneity. Since the *Trattato di tutta l'arte* betrays no general familiarity with Jacopo's *Tractatus*, a shared source for the mistake imposes itself.

The method *per giometria* is shared with the Picardian *Pratike de geometrie* [ed. Victor 1979: 489], which however does not propose a value for the height. Since references to genuine measuring (as distinct from measures already known) are extremely rare in the so-called "practical geometries", a connection (hardly direct descent) is none the less almost certain; cf. [Høyrup 2009c].

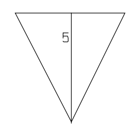
is somewhat more convincing.

The fallacious answer to the second question is shared, however. Jacopo confirms that he had no spatial intuition, nor experience with cutting cloth to fit a conic shape – and the compiler of **M**+**F** no more.[48]

The answer to the first question is found in agreement with Jacopo's basic formula for the circular area, semi-diameter times semi-periphery, characteristic of Provence. We may conclude that he found the problem here, and guess that his source shared his failing spatial competence.

Even the next problem (pp. 300, 445), dealing with the simple plane geometry of a "shield" shaped as an equilateral triangle,[49] appears to be copied without full understanding:

> A shield, that is, a triangle, which by the straight in middle is 5 cubits, say me, how much will the said shield be by each face. Do thus, multiply 5 times 5, they make 25, and divide 25 by 3, from which comes 8 and $^1/_3$, and join 8 and $^1/_3$ above 25, they are 33 and $^1/_3$, know that the said triangle will be root of 33 and $^1/_3$ by face. Find the root following the rule we have said, which root we say to be 5 and $^5/_6$ less $^{17}/_{54}$ not precisely, and so much will the shield be by face. I show you the diagram in order to understand better. Make thus all the similar. And this is understood about a shield which has faces equal in measure.



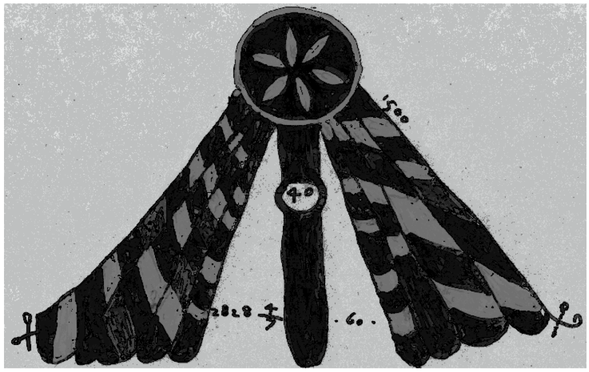
The first step (shared with **V**) presupposes awareness that the half-side of an equilateral triangle equals $^1/_{\sqrt{3}}$ times the height. This is not difficult to show – according to the Pythagorean rule the square on two half-sides equals the sum of the square on one half-side and the square of the height; but if Jacopo understood this, he would probably tell – when it is within his reach he likes to explain things that are not quite straightforward. The problem is thus likely to have been borrowed wholesale. The second part, the determination of the approximate square root, supports this conclusion, since it is *not* done according to the rule taught at an earlier moment.

Here, however, a comparison with the corresponding lines in **V** (p. 298) is informative:

---

48 This lack of spatial understanding did not lead him to invent a wrong solution – only to copy it uncritically. The same fallacy (and the same doubly mistaken formula for the area of a regular pentagon) is found in a geometry contained in the Latin manuscript Munich, Clm 26639 [ed. Kaunzner 1978: 36, 39]. As argued by Kaunzner, at least the geometrical part of the manuscript was written in the outgoing 15th century, but nothing in its texts betrays inspiration from the abbacus tradition.

49 The cyclopic dimensions of the "shield" (*schudo*) in question shows that the word serves as a semi-technical geometric term and does not refer to a real piece of armour.



Now find its root, that is, of 33 and $^1/_3$, which comes to be 5 and $^7/_9$ less $^4/_{18}$.

This looks suspicious. Why "$^7/_9$ less $^4/_{18}$" and not just "$^5/_9$"? Calculation according to the method that was taught would give

$$\sqrt{33\tfrac{1}{3}} \approx 5+\frac{8\frac{1}{3}}{2\cdot 10} = 5\,\frac{5}{6}\ .$$

If we use the same method but approximating from above, observing that $33\,^1/_3 = 36–2\,^2/_3$, we get

$$\sqrt{33\tfrac{1}{3}} \approx 6–\frac{2\frac{2}{3}}{2\cdot 6} = 5\,\frac{7}{9}\ .$$

So, $5\,^7/_9$ is a first approximation reached through approximation from above.[50] Now, $(5\,^7/_9)^2 = 33\,^{31}/_{81} = 33\,^1/_3+{}^4/_{81}$. The correct second approximation (again from above) would be

$$\sqrt{33\tfrac{1}{3}} = 5\frac{7}{9}-\frac{\frac{4}{81}}{2\cdot 5\frac{7}{9}}\ .$$

Instead, the author simply subtracts the excess, obviously not understanding why the usual approximation works. And then either he, Jacopo, or some further copyist on the way toward **V**, miswrites $^4/_{81}$ as $^4/_{18}$.

The compiler of **M**+**F** has obviously seen that something was wrong, and tried his own hand. The usual first approximation from below gives him $5\,^5/_6$. Since $(5\,^5/_6)^2 = 34\,^1/_{36} = 33\,^1/_3+{}^{25}/_{36}$, a correct second-order correction (from above) would be a subtraction of $(^{25}/_{36})/(2\cdot 5\,^5/_6) = {}^5/_{84}$. How this has become a subtraction of $^{17}/_{54}$ I cannot explain[51] – but at least we see that the compiler knows about the possibility of a second approximation,

---

50 It is possible that $33\,^1/_3$ has been reexpressed as $^{100}/_3 = {}^{300}/_9$, and the root of 300 = 324–24 then found by approximation from above as $18–{}^{24}/_{2\cdot 18}$. The outcome is the same.

51 In [2007: 22], calculating badly I suggested $(2\cdot 5\,^5/_6)/36$, which is indeed $^{17½}/_{54}$.

improving on the “closest root”.

Another problem about a “shield” follows, now with height 9 cubits. The first part of the calculation goes as before, leading to the extraction of $\sqrt{108}$, this time found as $10\frac{2}{5}$ according to the method that was taught for the “closest root”. **V** instead repeats the blunder of the first shield problem, subtracting the excess from the “closest root”.

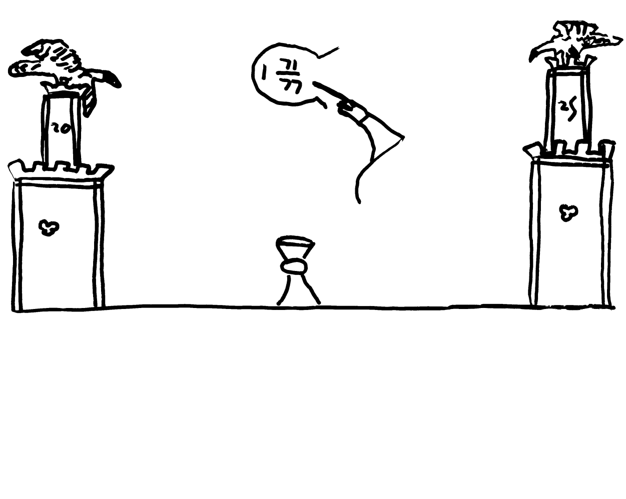
After this, Jacopo (in both versions) gets back to something he appears to understand (pp. 301, 446):

> Two lances which are stuck in one plane, and one lance is 10 cubits long, and the other is 17 cubits long, and from one lance to the other there are 20 cubits. Say me how many cubits there will be from one of the points to the other of the said lances. Do thus, detract 10 from 17, 7 remain, and multiply 7 times 7, they make 49. And similarly multiply 20 times 20, they make 400, and join together these two numbers, that is, 49 and 400, they make 449. And find the root of 449, which is 21 and $\frac{4}{21}$. And we shall we say that from one point of the lance to the other there are cubits 21 and $\frac{4}{21}$ of a cubit. And it is done. I show you the diagram.

The mathematics asks for no commentary – but we may take note of the dress which, like the towers with moat and the pavilion, reminds us that the artisans and merchants who sent their sons to the abbacus school, lived under the conditions of endemic warfare.

The next problem, also a rather simple application of the Pythagorean rule, is characteristic of Jacopo: it borrows a traditional dress but uses it for a different mathematical purpose (here, as repeatedly, a much simpler purpose; pp. 301, 446):

> There are two towers in a plane, as I show you drawn. And one tower is 20 cubits high and the other is 25 cubits. And in the middle between these two towers there is a goblet, as you see drawn. And from one tower to the other there are 100 cubits. And on top of each one of these towers there is a dove, which wants to go drink from this goblet. And from one tower to the other there is 100 cubits, and they set out at one and the same hour,



> and fly equally, one as the other. Say me how much earlier one will be there than the other to drink from the goblet. Do thus, say, because from one tower to the other there is 100 cubits, then divide 100 by half, from which comes 50, and multiply 50 times 50, they make 2500. And because one tower is 20 cubits high, then multiply 20 times 20, they make 400. And join 400 above 2500, and you have 2900. Now find the root of 2900, which is 54 less $^{4}/_{67}$. And in so much will the dove come to drink which is on the tower that is 20 cubits high, that is, in 54 cubits less $^{4}/_{67}$ of a cubit. If you want to know when the other dove will be there, then multiply 25 times 25, they make 625. And similarly join above 2500, they are 3125. And find the root of this, that is, of 3125, which is 56 and $^{1}/_{112}$, and in so much will the other dove be to drink of the goblet, that is, in cubits 56 and $^{1}/_{112}$. Now detract from 56 and $^{1}/_{112}$, 54 and $^{4}/_{67}$, 1 and $^{73}/_{77}$ remains, and in so much will one dove be earlier to drink of the goblet than the other, that is, cubit 1 and $^{73}/_{77}$ of a cubit, that is, the one that is on the tower of 20 cubits.

Traditionally, the dress of the two doves on two towers is used for a different purpose: instead of giving the position of the goblet, it is stated that the two doves not only set out but also arrive at the same moment – that is, the distances from the tops of the towers to the goblet are the same. The solution builds on application of the Pythagorean rule to two equilateral triangles that have one side (namely the hypotenuse) identical and the sum of two corresponding sides given (here the distances of the goblet from the towers). The same trick serves in the determination of the height of a triangle with given sides. That is observed by Mahāvīra in his ninth-century *Gaṇita-sāra-sangraha* (VII.201½–203½, ed. trans. [Raṅgācārya 1912: 249*f*]), whereas Paolo Gherardi has a correct but only halfway argued solution in his *Libro di ragioni* [ed. Arrighi 1987: 65–67]. The *Liber habaci* (equally Provençal, we remember) has a sham solution which only works for its specific parameters, while the *Columbia Algorism* (late 13th century, as we shall see on p. 170, and linked to the Ibero-Provençal region) prescribes a correct calculation without any argument (while replacing the doves by falcons and the cup by a duck). This may be too difficult for Jacopo (and most abbacus writers). Instead, as we see, he changes the problem in such a way that nothing but simple use of the Pythagorean rule is required. We may wonder that he measures time as length, but as familiar from 14th-century Aristotelian natural philosophy, velocity ("motion") was not a developed, quantified notion; so, after all, Jacopo's choice is the best he can make.

Less adequate are the determinations of the square roots. **V** (p. 303) claims the root of 2900 to be $53\,^{9}/_{106}$, while the usual "closest root" is indeed $53\,^{91}/_{106}$ (almost certainly a writing or copying error). **M** and **F** instead approximate from above, which should give $54–^{4}/_{27}$. **M** instead has an indubitable "54 less $^{4}/_{67}$", while **F** [ed. Simi 1995: 32] gives "4 less $^{4}/_{77}$". **V** correctly approximates $\sqrt{3125}$ from below as $55\,^{10}/_{11}$ (p. 303), while **M** as well as **F** try an approximation from above (in principle a good idea). This should give $56–^{1}/_{112}$ – but both write "56 and $^{1}/_{112}$". The difference found by both, which on the conditions of **F** should be $(56+^{1}/_{112})–(54–^{4}/_{77})$, is given as $56–(54+^{4}/_{77})$. We may conclude that the common archetype for these two manuscripts (not necessarily the text of the original compiler) had $^{4}/_{77}$ and not $^{4}/_{67}$ – but also that the familiarity of the compiler with approximations from above was more than counterbalanced by other shortcomings of his. In German, there are two terms for this kind of misrepair of a text: *Verschlimmbesserung* and *Verballhornung*. For some reason, there seems to be no established correspondingly colourful equivalent in English.

A problem follows in **V** as well as **M**+**F** (pp. 303, 447) about a rather impossible building – gutters (*piovetoi*) being put in a position where they cannot serve to collect the rain falling on the roof. Instead they serve as pretext for another application of the Pythagorean rule, leading to the extraction of $\sqrt{569}$. **V** approaches from below, which gives $23\,^{20}/_{23}$, and adds the same erroneous correction term as in the shield problems (here $(^{20}/_{23})^2 = {}^{400}/_{529}$). This time **M**+**F** do not venture into independent calculation but simply omit the additional term.

M+**F** close the chapter by showing how to extract the ("closest") root of 101. This has no counterpart in **V**.

All in all, Jacopo's "practical geometry", in either version, has little to do with the genuine practice of surveyors or with the use of their measurements in the determination of rent or taxes.

The coin list

The next section[52] brings us back to what is needed in commercial life – more precisely, in exchange reaching beyond local trade. Its introduction runs like this (pp. 331, 448):

> In the name of God, Amen. Here are written all modes of alloys of coins, and similarly all alloyings of gold and silver and copper, how are alloyed one coin or bullion of gold in ingots, or silver of all rates.
>
> And we begin thus. You shall know that one ounce of fine gold is 24 carats. And the baser the gold, the less carats are there in the ounce.[53] And the better the gold, the more carats are there in the ounce. And similarly happens with silver, but silver is alloyed at ounces, or indeed at *denari* of weight. And the silver that holds 12 ounces per pound is fine silver and good and pure.

A list of coins of 6 pages (fols 42^r–44^v in **M**) follows. It may at first astonish a modern reader that only the fineness and not the value in terms of some standard is indicated. The explanation is obvious, however: as long as the value of a coin was its metal value and not guaranteed by some central bank, the only thing that was certain was the fineness (unless the coin was counterfeited). The quantity of metal had to be controlled by weighing, since some small clipping (or simply honest wear at the touchstone) might have reduced it. We may look at an extract:[54]

fiorini of gold from Florence are alloyed at carats	24 per ounce
Augustales of gold are at carats	$20\frac{1}{2}$ per ounce
Perperi pagliolati are at carats	15 per ounce
Dobre dell'Amira — — are at carats	$23\frac{1}{2}$ per ounce
Dobre del Rascetto are at carats	$23\frac{1}{2}$ per ounce
Castellani of gold are at carats	$23\frac{1}{2}$ per ounce
Alfonsini of gold are at carats	$20\frac{1}{2}$ per ounce

---

52 "Next" in **M+F**; in **V**, the geometry and the coin list are separated by several chapters of algebraic character – see below, p. 185.

53 The established value of the carat was 4 grains, that is, $\frac{1}{24}$ of a Roman *solidus*, and fineness was measured as carats in a *solidus*, not of an ounce [Zupko 1981: 79]; cf. above, note 21. Since this was always used as a relative measure (different from when the weight of diamonds is given in carats), the definition given here actually changes nothing. The mistake, or whatever we will call it, seems to be of French-Provençal origin, cf. [Høyrup 2007: 123*f*].

54 The full lists of **M**+**F** as well as **V** (almost identical) are in [Høyrup 2007: 448–452, 331–336], the latter with translation. The full list is also transcribed in [Travaini 2003: 104–108]. Lucia Travaini further gives a full description of all the coins listed in this and a number of other coin lists (pp. 235–313); my commentary draws on this description.

Tornesi of gold are at carats . . . . $23\frac{3}{4}$ per ounce
Old Bezants of gold are at carats . . . . 24 per ounce
Old communal and intermediate *Perperi* of gold are
at carats . . . . 17 per ounce
Saracen Bezants of gold, of which 12 go per ounce,
are at carats . . . . 15 per ounce

– 26 more gold coins follow –

Here are written what all silver coins contain.
Tornesi grossi . . . . are at ounces $11\frac{1}{2}$ per pound
And it is to be understood that the *pound* is of 12 ounces of fine silver in all alloyings
Medals[55] from Tours, first class, are at ounces . . . . $11\frac{1}{2}$ per pound
Medals [from Tours], third class, are at ounces . . . . 11 per pound
Carlini and *mergugliesi* and *barzellonesi* are at ounces $11\frac{1}{4}$ per pound
Sterlings . . . . are at ounces . . . . 11 *denari* 2 per pound
Venetiani from Venice . . . . are at ounces . . . . $11\frac{3}{4}$ per pound

– 14 more silver coins follow –

Here are written the alloyings of small coins
Parigini of first class are at *denari* 5 and grains 18 of alloy[56]
Parigini of second class are at *denari* 4 grains 16 of alloy
Parigini of third class are at *denari* 3 grains 14 of alloy
Old *Tolosini* "with the cross" are at *denari* 6 grains 18 of alloy

– 50 small denominations follow[57] –

First of all we notice the wide commercial network implied by the coins: *Perperi* (from *hyperperon*) is a Byzantine coin; *pagliolati* refers to the dynasty of *Palaeologoi*, the "communal" were minted by the Nicaea dynasty during the crusader occupation of Constantinople during the first half of the 13th century. *Augustali* had been minted in Sicily by Frederick II of Hohenstaufen and Charles d'Anjou. *Dobre* ("double", originally double dinar) were minted in the Iberian Peninsula and in the Maghreb ("Amira" is Almeria). *Castellani* were Castilian emulations of the *dobre*. *Alfonsini*, minted by Alfonso VIII of Castile, were also emulations of a Moroccan *dobre*. "Bezant" was used about the

[55] *Medaglia*, from *medius*>*medialia*, is mostly a half-*denaro*, here however the half of a more valuable unit.

[56] "*Denari* of alloy" actually corresponds to ounces per (light) pound; cf. note 21.

[57] 34 of these, all dealing with Lombardian coin, are a secondary addition and absent from **F** as well as **V**. They have obviously been important in the environment where **M** (or some precursor later than the shared compilation of **M**+**F**) was produced.

hyperperon in the Latin world, but also about the many imitations from the crusader states and the Islamic Mediterranean (Jacopo's "old bezants" are probably from Egypt, his "Saracen bezants" from the Jerusalem Kingdom). *Tornesi*, as we remember, are from Tours in France. *Carlini* were minted in Naples, *mergugliesi* in Montpellier, *barzellonesi* in Barcelona. Sterlings were English (then as now), *Parigini* were from Paris, *tolosani* from Toulouse. The many coins left out above are mostly from the same region but add German areas.

Since the coin list is present in **V** as well as **M**+**F** (with minor variations, apart from the final addition in **M**), we may safely assume that it was also present in Jacopo's original. Jacopo is likely to have copied an existing list – after all, he was almost certainly neither a money-dealer not a banker. Since one of the coins, the *rinforzati* from Provence (not present in the extract) was only coined from 1302 onward, while other coins from 1303 are not included [Travaini 2003: 102], the list he copied must have been a quite recent list.

Alloying problems

The coin list calls for no mathematics beyond the numbers indicating fineness. The last chapter in the treatise, about alloying, does. Indeed, it contains *nothing but* mathematics: one will look in vain for technical advice about the refining or alloying of bullion, and also find nothing about assaying.

All calculations are straightforward and well explained, and there are few repetitions. The metrology is the same as in the coin list – ounces and carats for gold, pounds, ounces, *denari* and *grani* for silver and copper.

V contains an explicit transition between the two chapters (p. 337):

> Here end all the alloys of coins. Now we begin to make some computations of alloying.

M+**F** start directly with a problem (p. 452):

> I have 60 ounces of gold which is 16 carats per ounce, and I want put it in fire and refine it so much that it becomes of 21 carats per ounce. Say me how much these 60 ounces will become in weight, taken out of the fire when it is of carats 21, neither more nor less. Do thus, know how many carats of gold there are in the said 60 ounces which you put in fire before, and multiply 60 times 16, they make 960, and so many carats was the gold that you to put in fire before, that is, carats 960. Now if you want to know how much will become in weight, then divide carats 960 by 21, because you want it to become of carats 21, from which comes 45 and $^5/_7$, and they are ounces. And we shall say that the said 60 ounces which you put in fire at carats 16 per ounce, will become, when taken out of the fire, ounces 45 and $^5/_7$ of an ounce, and will be of 21 carats per ounce. And it is done.

This is followed in **M**+**F** by a strictly analogous problem (absent from **V**) where the resulting gold is requested to be of 24 carats. As we see, there is no reference to a general rule (for instance, the inverse rule of three, which would be fully adequate, cf. note 35); instead, the reasons for the single calculational steps are made clear.

After these two comes a mathematically simple problem of mixing, shared with **V** (pp. 338, 452):

> I have 7 ounces of gold, which is at carats $19\,^1/_2$ per ounce. And I have 9 ounces of it which is of carats 20 and $^1/_4$ per ounce. And I have 16 ounces of it which is of carats 21 and $^2/_3$ per ounce. And I also have 20 ounces of gold of carats 23 $^3/_4$ per ounce. Now I want to fuse all these four golds together and make an ingot of them, thus mixed together. Say me how much this whole ingot will be in weight, and of how many carats of gold per ounce it will turn out to be precisely. Do thus, firstly know how many carats of gold you have in the first 7 ounces, which is of $19\,^1/_2$ carats per ounce. And multiply 7 times $19\,^1/_2$ carats, which makes 136 and $^1/_2$, that is, which are carats. And we shall say that in the said 7 ounces there are $136\,^1/_2$ carats. [similarly for the other golds]. Now join together all these carats, that is, carats $136\,^1/_2$ and carats $182\,^1/_4$ and carats $346\,^2/_3$ and carats

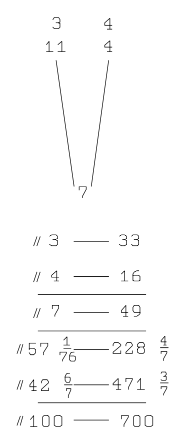
> 475, which in total are carats 1140 and $\frac{5}{12}$ of a carat. Now similarly join together all the gold, that is, ounces 7 and ounces 9 and ounces 16 and ounces 20, which in total are ounces 52. Now divide all these carats, that is, 1140 and $\frac{5}{12}$, by 52, from which comes carats 21 and $\frac{581}{624}$, which is well over $\frac{3}{4}$. And we shall say that this whole ingot will be ounces 52, and of carats 21 and $\frac{3}{4}$. And it is done.

In the end, **V** adds

> And thus all the similar computations are done. And if you might want to fuse together of 100 rates of gold and of different rates, then do always by this rule. And you cannot go wrong.

This is evidently the reason to illustrate the principle by means of four different alloys; the compiler of **M**+**V** either has not understood that purpose or, more likely, has found the observation superfluous. Apart from that we may have a look at the rounding (present only in **M**+**F**). Expressed as a decimal fraction, $\frac{581}{624}$ is 0.931..., certainly well above $\frac{3}{4}$, and much closer to 1. We may think the precision to be poor, but it may perhaps have a decent reason. In the coin list, the fineness of gold coins is mostly given with a precision of $\frac{1}{4}$ of a carat. "Well over" $21\frac{3}{4}$ carats thus ensures that the ingot is presented with no more than its actual value, while the closer approximation 22 carats would be fraudulent. Like the *welsche Praktik*, the safe rounding may thus have been inserted with the purpose of adapting the text to the conditions of the market.

The next problem (pp. 339, 453) is of the type referred to in note 3 as a model explaining the principle of the double false position:



> I have bullion which is at *denari* 11 of alloy[58] and bullion which is at *denari* 4 of alloy. Now I want to make a coin that is at *denari* 7 of alloy, neither more nor less, and I want to alloy 100 marks of it. Say me how much I should put of each of these two bullions in these 100 marks so as to get 100 marks at *denari* 7 of alloy. Do thus, say, the alloy which I want to make is at *denari* 7, and the highest bullion I have is at *denari* 11. We shall thus say, from 7 until 11 there is 4. And take marks 4 of the contrary bullion which is at *denari* 4 of alloy. And similarly say, from 7 until 4 diminishes 3, and take marks 3 of the contrary, that is, of the one which is at *denari* 11 of alloy. Now you have alloyed marks 7 of *denari* 7 of alloy. And you have put marks 4 of the bullion which is of *denari* 4 of alloy, and you have put marks 3 of the bullion which is of *denari* 11 of alloy. And we wanted to alloy 100 marks. Therefore multiply 3 times 100

[58] Cf. note 56 for the expression "of alloy". In the present calculations, the *denaro* is taken to be $\frac{1}{12}$ of a mark.

> marks, they make 300 marks, and divide 300 by 7, from which comes 42 and $^6/_7$. And so much is needed of the bullion which is at *denari* 11 of alloy. Now multiply 4 times 100, they make 400, and divide in 7, from which comes marks 57 and $^1/_7$ of mark, and so much is needed of the bullion which is at *denari* 4 of alloy. Now we have alloyed 100 marks of bullion, which is at 7 *denari* of alloy. And we have put 42 marks and $^6/_7$ of bullion that is of 11 *denari* of alloy, and he[59] has put there 57 marks and $^1/_7$ of bullion which is at 4 *denari* of alloy. Now join together marks 42 and $^6/_7$ and marks 57$^1/_7$, which are 100 marks. You have thus alloyed 100 marks of it. And by this rule you can alloy as much of it as you wish. Let us now verify whether we have alloyed well, and it is verified in this way. And say thus, in the said 100 marks that you have alloyed at *denari* 7 of alloy, there enter *denari* 700 of alloy. Now let us see whether we find again the said 700 *denari*. Say thus, we have alloyed and put there marks 42 and $^6/_7$ of a mark at *denari* 11 of alloy, in which there are 471 *denari* and $^3/_7$ of a *denaro*. And you have put there marks 57 and $^1/_7$ of bullion which is at *denari* 4 of alloy per mark, in which there are *denari* 228 and $^4/_7$. Now join these *denari* together, that is, *denari* 471 and $^3/_7$ and *denari* 228$^4/_7$, which in all are *denari* 700. We have thus alloyed well, since we precisely found again the said 700 *denari*. It would have been a pity if we had found more or less.

The accompanying diagram comes from **V** and is absent in **M** and **F**. Another striking difference is that **V** explicitly uses the partnership model for the determination of how much each sort should contribute to the 100 marks (a capital of 7 and a profit of 100 to be shared between partners having invested 4 respectively 3).

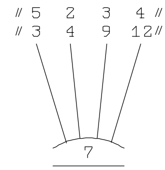
The charming closing remark is an innovation of the compiler of **M**+**F** – **V** closes with a reference to the diagram.

The following two problem are analogues of the ingot-problem and of the one just discussed. In the end (pp. 344, 455) comes this:

> This is a general alloying of four bullions, and in the said way we may alloy gold and silver and copper of whatever fineness they be and however much you may want to make the alloy. And in this way you may alloy however many bullions or coins it may be. And this we shall write hereby, and similarly we shall show it materially by diagram, how the said alloying is made and how the bullions are to be taken.
>
> First say, I have bullion of four kinds. The first is base bullion and is of *denari* 3 of alloy, the second is of *denari* 4 of alloy, and the third is at *denari* 9 of alloy, and the fourth is of *denari* 12 of alloy. And I want to make a coin that is at *denari* 7 of alloy, neither more nor less, and I want to alloy 30 marks of it. Say me how much I shall put into these

---

[59] This sudden shift to the third grammatical person is present in **M** as well as well as **F**. Abbacus masters in general were no great mathematicians, nor great masters of style. Unmotivated jumps into the third person are still rare, but vacillation between “I do”, “you do” and “we do” are pervasive – thus also in the present problem, and in its counterpart in **V**.



30 marks of each of these bullions so that the said 30 marks be alloyed at *denari* 7. Do thus, say, the alloy that I want to make is at *denari* 7, and the best bullion I have is at *denari* 12, therefore say, from 7 to 12 there are 5. And take marks 5 of the contrary, that is, of the basest bullion, which is at *denari* 3 of alloy. And similarly say, from 7 to 3 diminishes 4, and take marks 4 of the contrary bullion, that is, of the best, which is at *denari* 12 of alloy. And further say, from 7 until 9 there are 2, and take marks two of the bullion that is of *denari* 4 of alloy. And similarly say, from 7 to 4 diminishes 3, and take marks 3 of the bullion that is at *denari* 9 of alloy. Now we have alloyed marks 14 of bullion at *denari* 7 of alloy, having put there marks 4 of the bullion that is at *denari* 12 of alloy, and having put there 5 marks of the bullion that is at *denari* 3 of alloy; and having put there marks 3 of the bullion that is at *denari* 9 of alloy, and having put there marks 2 of the bullion that is at *denari* 4 of alloy. Now you have known how much is needed of each of these 4 bullions. And we wanted to alloy 30 marks. Do thus, join together all these marks, that is, 4 and 5 and 3 and 2, they are in all marks 14, and this is the divisor. Now, because you want to alloy 30 marks of it, then multiply 30 times 4, they make 120, and divide 120 by 14, from which comes 8 and $^{4}/_{7}$, and so many marks of fine silver will enter in the said 30 marks. Now multiply 3 times 30, they make 90, and divide 90 by 14, from which comes 6 and $^{3}/_{7}$, and so many marks are needed of the silver which is at *denari* 9 of alloy, that is, marks 6 and $^{3}/_{7}$ of a mark. [...]. Now join together all these marks which you have put together, and know whether they are 30 marks, that is, marks 8 and $^{4}/_{7}$ and marks 6 and $^{3}/_{7}$ and marks 4 and $^{2}/_{7}$ and marks 10 and $^{5}/_{7}$, which in all are marks 30. We have thus alloyed 30 marks of it. And in this way you may make all alloyings.

Explicit liber Tractatus algorismi. Deo gratias.

Evidently, this problem (shared with **V**) is strongly underdetermined; what is offered is a possible solution. Most noteworthy is perhaps the reference to a diagram, which is in neither **M** nor **F** but only in **V** – one of many indications that **V** is Jacopo's original version.

V, once again, has an explicit reference to the use of the partnership model. **M**+**F** has eliminated it, but this time a trace remains, the reference to "the divisor" (cf. above, p. 23).

The final explicit and reference to divine Grace is obviously not present in **V**, which goes on with a collection of 32 mixed problems. But the routine religious tone is not rare in abbacus texts. The commercial environment and its teachers may not have been much influenced by the ecclesiastical prohibition of usury (relative as it was) – but sinning (in this as in so many other environments and situations) did not prevent pious attitudes (cf. above, note 24).

This was one abbacus book among many – and because it seems to be a recast of an original, meant to adapt the text to the school environment, probably as representative as a single specimen can be – apart from the occasional Provençal colouring and from the absence of algebra, included in many abbacus books (as also in Jacopo’s original). However, even this absence reflects its adaptation to the school environment. In any case, we shall return to the algebra contained in **V** and to abbacus algebra in general in chapter IV.

III. Fibonacci and the Fibonacci story

From the references in the preceding chapter to analogues in earlier sources it is obvious that abbacus mathematics had roots in preceding mathematical cultures. Since Italian traders had direct interaction with neither Indian nor Chinese mathematics, the most important inspiration must have come from the Arabic world. There are admittedly some borrowings from Latin post-agrimensor *geometry*, and also some influence from the Byzantine world, to which we shall return; but the overarching importance of the Arabic influence is obvious already from the first part of the standard curriculum of the abbacus school: the teaching of the Hindu-Arabic numerals and their use.

It is widely claimed in popularizations (in print and on the web) that Fibonacci was the one who brought the Hindu-Arabic numerals to Europe.[60] Historians of mathematics know better – after all, Jacopo's title *Tractatus algorismi* refers to al-Khwārizmī's introduction to the topic, translated as *Dixit algorismi* rather early in the 12th century.[61] Even historians of mathematics, however, have tended to believe that abbacus mathematics descends from Fibonacci's writings. Thus, Elisabetta Ulivi, one of best scholars in the field, explained in [2004: 44] that

> the name "abbacus school" designates those secondary-level schools that were essentially dedicated to practical arithmetic and geometry and were in the tradition of Leonardo Pisano's *Liber abbaci* and *Practica geometriae*,

and in [2002a: 10] that *libri d'abbaco*

> were written in the vernaculars of the various regions, often in Tuscan vernacular, taking as their models the two important works of Leonardo Pisano, the *Liber abaci* and the *Practica geometriae*.

---

[60] Cf. this (which even mistakes Europe for "the world"):

> Fibonacci is considered to be one of the most talented mathematicians of the Middle Ages. Few people realize that it was Fibonacci that gave the world the decimal number system (Hindu-Arabic numbering system), which replaced the Roman numeral system.

(https://www.thoughtco.com/leonardo-pisano-fibonacci-biography-2312397, accessed 3 February 2020). It would not be difficult to put together a whole collar of idiocies, but this single pearl should suffice.

[61] According to André Allard [1992: xv], the manuscript of a descendant treatise was copied in Toledo around 1143. [Folkerts 1997] contains a critical edition with German translation of the treatise itself.

These are comparatively weak statements – after all, "in the tradition" and "take as models" are rather open claims. Much stronger was the assertion of Van Egmond [1980: 7] that all abbacus writings "can be regarded as [...] direct descendants of Leonardo's book", repeated [Van Egmond 2008: 303] in the statement that the "*trattati o libri d'abbaco* [were] clearly modeled after LEONARDO PISANO'S *Liber abbaci* of 1202". Raffaella Franci and Laura Toti Rigatelli said in [1985: 28] that "the abacus schools had risen to vulgarize, among the merchants, Leonardo's mathematical works", yet adding cautiously on p. 45 that

> in Florence, in the 14th century, at least two algebraic traditions coexisted. One of them was inspired by Leonardo of Pisa and was improved by Biagio the Old and Antonio de' Mazzinghi, the other, the beginning of which is unknown until now, has Gerardi [i.e., the above-mentioned Paolo Gherardi / JH] as its first exponent.

Arrighi [1987: 10] goes further in this direction, suspecting Paolo Gherardi's *Libro di ragioni* as well as the *Liber habaci – in toto*, not only Gherardi's *algebra* – to be either re-elaborations or translations of French (that is, Provençal) writings (there is no algebra in the *Liber habaci*).

The *Liber abbaci*, the autobiography, and the meaning of the title

As we shall see in the next chapter, this is a perspicacious observation. First, however, we shall take a closer look at Fibonacci's work – very famous, but not rarely misrepresented, for which reason this closer view needs to be quite extensive.

Nine complete or fairly complete manuscripts survive, listed in [Giusti 2020: xxix–xxxi]. The full edition made by Boncompagni in [1857] was based on a single manuscript from the 14th century[62]. The English translation made by Laurence Sigler and published in [2002] was based on this edition,[63] and Ji Zhigang's Chinese translation on that of Sigler. [Germano & Rozza 2019] is the first volume of an intended complete critical edition with accompanying Italian translation (so far containing only 5% of the complete text), while [Giusti 2020] (published when most of the draft for the present chapter was written) is a full critical edition. In what follows, if no other information is given, references to the *Liber abbaci* indicate the page numbers in [Boncompagni 1857] and [Giusti 2020] in the format [B*m*;G*n*], as already explained in note 30.

The *Liber abbaci* is usually taken to have been written in 1202, and then revised in 1228. All manuscripts containing the beginning of the work give the date 1202 for the

[62] Now Florence, BNC, ms. Conv. Soppr. C. I. 2616. One other manuscript is judged by Guiseppe Germano [Germano & Rozza 2019: 72, 79, 81] as well as Giusti [2020: xxix, xxxi] to be certainly from the 13th century, and two others from the 13th–14th century.

About one of the latter (Vatican, Pal. Lat. 1343) I have wrongly claimed at several occasions that it is incomplete, misled both by the electronic version I used in the Vatican Library reading room and the CD which I bought from the Vatical Library later. Apart from a few lacunae, it is actually complete – only the scan was incomplete. In the meantime the Vatican library has put a full high-quality scan on the web (which I have used when needed), at the address

https://digi.ub.uni-heidelberg.de/diglit/bav_pal_lat_1343?ui_lang=ger

(accessed 2.1.2020). Boncompagni, living in Rome in the Papal State, had studied this manuscript [1852: 32]; I do not know why instead he used a Florentine manuscript for his edition – possible explanations are the lacunae, or the omission of many of the marginal schemes and the sketchy character of many others.

When referring to this manuscript (henceforth $\mathbf{V_F}$), I shall make use of the most recent foliation.

[63] Unfortunately, Sigler died in 1997, leaving his work on floppy disks which were kept so long by the intended publisher without being used that their format was out of date when his widow Judith Sigler Fell claimed them back, one even being lost. What was published was thus the fruit of a recovered incomplete electronic text combined with immense work by Fell on Sigler's preliminary notes for the lost part and the formatting – see [Devlin 2017: 106–114]. Readers of Heinz Lüneburg's ungenerous review [2003] should take this into account (and also be aware that Lüneburg commits blunders more serious than anything he can reproach Sigler).

first edition, which can therefore be relied upon.[64] The precision of 1228 is subject to more doubt.[65] In any case, for my present purpose it is not important whether the year is precisely 1228, so I shall henceforth speak of the second edition as being from 1228.

All known manuscripts descend from the 1228 edition – with one exception. Giusti [2017] has discovered that in the manuscript Florence, Biblioteca Medicea Laurenziana, Ms. Gaddi 36 (henceforth **L**; containing only chapters 12–15), chapter 12 is quite different from the corresponding chapter in the other manuscripts . According to strong internal evidence, it is older. As argued by Giusti, it is likely to represent the original 1202 version; I shall henceforth speak of it as such; as with the "1228 version" with a proviso – here, that we cannot exclude that **L** was actually the result of an intermediate revision of which we know nothing (but see below, p. 84, on an observation that speaks against this possibility). In any case, the copyist appears to have used at first a manuscript of this early version and then, getting access to the revised version (a misfortune for historians!) switsched to that.

According to Giusti [2020: lxxxiii], all surviving manuscripts except chapter 12 of **L** seem to derive from a single archetype ω, since all "show a series of omissions and errors that cannot reasonable be attributed to the author". Comparison of the two version of chapter 12 (taking into account the critical apparatus of both editions) reveals, however, that the large majority of the ω-errors in chapter 12 are also found in **L**.[66] This leads to a different conclusion: Fibonacci conserved a master copy of the 1202 version, and inserted new material into it while removing what had become redundant or what he did not like at second thoughts (we shall encounter an example below on p. 73). All

---

[64] With a minor doubt: in medieval Pisa, the year was counted from the Incarnation, meaning that Pisa's year 1202 – likely to be the one Fibonacci thought of – began at Julian 25 March 1201 and ended at Julian 24 March 1202. "1202" is thus quite likely to be 1201.

[65] Two of the manuscripts of certain or possible 13th-century origin (and one more) state that "here begins the *Liber abbaci* composed by Leonardo the Pisan of the sons of Bonaccio and corrected by the same in 28", while the last manuscript of possible 13th-century date has "... corrected by the same in the year 1228" [Giusti 2020: xvii], which could be a copyist's interpretation of the shorter variant. No other manuscript gives discordant information, but indicating a year by *XXVIII* or *28* alone would be quite unusual.

[66] Leaving out the 32 cases where the passage in question in the 1228 version has no counterpart in **L**, there are 71 agreements or near-agreement between **L** and ω and only 19 agreements of **L** with the corrected 1228-text (many of which could be produced by an alert copyist discovering the mistake in his original).

manuscripts were made from this evolving master copy.[67]

In the introduction to the *Liber abbaci* (following a dedication to Michael Scotus, and thus probably the original prologue from 1202) we read the following:[68]

> After my father's appointment by his homeland [Pisa] as state official in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and, in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days. There, following my introduction, as a consequence of marvelous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business, I pursued my study in depth and learned the give-and-take of disputation.

From this, two things may be derived. Firstly, that Fibonacci cannot have been born much later than 1170 – he must have had time to be active before 1202 as a widely travelling merchant. Secondly, that he learned his mathematics not only in Béjaïa ("Bugia") in present-day Algeria but also in Syria, Egypt and Sicily (strongly connected to the Arabic world), and further in Constantinople ("Greece") and Provence.

From the *incipit* (see note 65) we learn that a fuller version of the surname "Fibonacci", namely "of the Bonaccio family", was already in use in the 13th century ("Leonardo Pisano" could obviously only be used to identify him outside Pisa). We also learn that the book was already supposed by then to deal with "abbacus" (Grimm's "study of calculation" renders *studio abbaci*). Fibonacci himself thought so too – in his *Pratica geometrie* [ed. Boncompagni 1862: 9, 24, 81, 148] he speaks about it as "the abbacus book in a larger manner", "our book on abbacus", and the "abbacus book". We cannot be sure, however, that the word meant the same to Fibonacci as to writers of the generations where the abbacus school had been established (and to Grimm). We have no traces of the terms *abacus/abbacus* except as referring to the reckoning board before Fibonacci, so he may well have grabbed it for his specific purpose.[69] The next time

---

[67] As we shall see in note 464, even Fibonacci's *Flos* and *Liber quadratorum* appear to have existed as master copies from which copies (dedicated to different patrons) were made.

[68] [B1;G4] – but Boncompagni, following his manuscript (which diverges from the others on this account), gives a text that only refers to Egypt etc. as "places of business", missing that Fibonacci had gone there as a merchant. The pure mathematician Georg Eneström, wishing Fibonacci to belong to his own kind, in [1906] used this as the basis for another brutal attack on Moritz Cantor.

The translation is due to Richard Grimm [1976], who made a critical edition of the passage and offers an extensive commentary.

[69] At least not in Italy. Two manuscripts of a commentary to *Elements* X, probably made by Gerard

it occurs seems indeed to be in a document from 1241 or slightly before stating that the Pisa authorities had assigned a pension to Fibonacci because of his “abbacus estimations and accounting” (*abbacandes aestimationes et rationibus*) in the service of the city and its authorities [Bonaini 1857: 241].[70] It is quite likely that the authorities had taken over Fibonacci’s word.

The *Liber abbaci* does not help much, but is gives some hints. Firstly, chapter 13 is stated [B318;G499] to deal with “the *elchatain* rule [the double false position], by which almost all abbacus questions will be solved”. The phrase “abbacus questions” points back to [B166;G285], where it describes the contents of chapter 12: mixed, largely recreational problems (see below, p. 78) – similar in genre to Jacopo’s collection of mixed problems (certainly in neither details nor level).

Secondly, a root extraction “in the abbacus way” is explained on pp. 53*f* as a combination of an algorithm for finding the integer part of the root of a multi-digit number (seemingly a transfer to paper of an algorithm developed for a dustboard where deletion and rewriting is possible) with the method for finding what later abbacus books speak of as “the closest root”. The example is $\sqrt{743}$, and the result is $27\,{}^{14}/_{(2\cdot 27)}$. Here, “abbacus way” clearly refers to a specific method and not to practical computation in general; the two methods were well known in the Maghreb – they are described together in al-Qalasādī’s *Kašf* [ed., trans. Souissi 1988: 57–60] (of later date, but an exposition of inherited knowledge). The best guess is therefore that Fibonacci’s “abbacus” was meant as an equivalent of Arabic *muʿāmalāt*.[71] That semantic equivalence, if ever known by anybody except Fibonacci himself, was soon to be forgotten in Italy; but abbacus

---

of Cremona, characterize it as *abbacus* [Busard 1997: 31]. The manuscripts are from the 13th and 14th centuries, but they may repeat a 12th-century term. There is evidently no reason to connect this to Fibonacci’s usage.

[70] This formulation should kill the hypothesis that he got it for holding an abbacus school. The title *magister* given to him in the document proves nothing, it was used too widely about anyone who was the “master” of others, whether of students, of serfs, of subordinate officials, artisans of a craft, etc. See [Du Cange 1883: V, 168–173] and [Niermeyer 1976: 624*f*]. If instead of dictionaries we trust Fibonacci’s own use of the term *magistraliter* (see imminently), it may simply designate him as a learned man, which he certainly was.

Evidently, *aestimationes* cannot refer to teaching. It should rather make us think of urban surveying for the city, which a number of 14th–15th-century abbacus masters are also known to have practised – three examples are mentioned by Ulivi [2004: 52*f*, 58]. *Rationes* is probably “accounting”, cf. present-day Italian *ragioneria* [also *ragioneria di stato*].

[71] In general usage “mutual, business, social relations”, whence “*muʿāmalāt* calculation” designates the mathematics of practical life – my thanks to Ulrich Rebstock for suggesting a more elaborate explanation than I had originally offered.

mathematics remained a close relative of *muʿāmalāt* mathematics.

Some general characteristics of Fibonacci's project

If "abbacus" was really meant to render *muʿāmalāt*, then Fibonacci's *Liber abbaci* is not only an indubitable parallel to the *Liber mahameleth* probably written in al-Andalus before the mid-12th-century and more or less freely translated into Latin by Gundisalvi or in his environment around 1260 (for this, see [Høyrup 2021a: 42–44]); even the title is in a certain sense the same.

The parallel, however, does not depend on whether Fibonacci *meant* "abbacus" to translate *muʿāmalāt*. Both the *Liber mahameleth* and the *Liber abbaci* had as their aim to apply a theoretical perspective on practical arithmetic – representing, in Felix Klein's words and title [1908], *Elementarmathematik vom höheren Standpunkt aus* ("Elementary mathematics from a higher vantage point"). At times, this aim shines clearly through Fibonacci's language, which distinguishes between doing something *secundum vulgi modus* ("according to the way of common people" – for brevity "in the vernacular way"[72]) and similarly, contrasted to doing it *secundum artem*, "according to the art" or *magistraliter*. At times, "the art" is specified as "the art of abbacus", in a way that points in the same way as the above-mentioned "abbacus way" to determine a square root, toward numerical methods used in the Maghreb. At one point [B215;G359], "*magistraliter* according to the same art" follows a few lines after a reference to *nostrum magisterium*, "our teaching" – namely of a numerical method of the same kind. But this instance is clearly not to be generalized, and the regular meaning is different.

Four examples will suffice (there are more). Firstly, on pp. 63*f*, the vernacular way of adding $^1/_3$ and $^1/_4$ is to find a reference magnitude whose $^1/_3$ and $^1/_4$ are integers, and then measure the sum by this same magnitude (cf. the tree-example above, p. 25). The alternative (here not given a name) is to find the sum as $^{1\cdot3+1\cdot4}/_{3\cdot4}$. Secondly [B115; G198], the vernacular way to multiply 5 ß $6\,^6/_{19}$ δ by 13 is to multiply the ß and the δ separately. Thirdly [B127:G219], a composite exchange of money can be done stepwise (the vernacular way) or by means of a scheme for composition of ratios (by art). Fourthly [B364;G563], $4+\sqrt{\sqrt{10}}$ is found first *secundum vulgarem modum*, specified to be *secundum propinquitatem*, "by approximation", $\sqrt{\sqrt{10}}$ being approximated as "less than $1\,^4/_5$"[73]. *Magistraliter*, instead,

$$4+\sqrt{\sqrt{10}} = \sqrt{16+\sqrt{10}+8\sqrt{\sqrt{10}}} = \sqrt{16+\sqrt{10}+\sqrt{\sqrt{40960}}} .$$

---

[72] In the *Pratica geometrie* [ed. Boncompagni 1862: 1] Fibonacci himself indeed explains *vulgaris* as being *quasi laicali more*, "so to speak as laymen do". The counterpart there is represented by "geometrical demonstrations".

[73] Most likely, $\sqrt{10}$ is found to be approximately equal to $3+{}^1/_{2\cdot3} = 3\,^1/_6 = {}^{114}/_{36}$. Next, $\sqrt{114}$ will have been found to be approximately $10+{}^{14}/_{2\cdot10} = 10\,^{14}/_{20}$. Therefore, $\sqrt{\sqrt{10}}$ will be approximately equal to $1\,^{94}/_{120}$, slightly less than $1\,^{96}/_{120} = 1\,^4/_5$.

The latter argument is accompanied by a line diagram, in which a line *abc* is divided into *ab* = 4, *bc* = $\sqrt{\sqrt{10}}$. Since nothing is done with this diagram, it seems to be there just because it belongs with the magisterial way.

Chapter 1 – introducing the Hindu-Arabic numerals

Fibonacci was not the first to teach Latin Europe about the Hindu-Arabic numerals; as we have seen, even abbacus writers like Jacopo were to draw on Sacrobosco, who continued a tradition inaugurated by the Latin translation of al-Khwārizmī's treatise on the topic. However, the foundation for this ascription of honour is obvious, namely that Fibonacci starts by introducing *the reader* to them, first showing them and then indicating (like Jacopo, cf. p. 11) their meaning in terms of what the reader would understand as numbers proper – that is, numbers written with Roman numerals. This is supplemented by a description of the system of finger-reckoning, followed by a corresponding depiction (absent from some manuscripts). In the end of the first chapter comes tables for addition and multiplication.

Chapter 2 – multiplication of integers

Chapter 2 [B7;G13] teaches the multiplication of integers with two or more digits; here, the preceding instruction in finger reckoning comes to serve when intermediate partial products are to be remembered. In the end comes, first, explanation of a purely mental method, and then [B.19;G31] a presentation of another method, "very praiseworthy in particular for the multiplication of large numbers".[74] This latter method was to be known in abbacus writings as multiplication *a scacchiera*, "in chess-board", and even Fibonacci refers to a chess-board. The method is close to what was known also in the Maghreb, and asks for addition in diagonal (apart from that, the basic principle is the one we use today). It is a paper algorithm, different from those inspired by the use of a dust- or clayboard, and as such the only thing in this chapter recalling later abbacus writings (but as we see a method which Fibonacci speaks of as already existing, and which for instance Giovanni de' Danti [ed. Arrighi 1985: 14] was to speak of in 1370 as *arte vecchia*, "old art".

[74] This passage is misplaced in Boncompagni's manuscript and hence also in his edition.

Chapters 3–4 – addition and subtraction

Chapter 3 [B18;G33] teaches the addition of multi-digit integers – including numbers of *libri*, *soldi* and *denari* arranged in columns. The ultrashort chapter 4 [B22;G39] teaches subtraction of a smaller from a larger integer (this time without monetary or metrological applications).

Chapter 5 – division

Chapter 5 [B23;G43] is stated to deal with division involving integers, but also covers auxiliary matters we would not automatically include under that heading – first of all the writing of fractions by means of a fraction line, and the "ascending continued fractions", like "four seventh, and one half of a seventh", written $\frac{1\ \ 4}{2\ \ 7}$.[75] The fraction line is a 12th-century invention of Maghreb mathematicians, already used in the *Liber mahameleth;* the notation for ascending continued fractions, also a Maghreb invention, is later and not yet used in the *Liber mahameleth*, which instead uses words to render these composite fractions, characteristic of Arabic mathematics.[76] Fractions of this kind can be continued *ad libitum* – Fibonacci elsewhere goes until 10 levels.

Ascending continued fractions are used throughout the *Liber abbaci* as well as in Fibonacci's other writings. At the present point[B23;G44], Fibonacci introduces several other notations (whether his own inventions or not is not clear):[77]

$$\frac{2\ \ 4\ \ 6\ \ 8}{2\ \ 5\ \ 7\ \ 9}\circ \text{ meaning } \frac{8}{9}+\frac{6}{7}\cdot\frac{8}{9}+\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}$$

$$\circ\frac{2\ \ 4\ \ 6\ \ 8}{3\ \ 5\ \ 7\ \ 9} \text{ meaning } \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}$$

$$\frac{\underline{1}\ \ \underline{1}\ \ \underline{1}\ \ 5}{5\ \ 4\ \ 3\ \ 9} \text{ meaning } \frac{5}{9}+\frac{1}{9}\cdot\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)$$

None of these seems very useful, and Fibonacci agrees in practice (using them very rarely[78]).

Tabulated divisions follow [B25*f*;G45–47], with divisors from $i = 2$ to 13 and dividends 1 to $10i$ – for $i \leq 4$ with indicated remainder, for higher i with omission of dividends that

75 Fibonacci takes over the Arabic right-to-left writing direction for numbers. In mixed numbers he also systematically writes the fraction to the left (in all his works). In order to facilitate the reading, in my translations I shall follow current habits, thus changing Fibonacci's $^1/_2 2$ into $2\,^1/_2$ – and in order to distinguish fractions from ratios understood as relations between two numbers, I shall write fractions with a slash. *Within* ascending continued fractions, for which no current habit exists, I shall follow Fibonacci.

76 Explained for example by al-Qalasādī in the *Kašf* [ed., trans. Souissi 1988: 48*f*]. The appearance of ascending continued fractions in various mathematical cultures until Fibonacci is described in [Høyrup 1990a], the story from Fibonacci until Christopher Clavius is told in [Vogel 1982].

77 In the Boncompagni edition, the small circle is spoken of but actually omitted in the second line; $\mathbf{V_F}$ (fol. 11^{v}) inverts the positions of the circles in the two first lines, but they seem to have been forgotten at first by the scribe and then pressed in afterwards. Later on, $\mathbf{V_F}$ (e.g., fol. 115^{r}) agrees with the other manuscripts [B312;G491].

78 I have observed the first type on p. [B61;G101] (which teaches how to multiply such fractions), the second on p. [B312;G491], [B313;G491] and [B339;G531]. [B77;G130], explaining how to add and subtract such numbers, speaks about both types. By mistake they are not written in Boncompagni's manuscript; in the others they are.

do not divide. Next it is explained how this serves for divisions of larger dividends by single-digit and then two-digit divisors. “With enough of division by two-digit numbers” [B36;G62], the way to divide by composite numbers by means of factorization follows.[79] Here the ascending continued fractions come to serve. For instance, on p. [B38*f*;G65], $^1/_{951}$ is shown to be $\frac{1\ \ \ 0}{3\ 317}$ (317 being prime – Fibonacci shows it has no adequate divisors, trying all primes below $\sqrt{317}$). When 749 has to be divided by 75 [B41; G70], at first $^1/_{75}$ is expressed as $\frac{1\ 0\ 0}{3\ 5\ 5}$. Division of 749 by 3 yields 249, with remainder 2; division of 249 by 5 yields 49, remainder 4; division of 49 by 5 yields 9, remainder 4. Therefore, the total result is $\frac{2\ 4\ 4}{3\ 5\ 5}9$.[80] This use of factorization was also familiar in the Maghreb – the same procedure is shown by al-Qalasādī in the *Kašf* [ed., trans. Souissi 1988: 42].

The chapter ends with a procedure for division by three-digit prime numbers that cannot be factorized – at this point Fibonacci speaks of them as *numeri asham* instead of *hasam*.

[79] The factorization of a composite number is spoken of as its “rule” (*regula*). Correspondingly it is explained [B30;G53] that non-composite numbers in Arabic are called *hasam*. This is indeed the technical-mathematical meaning of *aṣamm*, “deaf” in the Maghreb, though mostly not elsewhere in the Arabic world [Souissi 1968: 220*f*; Saidan 1974: 368]; for a 12th-century exception from Baghdād that suggests the Maghreb usage may have been more widespread than we know, see [Rebstock 1992: 130 n. 194]. In “Greek” (apparently contemporary, Byzantine Greek) they are *coris canon* (which must stand for χωρίς κανών, without declension), “We however call it ‘without rules [*sine regulis*]’” – an evident calque on the Greek expression, which according to Fibonacci was already current within an environment which he here characterized as “we” (when introducing a new term Fibonacci uses the future tense, “we shall call it”). The following lines show that Fibonacci also knows the Euclidean terms for prime as well as composite numbers.

[80] Boncompagni, and no doubt his manuscript, writes $\frac{244}{3559}$, without the necessary spaces. His text errs regularly on this account, the 14th-century copyist was obviously not too familiar with the notation for ascending continued fractions, and also did not follow the computations systematically. $\mathbf{V_F}$ (fol. 17^r) is correct.

Chapter 6 – multiplication of mixed numbers

Chapter 6 [B47;G79] deals with the multiplication of mixed numbers. “Mixed numbers” may contain an integer and one or several fractions – and here a fraction may be simple or an ascending continued fraction, or even the first type “with circle” (above, p. 67); for our present purpose there is no need to go into detail, since there is no specific connection to what can be found in abbacus books.

Chapter 7 – addition, subtraction and division of mixed numbers

Chapter 7 [B63;G107] takes up the addition, subtraction and division of mixed numbers, and the reduction of several fractions to one. On the whole, there is once again no reason to go into details – yet with one exception.

This exception concerns part 7 of the chapter, dealing with the disgregation of fractions into aliquot parts (also known as “unit fractions”). As well known, aliquot parts, including $^2/_3$, had been the standard way to express fractional quantities in Pharaonic Egyptian mathematics, and they were taken over in ancient Greek practical arithmetic, and hence also in the administration of the Byzantine Empire. For a while, the administration of Syria and Egypt was continued in Greek after the Islamic conquest, and after the switch to Arabic as administrative language in the outgoing seventh century [Robinson 2010: 209], accounting techniques including the writing of fractions may well have survived. Then, around 1100, the Norman rulers of Sicily adopted Egyptian administrative practices [Johns 2002]. So, among the places where Fibonacci states to have learned, at least Egypt and Sicily are possible sources for his interest in aliquot parts. Much more likely, however, is “Greece” (i.e., Byzantium). Six problems in the *Liber abbaci* either deal with something supposed to have taken place in Constantinople, or they are stated to have been presented to Fibonacci there. In three if them, the data contain no fractions [B190,274,276; G324,440,443]. On p. [B188;G319], the data contain these fractions: $^1/_9\ \ ^1/_3$ – $^1/_9\ \ ^1/_3$ – $^1/_2\ \ ^1/_7$ $^4/_9\ \ ^1/_2$. On p. [B203;G340], the data make use of the fractions $^1/_8$ – $^1/_{10}$. On p. [B249;G405], finally, we find $^1/_5\ \ ^2/_3$ – $^1/_{480}\ \ ^1/_6\ \ ^2/_3$ – $^1/_{688}\ \ ^1/_6\ \ ^2/_3$ – $^1/_{420}\ \ ^1/_7\ \ ^2/_3$ – $^1/_{810}\ \ ^1/_{27}\ \ ^1/_{10}\ \ ^2/_3$. If we remember that $^2/_3$ belonged since Pharaonic times to the category, this is quite striking. All in all, Byzantium is most likely to be the location where Fibonacci learned to find aliquot parts interesting. However that may be, *Egyptian-style fractions play no role in abbacus writings*, and there is no reason to pursue the matter.

Chapter 8 – the rule of three

Chapter 8 is told to deal with "finding the price of goods in the major way" (*per maiorem guisam*). In abbacus terminology, that way would be spoken of as the "rule of three", but Fibonacci does not use this expression. Instead he explains [B83;G141] that

> In all business, four proportional numbers are always found, of which three are known, and the last one unknown: the first of these three known numbers is the number in which any merchandise is sold, be it a number, a weight or a measure. A number may be a hundred hides, a hundred goatskins, and similarly; weight, either *cantari*,[81] or quintals, or pounds, or ounces and such; measures, *metri*[82] of oil [...]. The second is the price of that sale, that is, its first numbers, whether it be any quantity of *denari*, of bezants, of *tareni*, or some other current coin. The third how much is the quantity of merchandise sold, whose price, that is, the fourth number, is unknown.

Fibonacci goes on with the possibility that it is the quantity sold that is unknown while the price is known. He then explains how to insert the numbers in a rectangular scheme and perform a cross-multiplication followed by a division.

For comparison, we may look at how al-Karajī deals with the same matter in *al-Kāfī* [ed. Hochheim 1878: II, 16] (my translation from Hochheim's German):

> Know that in questions about commercial transactions you must have four magnitudes, which are pairwise similar, the price, the measure, the purchase amount and the quantity.
>
> The price is the value of a measuring unit that is used in trade [...].
>
> [...] Of these four magnitudes, three are always known, and one is unknown. [...].

As we see, the reference to four, not three magnitudes is shared. But that is as far as the similarity goes, al-Karajī is obviously not Fibonacci's source (he also does not refer to a scheme). Other Arabic scholarly authors are just as different from what we see in the *Liber abbaci*. Not least the scheme, emulating the writing on a dust- or clayboard, suggests that Fibonacci describes in his own terms what he has seen in practical use, and does not copy from a book.

After this introduction follows a large number of examples, sometimes complicated by the need to perform a preliminary conversion of units (in the present context made separately). First merchandise versus money is dealt with, then exchange of one coin (or weight of non-coined silver) into another one (involving the subdivision of both in £, ß

---

81 Borrowed from the Arabic unit *qintar*, equal to 100 *ratl*, whence the normal translation "hundredweight". In Italy varying between 100 and 250 pounds; the *ratl*, on its part, was borrowed as a *rotulo*.

82 From the Greek *metron*, a capacity measure used in the Southern and Central Italy (which were under strong Byzantine influence); depending on the locality between 10 and 30 litres.

and δ[83]). Length (specified on p. [B111;G191], as could be expected, to concern cloth) and other particular measures are dealt with in a number of problems, followed by a short section about partnership (the “partnership rule” for proportional sharing being indeed, as we remember, an application of the rule of three “in parallel”). The last part of the chapter deals with difficult metrological conversions – first from *rotuli*[84] of Pisan *cantari* to light pounds, given that one *cantaro* equals 100 *rotuli* but also 158 pounds.

[83] For example [B105;G180],

> A Pisan £, that is ß 20, is worth £ 1 Bolognese, and further 54 [meaning 54 δ = $4\frac{1}{2}$ ß], that is, ß 20 Pisan are worth ß $24\frac{1}{2}$ Bolognese. Therefore [ß] 20 Pisan are worth [ß] $24\frac{1}{2}$ Bolognese, and £ 20 Pisan are worth £ $24\frac{1}{2}$ Bolognese. And it is asked how many Bolognese are to be had for $11\frac{1}{4}$ Pisan. [...]

[84] Cf. above, note 81.

Chapter 9 – barter

In the Boncompagni edition and in most manuscripts, chapter 9 begins [B118;G205] with the words *Incipit capitulum nonum de baractis mercium atque earum similium*, "The ninth chapter about barter and similar matters begins". Before that, however, manuscript $\mathbf{V_F}$ (fol. 47^r) has the words *Hic incipit magister castellanus*, "Here begins the Castilian master". Already Boncompagni [1852: 38], who noticed this passage, stated that no other manuscript of the *Liber abbaci* known to him contains this passage; on the other hand, $\mathbf{V_F}$ is one of the oldest manuscripts and therefore to be taken seriously. Moreover there are indications in the text of books I–IV [Germano & Rozza 2019: 126] that this manuscript might descend from an early stage of what they call an "archetype in motion" – the "evolving master copy" argued for above, p. 59;[85] finally, it is hardly imaginable that a copyist should insert this reference to a Castilian master on his own initiative. There are thus fair reasons to accept the claim that this chapter (or its initial part) is copied from an older Castilian treatise; that Fibonacci should have deleted the reference during further work on the manuscript fits his habit of hiding his sources even he copies verbatim.[86]

The topic of the chapter is thus barter, together with mathematically analogous questions. Barter was not uncommon in late medieval trade. Cash currency was often in short supply, and banks could not step in in all places and in all kinds of trade with alternative financial instruments. A seller might therefore have to accept that a buyer paid in kind. Obviously, that might be inconvenient, and therefore the value of the merchandise serving as payment might be reduced compared to what it would be in cash trade. *How much lower*, however, would be a question of commercial power, of transport costs if the payment had to be realized in a different market, etc. Such questions, however, could and cannot be dealt with in the abstract, and Fibonacci (as many abbacus writers) would not mention them.[87] Instead, they give rules for "just" barter, justice being tacitly

---

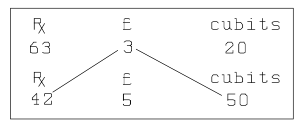
[85] On pp. 109*f*, Germano and Rozza express the opposite opinion, stating that all manuscripts used for the edition seem to depend on a single manuscript, derived from Fibonacci's autograph yet already secondary. They base this claim on number of points where all manuscripts agree on a formulation which the editors cannot believe Fibonacci would have made. None of them are convincing – seeing them as errors merely reflects the failing understanding of the editors; moreover, not all of them are shared. See [Høyrup, 1921b], written before the appearance of [Giusti 2020]; already comparison of Germano's and Rozza's edition with $\mathbf{V_F}$ reveals the shortcomings.

[86] The only exceptions to this rule are Euclid (often mentioned, cf. [Folkerts 2004]) and a single reference to Ptolemy and Aḥmad ibn Yūsuf on p. [B119;G206]. As to verbatim copying, which can only be proved when we happen to possess and to recognize the source, see [Høyrup 2001: 93] on the use of the *Liber mensurationum* in the *Pratica geometrie*, and note 142 below.

[87] Others do; the method is then that the seller, instead of accepting the merchandise of the buyer at reduced price, augments that of his own if payment in kind is offered – cf. [Pacioli 1494: 161^r*ff*] and, in general, [Tropfke/Vogel et al 1980: 521]. There is no guarantee that this corresponded

determined by the values of the respective merchandises *in the actual trading situation*.

The method is the one known elsewhere as the “rule of five”. Fibonacci, not giving a name to the rule of three, obviously gives none here. Instead, he offers a “universal rule” [B118;G205] for how to expand the calculational scheme so as to cover this more complicated case. It is illustrated by an example: 20 cubits of cloth are worth 3 £, and 42 *rotuli* of cotton are worth 5 £ (both kinds of £ specified as Pisan). What then is to be given for 50 cubits of cloth? The numbers are inscribed within a rectangular frame as shown in the diagram, and it is explained that the ratio of the cubits of cloth to that of *rotuli* of cotton is composed from the ratios 20 : 3 and 5 : 42. $5\cdot20$ cubits of cloth are indeed worth $5\cdot3$ £, and $3\cdot42$ *rotuli* of cotton are worth $3\cdot5$ £. In consequence $5\cdot20 = 100$ cubits are worth $3\cdot42 = 126$ *rotuli*. By the rule of three we get that 50 cubits are worth ${}^{50\cdot126}/_{100}$ *rotuli*. Following backwards the calculation we see that 50 cubits are worth ${}^{50\cdot3\cdot42}/_{5\cdot20} = 63$ *rotuli*, which is what is expressed in the scheme. In the end, Fibonacci points out that this composition of ratios is taught in the *Almagest* and by “Ametus filius” (Aḥmad ibn Yūsuf), and inverts the calculation, finding that 126 *rotuli* of cotton are worth 50 cubits of cloth. A number of examples follow, some of which concern the exchange of two coins, both of which known with respect to the same third coin, and a final example concerning five different coins “in cascade”. This is explained [B127;G219] to be made “in the vernacular way” by stepwise calculation, and according to art by composition of ratios (with extension of the preceding scheme).



A second part of the chapter deals with the exchange of bullions of different fineness, which leads to analogous calculations, and a third with seemingly recreational problems with the same mathematical structure and dealt with by means of similar schemes – for example [B132;G228], *a* horses eat *b sextarii* of grain in *c* days; in how many days will *p* horses eat *q sextarii*? This latter problem is used as the basis for a theoretical investigation of changes in what is given within composite ratios.

---

better to what real merchants would do: after all, the inconvenience of receiving payment in kind depended on the merchandise received. Medieval trade corresponded no better than its modern counterpart to the frictionless idealizations of neoclassical economics.

Antonio de’ Mazzinghi (presented below, p. 230) shows in his *Fioretti* [ed. Arrighi 1967a: 31, 33] how this could be used to construct intriguing mathematical problems of the second degree: for instance, two merchants exchange wool and cloth, both augmenting the price of their merchandise.

Chapter 10 – partnership

Chapter 10 is dedicated to partnerships. Proportional sharing (within a partnership or any similar structure) asks for nothing but addition of all shares followed by application of the rule of three "in parallel", and Fibonacci teaches how to organize even this calculation within a scheme, first [B135;G235] for partnerships with two participants, then [B139;G242] three, then [B142;G246] four. In the end comes a problem ($^1/_3$ and $^1/_4$ and $^1/_5$ and $^1/_6$ of something add up to 60), where the partnership structure is used explicitly as an abstract model (cf. note 26 and preceding text). Such explicit use of familiar structures (either commercial or recreational) as general models recurs repeatedly in the *Liber abbaci*.

Chapter 11 – alloying

Chapter 11 deals with alloying, and starts [B143;G249] by explaining the meaning of the expression "I have coin at so and so many ounces, let us say at two", namely that "in a pound of that coin we understand ounces 2 of silver to be contained". We have already encountered this use of the first person singular in problems about money exchange and alloying in Jacopo's *Tractatus* above (pp. 27 and 51), and the usage turns up not only in many other abbacus books [Høyrup 2019a: 787*f*] but also in a Castilian merchants' manual,[88] and further in a Byzantine arithmetic book from the early 14th (Ψηφηφορικὰ ζητήματα καὶ προβλήματα, "Calculation Questions and Problems", [ed. Vogel 1968: 21–27]), which appears not to be much influenced by the nascent Italian tradition. In this Byzantine book, the first person singular is also used for other problem types – mostly but not exclusively such as have to do with payment in gold coin. Fabio Acerbi (personal communication, 7 January 2019) tells me that other Byzantine practical arithmetics do as much. It seems most likely that the habit goes back to Byzantine money changers.

In his alloying *problems* Fibonacci does not use it, but at a later moment [B160;295], when he reduces a problem of type "lazy worker" to an alloying problem, he introduces the latter by the words "I have bullion at 26 and at 37 [...]"/*habeo monetam ad 26 et ad 37* [...]. As we shall see in a monent he also uses it when applying the model of simple alloying in sub-procedures. He thus expects the reader to recognize through this phrase the use of the alligation model – implying that the phrase was already in use.

Fibonacci divides the exposition in seven *differentiae*. In the first [B144;G250] it is asked how much coin of given fineness can be produced from a given amount of pure silver or copper by addition of copper respectively pure silver. In the second, for example [B145;G251], 7 pounds at 5 ounces is to be brought to 2 ounces by addition of how much copper? In the third, for example [B147;G255], how much pure silver has to be added to 9 pounds of silver at 2 ounces in order to bring it to 5 ounces per pound? In the fourth, for example [B148;G257], how much silver at 5 ounces has to be mixed with how much copper so as to produce 30 pounds at 2 ounces? And also [B149;G259], what results if several monies of given fineness are mixed in given proportion? In the fifth [B150;G260], how to obtain a given quantity of silver of, say, 7 ounces, from equal quantities of silvers of 4 and 3 ounces, respectively, by adding pure silver.

The sixth *differentia* deals with the problem type discussed in note 3: first [B151;G261] how to obtain silver at 5 ounces from silvers at 2 and 9 ounces, respectively, then going on with mixings of three, four and seven kinds of silver; these are obviously indeterminate problems, which allows Fibonacci to make one or more free choices, for instance as done in Jacopo's "general alloying" (above, p. 53). All of these indeterminate problems make use of the "I have" clause when a partial procedure is performed as a simple alloying.

[88] "And if they should say to you, I have three kinds of silver ...". Real Academia Española, Ms. 155, *De arismetica* fol. 151ʳ [Caunedo del Potro 2004: 45].

The seventh *differentia*, finally, shows 11 examples of how the alloying model can be used in other situations. With two exceptions (regarding simple numerical variations), all use “I have” to signal the application of the alloying model.

Chapter 12 – mixed “abbacus questions”

As quoted above (p. 61), chapter 12 is presented [B166;G285] as containing “abbacus questions”. The initial table of contents [B2;G5] instead says “on the solution of many problems that have been posed and which we call rambling[89]”. The collection of mixed problems is, however, made less rambling by being organized in nine parts, in as far as possible collecting structurally similar problems. Since it can be seen to be close to defining what Fibonacci may have meant by “abbacus”, it deserves fairly close attention.

12.1, summation of series

Part 12.1 [B166;G285] deals with “collections of numbers”, that is, with the summation of arithmetical series or of ascending squares.[90] First the principles are set forth abstractly, then they are applied in problems of pursuit (one traveller moving uniformly, the other with arithmetically increasing speed).

12.2, “proportions of numbers”

Part 12.2 [B169;G290] is about “proportions of numbers”, where *proportio* mainly stands for *ratio* in the strict sense of a relation between two numbers a and b (thus not for the fraction $^{a}/_{b}$); at times is stands instead for what we would call a proportion, that is, the identity of two ratios, but often Fibonacci here speaks about “proportional numbers”, *numeri proportionales*.[91] At first, the naming of ratios by the outcome of the corresponding division is explained, then [B170;G290] the finding of a fourth proportional – “if it is asked about 6, to which number it has the same ratio as 3 to 5”. The result is stated to be (5·6)÷3. This is Fibonacci’s basic way, unnamed but magisterial. Then follows:

> In our vernacular usage it is in fact habitual to state this same question in a different way: namely, if 3 were 5, what would then 6 be. And when it is stated like this, 5 is similarly multiplied by 6, and the outcome is divided by 3.

After this “counterfactual” formulation of the rule of three follows, as “another way about

---

[89] *erraticas*. Sigler [2002: 16] translates “On the solutions to many posed problems that we call false position”, but in the chapter itself *erraticus* is only to be found in the heading of section 7, which makes no use of a false position. *Erraticus* is thus no name for the single false position which, as we have seen on p. 25, Fibonacci does not really like (however much he makes use of it).

[90] The latter topic [B167;G286] is absent from the early version in **L**.

[91] On one occasion [B171;G292] Fibonacci explains that “such a proportion is called *proportionalitas*”, and twice [B171,222;G293,368] he refers to a continued proportion as *proportionalitas continua*. This usage agrees with Boethius’s *De institutione arithmetica* II.40 [ed. Friedlein 1867: 137].

proportions thus", a counterfactual *calculation* of the kind we encountered in Jacopo's *Tractatus* (above, p. 18):

> If it were proposed to you that 7 were the half of 12, how much would be the half of 10? This can in fact be understood in two ways, namely, when if it is said, if 7 were the half of 12; one may understand, either that the half of 12, which is 6, grows to 7; or that 7 is diminished to the half of 12, that is, to 6. Therefore, if 6, which are the half of 12, grows to 7; therefore also the half of 10 grows: and then you will need this rule: you shall multiply 7 by 10, and you divide by 12, $5\,{}^{5}\!/_{6}$ result for the half of 10. And if you want to understand that 7 is diminished to 6, that is, to the half of 12. In consequence the half of 10 is also diminished. And then you multiply the already mentioned 6 by the half of 10, that is by 5, they will be 30. Which you divide by 7, $4\,{}^{2}\!/_{7}$ result; and so much must then the half of 10 be. And in this way you can solve similar questions by which method you like, of the two methods described. However, we are accustomed to always answering those who ask according to the first method.

We observe, firstly, that Fibonacci presents the counterfactual calculation far away from the rule of three (introduced in chapter 8, we remember from p. 71, as the "major way"), not just as a secondary example. Secondly, his exposition of two different interpretations of the question and his ensuing acknowledgment that "we" only use the first of them makes it obvious that he refers to something familiar in his background environment (or, rather, a segment of his background environment[92]), which he then exposes critically.

On p. [B170;G291] the text goes on with another critical reinterpretation of the habitual (but here of a simple counterfactual statement), pointing out the connection to the rule of three:

> If ${}^{1}\!/_{3}$ were ${}^{1}\!/_{4}$, how much would ${}^{1}\!/_{5}$ be? This question is as if it was said, ${}^{1}\!/_{3}$ of a *rotulo* for ${}^{1}\!/_{4}$ of a bezant. How much are worth ${}^{1}\!/_{5}$ of a *rotulo*? Therefore it should be written in the way of a commercial transaction, and done according to what was done in similar cases in chapter 8,

℞	℞
$\frac{1}{4}$	$\frac{1}{3}$
$\frac{1\ \ 1}{2\ \ 10}$	$\frac{1}{5}$

accompanied by the apposite marginal scheme showing the organization within a rectangular frame ($\frac{1\ \ 1}{2\ \ 10}$, we remember, stands for ${}^{1}\!/_{10}+{}^{1}\!/_{2}\cdot{}^{1}\!/_{10} = {}^{3}\!/_{20}$).

From this critique of counterfactual statements and calculations Fibonacci moves to purely (unnamed) "magisterial" matters, apparently added in the 1228 edition (they are absent from **L**): How to construct a set of four integers in proportion – easy, since no

[92] Hypothetically it could seem that his basic exposition of the rule of three, with its suggested use of a *lawha*, was based on what he knew from Maghreb trade; the counterfactual formulation, on the other hand, appears to refer to an Iberian environment, cf. below. p. 180 onward. This would explain that the two ways are presented at a distance. In general, when Fibonacci's "we" can be located, it appears to point to the Iberian environment.

constraints are imposed; similarly, with six numbers, $a : b = c : d = e : f$; and [B171;G292] how to divide 10 into four proportional parts (which is where the term *proportionalitas* is mentioned, so as to distinguish it from a continued proportion); how to construct a continued proportion with as many members as wanted. And finally, with several examples, how to find numbers a and b such that $fa = gb$, f and g being given simple or composite fractions (e.g., $^2/_7$ and $^3/_8$, or $^1/_4+^1/_3$ and $^1/_5+^1/_4$), with extension to three or four numbers.[93]

12.3, "questions of trees ..."

Part 12.3 [B173;G296] is said to deal with "questions of trees and similar things, in what way they are solved". The beginning, about "a tree, of which $^1/_3$ $^1/_4$ is underground. And they are 21 palms", was already quoted above (p. 25); as we remember, Fibonacci introduces both his own explanation and the one habitually used by practical reckoners – namely the single false position. This double approach probably explains the slightly awkward heading.

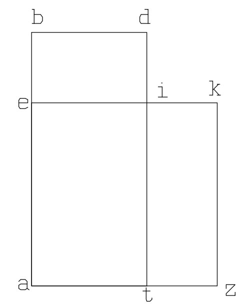
Next comes another problem about a tree, and then three more "about a tree or a number, to which was added" some fraction of itself. Thereby it should be clear that a method of general validity is taught – a method which is afterwards spoken of as the "rule of the tree".

The single false position (even when rationalized as done by Fibonacci) works primarily for problems of the first degree.[94] In the next problem [B175;G298] (absent from **L**, which therefore also omits the geometric diagram and argument) Fibonacci shows how to use it in a specific problem of the second degree: $^1/_3+^1/_4+^1/_5+^1/_6$ $(= {}^{57}/_{60})$ of a number equals the (square) root of that number – in symbols, $^{57}/_{60}\,n = \sqrt{n}$. This evidently leads to $(^{57}/_{60})^2n^2 = n$, and further to $n = (^{60}/_{57})^2$. Fibonacci does not indicate the intermediate calculations, but this is exactly his solution; what he does (forgetting his normal reinterpretation of the false position) is to posit that the number be 60. Since the present fraction is cumbersome, he takes advantage of the fact that $^{57}/_{60} = {}^{19}/_{20}$, whence

---

[93] In the case $^2/_7\,a = {}^3/_8\,b$, Fibonacci multiplies in cross by the denominators. This entails that $16a = 21b$, an evident solution to which is $a = 21$, $b = 16$. Fibonacci has a more complex argument, referring to proportion theory. Evidently, $a = {}^3/_8$, $b = {}^2/_7$ would already be a solution – but in context like this, Fibonacci prefers to understand "numbers" as (evidently positive) integers *if possible* (when 10 is be divided [B171;G292] to into 4 parts in proportion it is not).

[94] It should be remembered that the degree of a problem is intrinsic and does not depend on the tool used to solve it, which can be by guessing; by stepwise numerical approximation; algebraic; or geometric.



$^{19}/_{20}n = \sqrt{n}$, and now gives a geometric argument.[95] *ab* represents *n*, while *at* is 1; therefore the area *ad* is also *n*. $ae = az$ represents $^{19}/_{20}n$ – and since *ae* is also $\sqrt{n}$, *ak* must equal *n* and therefore *ad*. Subtracting *ai* from both, we see that $ib = tk$. This gives the proportion $ti : id = ei : ik$, whence $ti : (ti+id) = ei : (ei+ik)$. But $ei = 1$, $ti = ae$, and $ti+id = ab$, whence $ti : (ti+id) = 19 : 20$, while $ei+ik = ae = \sqrt{n}$. Therefore $19 : 20 = 1 : \sqrt{n}$, $\sqrt{n} = {}^{20}/_{19}$. Fibonacci stops here, without finding *n* itself.

This geometric argument, with its appeal to proportion theory (the first of many of its kind in the *Liber abbaci*) is absent from **L** where it should be expected [ed. Giusti 2017: 34]; it is thus probably evidence of an effort to raise the theoretical/magisterial level of the work.

A number of related problems follow, none of them provided with a geometric proof. Then follow (sometimes quite tangled) first-degree problems solved by means of the single false position, now formulated in the vernacular way; problems about combined works (about travellers meeting, emptying of casks, etc.; cf. above, p. 23). Even problems of type *leo in puteo* (above, note 44) are dealt with here, which is only adequate because Fibonacci (as later Jacopo and many others) misses the recreational prank.

For one of these first-degree problems [B190;G324], a "give-and-take" question, an alternative method by means of *regula recta* is introduced. The problem is that a first man (A) asks from a second (B) 7 δ, saying that then he shall have five times as much as the second has. The second asks for 5 δ, and then he shall have seven times as much as the first. Reducing it to a form where the "rule of the tree" can be applied is slightly intricate. In the 1228 version the argument is supported by a line representation

[95] Difficult to follow in the Boncompagni text, since the copyist has read *c* instead of *t* (the two letters may indeed be quite similar) in the lettering of the diagram, and also sometimes in the text. The letter sequence *a–b–t–d–...* is unique in the work, and one may ask whether Fibonacci's original sequence in the master copy was *a–b–c–d–...*, some copyists systematically misreading (at least the one who produced **L**) *c* as *t* (which suggests understanding of the argument), others instead misreading inconsistently. As we shall see, a letter sequence *a–b–c–b–...* would be firm evidence that Fibonacci constructed the proof himself.

(functioning much as an algebraic argument),[96]

a e g d b

where *ab* represents the shared possession, and *ag* the possession of A, *gb* that of B. *gd* is 7, and *eg* 5. If B gives 7 to A, he shall be left with *db*, while A shall have *ad*. Therefore, if *ad* is divided into 5 parts, each of these shall equal *db*, for which reason *db* is $^{1}/_{6}$ of *ab*. Similarly, *ae* is $^{1}/_{8}$ of *ab*. That is, if $^{1}/_{6}+^{1}/_{8}$ of *ab* is removed, we are left with 5+7 = 12 – which is solved by "the rule of a tree".

L instead gives a purely verbal argument running along the same lines. Line diagrams of the kind used in 1228 turn up repeatedly in chapter 12, but never in *L* – that is, they represent a tool which Fibonacci did not use in 1202, and perhaps did not yet know.

For us, the alternative by *regula recta* definitely looks algebraic: B is posited to possess *a thing* (*res*) and 7 δ. After having received 7 δ, A therefore has 5 *things*, originally thus 5 *things* less 7 δ. If instead B gets 5 δ from A, he shall have a *thing* and 12 δ, while A shall have 5 *things* less 12 δ. Therefore, a *thing* and 12 δ equals 7 times 5 *things* less 12 δ – etc.

Fibonacci explains that the *regula recta* is used by the Arabs, and is very praiseworthy. He clearly does not think of it as belonging with the art of algebra; as in the Arabic tradition and in the *Liber Mahameleth*, algebra is fundamentally a second-degree technique.[97]

---

[96] The use of the letter sequence *a–b–g–d–...* indicates that Fibonacci has taken over the argument from an Arabic source; in principle, a Greek source would be possible, but as we shall see all diagrams with this letter sequence where an origin can be tentatively established appear to be of Arabic origin, while those which Fibonacci appears to have constructed himself have the sequence *a–b–c–...* (reasons that we can really trust the lettering as indicator of origin are stated below, p. 140). Since the method is used in an addition from 1228 to a problem already present in the 1202 version, it seems likely that Fibonacci adapted a borrowed diagram to the actual numbers; but it cannot be excluded that his source had a problem with the same set of "interesting" parameters (5–7, 7–5).

In one respect, a line-based argument differs from rhetorical or symbolic algebra: It allows ratio-taking between lengths, but does not (inasfar as I have noticed) make use of coefficients beyond the rudiments we see here – we find nothing like "4 times *ac*". However, simple coefficients like "twice" and "thrice" (*bis*, *ter*) do turn up (e.g., [B212;G355] and [B213;G356]) in verbal problem solutions that could well be reformulations of line-based arguments. Below (p. 110) we shall encounter a use of line diagrams where products of segments (*not* regarded as rectangles) play a central role.

[97] One element of the characteristic *al-jabr* terminology does turn up at times in *regula recta* operations, namely *restaurare*, the operation that restores something subtracted on one side of the equation – for example on p. [B260;G421]. Mostly, however (as here), Fibonacci argues that "when equals are added to equals, the totals are equal" and "if equals are removed from equals, what remain

Fibonacci is not alone in the Latin world to speak about this rule. Under the name *regula* it is made profusely use of in the 12th-century *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304–371] as an alternative to the double false position; in that work, the unknown is called *census*, from Arabic *māl*, "possession" or "amount of money".[98] In the later fifteenth century, Benedetto da Firenze was to speak of it as *modo recpto/repto/recto* (naming the unknown *quantità* instead of *res*); two encyclopedic anonymous manuscripts from the same years do as much.[99] The changing names for the technique as well as for the unknown seem to indicate that Fibonacci did not borrow from the *Liber augmenti*, while Benedetto's deviating terminology could suggest that it had been handed down through a teaching tradition rather than in writing – fresh borrowings from the Arabic after 1450 seem unlikely.

Instead of coming from the algebra tradition, the method may well go back to classical Antiquity. Diophantos's *Arithmetica* I.15 [ed. trans. Tannery 1893: I, 36*f*] has exactly the same structure as the present problem – A and B are numbers, and if A receives 30 units it will be twice what remains of B; if B receives 50 units from A, it will be thrice what remains of A. Diophantos's method also has the same structure: he posits B to be 1 *arithmós* plus 30, etc. This does not mean that Fibonacci had read Diophantos; book I of the *Arithmetic* consists of (widely circulating) recreational problems or mathematical riddles deprived of their concrete dress.[100] Even the use of the *arithmós*, "number", as name for the unknown was not invented by Diophantos, see [Vogel 1930].[101]

The problem for which Fibonacci introduces the *regula recta* is said by him to have been proposed to him by some master in Constantinople. However, since the solution by *regular recta* was only added in the 1228 edition, there is no reason to doubt Fibonacci's words that his own direct source for it was Arabic. We should also remember that problems were presented as challenges, and that it was up to the receiver to find the solution – cf. Fibonacci's reference to having learned the "give-and-take of disputation"

---

will be equal".

[98] *Regula* is also used in the manuscript Paris, BN, Latin 15120, a small 13th-century collection of mathematical problems – see [Sesiano 2000: 78–82]. Since much of the text is borrowed verbatim from the *Liber augmenti et diminutionis*, this tells nothing new.

[99] Florence, BNC, Palatino 573; Vatican, Ottobon. lat. 3307. All three are discussed in detail below, p. 249 onward.

[100] [Tropfke/Vogel et al 1980: 610*f*] lists variants of the present structure, from first- and fifth-century China, from Bhaskara II, from the *Greek Anthology*, etc.

[101] The names for higher powers were also traditional (ἐδοκιμάσθη, "it has been approved"), as Diophantos tell himself [ed. Tannery 1893: I. 4]. That does not concern us here, but see [Høyrup 1990b: 211*f*]

on his travels (above, p. 60).

A number of similar but often more complicated problems follow which make use of the same techniques; often they are provided with metamathematical commentaries, for example about the existence of infinitely many solutions [B197;G332]. One [B203; G340], about somebody selling three pearls in Constantinople for bezants (thus likely also to have been encountered in Byzantium[102]), is similarly solved first by a false position, next by the *regula recta*. This time the second solution is also present in **L** [ed. Giusti 2017: 78], yet without introductory words. It appears that Fibonacci already knew the *regula recta* in 1202 but by then used it accidentally without noticing the need for an explanation. Since any systematic revision would naturally lead to the discovery that an explanation would be adequate, this observation (to which several parallels can be given) supports the assumption that **L** really reflects the 1202 version, and does not descend from an intermediate revision (cf. above, p. 59).

In 1228, the *regula-recta* solution stops after having found the *thing*, standing for the price of the first pearl. **L** instead finds the complete solution. The 1228 version, moreover, introduces yet another method [B203;G341], the *regula versa*, the "reverted rule", which here is nothing but a stepwise backward calculation.

In the end of part 3 come a large number of pure-number problems involving the principle discussed in the end of part 12.2, numbers a and b where $f_1a = f_2b$, f_1 and f_2 being given simple or composite fractions. Often, the false position is not made explicit but remains implicit in the numerical calculations that are performed.

12.4, finding a purse

Part 12.4 [B212;G355] is dedicated to a particular recreational problem type, "the finding of a purse". The first problem runs

> Two men, who have *denari*, find a purse containing *denari*. When they have found it, the first says to the second, "if I get the *denari* in the purse together with the *denari* I have, then I shall have three times as much as you". Against which the other answered,

---

[102] When Fibonacci says that a particular problem was presented to him by a Byzantine master [B188,190,249;G319,324,405], prices are mostly indicated in bezants – only one example [B190; G324] deals with unspecific "money"; when problems tell stories taking place in Constantinople [B161,203,274,296;G277,340,440,276,443], bezants are always used.

We cannot conclude with certainty that all problems referring to bezants are connected to Byzantium, however. Firstly, we have evidently no guarantee Fibonacci would always connect currency and locality in this way (cf. below, note ?); secondly, as mentioned on p. 49, bezants were also minted in Arabic and crusader countries, and we cannot be sure that Fibonacci would *always* specify when such variants are meant, as he does twice on p. [B137;G238] (*hyperperi or Saracen*, and *garbi*, from the Arabic "West"). Still, the appearance of bezants may be counted as circumstantial evidence.

> “and if I get the *denari* of the purse together with my *denari*, I shall have four times as much as you”.

The argument runs as follows – A stands for the possession of the first man, B for that of the second, and p for the contents of the purse (which is indeed spoken of in the argument simply as *bursa*, “the purse”):

$$A+p = 3B$$

whence

$$A+B+p = 4B$$

and thus

$$A+p = \tfrac{3}{4}\,(A+B+p)$$

This part of the argument makes use of a false position: if $A+p = 3$, then $A+B+p = 4$. The next part, in which the corresponding calculation is made for $B+p$, does without a false position and shows directly that

$$B+p = \tfrac{4}{5}(A+B+p)\ .$$

Now a new false position is made, namely that $A+B+p$ is a number of which $\tfrac{3}{4}$ and $\tfrac{4}{5}$ can be found, for which 20 is chosen. Then $A+p = 15$, $B+p = 16$, and therefore $(A+p)+(B+p) = (A+B+p)+p = 31$, whence $p = 11$, $A = 4$, $B = 5$. Alternatively, with the same position, $B = \tfrac{1}{4}\,(A+B+p) = 5$, $A = \tfrac{1}{5}\,(A+B+p) = 4$, $p = 20–4–5 = 11$.

So far Fibonacci has not observed that this is just one of many possible solutions. However, a third solution by *regula recta* (not named here, and not to be found in **L**) finds the ratios between A, B and p, and thus implies it.[103] It identifies A with the *thing*, and then operates with the *thing* and the *purse* on an equal footing, that is, with two unknowns.[104] Since *thing*+*purse* is thrice B, B must be $\tfrac{1}{3}$(*thing*+*purse*). Therefore,

---

[103] If anybody should doubt it, Fibonacci is thus fully aware that this indeterminate problem has as many solutions as requested. That he does not say so shows that *here and on this account at least* he obeyed the norm system of the recreational culture of mathematical challenges, where all that was asked for was the ability to find *a* solution. This certainly disagrees with the norms of present-day mathematics, and also with those norms which made Abū Kāmil ask for the complete set of solution to the “problem of 100 fowls” in the *Book on Fowls* [ed. trans. Rashed 2012].

We shall soon (p. 91) encounter a more complicated problem solution where Fibonacci finds the complete set of solutions.

[104] To my knowledge, this use of two indubitably *algebraic* unknowns (according to the criteria proposed in [Heeffer 2010: 61]) in the *Liber abbaci* has not been noticed so far. Heinz Lüneburg [1993], it is true, speaks much about equations with several variables, but does not observe the difference between his own equations and those which Fibonacci occasionally produces by means of the *regula recta*. In a note, Laurence Sigler [2002: 626] comes closer, but the final words of

if he gets the purse, he will have *purse* + $^1/_3$*purse* + $^1/_3$*thing*, which will be 4 *things*. Therefore 4 *purse* = 11 *thing*. In consequence, $p : A = 11 : 4$.

A number of variations follow, where the men may be three, four or five, and where they may find one, two, three or four purses. The principles serving the solution are the same, but obviously the calculations sometimes become much more intricate. In others, the sum of possessions comes into play; all of these are absent from **L** and thus added in the 1228 edition; they are even more difficult. One about four men [B225;G372] will serve us later (below, p. 300), and may conveniently serve as example:

> The first and the second with the purse have the double of the *denarii* of the third; and the second and the third the triple of the fourth, and then the third and the fourth the quadruple of the first, while the fourth and the first with the purse similarly have the quintuple of the second. The solution to this problem you will find by finding the ratio of the *denarii* of the purse to the *denarii* of the first in this way. Because the first and second with the purse have the double of the third, half of the *denarii* of the first and second and the purse is as much as the *denarii* of the third man. Similarly from the other propositions you will have that $^1/_3$ of the second and third man and of the purse is as much as the *denarii* of the fourth man, and $^1/_4$ of the third and fourth man and of the purse is the quantity of the *denarii* of the first, and $^1/_5$ of the *denarii* of the fourth and first man and of the purse is the quantity of the *denarii* of the second. And because $^1/_2$ of the first and second and of the purse is the quantity of the third, the third part of the first and second and purse, that it $^1/_6$ of them, is $^1/_3$ of the third man. Commonly are joined $^1/_3$ of the *denarii* of the second and purse: then will $^1/_6$ of the first and $^1/_2$ of the second and of the purse be as much as $^1/_3$ of the second and third and of the purse. But $^1/_3$ of the second and third and of the purse is the quantity of the *denarii* of the fourth man; hence $^1/_6$ of the first and $^1/_2$ of the second and of the purse are the quantity of the *denarii* of the fourth man. Therefore $^1/_4$ of $^1/_6$ of the *denarii* of the first, that is, $^1/_{24}$, and $^1/_4$ of $^1/_2$, thus $^1/_8$ of the *denarii* of the second and of the purse, are $^1/_4$ of the *denarii* of the fourth

---

the note shows him not to distinguish between unknown *entities* and *algebraic* unknowns.

It is no sensation, however: in the *Flos* [ed. Boncompagni 1862: 236], Fibonacci was already known to have made use of the two unknowns *res* and *causa*. Neither there nor in the present or following cases in the *Liber abbaci* is there any hint that Fibonacci believed to have introduced something remarkable.

In Arabic (post-al-Khwārizmīan) first-degree algebra, it was customary to use *thing* (*šay'*) for the first unknown and coin names for the following ones – in Abū Kāmil's *Book on Fowls* [ed. trans. Rashed 2012: 736–755] *dinar*, *fels*, and *khatem*. This method was not totally unknown in the Latin world: a couple of times, the *Liber Mahameleth* [ed. Vlasschaert 2010: 209*f*] uses *res* and *dragma*. Having learned the *regula recta* somewhere in the Arabic world Fibonacci may also have learned about the variant with two unknowns there; we cannot know for sure, but below (p. 91) we shall encounter corroborating evidence.

> man. Commonly are added $^{1}/_{4}$ of the third and of the purse: then $^{1}/_{24}$ of the first with $^{1}/_{8}$ of the second and with $^{1}/_{4}$ of the third and $^{3}/_{8}$ of the purse will be as much as $^{1}/_{4}$ of the *denarii* of the third and fourth and of the purse. But $^{1}/_{4}$ of the third man and the fourth and of the purse is the quantity of the first. Therefore $^{1}/_{24}$ of the first and $^{1}/_{8}$ of the second and $^{1}/_{4}$ of the third and $^{3}/_{8}$ of the purse are as much as the *denarii* of the first. Then their fifth part, that is $^{1}/_{120}$ of the first and $^{1}/_{40}$ of the second and $^{1}/_{20}$ of the third and $^{3}/_{40}$ purse, are $^{1}/_{5}$ of the *denarii* of the first. Commonly are added $^{1}/_{5}$ of the fourth man and the purse: then $^{1}/_{120}$ of the first and $^{1}/_{40}$ of the second and $^{1}/_{20}$ of the third and $^{1}/_{5}$ of the fourth and $^{11}/_{40}$ of the purse will be as much as $^{1}/_{5}$ of the fourth man and the first and of the purse. [...]

The final omission [...] is as long as the part that was translated. It leads to

> Hence $^{79}/_{600}$ and $^{1}/_{150}$ of the first, that is $^{83}/_{600}$ of the same, with $^{1}/_{25}$ of the purse, are $^{29}/_{200}$ of the purse. Commonly are taken away $^{1}/_{25}$ of the purse. Remain $^{83}/_{600}$ of the first, as much as $^{21}/_{200}$ of the purse. Then two numbers should be found so that $^{83}/_{600}$ of the first are $^{21}/_{200}$ of the second, they will be 63 and 83. Then if the first man has 63, the purse is 83. [...].

If we admit the identity of “the *denari* of the first/first man”, “the quantity of the *denari* of the first man”, “the quantity of the first man” and “the first man”, this is rhetorical algebra with five unknowns: we observe the additions and subtractions performed “commonly”, that is, from both sides of an equation, and the complicated substitutions.

It is difficult for us to follow the argument without making algebraic notes. There are also traces in the text that Fibonacci *described* a procedure performed by other means. Several errors are of the type that might occur when such a procedure is transferred: “$^{1}/_{150}$” instead of “$^{1}/_{150}$ primi” and “denariis secondi” instead of “denariis primi”. Both are ω-errors (above, p. 59), that is, they belong to Fibonacci’s evolving master copy. So, when Fibonacci describes the procedure in rhetorical algebra he appears to copy from somewhere, and with high probability from his own calculation. This *could* be a solution by rhetorical algebra made separately, but it could also be (might rather be) an argument by means of line diagrams of the type which was discussed on p. 82. These procured a tool which Fibonacci introduced in the 1228 version and may have been used even when no diagram is drawn.[105]

One problem [B227;G374] is shown to be insolvable, and another one [B216;G359] to be so unless one of the men has a debt. In one [B222;G368], where the contents of

---

[105] Where they appear, these diagrams are invariably lettered $a - b - g$ ..., meaning that the problem solutions where they appear are faithfully taken over from an Arabic source (cf. above, note 96). Knowing the technique Fibonacci may well have used it privately without feeling the need to show the diagrams when calculating on his own.

four purses is stated to be in continued proportion, this proportion can be chosen freely, thus adding no particular difficulty.

12.5, buying a horse

Part 12.5 [B228;G375] deals with another illustrious recreational problem type, the "purchase of a horse". Normally at least three buyers are involved, but Fibonacci's first example runs like this:

> Two men having bezants found a horse for sale. Wanting to buy it, the first said to the second, "if you give me $^1/_3$ of your bezants, I shall have the price of the horse". The other asked him for $^1/_4$ of his bezants, and then he would similarly have the suggested price. The price of the horse is asked for, and the bezants of each.

At first a unargued rule is given:

> Write in order $^1/_4$ $^1/_3$, and detract 1 which is above 3 from these same 3, 2 remain; which you multiply by 4, they will be 8 bezants; and as much had the first. Similarly when 1 which is above 4 is detracted from these same 4, 3 will remain; which, when multiplied by 3, give back 9 bezants; and as much had the other. Again, multiply 3 by 4, they will be 12; from which take away 1, which results from the multiplication of the 1 which is above 3 by the one which is above 4, 11 bezants remain for the price of the horse.

Then comes the explanation:

> This rule indeed follows from the rule of proportions, namely from the finding of proportion of the bezants of the one to the bezants of the other. Which proportion is found thus: Since the first with $^1/_3$ of the bezants of the second has as much as the second with $^1/_4$ of the bezants of the first, if in common $^1/_3$ of the bezants of the second is removed will remain the first equal to $^2/_3$ of the bezants of the second, and $^1/_4$ of his own bezants. Likewise, if in common is removed $^1/_4$ of the bezants of the first, will remain $^3/_4$ of the bezants of the first as much as $^2/_3$ of the bezants of the second.

Thereby Fibonacci has reached a situation he has dealt with before in part 12.2, about "proportions of numbers" – see note 93 and preceding text. The method taught there leads to the solution that the first has (3–1)·4 bezants, the second (4–1)·3 bezants, as stated in the present rule. Alternatively, solution by means of the *regula recta* is taught, here spoken of not as a rule but as "the Arabic way" (*per modum arabum*).[106]

In the case that the price of the horse is given, the (unnamed) partnership rule is applied.

Most problems about the "purchase of a horse" involve three or more buyers. They

---

[106] Neither the explanation nor the "Arabic" alternative are in **L**. They are thus added in the 1228 version.

are obviously more intricate. In Fibonacci's first example of this, the first of three men asks the second for $^1/_3$ of the possession in order to be able to buy the horse, the second asks for $^1/_4$ of what the third has, and the third asks for $^1/_5$ of the possession of the first – summarized in symbols thus

$$a+{}^1/_3b = b+{}^1/_4c = c+{}^1/_5a \ .$$

From this it is concluded that

$$a = {}^2/_3b+{}^1/_4c \ , \quad b = {}^3/_4c+{}^1/_5a \ , \quad c = {}^4/_5a+{}^1/_3b$$

(the expression in words obviously takes much more space, but the calculations are the same). From the last equation follows that

$$^1/_4c = {}^1/_5a+{}^1/_{12}b \ ,$$

and when this is inserted in the first equation we find that

$$a = {}^2/_3b+{}^1/_5a+{}^1/_{12}b = {}^3/_4b+{}^1/_5a \ ,$$

whence

$$^4/_5a = {}^3/_4b$$

once more the situation dealt with in note 93, which shows that the $a : b = 15 : 16$. Similar arguments lead to $b : c = 48 : 52$, and thus $a : b : c = 45 : 48 : 52$.

A rather large number of similar problems follow. The number of men varies, and may go until 7, while the number of horses may go until 4; when several horses are involved, the differences between their values are given. Each man may ask cyclically from a neighbour (as in the examples we have looked at), from two successors in the cycle, or from all the others; at times, several men together (in the usual cyclical order) make the question. The fractions obviously vary, and in one case the purchase of a horse is replaced by the renting of a ship; except for the single initial *regula-recta* example, all solutions are built on the same basic ideas (which are also those that are normally applied to give-and-take and purse problems).

12.6, repeated travels with gain and expenses

Part 12.6, on its part, introduces not only a new problem type (the repeated travel with gain and expenses) but also a new method. At first comes this [B258;G417]:

> Somebody proceeding to Lucca made double there, and disbursed 12 δ. Going out from there he went on to Florence; and made double there, and disbursed 12 δ. As he got back to Pisa, and doubled there, and disbursed 12 δ, nothing is said to remain for him. It is asked how much he had in the beginning.

Who is familiar with earlier medieval recreational mathematics will recognize the type. Because the traditional versions do not speak of costs but of religiously imposed gifts, I have elsewhere called it the "pre-Modern merchant's nightmare". The earliest known instance is found in Ananias of Shirak's arithmetical collection [ed. Kokian 1919: 126]:

> A man entered three churches, and asked God, firstly, give me as much as I have, and I shall give you 25 *dahekan*. Similarly, the second time he gave 25; and similarly the third time. And he was left with nothing. Now find out how much he had at first!

In Fibonacci's world, as we see, nobody would believe merchants to be so respectful of their religious duties, but they would still incur costs and risk bankruptcy.

Whether formulated about costs or about gifts to God or the poor, such problems were mostly solved step by step backwards. Fibonacci's question would then be solved in this way: before disbursing 12 δ in Pisa, the merchant had 12 δ, that is, coming to Pisa he must have had 6 δ, which have been left over in Florence after he disbursed 12 δ there. Before disbursing 12 δ in Florence he therefore had 18 δ, and coming to Florence hence 9 δ. Etc.

Fibonacci chooses as different way, which will also serve him in the sophisticated variants which he is going to present. He makes the tacit false position that the initial capital is 1. He prescribes a sequence of unargued numerical steps, whose underlying explanation is this: Without disbursements, an initial capital of 1 δ would grow to a "Pisa value" of $2{\cdot}2{\cdot}2$ δ = 8 δ. However, it *should* grow to equal the Pisa value of *the disbursements*, which – also doubled at each change of city – is $(2{\cdot}2+2+1){\cdot}12$ δ = 84 δ. The basic ideas behind these two calculations are those of composite interest and discounting, both familiar in the commercial ambience. Since the Pisa value of the initial capital should equal the Pisa value of the expenses, the initial capital itself must be $((1+2+2{\cdot}2){\cdot}12$ δ$)/8$ δ = 10½ δ.

Then follow sophisticated variations: the rate of gain or the disbursements may vary; instead of the initial capital, the disbursement may be unknown though constant; etc. Sometimes solutions by *regula recta* are given. The basic idea underlying the solutions remains the same.

However, on p. [B264;G426] comes a problem where this will not suffice:

> Again, in a first travel somebody made double; in the second, of two, three; in the third, of three, 4; in the fourth, of 4, 5. And in the first travel he expended I do not known how much; in the second, he expended 3 more than in the first; in the third, 2 more than in the second; in the fourth, 2 more than in the third; and it is said that in the end nothing remained for him. And let the expenditures and his capital be given in integers. We therefore posit by *regula recta* that his capital was an *amount* [*summa*], and the first expenditure a *thing*.

Applying the technique used in the preceding problems, we would have to reduce the initial capital as well as the expenditures to final value, which inasfar as expenditures are concerned becomes somewhat arduous and at any rate involves the first unknown expenditure. So, this time Fibonacci applies the *regula recta* (mentioned by name) with two unknowns, positing explicitly *amount* and *thing* as algebraic unknowns and making a stepwise calculation. Knowing the problem to be indeterminate, Fibonacci asks for a solution in integers.

After the first travel, our merchant is seen to possess 2*amount*–*thing;* after the second, he has 3*amount*–$2\,{}^{1}/_{2}$ *thing*–3δ; after the third, 4*amount*–$4\,{}^{1}/_{3}$ *thing*–9δ; and after the fourth, 5*amount*–$6\,{}^{5}/_{12}$ *thing*–$18\,{}^{1}/_{4}$ δ. In this way we end up with the indeterminate equation

$$5\mathit{amount}-6\,{}^{5}/_{12}\,\mathit{thing}-18\,{}^{1}/_{4}\,\delta = 0$$

or, "if all-over $6\,{}^{5}/_{12}$ *thing* and $18\,{}^{1}/_{4}$ δ are added",

$$5\mathit{amount} = 6\,{}^{5}/_{12}\,\mathit{thing}+18\,{}^{1}/_{4}\,\delta$$

with the request that *amount* and *thing* have to be integers. With an astute stepwise procedure Fibonacci finds as possible solution the *amount* to be 46, and the *thing* to be 33. In the end (since the equation can be transformed into 60*amount* = 77*thing* + 219 δ), he points out that other solutions can be found by adding

> as many times as you will 60 to the first expenditure, that is, to 33, and as many times 77 to the capital that was found, that is to 46, and you will have what was asked for in ways without end.

Some variants follow, the last of which states the traveller to have a net profit of 12, that is, that he ends up with his initial capital and 12 more. Here, Fibonacci uses the opportunity to show how the *regula versa* may be applied in this complex case, using the same two algebraic unknowns: being left in the end with 1*amount* + 12, after disbursing 1*thing* + 7, he must have had before disbursing 1*amount* + 1*thing* + 19; in the fourth travel he must therefore carried ${}^{4}/_{5}$ of this, that is ${}^{4}/_{5}$ *amount*+${}^{4}/_{5}$ *thing*+$15\,{}^{1}/_{5}$, etc. Out of this comes the equation 1*amount* = ${}^{1}/_{5}$ *amount*+${}^{77}/_{60}$ *thing*+$6\,{}^{1}/_{20}$ (both express the initial capital).

Whereas the solution of the purse problem by means of two algebraic unknowns (see note 104 and surrounding text) is absent from **L** (as is the presentation of the *regula recta*), the present use of two algebraic unknowns (in *regula recta* as well as *regula versa*) is already in **L** [ed. Giusti 2017: 134–137]. The way it is introduced – "let us therefore posit by the *regula recta* that the capital was an *amount* and the expenditure a *thing*" – looks as if the technique was quite familiar. Fibonacci does not seem to be aware that he is proposing something new, and we may conclude that he was not.

In the last travel problem [B266;G429], the number of travels the traveller undertakes before being bankrupt is unknown. But it is known that initially he has 13δ, that at each travel he doubles his capital and expends 14 δ. Calculating stepwise Fibonacci finds that the traveller's net loss grows geometrically (the term is not used), as 1 δ after the first travel, 2 δ after the second, 4 δ after the third, and 8 δ after the fourth, where only 6 are available. That is (tacitly supposing the profit and the expenditure to have linear growth within the last travel), the number of travels must be $3\,{}^{3}/_{4}$. Observing that this is incongruous, Fibonacci adjust the expenditure so as to produce an integer number of travels.

Last in part 12.6 come problems that are "similar" to the travel problems – namely similar in mathematical structure. Most of them deal with a loan with interest that is amortized by the rent of a house. As Fibonacci explains [B267;G430], the annual growth of the debt corresponds to the gain in a travel, and the rent that is discounted from it each year to the expenses incurring in the travel.

12.7, "rambling problems"

Part 12.7 is said to contain "rambling" problems (cf. note 89). First [B276;G442] comes an analogue of Jacopo's problem about freight of wool paid in kind (above, p. 27), solved in the same way. After a couple of variations on this principle comes [B278; G445] a recreational classic. Somebody enters a garden with 7 gates and picks apples. When leaving he has to give at each gate to the guardian half of the apples he carries, and one more. In the end he leaves with one apple. From a modern mathematical point of view, this is a strict analogue of the repeated travels with gain and expenses, only with the "gain factor" being smaller than 1. This time, however, Fibonacci offers the stepwise backward calculation as his first method. Alternatively, he shows how the problem can be solved by means of the *regula recta* (with a single unknown).

After two problems about operations with mixed numbers [B279;G446] comes what Leonhard Euler [1774: 489] (who knew the problem not from the *Liber abbaci* but from the later tradition) was to characterize as a "question of a quite particular nature", and which I shall speak of in the following as the "unknown heritage":

> Somebody coming to his end instructed the oldest of his sons, saying: Divide my possessions among yourself in this manner. You take one bezant, and the seventh of what is left; but to the next one of the sons he said, and you take 2 bezants, and the seventh of what is left. But to the next one, that he should take 3 bezants, and take control of $^{1}/_{7}$ of what was left. And in this way he called all his sons in order, giving each one more than the others; and afterwards always $^{1}/_{7}$ of what was left; the last however had the rest. It turned out, however, that all had equally of the possessions of the father on the said condition. It is asked, how many were the sons; and how much he owned. Indeed you do like this: for the seventh, which he gave to each, you retain 7; from which you extract 1, 6 remain. And so many were the sons; which 6 you multiplied in itself; and so many were his bezants. And if the first of the sons had had $^{1}/_{7}$ of the possessions of the father, and afterwards 1 bezant; and the second had had $^{1}/_{7}$ of the rest, and two bezants; and in this way it would have gone on for the other sons, adding for each one in order 1 bezant; then the sons would similarly be 6, and the bezants 6 seven times, that is 42. And if in each question the first should have 3 bezants, the second 6, and the rest similarly their bezants in ternary ascension; then the sons would similarly be 6, and the amount of the

bezants would be the triple of the said amounts, that is, of 36 and of 42.

Fibonacci is the first known source for the problem. It appears to have been unknown in the Arabic tradition, and there are strong reasons to believe it came from Byzantium or late Greek Antiquity – see the complete analysis in [Høyrup 2008], to which may be added that the appearance of the bezant as monetary unit fits a Byzantine origin.

Euler, in his elementary treatise on algebra, gives an algebraic solution, which however presupposes that this strongly overdetermined problem does have a solution. As we see, Fibonacci so far gives no arguments for his solutions.[107]

However, he does not stop here; in the sequel he avoids the absurdity of fractional *sons* by asking instead about the division of a *number* in shares under various conditions. At first he just takes the fraction to be $^2/_{11}$, and gives a solution corresponding to the transformation of this into $^1/_{5½}$, which yields 4 shares and a half-share, and totals 4½·4½ respectively 4½·5½ (the fraction of the remainder being taken after respectively before the absolute 1, 2, ...) – still without explaining why this is the solution, and even without checking. Then, however, he jumps to more sophisticated variants: first with absolute contributions 2, 5, 8, ... and fraction $^6/_{31}$ (absolute contribution before respectively after the fraction); next with absolute contributions 3, 5, 7, ... and fraction $^5/_{19}$.

For the first of these, Fibonacci produces a solution by means of *regula recta* [B280; G447], taking as *thing* the number to be divided. He then calculates the first and the second share and equates these, which gives him a correct solution (provided there *is* a solution[108]); next he claims to extract from this calculation a rule.

Comparison of the calculation with the rule shows that it is *not* extracted – see [Høyrup 2008: 618*f*] for the full analysis. If T is the number to be divided, the fraction $^p/_q$, and the absolute contributions α, $\alpha+\varepsilon$, $\alpha+2\varepsilon$, etc., then Fibonacci's *rule* expressed as a letter formula is

$$(*) \qquad T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]\cdot(q-p)}{p^2}\;.$$

Instead, Fibonacci's *calculation* would lead to

[107] The reader may see why the solution is possible and correct by drawing a 7×7-square or arranging 7×7 small objects in a square. This is what I did spontaneously myself when encountering the problem for the first time in the Vatican manuscript of Jacopo's *Tractatus*, and seems also to be Planudes's [ed., trans. Allard 1981: 191–194] underlying structure for his arithmetical arguments – see [Høyrup 2008: 621].

[108] Fibonacci may have been aware that what he obtains from the *regula recta* is only a possible solution (according to his calculation, the *only possible* solution). In any case he makes a complete calculation, showing that his solution is really one.

$$(\dagger)\qquad T = \frac{q^2(\alpha+\varepsilon)-(q-p)q\alpha-(q-p)p\alpha-(\alpha+\varepsilon)pq}{p^2}\ .$$

Since both are correct, they are obviously algebraically equivalent, but that is not easily seen without symbolic algebra and with tools at Fibonacci's disposal. We must presume that Fibonacci borrowed his rule from elsewhere, even though he was able to produce his own solution.

That is confirmed by the following three variants. For these Fibonacci only offers rules, no calculation of his own. His rule for the second of these (the one with absolute contributions 3, 5, 7, ... and fraction $^5/_{19}$) cases is

$$(**)\qquad T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot(q-p)}{p^2}\ ,$$

which would be the same as (*) if only negative numbers were within the horizon; since they were not, the formula to be used has to depend on whether $\alpha<\varepsilon$ or $\alpha>\varepsilon$.

However, this makes no difference in (†). If Fibonacci had transformed his own calculation into a rule, why should he have reduced it to a form that cannot be used universally?

Whereas the simple variants of the problem appear to have been created in the late ancient Greek or the Byzantine world and to have spread from there,[109] the most likely point of origin of the sophisticated versions is al-Andalus – and since they appear never to have reached the broader Islamic world, to be creations of the 12th century, the time of Ibn Rušd, who also had great influence on Latin and Hebrew but very little in Islamic philosophy. Even for this, the reasons are too complex to be recapitulated here, but see [Høyrup 2021a: 34–42]. Fibonacci obviously does not know how the rules were originally derived and proved to work; a possibility is the use of line diagrams – see [Høyrup 2008: 627*f*].

After this only partially understood visit to the area of higher arithmetic follows an interlude dealing with simpler matters, and then something more advanced, having to do with the "Chinese remainder theorem" – presented, obviously, not as a theorem but as a sequence of 3 problems. The first of these [B281;G450] asks for a number which leaves 1 as remainder if divided by 2, 3, 4, 5 and 6, and 0 if divided by 7. Fibonacci determines the least common multiple of 2, 3, 4, 5 and 6 to be 60, and argues that $p\cdot 60+1$ leaves the requested remainders with 2, 3, 4, 5 and 6, and then tries successive p-values until 7 divides, finding $p = 5$ to fulfil the condition; the requested number is thus $5\cdot 60+1$ = 301. Next come two problems where division by n leaves $n-1$ for n = 2, 3, 4, 5, 6 respectively 2, 3, 4, 5, 6, 7, 8, 9, 10 while 7 respectively 11 divide. Here, he subtracts 1 from successive multiples of 60 respectively 420. The methods used cannot serve for

[109] Not generally through Fibonacci, for reasons it would lead to far to list here.

other, less particular values of the remainders, and it is thus mistaken to state that this "is" what is known in modern number theory as "the Chinese Remainder Theorem" [Manin & Panchichkin 2005: 14]; the same can be said, however, about all the earlier instances analyzed in Ulrich Libbrecht's "monograph" on the theorem [1973: 214–413, pertinent 214–243]. Fibonacci most certainly borrowed at least the first problem from Arabic mathematics – Ibn al-Haytham's *On the Solution of a Number Problem*, also deals with it [ed. trans. Rashed 1984: 238–241]. However, when using the same approach as Fibonacci (which he refers to as "canonical"), Ibn al-Haytham takes the product instead of the least common multiple, and therefore does not find the smallest solution;[110] whether Fibonacci devised his shrewder way himself or learned it from later Arabic mathematics is difficult to know.[111] At least it is clear from his exposition that he understood it to the full.

After another elementary interlude [B283;G452] illustrating and exposing a temptation to apply the partnership rule mistakenly[112] follows yet another bit of theoretical arithmetic [B283;G452]: an explanation of what perfect numbers are and a rule for how to construct them stepwise. The rule that is given is that of *Elements* IX.36, but in the terminology which Fibonacci has borrowed from contemporary Byzantine Greek (prime numbers being numbers "without rule" – see above, note 79). A Byzantine inspiration is thus likely.

Next [B283;G453] comes a question which undeservedly has become the most famous piece of the work. A pair of rabbits is supposed to engender another pair each month,

---

[110] Rashed [1984: 228*f*] reads into Ibn al-Haytham's text a reference to Wilson's theorem, according to which $(n–1)!+1$ is divisible by n if and only if n is prime. Unfortunately, Rashed's own translation contradicts him: having found 720 as $2·3·4·5·6$ Ibn al-Haytham concludes (p. 238) that "if one divides seven hundred and twenty-one by each of these numbers, one always remains, and seven hundred and twenty-one is divided by seven because it has a seventh" – not "because seven is a prime number", and seven being prime has not even been mentioned. That seven hundred and twenty-one has a seventh can be stated as obvious (as it is), since seven hundred as well as twenty-one have it.

After giving this canonical solution (a term that shows that Ibn al-Haytham did not invent the problem nor this procedure) Ibn Al-Haytham gives a different, less elegant and more laborious calculation, which yields the smallest solution and shows that the problem possesses an infinity of solutions (which Fibonacci does not state explicitly but shows to know).

[111] Nothing like Fibonacci's second and third problem seems to be in known Arabic texts (Ahmed Djebbar, personal communication). Unless Fibonacci invented these himself (which nothing excludes), my guess (nothing but a guess) is that they were developed in al-Andalus, not least because of the vicinity in the *Liber abbaci* of this topic to the sophisticated versions of the unknown heritage.

[112] See below, p. 266.

which on its part conceives after one month and gives birth after another month. An initial pair thus becomes two in the next month; after three months, both of these produce. This evidently gives the "Fibonacci sequence" 1 – 2 – 3 – 5 – ..., characterized by the growth formula $n_p+n_{p+1} = n_{p+2}$. Fibonacci makes a painstaking step-by-step calculation for 12 months, and gives a commentary to a marginal diagram that corresponds to the growth formula. In the end he states that one may go on in this way for an unlimited number of months. But there is nothing about the convergence of the ratio between successive members of the sequence, nor *a fortiori* about the "golden section".[113] If [*if!*] there was any theory behind the rabbit problem, Fibonacci did not know.[114]

After this overvalued piece of arithmetical fun we find [B284–286;G454–456] a sequence of four problems superficially related to the same extended family as the give-and-take and purse problems (etc.) – first:

> There are four men, the first and second and third of which have δ 27. Similarly, the second, the third and the fourth have δ 31; the third, the fourth and the first have δ 34. However, the fourth, and the first and the second have δ 37. It is asked how much each one had. Add these 4 numbers into one, they will be 129; which number is the triple of the whole sum of the δ of these 4 men. Namely because in this amount each of them has been counted thrice. Therefore, if it is divided by 3 it gives 43 for their sum; from which, if you detract the δ of the first, the second and the third man, that is 34, remains for the fourth man δ 16. ... .

The next problems from the sequence have similar cyclical conditions, and the solutions follow from similar considerations. In the case [B284;G454] where the total possessions of two neighbours in circle with four participants (say, $a+b$, $b+c$, $c+d$ and $d+a$) are given, it is pointed out that this problem has no solution unless $(a+b)+(c+d) = (b+c)+(d+a)$. An impossible and a solvable instance are then shown.

Other problems follow that are less easy but still solved by similar methods (the methods that also served for the give-and-take and purse problems).

---

113 This section Fibonacci will only have known as division in extreme and mean ratio – the notion of a "golden section" belongs to the 19th century [Herz-Fischler 1987: 168*f*]. Late is also the belief in its importance in architecture, pictorial arts and mysticism.

114 Idealized calculations of breeding animals have an old history – whether they constitute an unbroken tradition is so far undecidable. The earliest known example (dealing with cattle, finding also the quantity and monetary value of the milk fat the cows produce) goes back to the 21st century BCE [Nissen, Damerow & Englund 1993: 100–102]. It is dressed up as a genuine piece of accounting, but the animals never die, the reproduction and the sex ratio of calves is constant, and so is the productivity of each cow. These complex calculations could obviously only be devised and transmitted in a literate school; the simplicity of the rabbit problem would allow it to be devised and to survive in a less literate environment.

Particularly intriguing (but ultimately to be solved by the same techniques as the others) is this [B293;G465] (henceforth, I shall refer to this problem type as the “grasping problem”):

> Three men had I do not know how many pounds of sterlings,[115] of which the half belonged to the first, the third to the second, and the sixth to the third. When they wanted to have them in a safe place, each of them grasped some quantity of these sterlings; and of the quantity he got the first put in common the half; and of what he got the second put the third part; and of what he got the third put the sixth part; and of what they put in common each received the third part; and in this way each had his share.

A first solution starts in this way:

> Since the first put in common $^1\!/_2$ of that which he got; of which $^1\!/_2$ he got back the third part, that is, $^1\!/_6$ of all he got: then remained for him from that which he took $^1\!/_6$ $^1\!/_2$, that is, $^2\!/_3$; and from that which the second put the first got $^1\!/_9$, since the second put the third of what he took, and of this $^1\!/_3$ the first got $^1\!/_3$, that is, $^1\!/_9$; and of that which the third put he got the third of the sixth part which this third put, that is, $^1\!/_{18}$. Therefore the half of the amount of all the sterlings, that is, the share of the first man, was $^2\!/_3$ of what the first took and $^1\!/_9$ of that which the second took, and $^1\!/_{18}$ of that which the third took.

In letter symbols:

$$^1\!/_2\ (A+B+C) = {}^2\!/_3A + {}^1\!/_9B + {}^1\!/_{18}C\ .$$

Similarly it is calculated that

$$^1\!/_3\ (A+B+C) = {}^1\!/_6A + {}^7\!/_9B + {}^1\!/_{18}C\ .$$

The latter equation (expressed in words, and without algebraic position) is transformed by addition of $^1\!/_2$ of all members into

$$^1\!/_2\ (A+B+C) = {}^1\!/_4A + \frac{1\ 10}{2\ \ 9}B + {}^1\!/_{12}C\ ,$$

whence

$$^1\!/_6A + {}^7\!/_9B + {}^1\!/_{18}C = {}^1\!/_4A + \frac{1\ 10}{2\ \ 9}B + {}^1\!/_{12}C\ .$$

“Detracting on both sides” (“de utraque parte” – the equation thinking is indubitable) $^1\!/_4A+{}^1\!/_9B+{}^1\!/_{18}C$, Fibonacci arrives at

$$^5\!/_{12}A = {}^{19}\!/_{18}B + {}^1\!/_{36}C\ .$$

Application of the perspective of the third man yields

$$^1\!/_6\ (A+B+C) = {}^1\!/_6A + {}^1\!/_9B + {}^8\!/_9C\ ,$$

---

[115] Silver coins in use in Italy as well as England, valued as bullion; we already encountered them in Jacopo’s coin list (above, p. 50).

transformed by means of another proto-algebraic operation into

$$\tfrac{1}{2}\ (A+B+C) = \tfrac{1}{2}A + \tfrac{1}{3}B + \tfrac{8}{3}C\ .$$

Combining this with the first equation,

$$\tfrac{1}{2}\ (A+B+C) = \tfrac{2}{3}A + \tfrac{1}{9}B + \tfrac{1}{18}C\ ,$$

and performing another proto-algebraic subtraction on both sides Fibonacci gets

$$\tfrac{1}{6}A = \tfrac{2}{9}B + \tfrac{47}{18}C\ .$$

Applying the rule of the fourth proportional (we would prefer, multiplying by $2\tfrac{1}{2}$) he transforms this into

$$\tfrac{5}{12}A = \tfrac{5}{9}B + \tfrac{235}{36}C\ .$$

But he already had

$$\tfrac{5}{12}A = \tfrac{19}{18}B + \tfrac{1}{36}C\ .$$

Therefore,

$$\tfrac{19}{18}B + \tfrac{1}{36}C = \tfrac{5}{9}B + \tfrac{235}{36}C\ .$$

Subtracting on both sides again and reducing the resulting fractions, Fibonacci finds

$$\tfrac{1}{2}B = 6\tfrac{1}{2}C\ ,$$

whence

$$B = 13C \quad \text{and} \quad A = 6\cdot(\tfrac{2}{9}B + \tfrac{47}{18}C) = 33C\ .$$

Since the problem is indeterminate, Fibonacci chooses $C = 1$, and gets $A = 33$ and a total of 47.

Apart from the lack of position of *distinct representatives* of what each of the three has grasped, this must be characterized as a perfect algebraic procedure, and thus as a demonstration that the border between arithmetical and algebraic solution at least of first-degree problems is far from sharp.

A second solution starts midway in the preceding one, but builds on the same principles. A third procedure by double false position is proposed in chapter 13, cf. below, note 136. Some variants with different numerical parameters follow (one with four men); they teach us nothing new.

The whole sequence is also in **L** [ed. Giusti 2017: 176–183], in practically the same words. Its first problem is also found in the *Flos* [ed. Boncompagni 1862: 234–236], with a slightly different formulation of the statement. Here Fibonacci tells that the problem was presented to him by Giovanni di Palermo in the presence of Emperor Frederick II – which can hardly have been at any other occasion than Frederick's visit to Pisa, taking place to all we know in 1226 [Giusti 2020: xix]. The presence of the problem in **L** appears to show that Fibonacci was well prepared, having already solved the problem in writing.

There is no reason to wonder, the problem was familiar in the Arab world. Al-Karajī [ed. trans. Woepcke 1853: 141] solves it in the *Fakhrī* (with the same numerical parameters) by means of two algebraic unknowns representing what I have designated *A* and *B*, taking advantage of the indeterminate character of the problem to identify *C* with 1 dirham. The order of operations is not precisely the same as that of Fibonacci, but on the whole Fibonacci's procedure might be a translation into words of al-Karajī's procedure or something similar – and since we have no other indications that Fibonacci knew the *Fakhrī* directly (cf. below, p. 148), the best guess seems to be that he had encountered it in an Arabic source descending from or related to al-Karajī.

The *Flos* tells the reader that Fibonacci had already given three solutions to the problem in "the book I put together on numbers" (that is, the *Liber abbaci*); here, however, he wants to present an "extremely beautiful way", which he presents to the Emperor. This way is a *regula recta* procedure, where the *thing* is posited as $^{1}/_{3}$ $(A+B+C)$. There is no reason to elaborate.

A group of two problems follows which, in mathematical future perfect, represent the first steps in partition theory.

The first of them [B297;G471] is "Bachet's weight problem" (see [Knobloch 1973]), first known from Mohammad ibn Ayyūb al-Ṭabarī's *Miftāḥ al-muʿāmalāt* from c. 1100:[116]

> Somebody had 4 weight pieces,[117] by which he weighed whole pounds of his merchandise from one pound until 40 pounds. The weight of each of these weight pieces is asked for. Then it is necessary that the first be of one pound; so that by it one pound can be measured. The second must be its double, with one added, or the triple of the same first; with these two weight pieces can be weighed from one pound until 4. But the weight of the third is one more than the double of both the others, that is, the triple of the second, namely 9; but the weight of the fourth is 1 more than the weight of the other three, that is, the triple of the third, namely 27; the weights of which joined together make 40. So, if you want to know how you may weigh with these weight pieces any number of pounds from one pound to 40 pound, let us say 14, then the fourth weight piece is put into one scale pan, and the rest is put in the other; and if you put the same fourth weight piece together with the first, and if you put in the other the rest, namely 9 and 3, then 16 pounds may be weighed [...]. And if you add a fifth weight piece, whose weight is the triple of that of the fourth, namely 81, with these five weight pieces may be weighed any number of pounds from one pound until 121 pounds; and thus in the same order weight pieces

[116] [Tropfke/Vogel et al 1980: 634], date according to [Hockey, Trimble & Williams 2007: 1149].

[117] In English it is unfortunately impossible to distinguish elegantly Fibonacci's *peso* (borrowed from Italian), here translated "weight pieces", from his *pondus*, their weight as quantity.

> may be added without end.

The final clause is very close to that of the rabbit problem[118] – closer than can be explained as an accident. The decisive difference is that in the rabbit problem the possibility to continue additions n_p+n_{p+1} is a triviality, since nothing is said about the properties of the resulting sequence. In the present case, instead, it is obvious from Fibonacci's intuitive argument by induction that what results is the optimal partition (with subtraction) of integers.

Theoretically seen, the other problem in the cluster ([B298;G471] – rather an instantiation of a theorem than a problem) is the purely additive counterpart:

> Somebody gave somebody for his daily work 1 mark of silver, which he paid by means of five cups that he had, so that none of them was broken; and this he did for 30 days. The weight of the first cup was 1, whose double, namely 2 mark, was the weight of the second. The weight of the third was 4, namely the double of the second. But the weight of the fourth was the double of the third, namely 8. When the weight of these 4 cups are joined together, they make 15 mark. When these are extracted from 30 mark, 15 mark remain for the weight of the fifth cup. On the first day he gave him the first vase. On the second he received from him this same first, and gave him the second. On the third the lord gave the worker this same first. On the fourth the lord received from the worker the first and the second, and gave him the third. And thus in the said order he paid him daily, until 30 days.

The underlying theorem is evidently that any integer can be expressed unequivocally as a sum of powers of 2 (the final 15 instead of 16 being chosen as a pragmatic shortcut allowing the worker to leave with all the cups). This is no deep insight, it had already been used in the Pharaonic standard multiplication algorithm.[119] Most interesting – not least because of the vicinity to the weight problem – is the term used for the cup. In Boncompagni's manuscript it is *sisphos* and *ciphus*, in **L** [ed. Giusti 2017: 184] it is *sciphos* and *scifis*. This is not Latin, nor a borrowing from any Romance language. It renders *spoken* Byzantine Greek, namely the way σκύφος was pronounced (better in **L** than in the later manuscripts, where the spelling is further influenced by Tuscan pronunciation). That is, Fibonacci encountered the problem in oral interaction in Byzantium.[120] Since he sees it as belonging together with the preceding, more sophisticated

---

118 "et sic posses facere per ordinem de infinitis numeris mensibus" respectively "et sic eodem ordine possunt addi pesones in infinitum".

119 In order to see it we only need to notice that division by 2 leaves either remainder 0 or remainder 1 (given the familiar laws of associativity and distributivity).

120 This would agree well with the appearance of a closely related problem in a Byzantine problem

problem, we may presume that *both* came from the same source; and since they share the essentials of the closing formula with the rabbit problem,[121] on its part close to the rule for production of perfect numbers with its Byzantine terminology, even this may be assumed to have been borrowed from Byzantium.

A group of problems about the sale of apples follows – and since apples are supposed to be sold whole, and the prices are also, *qua* numbers of δ, supposed to be integer, these are Diophantine problems. The first [B298;G472] begins in this way:

> One of two men had 10 apples, the other 30; and as they were in a market, each sold from his apples I do not know how many; but the prices were the same. And when they came to another market, they sold the rest, similarly at the same price; and that which the first had for his 10 apples was as much as that which the second had. The price of the apples in each market is asked for, and also how many were the apples each of them sold in each market. Divide in two parts the apples of the first man, namely 10, so that, when the first part is detracted from the number of apples of the other, namely from 30, remains a number that is divided integrally by the second part; and what comes out of the division is the price of each apple sold in the second market. [...]

If a_1 designates the number of apples sold in the first market by the first man, a_2 those sold by him in the second market, b_1 and b_2 correspondingly for the second man, and f and s the price of an apple in the first respectively the last market, then we have

$$a_1+a_2 = 10 \ , \ \ b_1+b_2 = 30 \ ,$$
$$f a_1+s\,a_2 = f b_1+s\,b_2 \ .$$

This is obviously strongly underdetermined – there are six unknowns but only three conditions. The latter condition can be re-expressed

$$s\,a_2 = f b_1+s\,b_2-f a_1 \ .$$

Fibonacci's first choice is to take a_1 so that

---

collection from the early 14th century [ed. trans. Vogel 1968: 112*f*], which seems not to be influenced by Italian material.

[121] The cup problem ends "et sic predicto ordine persoluit eum cotidie, usque in diebus 30", related to but since no unlimited procedure is promised not the same – cf. note 118. Only two other problems have somewhat similar closing formulae:

– [B340;G529]: "denarios vero tercii hominis reperies ordine suprascripto".
– [B384;G590]: "eademque via et ordine poteris operari in reperiendis radicibus cubicis numerorum decem vel plurium figurarum".

Though no definitive proof, the similarity between the closing formulae of the rabbit and the weight problem is a strong suggestion.

$$\frac{b_1+b_2-a_1}{a_2} = s$$

that is,

$$sa_2 = b_1+b_2-a_1 \ .$$

There is no hint of why this choice is made; the problem (and those that follow) have no pedagogical purpose, they show (and are almost certainly intended to show) the brilliance of the author. There is no reason in the present context to pursue the analysis.

Last in part 7 comes a problem somewhat similar in mathematical structure to the "purchase of a horse", formulated however as dealing with 5 *numbers*. In letter symbols:

$$\begin{aligned} a+b+c &= (1+{}^1\!/_2)d \\ a+c+d &= (2+{}^1\!/_4)e \\ a+d+e &= (3+{}^1\!/_5)b \\ a+e+b &= (4+{}^1\!/_6)c \end{aligned}$$

The problem is indeterminate, and Fibonacci starts by positing $a+b+c = 1\,{}^1\!/_2$, whence d = 1. He goes on with linear operations of the same nature as those used in the horse-, purse- and give-and-take problems.

12.8, divinations

12.8 [B303;G478] deals with "certain divinations". What this means can be illustrated by the beginning of the first problem:

> When however somebody has put a number in his memory and want you to find it: instruct him that he put the half of the number above the same number. And if some broken half occurs, instruct him to make it whole. The half of which total number you put above that same number; and if some broken half occurs, let him again make it whole. [...].

Afterwards, the trick is to ask for the subtraction of 9 as many times as possible, and to combine this with knowledge of when fractions had been repaired, and thus to reconstruct the number. Other similar problems follow, some of them with a method which presupposes an upper bound for the number, together with one where questioning allows to find out the points on three dice that have been thrown, and a few others. Some of them might as well have been presented as number problems elsewhere, without an imaginary partner – evidence of the lack of a strict boundary between mathematical problems, mathematical amusement, and riddles.

12.9, chess-board and other geometric series

Part 12.9 [B309;G486] deals with "the duplication of the chess-board, and some other rules".

The chess-board problem is one of the few recreational problems we can trace back to the early second millennium BCE. Originally the doublings were "until 30", then, after the invention and diffusion of chess, this was outcompeted by 64.

The oldest representative of the family we know is from Old Babylonian Mari, in north-eastern Syria [Soubeyran 1984: 30–35]; it doubles barley grains, and when their number becomes large it interprets the grain as a weight unit and uses larger measures. The next we know about is a Greco-Egyptian papyrus, perhaps to be dated to the Roman epoch. It starts from 5 shekel of silver, also goes until 30, and also expresses the higher multiples in adequate weight units. In the Latin collection *Propositiones ad acuendos iuvenes* ascribed to Alcuin of York [ed. Folkerts 1978: 51*f*], a king sends a servant successively to 30 manors, from each taking as many new men as he brought – here, obviously, metrology does not come into play. A few decades later (if we trust the ascription to Alcuin), al-Khwārizmī wrote a mathematical analysis of the problem in chess-board version, which we know from an extract or (rather) a paraphrase in Abū Kāmil's *Algebra* [ed. trans. Rashed 2012: 724–728], to which we shall return presently.

All of these have a simple doubling in each step. However, as in the interpretation of the counterfactual calculation (above, p. 79), Fibonacci suggests two different ways to understand the problem – and the passive *proponitur* suggests both already existed, Fibonacci did not share the modern way to signal or feign objectiveness by hiding behind a grammatical passive:

> The doubling of the chess-board is proposed in two ways, of which one is that the following square is double its antecedent; the other, when the following square is the double of all its antecedents.

Both possibilities are explored with details and perspective. The first begins in this way:

> The first doubling can be made in two ways, namely if we operate by doubling from square to square until the last square. The other way is that you double as much as the first square, and you have two; which two multiply in itself, they will be 4; which 4 are 1 more than the doublings[122] of the two squares. For example: In the first square put 1. In the second 2; which joined, make 3; the above-written 4 are 1 more than these three; when these 4 are multiplied in themselves, they make 16; which number is one more than the doublings of the double of the first two doublings, that is, of 4 squares. For example: In the first there is 1. In the second 2. In the third, 4. In the fourth 8; which, joined together, make 15; which is 1 less than 16. Further multiply 16 in itself, they make 256; which are 1 more than the number of doublings of the double of the above-written squares, that is, of 8 squares which occupy the first row of the chess-board. For example, in the first there is one. In the second 2. In the third 4. In the fourth 8. In the fifth 16. In the sixth 32. In the seventh 64. In the eighth 128; which joined together make 255; which the above-

[122] From square 2 onward, the contents of a square legitimately can be spoken of as a "doubling" (*duplicatio*); Fibonacci extends the usage to the first square.

> written 256 exceed by 1, as we have said: therefore multiply 256 in itself, they make 65536, one more than the doublings of the first two rows, namely of 16 squares.

Fibonacci then finds “one more than the doublings” of the first four rows, then of all eight lines of the chess-board, and then of two chess-boards. “And multiplying thus we can go on until infinity”.

So far so good. All this depends on well-known properties of continued proportions (not referred to by Fibonacci here, it is true).

Next, unfortunately, he goes on with a pedagogical explanation because the resulting huge numbers may be difficult to grasp. He suggests to fill a *chest* with the contents of the first two rows (augmented by 1, he forgets), that is, 65536 bezants. Then the first square of the third row contains 2 chests, he claims; it should obviously be 1 chest, even according to his own preceding text. Going on with doublings he claims that the second contains 4 instead of 2 chests, etc., until the last square of the fourth row, supposed to contain 65536 chests, reinterpreted as a *house*. Further, 65536 houses make up a *city*. The last square of the last row is then supposed – the same error persisting – to contain 65536 cities.

Another pedagogical illustration of the immensity of the number follows: if each unit represents a grain of wheat, identified with the weight unit *grain*, how many standard ships can be filled? The outcome, 1525028445[123] ships plus a fraction, is rightly said to be “like infinite, and uncountable”.

A comparison with Abū Kāmil’s paraphrase of al-Khwārizmī explains the mistake in Fibonacci’s chest-explication. Abū Kāmil [ed. trans. Rashed 2012: 726*f*] relates that

> Muḥammad ibn Mūsā [al-Khwārizmī] – may God be satisfied with him – has made this easy and accessible by saying: you put down the first, two, he put down the first as two in order to liberate himself from adding one; if he multiplies it with itself, one has four, which is the second. And if one multiplies four by itself, one has sixteen, which is the fourth. [...] If you want to double and add the squares of the chess-board, multiply the eighth, which is two hundred fifty-six, by itself. What you obtain is the sixteenth. Multiply the sixteenth square by itself, what you obtain is the thirty-second square. [...].

There are no chests here, but we find the idea of using 16 squares, that is, two rows, as a basis for simplified calculation. Fibonacci, when borrowing either from Abū Kāmil or some later writing depending on him (or possibly some other source depending directly on al-Khwārizmī),[124] has obviously not only overlooked that his source starts with

---

123 All manuscripts including **L** actually write 1725028445 [Giusti 2020: 488, apparatus; Giusti 2017: 206, apparatus].

124 In any case, there can be no doubt that this time Fibonacci depends on a written source – the details that connects him to Abū Kāmil are not of the kind that would survive oral transmission.

2 in the first case but also not discovered that the consequences he draws from it are wrong and contradict what he has said just before. The mistakes are also in **L**, showing that Fibonacci did not make a complete critical reading of his master-copy when preparing the final edition in 1228.

Is the first part of the “other explanation” then Fibonacci’s own?

Almost certainly not. As we see, its basic trick is also in Abū Kāmil’s text. Moreover, it was discussed in much more detail as a “practical way other than what most people are accustomed to do” in Damascus in 952 by al-Uqlīdisī [ed., trans. Saidan 1978: 338]; in 1449, al-Qalaṣādī [ed. trans. Souissi 1988: 75*f*] also described it – with the further observation that

> the number placed in the 9th [square] is equal to the sum of the numbers of the first 8 squares plus 1. [...] taking the square of the number in the 9th one gets the one in the 17th; taking the square of the latter one gets the one in the 33rd; doing the same with the latter one gets the number of the 65th, that is, the sum of the first 64 numbers plus 1, which is the first term.

So, the approach was widespread among Arabic mathematicians, and too close in the details to make us believe that Fibonacci made an independent exposition.

It is tempting to see the ship calculation, with its grains and conversion to higher metrological units, as pointing to the Mari and the silver problem; but this is probably a temptation that is better resisted until, *possibly but unlikely*, an intermediate text should be dug up in some library.

The alternative interpretation of the doubling problem (“the following square is the double of all its antecedents”) determines this sequence by stepwise calculation:

$$1 – 2 – 6 – 18 – 54 – 162 – 486 – 1458 – 4374$$

Fibonacci does not point out that the step factor from 2 onward is constantly 3, as follows from inspection, and as can also easily be argued: if square n holds C_n, the sequence being determined from

$$C_n = 2\cdot(1+...+C_{n-1}) \ ,$$

then

$$C_{n+1} = 2\cdot(1+...+C_n) = 2C_n+2\cdot(1+...+C_{n-1}) = 2C_n+C_n = 3C_n \ .$$

It would not be difficult to formulate this in words instead of symbols, just more lengthy. Instead Fibonacci observes that

$$(1+2+6)^2 = 81 = 1+2+6+18+54$$

while

$$(1+2+6+18+54)^2 = 6561 = 1+2+6+16+54+162+486+1458+4374 \ .$$

This rule is claimed with no hint of an argument to go on corresponding to squares no.

5, 9, 17, 33 and 65. It is obviously a parallel to what was used in the “first interpretation”, and would be evident if we knew the *sums* (not just the contents of the single squares) to be in geometric progression. In symbols, and if we take into account that $C_{n+1} = 2{\cdot}3^{n-1}$, it is easily established that they are: since, for $n{\geq}2$, from

$$C_{n+1} = 2\cdot\sum_1^n C_i$$

follows

$$\sum_1^n C_i = 3^{n-1} \ .$$

Moreover, for n taking on the values 5, 9, 17, 33 and 65, n–1 equals successive powers of two. A skilled medieval arithmetician would probably be able to establish it using words, perhaps (as in *Elements* VII–IX) supported by letter-carrying line segments. However, Fibonacci seems not to posses the building blocks for the argument, and therefore offers none.

Instead, he merely uses the rule to find

$$\sum_1^{65} C_n = 3{,}433{,}683{,}820{,}292{,}512{,}484{,}657{,}849{,}089{,}281 \ ,$$

from which he concludes that

$$\sum_1^{64} C_n = \tfrac{1}{3}\cdot\sum_1^{65} C_n = 1{,}144{,}561{,}273{,}430{,}837{,}494{,}885{,}949{,}696{,}427 \ ,$$

Even the latter result is correct – namely (since $C_{n+1} = 3C_n$ for n≥2) because

$$\sum_1^{65} C_n = 1{+}2{+}\sum_3^{65} C_n = 3{+}3\sum_2^{64} C_n = 3\sum_1^{64} C_n \ .$$

However, Fibonacci’s argument (should his words be meant as an argument) is quite opaque.

Given the absence of genuine arguments, even this second interpretation of the chess-board problem and the way to deal with it are almost certainly borrowings – seemingly made with little understanding. The similarity of the 5–9–17–33–65-argument with what we saw in the treatment of the “first interpretation” indicates that Fibonacci took both from the same source – which makes it doubtful that he borrowed *directly* from Abū Kāmil.

From where he borrowed is a guess, but since we know that Abū Kāmil’s *Algebra* circulated in al-Andalus in the 12th century, and since the sophisticated version of the “unknown heritage” seems to have come from there (together with the secondary stratum of chapter 14 and the bulk of part 15.1, as we shall see), the best guess would be al-Andalus.

A further number of problems about geometric growth or decrease follows:

– [B311;G489] One δ lent at composite interest becoming 2 δ in five years, followed over five-year periods until 50 years, and then squared to give the outcome of 100 years; with the variant that 20 hides are sold, the first for 1 δ, the second for 2 δ, the third for 4 δ, etc.

– [B311;G489] Seven old women go to Rome, each carrying 7 pilgrim's staffs, each staff carrying 7 small sacks, each sack containing 7 breads, each bread with 7 knives, each knife provided with 7 sheaths. The sum is found by stepwise calculation, showing that if Fibonacci knew the sum formula for geometric series, he did not think of it in this connection. There is no reason to doubt a connection to the similar problem in the "inventory of a household" in the Rhind Mathematical Papyrus [ed. trans. Peet 1923: 121]: 7 houses, 49 cats, 343 mice, 2301 [miswritten for 2801] spelt. As said by Peet, "evidently based on a nursery problem [...] seven houses, in each 7 cats, [...]".
– [B312;G490] A tree with 100 branches, on each 100 nests, in each nest 100 eggs, in each egg 100 birds.
– [B312;G490] 100 £, each 4 growing with profit to 5 in a year, followed over 18 years.
– [B313;G491] Somebody originally possessing 100 bezants travels through 12 cities, in each spending $^1/_{10}$ of his money.
– [B316;G495] From a cask containing 100 measures of wine, each month $^1/_{10}$ is drawn; explained to be a parallel to the preceding question, for which reason the remainder after ten months can be taken over from there.

Finally comes a problem of a different kind [B316;G495], namely a variant of the garden problem (above, p. 26), where somebody leaves a city with 10 gates, paying at the first $^2/_3$ of his bezants plus $^2/_3$ of a bezant; at the second $^1/_2$ of what he has left, plus $^1/_2$ of a bezant; at the third $^1/_3$ of what he has left, plus $^1/_3$ of a bezant; ... ; at the tenth, $^1/_{10}$ of what he has left, and $^1/_{10}$ of a bezant. After which he is left with 1 bezant. Two procedures are explained: firstly, stepwise backward calculation, which because of the specific numerical parameters is fairly easy – going backwards gate by gate, what he had left was $^{11}/_9$, $^{12}/_8$, $^{13}/_7$, etc.; secondly, the method introduced with the repeated travels (above, p. 89), reduction of all disbursements to final value.

The appearance of bezants does not demonstrate that Fibonacci knew the problem from Byzantium, even though it suggests so (cf. note 102); the use of the sequence of aliquot parts including $^2/_3$, however, points in the same direction (cf. p. 70). Combined, the two observations makes it highly plausible that the problem (in this specific form and with these parameters) had been presented to Fibonacci in Byzantium – the challenger (if presented as a challenge) probably knowing that it could be solved in the first way, since it fits the sequence of aliquot parts so nicely. The second method, close to what Fibonacci has used before and not fitted to the specific parameters, is likely to be his own contribution.

Chapter 13 – *elchatayn* rule

Chapter 13 deals with "the *elchatayn* rule, and how by means of it almost all abbacus questions can be solved", where *elchatayn* is the Arabic term for the double false position (as explained in the very beginning of the chapter),[125] and "abbacus questions" are such as are presented in chapter 12. (On the rule itself, see note 3.)

After this terminological clarification follows a presentation. It does not refer to its connection to the alligation rule; this is not excepional, nobody seems to do so.[126] Instead it is said [B318;G499] that the solution

> is found according to the proportion of the difference from one position to the other, that is that it falls under the rule for the fourth proportional, in which three numbers are known; by which the fourth unknown number is found, that is, the truth of the solution, is found; of which the first number is the difference between the number of one false position and the other. The other is how one gets closer to the truth by that same difference. The third is what is lacking in the approximation to the truth.

Unfortunately Fibonacci does not tell what the fourth proportional represents (namely, how much one has to go beyond the second position before reaching the correct value[127]) but leaves that to the ensuing example, which also shows how to proceed if one error is in deficit, the other in excess.

The question is, 100 *rotuli* are worth 13 £, what is 1 *rotulo* worth? The two positions made are that the value is 1 ß, and 2 ß. With the former position. 100 *rotuli* would be worth 5£, with the latter 10 £. That is, increasing the position from 1 ß to 2 ß, we approximate the truth by 5 £. But from the second position we still have to approximate it by 3 £ more, whence the proportion spoken of in the quotation.

Afterwards, two positions are proposed that both lead to an excess (7 £ and 3£), and two where on leads to a deficit and one to an excess (3 £ and 2 £). The principles of these calculations are similar. We observe that none of this has to do with the proper method of two false positions – they make linear extra- or interpolations from one of the two positions. However, on p. [B319;G501] comes an observation that

> there is another way for the *elchataym;* which is called the rule of augmentation and diminution. And the first error is multiplied by the second position; and the second error by the first position. And if the errors are both diminished, or both added, the smaller

---

125 Namely the genitive dual *khata'āni* of *khaṭa'*, "error, mistake" – thus "of two errors". The not uncommon derivations from *Khiṭāi*, Central-Asiatic Turkish for China, can be safely disregarded, cf. the explanation in [Needham 1959: 118 n. b].

126 However, cf. below, p. 110, on Liu Hui.

127 The precise words presuppose that both guesses err by deficit, and that the larger position gives the better value – otherwise some formulations have to be twisted.

> outcome of the aforementioned multiplication is subtracted from the major, and the remainder is divided by the difference between the errors; and in this way the solution to the question is found. And if one of the errors be added, and the other diminished, then both multiplications are added together, and the outcome is divided by the errors joined.

This is then applied to the same example, after which follows proofs by means of line diagrams. In the first [B320;G502], for both errors being in deficit, this diagram is used:

$$\underline{a \qquad g \qquad d \qquad b}$$
$$\underline{e \qquad i \qquad z}$$

Here, *ag* represents the first position (for our convenience P_1), *ad* the second position (P_2), *ez* the first error (E_1), *iz* the second error (E_2). The solution "according to the proportion of the difference from one position to the other" can then be expressed

$$\frac{E_2(P_2-P_1)}{E_1-E_2}+P_2 = \frac{iz\cdot gd}{ei}+ad$$

Fibonacci speaks about multiplication and division, that is, he deals with the segments as numbers.

More interesting is the proof of the solution by means of "augmentation and diminution",

$$\frac{E_1\cdot P_2-E_2\cdot P_1}{E_1-E_2} ,$$

in terms of the line segments

$$\frac{ez\cdot ad-iz\cdot ag}{ei} = \frac{(ei+iz)\cdot ad-iz\cdot ag}{ei} = \frac{(ei+iz)\cdot ad-iz\cdot ag}{ei} .$$

Now, $ad\cdot iz = (ag+gd)\cdot iz = ag\cdot iz+gd\cdot iz$. Moreover, since increase in the position is proportional to decrease in the error,

$$\frac{ei}{iz} = \frac{gd}{db} ,$$

whence $iz\cdot gd = ei\cdot db$. Therefore $ad\cdot iz = ag\cdot iz+gd\cdot iz = (ag+gd)\cdot iz = ag\cdot iz+ei\cdot db$. Inserting this we get

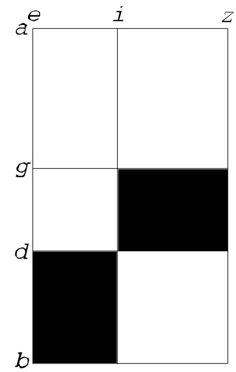
$$\frac{ez\cdot ad-iz\cdot ag}{ei} = \frac{ad\cdot ei+ag\cdot iz+ei\cdot db-ag\cdot iz}{ei} = \frac{(ad+db)\cdot ei}{ei} = \frac{ab\cdot ei}{ei}.$$

That is, Fibonacci can conclude that

$$\frac{E_1\cdot P_2-E_2\cdot P_1}{E_1-E_2} = ab ,$$

“as was to be shown” (*quod opportebat ostendere*).

This Euclidean phrase (used regularly in the Latin translation made directly from the Greek [ed. Busard 1987][128]) is quite fitting. The demonstration is, in Euclidean style, a piece of synthesis, showing that the rule is correct, but giving the reader no idea about how it was devised. The tools made use of are very close to being algebraic (cf. also note 82). The reference to products of segments is noteworthy. In classical mathematics, these would have been dealt with as rectangles, as shown here, and the final division as finding a side of a rectangle from its application to the other side; Fibonacci makes no attempt to respect this canon, even though it would have been easy to do so (and even though he uses procedures elsewhere which do respect them – we may assume that his choice of style depended on his source[129]): since the two black rectangles are equal, the difference between the rectangles $ez \times ad$ and $iz \times ag$ is seen immediately to equal the rectangle $ei \times ab$. Easier, one would say, than the quasi-algebraic proof actually given.



Quasi-algebraic proofs based on line segments are also given for both errors being in excess, or for one being in deficit, the other in excess.

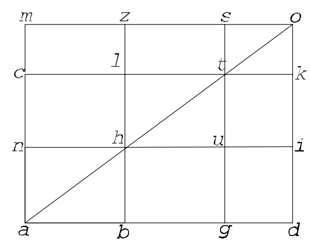
Before we address what Fibonacci does with this when dealing with problems, it may be fitting to look at some elements of the earlier history of the method of the double false position.

The earliest appearance of the rule in known sources is in the seventh chapter of the Chinese *Nine Chapters* [ed., trans. Chemla & Guo 2004: 549–597], to be dated to the first century CE. The chapter deals with the method of “excess and deficit”; the text only gives a numerical prescription explaining how the numbers are to be placed on the counting board, not very different from the marginal schemes used by Fibonacci (also reflecting the used of a board, probably a clayboard). A commentary by Liu Hui from 267 CE then explains how the errors are to be balanced, not referring to alloying but following exactly the same principle [Chemla & Guo 2004: 549*f*].

[128] That Fibonacci knew this translation is nothing new, see [Folkerts 2004: 109*f*] and [Busard 1987: 18*f*].

[129] We notice the letter sequence *a–b–g–d–...*, pointing to the use of an Arabic source. Other arguments which similarly seem to be of Arabic origin follow the classical canon; but there is evidently no reason to assume that Fibonacci drew on a single Arabic source for all arguments involving lettered diagrams.

The *Nine Chapters* make more advanced use of the method than done elsewhere;[130] that does not concern us at this point. Let us instead turn to the earliest known Arabic presentation of the rule – actually a proof of its validity, no mere presentation, the rule itself is supposed to be already familiar. It was written by Qusṭā ibn Lūqā in the second half of the ninth century. A first sketched proof [ed. trans. Suter 1908: 113*f*] is similar to Fibonacci's first proof, though formulated without reference to proportions and not using a line diagram: the positions are supposed to be $P_1 = 4$ and $P_2 = 8$, and the respective errors $E_1= 7$, $E_2= 4$; increasing the position by 4 therefore reduces the error by 3, and therefore increasing the position by 1 decreases the error by $^3/_4$. In order to eliminate the error we must therefore increase the position from 8 to $8+4\div{}^3/_4 = 13\,^1/_3$.The rule is then argued to agree with this calculation.



Qusṭa does not leave the matter there, however, but gives a strict proof, based on diagrams (lost in the surviving manuscript, but reconstructible from the text – here after Heinrich Suter). *ab* and *ag* represent P_1 and P_2, *od* the true value, and *hz* and *ts*, respectively E_1 and E_2 (in the best Euclidean style, Qusṭā performs a construction). Then

$$ad : do = ag : gt = ab : bh.$$

Then, $E_1{\cdot}P_2 = \sqsubset\!\sqsupset mu$, $E_2{\cdot}P_1 = \sqsubset\!\sqsupset ml$, for which reason $E_1{\cdot}P_2 – E_2{\cdot}P_1$ equals the gnomon *nuszlc*. But according to *Elements* I [prop. 43], $\sqsubset\!\sqsupset zt = \sqsubset\!\sqsupset ti$, for which reason the gnomon equals $\sqsubset\!\sqsupset ci$. But $cn = E_1 – E_2$, whence

$$ni = \sqsubset\!\sqsupset ti \div cn = \frac{E_2{\cdot}P_1 - E_1{\cdot}P_2}{E_1 - E_2}.$$

For the other cases (both errors in excess, and one in excess and one in deficit), similar

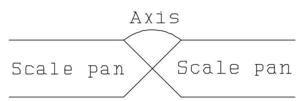
130 For example [ed., trans. Chemla & Guo 2004: 559],

> Let us assume that something is bought in common and that, if each one pays 8, there is an excess of 3, and if each one pays 7, there is a deficit of 4. It is asked how much is the quantity of persons and the price of the thing.

rigorous proofs are given.[131]

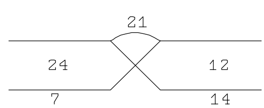
This, obviously, does not present us with the invention, it is a justification of a method that is already known. The initial proof-sketch appears to reverberate in Fibonacci's first proof, but obviously is not Fibonacci's direct source.

From the Maghreb, where we might believe Fibonacci to have picked up his direct inspiration, we know a different approach – exemplified by Ibn al-Bannā’ ’s *Talkhīṣ* (a highly appreciated summary of calculation techniques from the decades around 1300). Ibn al-Bannā’ [ed. trans. Souissi 1969: 88] speaks of the method of "scale pans", illustrated by this drawing (in the edition of the Arabic text, no words are written into the diagram):



> You put the known number of the hypothesis on the axis; you take for one of the scale pans whatever number you like; you submit it to the operations indicated in the hypothesis, addition, reduction, or otherwise; then you compare the result with the number put on the axis. If you find it exactly, then this scale pan is the unknown number. If the result is wrong, write the error above the scale pan if it is in excess, and below if it is in deficit. Then take for the other scale pan what even number you like, except the first one; operate in the same way as with the first one. [...].

In the Maghreb, this remained a favourite way to arrange the calculation. In 1449, al-Qalaṣādī [ed. trans. Souissi 1988: 68] would still explain it, filling in numbers for clarity,



representing the solution of the problem "to find a number such that the sum of its third

---

[131] Rigorous only, as it stands, in the situation where a position 0 would give an outcome 0; however, if *od* represents the excess between the requested outcome over the outcome for position 0, the argument remains valid. In the language of analytic geometry, the argument as given holds only for problems $y = \alpha x$ (where a single false position would suffice); the reinterpretation of *od* allows the argument to hold for the general first-degree equation, $y = \alpha x+\beta$.

Rigorous moreover, as Qusṭā points out, only in problems where no square and cube roots appear. Qusṭā translated Diophantos into the language of Arabic *al-jabr* and also here speaks of the unknown number as a *māl*, a "possession"; that is, for him, problems of the second and third degree deal with the *māl* and its square or cube root.

and it fourth is 21", with positions 24 and 12.[132]

It should be obvious that Fibonacci did *not* borrow this, in spite of his reference to the method by an Arabic name. Is it possible to go beyond this negative conclusion?

Firstly, as noticed, Qusṭā's first calculation might seem to reverberate in Fibonacci's text. That is not impossible, the manuscript used by Suter was copied in India in 1722, so Qusṭā's ideas may have circulated well and inspired widely. The proof by means of line segments is also a borrowing – proofs constructed by Fibonacci himself use the Latin lettering sequence *a–b–c–...*, as pointed out in note 96 ald elsewhere. The sequence *a–b–g–...* must be based on either Greek or Arabic material. From Byzantium, however, we do not know of a similar use of line diagrams (nor in material in the *Liber abbaci* which for other reasons can be linked to Byzantium); in the *Liber mahameleth*, on the other hand, they abound (always with sequence *a–b–g*). Since Fibonacci's proofs for double false position are not to be seen in what we know from the Maghreb, the best guess is therefore that Fibonacci took over his line proofs from al-Andalus.

The rule itself is likely to have spread over the whole region between China and the Mediterranean among traders and similar groups of practical calculators, and to have been taken up by "mathematicians", that is, those engaged more centrally with mathematics and mathematics teaching: those in China approaching it with *their* conceptual and practical tools, those in Arabic and Mediterranean areas with theirs – the Arabic world not learning from the Chinese teaching of officials, nor (already for obvious reasons of chronology) the Chinese teachers learning from Qusṭā and his kin. That was already proposed by Randy Schwarz [2006: 292]. The way "mathematicians" East and West justified solutions to problems belonging to the family "purchase of a horse" is a parallel example – see [Høyrup 2016: 465–469] – evidence that this kind of diffusion *cum* local justification is possible.

13.1, problems already dealt with

Let us return to the *Liber abbaci*. After the proofs [B322;G505] onward come applications of the rule to problems select solved (sometimes with different numerical parameters) in the preceding chapters by other methods. At first it is asked how to mix silver at 3 ounces with silver at 6 ounces in order to get silver at 5 ounces. Strikingly, instead of using the double false procedure, after having made two positions and found the corresponding errors, Fibonacci makes a detailed calculation along the lines of his first proof – yet without reference to proportions, using instead the rule of three (as usually not identified by any name). A parallel example [B323;G506] instead applies the standard

---

132 Interestingly, a treatise written in *ca* 1575 by a Morisco and combining a Castilian treatise with material drawn from the Maghreb tradition uses the scala-pan diagram for an alligation problem [Ageron & Hedfi 2020: 40*f*]; there must then have been some awareness in the Maghreb (the diagram is not in the Castilian treatise) that the double false and alligation calculation are analogous.

procedure referred to earlier as “augmentation and diminution”.

After this [B323;G5+7] comes what Fibonacci himself calls a “noteworthy question” of type “lazy worker” (cf. above, p. 26):

> Some worker should receive 7 bezants in a month if he works; and if not, he should give back to the master of the undertaking 4 bezants at monthly rate. And sometimes he worked, and sometimes not. So that, finished the month, he was to receive from the master 1 bezant.

Exactly the same problem is solved in the seventh *differentia* of chapter 11 ([B160;G275]; cf. above, p. 77), as one of the examples of how the alloying model can be applied to other questions.[133] Here, the problem is solved by the standard procedure, complicated only by the need to convert the monthly rates into payment per day.

The following problem [B324;G508] is a slight numerical variation on a tree problem from [B174;G298]: $\frac{1}{4}$ $\frac{1}{3}$ of a tree is under the ground, and 20 ells instead of 21 palms are above. Again, the standard procedure is followed, and Fibonacci teaches how to make convenient positions that eliminate the fractions of the problem statement (P_1 = 12 ells, P_2 = 24 ells). Then [B324;G508] comes an exact repetition of a complicated wage problem first solved [B186;G317] by means of the technique for combined works, and then an equally exact repetition of the give-and-take problem [B190;G324] which already served to introduced the *regula recta* (above, p. 81). A problem about four men finding a purse [B326;G511] changes the phrasing but not the numerical parameters of a counterpart from [B218;G362]. Five men buying a horse [B327;G513] is an exact repetition of a problem on p. [B234;G384]. The first problem about repeated travels ([B258;G417], cf. above p. 89) is repeated on p. [B329;G515] in slightly more compact words but the same parameters; similarly, a house-renting problem (see above, p. 92) from [B270;G434] turns up in changed words but with the same parameters on p. [B329;G516]. The first of the “rambling” problems ([B276;G442], cf. above, p. 92) is repeated with slightly different numerical parameters on p. [B330;G517]. On p. [B330;G518], a problem about men having money changes their number to 6, while a corresponding problem on p. [B285;G455] from speaks of 5 men.[134]

The situation of the next problem [B331;G519n] is slightly more puzzling (not present in all manuscripts). It is the traditional version of the two-tower problem, in which the two birds arrive at the same moment (see above, p. 46). In the 1228 edition, the same problem is solved in the “geometric” part of chapter 15 [B398;G611], and thus not where “abbacus” problems are supposed to be. As it turns out, the early version of chapter 12

---

133 At a first glance, that solution seems suspicious, since it adds the 30 days to the 1 bezant. Actually, what Fibonacci does is blameless, apart from the excuse to add 30; all he does is to shift the zero so as to avoid coin of –4 ounces per mark.

134 In both cases, the sum of the total possession less that of each of the men is given.

in manuscript **L** actually contains a corresponding problem where it would be expected [ed. Giusti 2017: 192]. When preparing the final version, Fibonacci must have moved it while editing it (or perhaps while borrowing from a different source); since the problem as it looks here and as it turns up in chapter 15 are identical (both deviating from what is found in **L**) Fibonacci must have been fully aware of what he was doing. Giusti suggests the problem to belong "to a first version of chapter 13" hypothesis [2020: cvii] – but if so, then to an intermediate version, later than the one represented by **L** chapter 12, and thus evidence that the *Liber abbaci* was an ongoing project that cannot be reduced just to an original from 1202 and single revision from around 1228.[135]

The first part of chapter 13 closes with four more problems that repeat (in one case with a small numerical variation) problems that have been solved before.[136] One, [B334;G522], dealing with four men buying a horse, is noteworthy for the need to introduce a layered structure and a particular terminology: for each of two "universal positions", a second *elchataym* procedure with "particular" positions is introduced. They are introduced by the phrase "you will call it"/"it will be called" (*appellabis*/ *appellabitur*), suggesting that this terminology is Fibonacci's own invention, not something already existing.[137] As Fibonacci points out, these nested procedures are to be considered with not little care.[138] I have observed the technique in only one abbacus text preceding printing (below, p. 291), where I shall return to how it works; the appearance in two printed books will be mentioned on pp. 329 and 346.

135 It can also be imagined, however, and may all in all be more plausible, that Fibonacci at first simply left out the two-tower problem when preparing what became the 1228-version from chapter 13 part 1 and then, in a second instance, when he had inserted the problem in chapter 15 part 2, reestablished the problem in chapter 13 part 1, but now with the new words and parameters. We would still have evidence of an ongoing project, but now the Boncompagni manuscript and its family end up being the final, not the first variant of the 1228 version. In any case it seems we have positive evidence that we need to speak of a "1228 family", not simply of a single "1228 version" represented by a number of manuscripts with different copying errors, omissions, corrections gone wrong, and contaminations (and whatever else may happen within manuscript family descending from a single archetype). Cf. below, note 153.

136 [B332;G519] repeats [B198;G334] with a minor variation; [B333;G521] repeats [B214;G357]; [B334;G522] repeats [B245;G400]; [B335;G524] repeats [B293;G465].

137 In contrast we encounter on p. [B352;G547] the expression "which are called" (*dicuntur*), signalling that a concept already exists and has been adopted by Fibonacci (see below, p. 117).

138 *[...] in hac questione etiam et in similibus plures elchataieym necessarii sunt, in quibus non modicum considerandum est.*

13.2, new problems

Part 2 is announced [B336;G526] as applying the double false position to problems that are *not* dealt with elsewhere in the book. The types are not new, however, they still represent "abbacus questions".

The starting point [B336;G526] is a give-and-take problem, solved first by means of two false positions, afterwards also in the way taught earlier on. Seven more of similar types follow. Two of them [B340,342;G531,533], both about five men, each of whom after having received given fractions of the possessions of the other will have a given amount, make use of nested *elchataym* procedures (the second of them even in three levels). Four explain alternative procedures, of the kind taught for similar problems in chapter 12. Of particular interest is the first of two alternative solutions to the problem on p. [B338;G529], which in symbols can be expressed

$$A+{}^{1}\!/_{3}(B+C) = 14\ ,\quad B+{}^{1}\!/_{4}(C+A) = 17\ ,\quad C+{}^{1}\!/_{5}(A+B) = 19\ .$$

Without referring to the *regula recta*, Fibonacci posits B+C to be a *thing* (*res*), and C alone to be *part of a thing*, afterwards spoken of simply as *part* (*pars*). This leads him to the two equations

$$^{11}\!/_{12}\,thing-{}^{3}\!/_{4}\,part = 13\,{}^{1}\!/_{2}\ ,\quad {}^{4}\!/_{5}\,part+{}^{2}\!/_{15}\,thing = 16\,{}^{1}\!/_{5}\ .$$

Multiplying the latter by ${}^{5}\!/_{6}$ he gets the same number at the right-hand side,

$$^{11}\!/_{12}\,thing-{}^{3}\!/_{4}\,part = 13\,{}^{1}\!/_{2}\ ,\quad {}^{2}\!/_{3}\,part+{}^{1}\!/_{9}\,thing = 13\,{}^{1}\!/_{2}\ ,$$

which allows him to find *thing* : *part* = 51 : 29, whence (*thing*–*part*) : *part* = (51–29) : 29 = 22 : 29, that is, B : C = 22 : 29. Inserting instead the ratio *thing* : *part* into the first equation allows Fibonacci to find that 22 : (*thing*–*part*) = 56 : 27; etc. Being much less versed in proportion techniques than Fibonacci, we will probably find the way the equations are solved clumsy, but the use of these is what allows Fibonacci to speak of the procedure as "according to an investigation of proportions", similarly to what he did done in the solution of the first problem about the "purchase of a horse", cf. above, p. 88. Once more, Fibonacci obviously sees nothing remarkable in the use of two algebraic unknowns.

Fibonacci appears indeed to have borrowed not only the general idea of using two algebraic unknowns but also the name "part" for the second unknown. The two algebraic unknowns used by al-Karajī when he solves the grasping problem (above, p. 98) are indeed *šai'* ("thing") and *qasm* ("part"). In a give-and-take problem [ed. trans. Woepcke 1853: 139], al-Karajī uses *šai'* and *qist*, "share"/"measure" as his unknowns; even *qist* can thus have given rise to a translation "part" (as we see, here at least al-Karajī does not follow the habit to use coin names for unknowns beyond the *thing*). Since Fibonacci does not solve the grasping-problem by means of explicit algebra, he is not likely to have known the *Fakhrī* directly, but al-Karajī's term may reflect more general ways unknown to us, or it may have been borrowed from him by later Arabic writers. Fibonacci's explanation of *part* as "part of a *thing*", which as no counterpart in al-Karajī's text, points in the same direction.

Chapter 14 – square and cube roots

Chapter 14 deals with the finding of and operation with square and cube roots, and with such binomials (*binomi*) and apotomes (*recisi*) as are dealt with in *Elements* X (and a few more).

A puzzling preamble

At first [B352;G547] comes this preamble:

> Let it be me permitted to insert in this chapter about roots certain necessary matters, which are called keys [*claves*]; since they are all proved by clear demonstrations in Euclid's Second, it will suffice beyond their definitions to proceed by means of numbers. The first of which is that, when a number is divided into any number of parts, then the multiplications of these parts in the whole divided number, joined together, equal to the square of the divided number, that is, the multiplication of the same number in itself.[139] For example: let 10 be divided into 2, and 3, and 5. I say that the multiplications of the two, the three, and the five in 10, evidently 20, and 30, and 50, equal the multiplication of 10 in itself, that is, 100. [Similar versions follow of *Elements* II.1; II.4; and the corollary $2a{\cdot}(a+b)+b^2 = a^2+(a+b)^2$]. Further, if a number is divided into two equal parts, and also into unequal parts, then the multiplication of the smaller part by the larger, together with the square of the number which there is from the smaller part until the half of the whole divided number will be equal to the square of the said half [*Elements* II.5; follows a numerical example and a similar version of II.6]. To the latter two definitions are reduced all questions from *aliebra* et *almuchabala*, that is, in the book of *contemptio*[140] and *solidatio*. Then, finished this, this chapter is divided into five parts. Of which the first is about the finding of roots; the second about their multiplication in each other, and that of binomials.[141] The third about their addition. The fourth about their mutual detraction. The fifth about the division of roots and of binomials.

As shown by the present tense "are called" (*dicuntur*), the notion of "keys" is borrowed, not introduced by Fibonacci himself, who in that case would use the future tense – see above, p. 115. The reference to *aliebra almuchabala* that closes the presentation of the keys leaves no doubt that it is adopted from an Arabic source. However, this use of "keys" seems not to be known from extant sources – in these, as exemplified by al-

---

[139] *Elements* II.1, applied to two equal lines; or, if we prefer, *Elements* II.2 generalized to division into several parts.

[140] As pointed out by Enrico Narducci [1858: I, 23], the word however spelled will certainly be a mistake for *contentio*, "comparison/contrast/struggle".

[141] The latter clumsy phrase (clumsy also in Latin) does not correspond precisely to either of the headings for a designated part 2 – see imminently.

Kāšī's *Miftāḥ al-ḥisab*, the "key" is that which unlocks a subject. We thus have to think of a region whose theoretical knowledge did not spread to the rest of the Arabic world, which once again leads us to al-Andalus; the regular use of the "key"-version of *Elements* II.5–6 in the *Liber mahameleth* (and in chapter 15 of the *Liber abbaci*, parts 1 and 2, as we shall see) confirms this inference, even though "keys" are never spoken of there.

On the other hand, the translation offered for *aliebra almuchabala* is puzzling. After Narducci's emendation it is quite adequate – but wholly different from what we encounter elsewhere at the time, and also from the translation given by Fibonacci when he presents the topic (see below, p. 141). The only plausible explanation is that this translation is part of the borrowed text, and thus that Fibonacci (at least here, but quite likely also elsewhere when he borrows from al-Andalus) takes advantage of an existing Latin translation which has now been lost.[142] It is perhaps worth noticing that Fibonacci includes no Iberian

---

[142] This is not the only occasion where Fibonacci copies an existing translation rather closely without betraying his source. One source which we possess is Abū Bakr's *Liber mensurationum* as translated by Gerard of Cremona. An example that illuminates Fibonacci's way to use a source is this problem about a rectangle, for which the sum of the two sides and the area is 62, while the difference between the sides is 2. In Abū Bakr's formulation [ed. Busard 1968: 94; ed. Moyon 2017: 172] the solution runs

> The way to find this will be that you diminish 62 by 2, and 60 remains, then join 2 to the half of the number of sides, and 4 results. Join this to 60, and 64 results. Thus take its root, which is 8. This is indeed the longer side. And if you want the shorter, diminish 8 by 2, and 6 remains, which is the shorter side.

Fibonacci [ed. Boncompagni 1862: 66] prefers that "64 result" and that "6 remain", seeing numbers as collections of units and not as single entities; but apart from that his text is word for word the same (also in Latin).

The identification of 4 as 2 plus half of the number of sides is if not fallacious then at least misleading. Jean de Murs' *De arte mensurandi* [ed. Busard 1998: 187*f*] betrays the underlying geometric idea (a reduction to the problem of a square plus its four sides), which corresponds to this symbolic calculation (a and b being the two sides):

$$ab+a+b = 62\ ,\quad a = b+2$$
$$(b+2)b+b+b = 60$$
$$b^2+2b+b+b+2+2 = 60+4$$
$$b^2+4b+4 = 64$$
$$(b+2)^2 = 64$$
$$a = b+2 = \sqrt{64} = 8$$

Fibonacci probably does not understand why this works; indeed it only does because $a–b = 2$; in the general case where $a = b+n$, what should be added in the third line is $(^{n+2}\!/_2)^2$. Probably for this reason he adds this explanation by means of algebra (explanation of the terminology follows below, p. 142):

> posit the smaller side as a thing, then the larger will be a thing and two dragmas. From the multiplication of this shorter side by the longer results the area. Therefore multiply the thing,

locations (neither al-Andalus nor Castile) when listing the places where he had learned after Bejaïa – "Egypt, Syria, Greece, Sicily, and Provence", see above, p. 60. This might imply that he knew not only the barter treatise (above, p. 73) but also what he had taken from al-Andalus from written material, not from direct confrontation (cf. also note 165 below).[143]

"Keys" of a similar kind return in Pacioli's *Summa* [1494: 88r–89v] – see below, p. 341; there, they are statements (theorems not supported by proofs) about numbers in continued proportion. I know of no intermediate source mentioning them, and it is next to certain that Pacioli was not inspired by Fibonacci on precisely this point;[144] yet since Pacioli makes critical observations to some of them, he must be borrowing the group as a whole from some preceding treatise.[145]

> that is the smaller side, by the thing and by two dragmas, and you will have a possession and two roots as the expanse; which, if you add to them the two sides, namely 2 roots and 2 dragmas, will be a *census* and 4 roots and 2 dragmas, which equal 62 dragmas. Remove 2 dragmas in each place, and a *census* and 4 roots remain, which equal 60, and so on.

That is, at the point where it is clear that what should be added is not 2+2 but $(^{2+2}/_{2})^2$ Fibonacci stops, hiding that he does not understand.

143 One barely possible identification of Fibonacci's source for the "keys" should be mentioned. The *Liber mahameleth* refers repeatedly to an algebra chapter that has been lost in all extant manuscripts but must have been there originally (at least in the Arabic original). If *restauratio*/*restaurare* or *opponere*/*oppositio* were used in the standard way in the rest of the book this would be excluded. But they are not. *Opponere* occurs in a reader's marginal commentary [ed. Sesiano 2014: 146 apparatus], *restaurare* once in a *regula recta* solution of a first-degree problem [ed. Sesiano 2014: 243] and once where it designates a *multiplicative* completion (a normalization) [ed. Sesiano 2014: 354], for which it also serves a couple of times in Abū Bakr's *Liber mensurationum* [ed. Busard 1968: 88, 99] but never in al-Khwārizmī's nor in Abū Kāmil's algebras. *Restauratio* and *oppositio* are wholly absent from those parts of the *Liber mahameleth* which we possess; nothing thus excludes that what Fibonacci draws on here could be the lost algebra presentation from the *Liber mahameleth*. But positive evidence is completely absent.

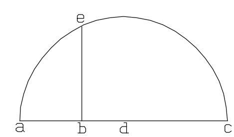
144 [Pacioli 1494: 106v–111r] indeed contains a list of no less than 66 *conclusiones seu evidentiae* (also theorems not supported by proofs), which starts with close analogues of Fibonacci's keys (the first of them, however, closer to *Elements* II.1 than Fibonacci). They are *not* called "keys".

145 If we suppose Pacioli's source to have been written in Italian, one is tempted to point to Antonio de' Mazzinghi (presented below, p. 230) since he is the only algebraist know by name from the preceding centuries whose level would have permitted him to produce this set of "keys", and since he was furthermore interested in continued proportions; but care should be taken, the anonymous author of the manuscript Florence, BNC, fondo princ. II.V.152 (below, p. 240) was also a brilliant algebraist, as we shall see, and there may well have been others. Any historian working on matters preceding book-printing (and quite a few others too) should realize that there is much less cheese

The copied section ends with the puzzling Latin translation. Indeed, "finished this", the description of the contents of the chapter follows.

14.1, extraction of square roots

After the preamble come examples of the algorithm for extracting square roots, presented as the "abbacus way" (*secundum abaci materiam*) – which turns out to be what Jacopo and other abbacus authors speak about as the "closest" approximation (above, p. 36); however, Fibonacci knows and shows in his first example [B353;G548] that further approximation is possible, where the first approximation to $\sqrt{10}$ is found to be $3\,{}^{1}\!/_{6}$, while iteration gives $3\,{}^{1}\!/_{6}-{}^{1}\!/_{228}$ as second approximation. Further [B353;G549], $\sqrt{743}$ is shown to be approximately $27\,{}^{14}\!/_{2\cdot 27} = 27\,{}^{7}\!/_{27}$; similarly for $\sqrt{8754}$, $\sqrt{12345}$, and $\sqrt{927435}$. In the latter case [B355;G551] Fibonacci also describes how to make the second approximation by iteration of the procedure, without however performing the appurtenant tedious computations.



Between the finding of $\sqrt{10}$ and the following examples Fibonacci inserts an observation concerning the relation between the quantity of digits in a number and in its square root, and a geometric construction of a square root, building on either *Elements* II.14 or *Elements* VI.13; no proof is offered, it is just said to be "clearly demonstrated in geometry". The lettering being *a–b–c–d–e*, it is likely to have been produced anew by Fibonacci himself.[146]

After calculating $\sqrt{927435}$ Fibonacci offers [B355;G552] an explanation of how to find roots with higher precision by an alternative method based on an insight already explained in al-Khwārizmī's algebra [ed. trans. Hughes 1986: 243*f*]; the example used is

$$\sqrt{7234} = \frac{1}{100}\cdot\sqrt{10000\cdot 7234} = \frac{1}{100}\cdot 8505+\frac{4975}{2\cdot 8505} \approx \frac{1}{100}\cdot 8505\frac{1}{4},$$

in the end expressed in $\mathbf{V_F}$ as $85\ {}^{1}\!/_{20}\ {}^{1}\!/_{400}$ (in this unusual order). Other manuscripts have

than holes.

A reference to *Fibonacci's* keys is found in the mid-15th-century "abbacus encyclopedia" Florence, BNC, Palatino 573, fol. 317ʳ [ed. Arrighi 2004/1967: 188*f*]; the words leave no doubt that the author has read Fibonacci's text.

146 Euclid's corresponding diagram in *Elements* VI.13 is indeed different [ed. Heiberg 1883: II, 111] – also in the translation made directly from the Greek, which Fibonacci is known to have used [ed. Busard 1987: 134] (the latter, by the way, has the lettering *a–b–g–d*). Nor does any of the translations from the Arabic agree with Fibonacci's diagram.

various errors, Boncompagni’s thus $\frac{1}{400}\frac{1}{20085}$.[147]

On p. [B356;G553] begins a new section, a presentation of matters from *Elements* X, to which Fibonacci refers explicitly; but whereas Euclid deals with them in a “book about geometry”, here they are “shown according to number”.[148]

14.2a, the multiplication of roots and binomials

In Boncompagni’s manuscript this section carries the heading *pars secunda quartidecimi capituli de multiplicatione radicuum et de binomiorum*, “the second part of the 14th chapter on the multiplication of roots and of binomials”, which corresponds to what is promised in the introduction to the chapter. The heading is absent from the other manuscripts, all of which, however, also promise it in the introduction.[149] There can therefore be no doubt that it was intended by Fibonacci, but perhaps at first forgotten. Since another “part 2” starts a little later, I shall refer to the present one as “part 2_a”

The terminology suggests that Fibonacci is familiar with the translation made directly from the Greek [ed. Busard 1987] – *riti* for Greek ρητός, “rational”, *potentia* for Greek δυνάμει, “in power” meaning “in square”. However, the use of this translation is restricted; as soon as we come to the presentation of the thirteen kinds of irrational lines dealt with by Euclid, the language diverges not only from that of the translation made directly from the Greek but also from all other 12th-century Latin translations [ed. Busard 1967; 1983; 1984; 1992: 2001], and also from the commentary to *Elements* X probably made by Gerard of Cremona [ed. Busard 1997]. The whole structure and purpose is also different from the Euclidean text; either Fibonacci himself makes a very free paraphrasing and reinterpreting commentary, or he uses an existing work of that kind which we do not know about, either Arabic or translated into Latin.

There are indeed good reasons to paraphrase and reinterpret. Fibonacci’s aim is to discuss operations with roots. Since irrational roots in Fibonacci’s conceptual world are *not* numbers (though he shall soon speak about them as such, see imminently), and since

---

147 Since fractions were written with the denominator on the line and the numerator and the fraction line above, all that is needed for this misunderstanding (apart from the extra 0) is a fraction line that extends too far to the right.

148 Readers who are not at least superficially acquainted with *Elements* X may find the following paragraphs difficult. Since this part of the *Liber abbaci* appears to have left modest traces in the abbacus tradition except in the Florence encyclopedias, and since an explanation of what *Elements* X is about would easily become another book, I shall not try to untangle the knots (knots untangled, as is well known, become much longer pieces of string).

149 See the critical apparatus in [Giusti 2020: 548, 789*f*]. I consulted $\mathbf{V_F}$ myself.

he needs the notion of "commensurability *potentia*" or "in power",[150] he cannot give up the underlying representation of lines and the squares on them; but everything is provided with numerical examples, whence rationality becomes absolute and not relatively to an arbitrarily assigned standard line and its square.

Elements X.21 [ed. trans. Heath 1926: 49] defines the first line that is irrational even *in potentia* in this way:

> The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial.

Here, in agreement with definition 3, "rational straight lines" are those that are commensurable in power with a given standard measure. If the length of this standard is defined as 1, the lengths of the two lines thus become $\sqrt{a}$ and $\sqrt{b}$, where not both a and b can be squares (if they were, they would be commensurable in length too). This leads to Fibonacci's transformation, [B356;G553]:

> Of the thirteen irrational lines the first is the simple, called medial, whose power is the irrational called medial surface; because it is the mean proportional between two surfaces only commensurable in power, it will be understood that the line is the root of the root of a number, whose power is the root of a non-square number.

Then follow the definitions of the various kinds of binomials. The first two are defined thus in *Elements* X (definitions II, 1 and 2) [ed. trans. Heath 1926: III, 101],

> 1. Given a rational straight line and a binomial, divided into its terms, such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a first binomial straight line;
> 2. but if the lesser term be commensurable in length with the rational straight line set out, let the whole be called a second binomial.

In Fibonacci's number version, the first of these looks simpler [B357;G553]:

> The first binomial is the conjunction of a number and a root; and the power of the number exceeds the power of the root according to the quantity of some square number; as if the first name[151] were 4, the second root of 7; 16 are namely the power of 4, which add

[150] That two lines are commensurable "in power" (δυνάμει) means that the squares constructed on them are commensurable in area. In agreement with Fibonacci's ambiguous understanding we may say that they are "commensurable in square", thinking either of these geometric squares or of the arithmetical squares.

[151] Literally, *binomium* means "of two names". The "names" are thus the terms of the expression.

9 to 7,

since the binomial has the shape $a+\sqrt{b}$, where a and b are rational numbers; what is commensurable with a is obviously also commensurable with 1. The second [B357;G554], however, with shape $\sqrt{a}+b$, looks more opaque than the Euclidean definition, since here the excess is related to an irrational number:

> The second binomial is composed of a root and a number. And the power of the root adds a number similar to itself over the power of the smaller number, that is, over the power of the number. As if the major name were root of 112, and the minor name were 7. The power of the root of 112 exceeds indeed 49 by 63; and the number 63 is similar to 112, since their ratio is as that of the square number 16 to the square number 9.

I shall abstain from making a full analysis of Fibonacci's transformation of the Euclidean theory – it will be evident from these excerpts that it would go far beyond the limits of this book.[152] For further use I shall just list Fibonacci's examples for the six types of binomials:

1st: $4+\sqrt{7}$, $4^2–7 = 9$
2nd: $\sqrt{112}+7$, $112–7^2 = 63$, $63:112 = 3^2:4^2$
3rd: $\sqrt{112}+\sqrt{84}$, $112–84 = 28$, $28:112 = 1^2:4^2$
4th: $4+\sqrt{10}$, $4^2–10 = 6$, $6:4^2$ not as square number to square number
5th: $\sqrt{20}+3$, $20–3^2 = 11$, $11:9$ not as square number to square number
6th: $\sqrt{20}+\sqrt{8}$, $20–8 = 12$, $12:20$ not as square number to square number

After the definition of the six kinds of binomials (now spoken of as *numeri*, "numbers") accompanied by the examples just given Fibonacci shows [B357;G554] that the square of any of them is a "first binomial" – the counterpart of *Elements* X.60 [ed. trans. Heath 1926: III, 132],

> The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial,

---

As we shall see (below, p. 216), in abbacus algebra, "names" came to refer to the sequence of algebraic powers.

152 Lüneburg [1993: 259–272], though no complete analysis, is a praiseworthy courageous beginning.

In contrast, Moritz Cantor (within what he himself characterizes as an "almost insupportably detailed description of the *Liber abbaci* [Cantor 1892: 31; 1900: 34], indeed 28 pages long), only says [1892: 28; 1900: 28] that Fibonacci "follows the progress [*den Gang*] of Book X of Euclid's *Elements* rather closely"; this is simply wrong, Fibonacci borrows some definitions and results but nothing from the structure or the progress.

Sigler [2002: 631] has absolutely nothing to say about the matter in his notes.

but simpler because Fibonacci speaks of these squares as numbers, not areas to be applied to a line. Last in this introduction to the world of *Elements* X come the subtractive counterparts of the binomials, the apotomes (*recisi, seu apothami*) [B358;G555], still identified as *numeri*. Here, the exposition is more compact – only the first is exemplified, namely by $4–\sqrt{7}$ (as we and the supposed reader see, $4^2–7 = 9 = 3^2$).

14.2b, multiplying roots by roots

On p. [B358;G556] begins another part 2 (henceforth 2_b), "about the multiplication of roots in roots and numbers". One may assume that the part 2_a was absent from the 1202 version, and thus that the systematic presentation of some fundamentals from *Elements* X was added in the revised version.[153] Since binomials and apotomes turn up regularly here and in the following parts, we seem to be confronted with a parallel to what happened to the *regula recta*, which was used occasionally in the 1202 version but taken for granted by then, not explained (see above, p. 84). When preparing the revised version, Fibonacci found that even this rule was in need of an introduction and an explanation.

Since we do not possess the 1202 version of chapter 14, it cannot be established with full certainty that this is what happened; but we may notice that the proof that the square of any binomial is a first binomial is repeated on p. [B362;G560] within the third part, this time supported by a line diagram lettered *a–b–g–z–e–*... – not the proof offered on p. [B357;G554] (which is purely arithmetical and proceeds case by case), nor however the proof of the corresponding theorem in the translation of the *Elements* directly from the Greek [ed. Busard 1987: 260]. The latter evidently considers segments and areas contained by segments, while the present proof represents products of segments by segments.

10	20
a	b
g	d

Part 2_b thus deals (according to its heading as well as actually) with the multiplication of roots by roots (the "numbers" of the heading being forgotten). At first it takes the example [B358;G556] $\sqrt{10}\cdot\sqrt{20} = \sqrt{(10\cdot20)} = \sqrt{200}$, provided with a simple but noteworthy proof:

153 This would explain that Fibonacci planned part 2_a and promised it in the introduction to the chapter (plausibly also either added or adjusted in 1228), and then at first forgot to provide the inserted explanation with the corresponding heading.

Alternatively, since the heading for part 2_a is present in the same manuscript family as the revised two-tower problem (above, p. 115), Fibonacci included it in the hypothetical intermediate version; then, in the ongoing process, he discovered to have two headings for part 2, and eliminated the first of them (or a copyist, on the way to the remaining manuscripts, did so). Against this assumption speaks, however, that the heading for part 2_a is absent from $\mathbf{V_F}$, the manuscript which refers to a Castilian master, and therefore would also seem to represent an early branch of the "1228 family". Cf. above, note 135.

> Let *a* be the root of 10, and *b* that of 20: and beside *g* equal to *a*, *d* equal to *b;* therefore *g* is root of 10, and *d* of 20. Therefore, when I multiply *g* in *a*, that is, *a* in itself, result 10; and when I multiply *d* in *b*, that is, *b* in itself, they make 20. Therefore, when I multiply 10 in 20, then I multiply the product [*factum*] of *g*, *a* in the product of *d* , *b;* therefore the multiplication of the product of *g*, *a* in the product of *g*, *b* is 200. But the multiplication of the product of *g*, *a* in the product of *d*, *b* equals the multiplication of the product of *a*, *b* in the product of *g*, *d;* therefore the multiplication of the product of *a*, *b* in the product of *g*, *d* is 200. But the product of *a* in *b* equals the product of *g* in *d*; therefore the product of *a* in *b* but the product of *g* in *d* equals the multiplication in itself of the product of *a* in *b*. Therefore the multiplication in itself of the product of *a* in *b* makes 200. In consequence the product of *a* and *b*, namely of the root of 10 in the root of 20, is the root of 200; as was to be demonstrated.

The letters, we observe, do not designate lines as they do in the proofs of *Elements* VII–IX[154] – the proof is as close to being an instance of symbolic algebra as possible when only the variables (here really variables, not representatives of specific though unknown numbers) and not the operations are written by symbols. It also moves beyond the classical limitation that products should consist of no more than three factors.[155] We observe that the letter order is *a–b–g–d*, which indicates that the proof (and presumably its whole context) is borrowed – with or without lines carrying the letters. In the following pages, a number of similarly lettered proofs turn up, but none where letters stand directly for numbers. Should we conclude that in the first proof Fibonacci slips into unwillingly into mathematical modernity without thinking about it but the corrects his ways?

After an analogous example ($\sqrt{30}\cdot\sqrt{40}$) follows an explanation that a product of type $\sqrt{a}\cdot\sqrt{b}$ will be rational if a and b have the ratio of two square numbers (this time the letters are mine). After that $(3\sqrt{10})\cdot(4\sqrt{20})$ is reduced to $\sqrt{(9\cdot10)}\cdot\sqrt{(16\cdot20)}$, and a visualization (*si ad oculum deprehendere vis*, "if you want to indicate to the eyes") of the transformation $4\cdot\sqrt{20}$ into $\sqrt{(16\cdot20)}$ is offered, based on a diagram whose lettering involves *c*, and which can therefore be considered to be of Fibonacci's own making.

The rest of part 2_b deals with multiplications involving roots of roots – in part through

---

154 Obviously these lines are not geometric entities but simply representatives of unspecified numbers; the crux is that a letter in itself cannot do that, it can only serve as a name distinguishing something felt to be more substantial.

155 Not absolutely respected in classical mathematics, it is true. In the *Dioptra* [ed. trans. Schöne 1903: 282*f*], Heron allows himself something very close to what is done here when proving "his" formula for the area of a triangle; but when he transfers the proof to the *Metrica*, an attempt to represent practical geometry "from a higher vantage point", he has to justify the operations in a lemma [ed., trans. Schöne 1903: 16–19].

numerical examples only, in one case (to find two roots of roots whose product is rational) supported by another letter-based argument (with sequence *a–b–g–d–e*).

14.3, addition and subtraction of monomials, binomials and apotomes

Part 3 [B361;G559] is dedicated to "the mutual addition and detraction of roots, and of the other two simple numbers", that is, of binomials, apotomes and monomials (roots of irrational roots are included here under "simple numbers"). That is where it is proved again that the squares of all types of binomials are first binomials (cf. above, p. 124). It is shown [B362;G560] that $\sqrt{12}+\sqrt{10} = \sqrt{(22+\sqrt{480})}$ (the former expression, however, being deemed "more beautiful"), while $\sqrt{18}+\sqrt{32}$ (where the two terms are commensurable) is better expressed as $\sqrt{98}$ [B363;G562].

Further it is shown ([B363;G561], with a line-based proof, *a–b–g*) that the square on any apotome (number less root, root less number, or root minus root with radicands that are not commensurable in power) is a first apotome, with application of the argument to specific examples. It is also in this part ([B364;G563]; cf. above, p. 62) that we find the calculation of $4+\sqrt{\sqrt{10}}$ *secundum vulgarem modum*, specified to be *secundum propinquitatem* ("in the vernacular way, [...] according to approximation"), $\sqrt{\sqrt{10}}$ being approximated as "less than $1\,^4\!/_5$". The "magisterial" alternative,

$$4+\sqrt{\sqrt{10}} \;=\; \sqrt{16+\sqrt{10}+8\sqrt{\sqrt{10}}} \;=\; \sqrt{16+\sqrt{10}+\sqrt{\sqrt{40960}}}\;.$$

is accompanied by a diagram lettered *a–b–c*, which serves names-giving only.

Addition of two roots of roots is shown (with purely verbal arguments) to yield sometimes the root of an expression with three "names", sometimes the root of a binomial of one or the other kind – in our formalism

$$\sqrt{\sqrt{a}}+\sqrt{\sqrt{b}} \;=\; \sqrt{\sqrt{a}+\sqrt{b}+2\sqrt{ab}}\quad ,$$

which under specific conditions can be further transformed.

14.4a, mutual division of monomials

Part 4 ([B365;G565] – henceforth 4_a) first explains "the mutual division of three simple numbers", that is, number by root, root by number, and root by root, giving the rule to square the dividend and the divisor, leaving it to the ensuing examples to explain that afterwards the root is to be taken of the outcome – first $30\div\sqrt{10} = \sqrt{(900\div10)}$. Similar examples (with similar rules) follow that involve roots of roots.

Next [B367;G567] Fibonacci turns to products of fourth, fifth and sixth binomials – or so he says. The actual topic has nothing to do with the heading under which it is placed, and most likely it represents another addition from 1228 – a kind of continuation of part 2_a, inserted here because this seemed the most adequate place. Actually the topic is restricted to the products of forth, fifth and sixth binomials with their respective apotomes.

First, however, comes the claim that

> The root of a fourth binomial is composed of two lines, of which one is the root of a fourth binomial, and the other is the root of the apotome having the same name. Of which lines the first is called a major, the second a minor, and the conjunction from them, that is, the root of the fourth binomial, is similarly a major; and it is called a major because the major name it has as power is a number.

The concluding explanation of why a certain line is called a major is evidently wrong – a folk etymology.[156]proper definition is found in *Elements* X.39 [ed. trans. Heath 1926: 87]

> If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole straight line is irrational: and let it be called major.[157]

What Fibonacci tells corresponds instead to the propositions *Elements* X,57 and X.94. X.57 transferred to numbers states indeed that the square root of a fourth binomial is a major,[158] while X.94 says that the square root of a fourth apotome is a minor. So, $M = \sqrt{(a+\sqrt{b})}$ is a major and $m = \sqrt{(a-\sqrt{b})}$ the corresponding minor, provided that a^2-b is no square number. Some calculation then gives

$$(M+m)^2 = 2a+2\sqrt{a^2-b} \quad ,$$

either a first or a fourth binomial. Moreover,

$$(2a)^2-(2\sqrt{(a^2-b)})^2 = 4b \ .$$

But b is no square number, nor is therefore $4b$. In consequence, $(M+m)^2$ must be a fourth binomial, and so $M+m$ is indeed the square root of a fourth binomial.

Fibonacci offers a corresponding line-based proof, lettered *a–b–g–d–e–z–i*, which ends up with statement that the product of any binomial (say, $p+q$) with the corresponding

156 And even wrong, since the square on any binomial is a first binomial, which also fits this description.

157 Evidently, that does not tell much about the reasons hiding between the names "minor" and "major". A plausible explanation linking them to the geometry of the regular pentagon was proposed by Marinus Taisbak [1996].

158 "If an area be contained by a rational straight line and the fourth binomial, the side of the area is the irrational straight line called major". X.63 shows the reverse, "The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial" [ed. trans, Heath 1926: III, 125, 63]. Fibonacci's identification of the major with the square root of a fourth binomial is thus blameless.

apotome $p–q$ equals $p^2–q^2$, which serves in the following.

The insertion closes with a similar calculation [B368;G568], the determination of $(\sqrt{40}+\sqrt{5})\cdot(\sqrt{40}-\sqrt{5})$ and thus concerning a 6th binomial-apotome-couple. This calculation is supported by a proof based on a subdivided rectangle, still lettered *a–b–d–g–...* .

After that Fibonacci turns to something much simpler and in better agreement with the heading of part 4_a. Since he now makes use of a diagram that corresponds to his habits elsewhere in the book, we may assume that he has now returned to his own work, probably as it looked already in 1202:

> If you want to multiply 4, and root of 7, by 5 and root of 20, put the number under the number, and the square root under the square root, as shown in the margin. And multiply 4 by 5, and root by root, namely 7 by 20, the outcome is 30, and root of 140. And multiply contrariwise 4 by root of 20, and 5 by root of 7, the outcome is four roots of 20, and five roots of 7, that is, root of 320, and root of 175. [...].

4	7
5	20

Examples of multiplication of binomials consisting of number and root of root follow. One of them [B369;G571] gives Fibonacci the occasion to formulate the "sign rules":

> when something diminished is multiplied by diminished, then this multiplication increases; and when added are multiplied with each other, then even their product is to be augmenting; but when added is multiplied by diminished, then their product is to be diminished, as shall be shown in the following

– namely by means of a meticulous explanation of a rectangle divided by means of intersecting lines parallel to the sides.

[B371;G572] shows by means of a line proof (*a–b–g–d*) that a binomial multiplied by its apotome yields the difference between the squares on the terms – already shown in the insertion, we remember, which confirms that the suspected insertion is really one, and that Fibonacci did not rewrite the subsequent text after having made it.

Next, with a return to the categories of *Elements* X, it is derived that if the binomial in question be a third or sixth binomial, the product will be rational; the restriction to third and sixth binomials is superfluous and puzzling.

14.4b, division of binomials and apotomes

[B372;G575] opens another "part 4", which we may refer to as 4_b (this time both "part-4" headlines are present in all manuscripts[159]), presented as dealing with "the division of binomials and apotomes by rational and irrational numbers, and the contrary". Initially, for the division by rational and irrational numbers, it teaches to perform the

[159] Giusti removes the words *Pars quarta*, but his critical aparatus shows them to be present in all manuscripts.

division term by term. The “contrary” operation, the division of a rational number, a root or a root of a root by a binomial or an apotome, is taught from [B373;G575] onward; the method, as will be guessed, is to multiply divisor and dividend by the corresponding apotome respectively binomial, which gives a number as divisor “as has been shown” (the superfluous restriction to third and sixth binomials and apotomes being forgotten). The division of 10 by $2+\sqrt{\sqrt{3}}$ asks for an iteration of the procedure. [B376;G579] teaches the division by a trinomial[160] – for instance, $10\div(2+\sqrt{3}+\sqrt{5})$ – again making use of a procedure in two step. A rather simple line proof is lettered *a–b–c–d–e*, and is thus likely to be of Fibonacci’s own making.

A final part of part 4_b deals with the roots of binomials. Expressed in modern symbols, $\sqrt{(a+\sqrt{b})}$, where $a>\sqrt{b}$, can be reduced if rational numbers p and q can be found such that $p+q = a$, $pq = \frac{1}{4}b$. Then, indeed

$$\sqrt{a+\sqrt{b}} = \sqrt{p+q+2\sqrt{pq}} = \sqrt{(\sqrt{p})^2+(\sqrt{q})^2+2\sqrt{p}\sqrt{q}} = \sqrt{p}+\sqrt{q}\ .$$

To find p and q is a problem of the kind for which the “key” version of *Elements* II.5 was traditionally used, and the line proof (*a–b–g–d–e–f*) actually draws on that. It is observed that if $a+\sqrt{b}$ is a first binomial, then the solution is indeed rational.[161] Similar discussions for other classes of binomials follow.

14.5, cube roots

Part 5 [B378;G582] deals, thus its heading, with “the finding of cube roots, their addition and multiplication, and the extraction or division of the same”.

Initially it evidently explains what cubes and cube roots of numbers are and how to calculate cubes in the place-value system.

Next it describes an algorithm for extracting approximate cube roots of non-cube numbers. As a first step it finds (supported by a line diagram *a–b–g–d*) the difference

160 This obviously goes beyond *Elements* X. Including it may be Fibonacci’s own facile independent generalization but need not be. That Apollonios had discussed trinomials was well known in the Arabic world, for example from Pappos’s commentary to *Elements* X [ed. trans. Thomson & Junge 1930: 85]:

> We should also recognise, however, that not only when we join together two rational lines commensurable in square, do we obtain a binomial, but three or four such lines produce the same thing. In the first case a trinomial (trinomium) is produced, since the whole line is irrational; in the second a quadrinomial (quadrinomium); and so on indefinitely. The proof of the irrationality of the line composed of three rational lines commensurable in square is exactly the same as in the case of the binomial.

161 Since $(p+q) = a^2$ and $4pq = b$, $(p-q)^2 = a^2-b$, which is indeed a square if $a+\sqrt{b}$ is a first binomial.

between subsequent cube numbers, then the outcome is used to explain the algorithm.

Then [B384;G590] follow examples for the multiplication of cube roots or numbers with cube roots, and similarly; as was also to be the habit concerning square and cube roots in abbacus mathematics, the outcomes are "reduced to cube root", that is, for example, $\sqrt[3]{1080}$ instead of $3\sqrt[3]{40}$.[162] Next two (rather trivial) ways to produce two cube roots whose product is rational, and then [B384;G591] division of cube roots by cube roots (or expressions that can be "reduced to cube root").

Then follows [B384;G591] an explanation that cube roots, just like square roots, can sometimes be aggregated or detracted one from the other (the term *disgregare* is also used), and sometimes not. So, $\sqrt[3]{16}$ and $\sqrt[3]{54}$ can be aggregated because 16 and 54 are in the ratio of two cube numbers (8:27), while $\sqrt[3]{32}-\sqrt[3]{4} = (2-1)\sqrt[3]{4}$. The latter calculation is supported by an extensive line argument considering the decomposition of the cube on the line

$$\underline{a\qquad\quad b\quad\; c}$$

where *ab* represents $\sqrt[3]{32}$ and *bc* $\sqrt[3]{4}$; as suggested by the lettering as well as by the agreement with the explanation of the algorithm for the extraction of cube roots, the argument is likely to be of Fibonacci's own making.

The chapter closes by the observation that [cube] roots whose radicands do not communicate (that is, are not in the ratio of a cube number to a cube number) can neither be aggregated nor disgregated: $\sqrt[3]{5}-\sqrt[3]{3}$ "cannot be said more beautifully".

In the *Flos* [ed. Boncompagni 1862: 228] Fibonacci explains that a certain question about a cubic problem inspired him to study *Elements* X more accurately, and that, "because it is more difficult than those books that precede or come afterwards, I began to gloss upon this same Tenth Book, reducing its understanding to number, which in itself is proved by lines and surfaces". In a mid-15th-century manuscript[163] this has developed into a claim that Fibonacci wrote "a book about the 10th of Euclid"; since Fibonacci elsewhere refers to "books" of his when we know they existed, we may take from his words that the 15th-century admirer extrapolated, and that Fibonacci indeed restricted himself to glossing but did not write a genuine "commentary".[164]

---

162 It may be observed (even though that is hardly the explanation for this choice of mathematical aesthetics) that this choice improves the precision of approximations if only (as mostly) the rational multiplyer is larger than 1.

163 The "abbacus encyclopedia" Florence, BNC, Palatino 573, fol. 433^{v}–434^{r} [ed. Arrighi 2004/1967: 193].

164 In the *Summa*, Pacioli [1494: 119^{v}–142^{v}] has a much more thorough and systematic exposition of *Elements* X in number interpretation, and one might ask whether he based this on Fibonacci's supposed book – not least because he seems to have inspected Fibonacci's chapter 14 or perhaps

The inspiring question, thus it is told by Fibonacci [ed. Boncompagni 1862: 227], was asked by Giovanni di Palermo in the presence of Frederick II, that is, in 1226 (above, p. 98). This confirms that the three insertions in the present chapter (the preamble, part 14.2a, and the matter added to part 14.4b), here identified according to internal criteria, were indeed added in the 1228 version.[165] Moreover, the constant use of proofs lettered *a–b–g–...* in these shows that he drew on borrowed material, while the unorthodox translation of *al-jabr wa'l-muqābalah* in the preamble is strong evidence that he used an existing Latin translation, now lost (at least for the preamble, probably for all the insertions) – how creatively it is hard to know.

The answer given in the *Flos* to the problem shows beyond doubt that Fibonacci knew much more about *Elements* X than he put into the *Liber abbaci*.[166]

---

a larger treatise from which Fibonacci drew in the *Liber abbaci* what he found fitting (on fol. 144r he explains that *almucabala* means *oppositio* or *contemptio*, the latter with Fibonacci's characteristic mistake).

However, of Pacioli's numerical illustrations of the binomials (fol. 120r), only one, that for the first binomial, coincides with that of Fibonacci; given its simplicity, $4+\sqrt{7}$, this cannot be taken as evidence of inspiration (only $2+\sqrt{3}$, $3+\sqrt{5}$ and $3+\sqrt{8}$ are simpler). As it becomes obvious if the lettering of Pacioli's diagrams in the geometric part is compared with their counterparts in Fibonacci's *Pratica geometrie*, Pacioli is no less faithful to his model when he has one than Fibonacci. It seems most likely that Pacioli, perhaps inspired by what little is found in Fibonacci's chapter 14, produced his own numerical interpretation of *Elements* X.

165 This dating, by the way, supports the assumption that Fibonacci here used written material and did not draw on what he had learned while travelling.

166 See, apart from the text itself, for example [Woepcke 1854], [Rashed 2003: 57–60] and [Picutti 1983: 342–351].

Chapter 15 – theory of means, rules of geometry, and algebra

Chapter 15 "about pertinent rules of geometry, and about questions of *aliebra et almuchabala*", is explained in the very beginning to consist of three parts – the first dealing with "proportions of three and four numbers, to which many questions pertaining to geometry are reduced" (thus elucidating what is meant in the chapter heading); the second with "the solution of certain geometrical questions"; the third with "the way of *algebra and almuchabala*".

15.1, an investigation of means

As a matter of fact, the contents of part 1 has little to do with geometric questions. Since I have analyzed it in depth elsewhere, I shall only recapitulate here, leaving substantiation to [Høyrup 2011a: 97–100].[167]

First [B387;G595] come three questions (#1–3 in the numbering I introduced in [Høyrup 2011a]) about three numbers in continued proportion P:Q:R, where the sum of two of them (P+Q, Q+R or P+R) is given together with the third. In all cases, line diagrams lettered *a–b–c–...* are used for an argument where proportion operations[168] transform the question in such a way that the "key" version of *Elements* II.6 can be applied. For example, in the first question, where $P+Q = 10$, $R = 9$,

$$\frac{P}{Q} = \frac{Q}{R} \;\Rightarrow\; \frac{P-Q}{Q} = \frac{Q-R}{R} \;\Rightarrow\; Q\cdot(Q+9) = 90 \;.$$

These are followed by two questions (#4–5), still about three numbers in continued proportion, but now Q–P and R respectively R–P and Q are given. Now, the line diagram is lettered *a–b–g–d–...*, but in the ensuing argument even the letter *c* turns up, indicating that Fibonacci has used but transformed borrowed material. In the former, the "key" version of *Elements* II.5 is used, in the latter that of II.6. This first section of part 1 closes by an aside (#6) explaining that squares or cubes of four numbers in proportion are also

[167] More concisely also in [Høyrup 2009d: 62–65].

[168] In the present description of chapter 15 of the *Liber abbaci* I shall switch to a fraction-like notation for proportions. Then, the "proportion operations" on $\frac{a}{b} : \frac{c}{d}$ are

e contrario:	$\frac{b}{a} : \frac{d}{c}$
permutata:	$\frac{a}{c} : \frac{b}{d}$
conjuncta:	$\frac{a-b}{b} : \frac{c-d}{d}$
disjuncta:	$\frac{a-b}{b} : \frac{c-d}{d}$
conversa:	$\frac{a}{a-b} : \frac{c}{c-d}$
eversa:	$\frac{a}{a-b} : \frac{c}{c-d}$
aequa:	$\frac{a}{b} : \frac{a-c}{b-d}$

(from the Campanus *Elements* [ed. Busard 2005: 171*f*]); to these comes the "product rule" $a{\cdot}d = b{\cdot}c$.

in proportion. It has nothing to do, neither with what precedes or with what follows immediately. In section 3 of part 1 comes a problem (#50) where it is used, but there is neither forward nor backward reference, so Fibonacci has not noticed the connection.

Section 2 of part 1 (#7–38) is by far its larger portion, and its central piece. It is indeed a coherent piece of theory inspired by the ancient doctrine of means.

The concept of means had developed gradually. Plato's contemporary Archytas[169] knew three: the arithmetic, the geometric, and the harmonic mean. In the Early Common era their number had grown to around ten – Nicomachos (*De institutione arithmetica* II.xii–xxviii, ed. [Hoche 1866: 122–144], trans. [d'Ooge 1926: 266–284]) and Pappos (*Collection* III.xii–xxiii, ed. trans. [Hultsch 1876: I, 70–105]) each have a list of ten, but only nine coincide. They can all be defined by means of proportions, as shown in this

[169] Fragment 2 (generally accepted as genuine), [ed. trans. Diels-Kranz 1959: I, 435*f*].

scheme, which also indicates how Fibonacci's part 1 section 2 fits in:

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P} = \frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P} = \frac{R}{Q}$ or $\frac{R-Q}{Q-P} = \frac{Q}{P}$	P2	N2	27–29
$\frac{R-Q}{Q-P} = \frac{R}{P}$	P3	N3	7–9
$\frac{R-Q}{Q-P} = \frac{P}{R}$	P4	N4 (but inverted)	10–12 (inverted)
$\frac{R-Q}{Q-P} = \frac{P}{Q}$	P5	N5 (but inverted)	#34–36 (inverted)
$\frac{R-Q}{Q-P} = \frac{Q}{R}$	P6	N6 (but inverted)	#20–22 (inverted
$\frac{R-P}{Q-P} = \frac{R}{P}$	absent	N7	#16–18
$\frac{R-P}{R-Q} = \frac{R}{P}$	P9	N8	#13–15
$\frac{R-P}{Q-P} = \frac{Q}{P}$	P10	N9	#30–32
$\frac{R-P}{R-Q} = \frac{Q}{P}$	P7	N10	#37–38
$\frac{R-P}{R-Q} = \frac{R}{Q}$	P8	absent	#23–25
$\frac{R}{Q} = \frac{R-P}{Q-P}$	absent	absent	#26

Everywhere, it is assumed that $P \leq Q \leq R$, and only the last line does not presuppose that $P < Q < R$. That is indeed the condition that Q can sensibly be considered a mean; it is therefore not strange that this last case is not included by the ancient writers. The definition of the arithmetic mean in term of a proportion is evidently a clumsy artifice, the normal and reasonable definition being $R–Q = Q–P$.

What we find in part 15.1 section 2 of the *Liber abbaci* seems to be a new theory, inspired by the ancient list of means. Having skipped the idea of Q being a mean, its creator has added the case in the last line for the sake of theoretical completeness and coherence. For similar reasons of consistency he has left out the frivolous interpretation of the arithmetical mean in terms of a proportion. For each type of mean (except the pseudo-mean in #26, where Q turns out to coincide by necessity with R), it is shown how any of the three terms P, Q and R can be determined from the two others.

The creator is certainly not Fibonacci. That is indicated already by the constant use of line diagrams lettered *a–b–g–...*, but also by the failure to point out that #27–29 deal with the geometric mean, which he has already treated himself in #1–5 with line proofs of his own making.

The line proofs of section 2 mostly use proportion-based transformations so as to make possible the application of *Elements* II.5–6, invariably in "key" version, with no reference to Euclid, against Fibonacci's normal habit; those questions that are of the first degree are mostly solved by application of proportion techniques alone. Fibonacci must have drawn on the source from which he had also taken over the "keys", that is, a source located somewhere in the Arabic world. Moreover, line diagrams and "keys" are applied in strikingly similar ways in the *Liber mahameleth*, which makes it next to certain that this "somewhere" was al-Andalus: the same as the likely source for the sophisticated versions of the "unknown heritage", cf. above, p. 94, and the added components of chapter 14 (above, p. 118).

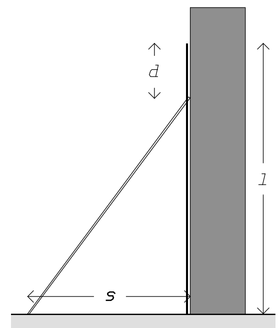
That the same methods are used in section 1 in connection with diagrams fully or in part of Fibonacci's own making does not contradict this conclusion; it only confirms that here (as mostly) he has understood what he borrows and is able to use it creatively.

Section 3 [B395;G607] deals with four numbers in proportion, $\frac{P}{Q} = \frac{R}{S}$. #39 shows that any of the four numbers can be found from the three others. This is evidently not new to the *Liber abbaci*, is has been explained and used in connection with (Fibonacci's stand-in for) the rule of three – see above, p. 71. #40–45 show that all can be determined if the sum of two of the numbers is known together with the other two individually – all six possibilities are covered. #46–49 do the same for differences (omitting two, but simple left-right shifting of the proportion reduces them to what was already treated). In #50, finally, P^2+Q^2 is given together with R and S. The solution makes use of what was explained in #6 (see above, p. 133), but as mentioned there is no cross-reference. All proofs are based on line diagrams lettered *a–b–g–d*, and this together with the obvious similarity of the question types shows that this group has been taken over from the same source and together with section 2. The absence of cross-references (even of the generic type which abound in the work) indicates that Fibonacci has taken over fairly faithfully, making no effort to edit the material (as he seems to have done in #4–5).

15.2, geometric and other questions

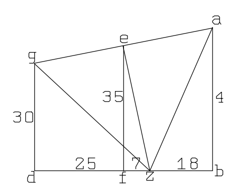
The description of part 2 as “the solution of certain geometrical questions” is also somewhat misleading. It opens, however, with two classical geometric recreational problem types: the “pole against a wall” and the two-tower problem (see above, p. 46).

The pole (here a spear or javelin, *asta*) against the wall deals with a reed, pole or spear of length *l*, at first standing against a wall; next the foot slides out a distance *s*, which makes the head slide down a distance *d*. The problem is known first from an Old Babylonian tablet (BM 85196, 17th c. CE) in the simple variants where only direct application of the Pythagorean rule is needed (namely because *l* is given together with either *d* or *s* (further references in [Høyrup 2002b: 13, 15]). In Seleucid and Demotic sources we find these together with the more intricate question where *d* and *s* are given. Fibonacci discusses the two simple variants only, with a reference to “the second-last of Euclid’s first book” (the Pythagorean teorem, indeed). A diagram lettered *a–b–c–...* tells us that Fibonacci, though repeating a problem of venerable old age, argues on his own.[170]



A second problem deals with two spears, at first planted vertically, afterwards one leaning toward the other. It calls for no further commentary.

The two-tower problem is more intriguing. The two towers (*ab* and *gd*, respectively) are 40 and 30 paces high, and their distance is 50 paces.[171] The two birds arrive at the same time to the fountain *z*, whence *az* and *gz* must be equal. At first a geometric solution is offered: a perpendicular to *ag* is erected in its midpoint. It hits the ground *bd* in *z*, and since the triangles *aez* and *gez* are both right, *ez* being shared and *ae* = *eg*, even *az* and *gz* must be equal.



This construction does not lead to a numerical value for the position of *z*. Therefore,

---

[170] The problem is also dealt with in the *Liber mahameleth* [ed. Vlasschaert 2010: $_{\mathrm{e}}$402*f*; ed. Sesiano 2014: 543*f*, trans. *ibid*. 1046*f*]. There a ladder (*scala*) is spoken of; the numerical parameters are different; the lettering of the diagram is also different; and an extra, algebraic solution is offered. Fibonacci has certainly not used this work, which in all probability he did not have access to.

[171] The unit *passus* is used very rarely in the *Liber abbaci*, which might help identify the source: in the *elchataym* version of the same problem ([B331;G519n], see above, p. 114; almost certainly copied from here); on p. [B179*f*;G306] (a hound pursuing a fox); and on p. [B182;G311], a “lion in the pit” problem about two ants.

“if you want to proceed by numbers”, a calculation is offered.

The trick behind this numerical solution is the observation that $gd^2+dz^2 = ab^2+bz^2$, whence $ab^2–gd^2 = dz^2–bz^2$. Now $dz^2–bz^2 = (dz+zb)\cdot(dz–zb)$; there should be no need to reduce the left-hand side of the equation. However, Fibonacci does not explain this background but presupposes without explanation that

$$\frac{ab+gd}{2}\cdot\frac{ab-gd}{2} = \frac{dz+zb}{2}\cdot\frac{dz-zb}{2} .$$

More precisely, he finds fz (= $(db–bz)/2$) as

$$\left(\frac{ab+gd}{2}\cdot\frac{ab-gd}{2}\right)\div\frac{dz+zb}{2} .$$

Fibonacci offers no explanation of how he comes to the equation, which suggests that he simply copies; alternatively (less in accordance with the many pedagogical explanations offered) he takes care not to leave any hint to the reader.

The geometric solution justifies that the problem is moved from chapter 12 to its present location. The original version, as shown by manuscript **L**, had no geometry and different numerical parameters. The numerical algorithm, on the other hand, is the same (and even there without explanation of its foundation. The *a–b–g–...* lettering of the diagram shows the geometric proof to be borrowed. In 1202, thus Fibonacci had the numerical solution (and even then apparently did not know why it works). Then, in the revised version, he adopted a geometric proof from a source where it solves a problem with different numerical parameters. Taking over even these parameters he had to adjust the numerical calculation while keeping the algorithm intact.

On p. [B399;G612] begins a sequence of problems about repeated travels, different in mathematical type from those of part 12.6 (above, p. 89). In the first, somebody starting with a capital of 100£ makes two travels, earning at the same proportion, and ends up with 200 £. The possession after the first year is argued to be the mean proportional between 100£ and 200£, approximated as 141£ 8ß $5\frac{1}{8}$δ.

In the next problem, on the same pages, somebody enters in partnership with 100£, and after the second travel the total possession of the partnership is 299£. This is a mixed second-degree problem, solved by means of a line diagram and *Elements* II.6 in “key” version (neither Euclid nor the notion of “keys” being mentioned).

The argument of the first of these two problems makes use of three numbers identified by line segments carrying the letters *a*, *b* and *g*. That of the second is based on a proper line diagram lettered *a–b–g–c–d–e–z*. Fibonacci thus seems to have taken inspiration for the basic diagrams from an Arabic source, but to have elaborated the arguments on his own.

The third problem [B399;G613] is an obvious extrapolation from the first, replacing the two travels with three but making no other changes. The argument, based on and referring to the *Elements*, is likely to be Fibonacci's own; at least he ventures rather widely into the theoretical topic.

In the last problem of the group [B401;G615], with two travels, the initial capital is unknown, the possession after one year is 80 bezants, and the ratio between the capital and the final possession is stated to be as 5^2 to 9^2. Since the mean proportional between 5^2 and 9^2 is 45, an argument by proportionality or by the rule of three (multiplication by 80 and division by 45) leads to the result. The reference to bezants (the previous problems speak about *lire*) suggests (does not prove) that Fibonacci was confronted with this question in Byzantium. When he says that this has happened he usually offers a solution of his own (so he tells or lets shine through); this could also be the case here. The solution is followed immediately by the observation that the same rule can be used to find two numbers (say, a and b) where $\frac{1}{5}a = \frac{1}{9}b$, $ab = 80$ (cf. above, note 93), with further variations of the numerical parameters. Even this seems to be Fibonacci's own elaboration of the answer to the bezant-problem.

More pure-number problems follow, first [B401;G615] about a way to produce Pythagorean triples – formulated however as "to find two roots in integers, whose squares joined together make a square number, that is, having a root". The underlying idea is the identity

$$\left(\frac{m^2-n^2}{2}\right)^2+(mn)^2 = \left(\frac{m^2+n^2}{2}\right)^2 ,$$

but Fibonacci generalizes by replacing m^2 and n^2 by two numbers that are in the ratio of square numbers (and, in order to get integers, asks that both be even or both be odd). He offers a proof based on the key version of *Elements* II.5, supported by a line diagram carrying no letters but only the numbers belonging with a corresponding numerical example.[172] In so far, it might be of Fibonacci's own making, no borrowing (though of course not going beyond existing knowledge – see imminently). On the other hand, the reference to "roots" and not to "numbers" seems suspicious, being far from what Fibonacci does elsewhere and not very far from the ways of Arabic algebra – once again it seems possible that Fibonacci borrowed a question and gave his own answer.

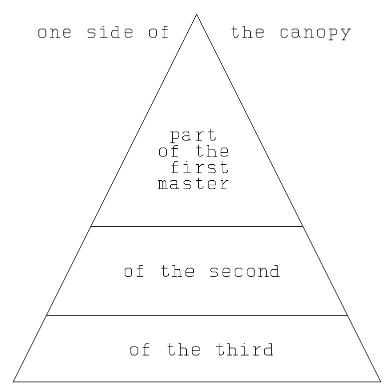
Another problem [B402;G617] asks for two "roots" whose "multiplications [each with itself, that is, their squares, together] make 41. Actually, given that $4^2+5^2 = 41$, what is asked for is a *different* pair. The problem is solved by a purely numerical prescription, followed however by the remark [B403;G618] that "it is shown by geometry in the booklet I composed about squares from where these foregoing inventions come" (the *Liber*

[172] Present in Boncompagni's manuscript, absent from $\mathbf{V_F}$; Giusti leaves it out from his edition.

quadratorum[173]), where it is indeed found [ed. Boncompagni 1862: 256] with a geometric proof lettered (like most lettered proofs in that treatise) *a–b–g*... . The previous question is dealt with slightly earlier in the *Liber quadratorum* [ed. Boncompagni 1862: 255], with an *a–b–g–d* line diagram and a correct reference to *Elements* X.[174] In both case, the *Liber quadratorum* speaks about *numbers*, not *roots*. Could this indicate that the problems were already in the 1202 version of the *Liber abbaci* and then later adopted into the *Liber quadratorum*?

After this excursion into the realm of numbers Fibonacci returns to geometry [B403; G618], at first with a problem about a piece of cloth long 100 ells and broad 30 ells, from which linen cloths long 12 ells and broad 5 ells have to be made. The calculation is simple – multiplication 100·30, division by 5·12. No notice is taken of the difficulty that some of the resulting 50 cloths will be instead in pieces of 6 by 10 ells and will need to be cut and sewn.

Next [B403;G618] comes an analogue of Jacopo's chest problem (above, p. 32), still with two cubic chests of sides 16 palms and 4 palms respectively, and no explicit hint of fraud. Fibonacci goes on with the cistern problems already discussed above (p. 41) in connection with Jacopo's fallacious solution of a similar problem. They are five in number, and ask for the determination of the volume of a cube, a cylinder, a cone, a double cone, and a sphere, and conversion between volume (cube feet, spoken of as *pedes quadrata*) and hollow measure (barrels).



The last geometric problem in part 15.2 speaks of a canopy (*ciborium*) composed of four (isosceles) triangles, each having the base equal to 30 palms, being high 36 palms along the side. Three master painters are to share the work, their shares being separated by lines parallel to the base. The heights of the delimitations of the shares are asked for, and found (with a generic reference to what "we have demonstrated above") to be

$$\sqrt{\tfrac{1}{3}\cdot 36^2} \text{ and } \sqrt{\tfrac{2}{3}\cdot 36^2}\ ,$$

the width of the base being irrelevant, as pointed out by Fibonacci.

The "geometry" part closes with yet another number problem having no obvious link

173 Discussed in some detail below, pp. 319 onward.

174 Lemma 1, ed. trans. [Heath 1926: III, 63],

To find two square numbers such that their sum is also square.

to geometry [B405;G622]: to find three numbers (say, a, b and c) such that $\frac{1}{2}a = \frac{1}{3}b$, $\frac{1}{4}b = \frac{1}{5}c$, $a \cdot b \cdot c = a+b+c$. The solution is found by means of a false position, namely $(a,b,c) = (8,12,15)$. With this position $a+b+c = 35$, $a \cdot b \cdot c = 1440$. Therefore, the positions have to be reduced by a factor $\sqrt{35/1440} = \sqrt{7/292}$.

Surprisingly, the numerical squares of a, b, and c are spoken of as tetragons (*tetragonus*). This Greek term is used regularly in the *Liber abbaci* about geometric squares (once, in a cistern problem, about a cube). It is never used except here about the square of a number. This could mean that here Fibonacci builds on a Greek, ultimately Byzantine source (which he might have encountered in Sicily as well as in Byzantium). On the other hand, a Latin translation from the Arabic *might* also have used it for *murabbaʿ*.

In any case, a borrowing is obvious, whether from the Greek or the Arabic. The way it is done illustrates how Fibonacci's deals with adopted material. The treatment of the problem consists of three sections. In the first of these, the term *tetragonus* is used 11 times, while *quadratus* is absent. In the second, which explains why a square root has to be taken, *quadratus* is used 5 times, *tetragonus* never; this is obviously an explanation added by Fibonacci, in which he uses his own language. In the last section, which verifies the outcome and which can be presumed also to be borrowed, *quadratus* is absent, while *tetragonus* is used 16 times.[175] As we see, Fibonacci is highly faithful to the original when borrowing (cf. also note 142), but he does not emulate its style in added material or commentaries. What we discern is faithfulness coupled to *deliberate* avoidance of imitation – it would have been only too easy to carry over the *tetragonus* to the commentary in the middle.

This observation should be taken into account when we interpret the lettering of diagrams. One might object to use of the sequence *a–b–g–...* as evidence of borrowing that Fibonacci could have tried to emulate the style of translations in constructions he had made himself. The use of the sequence *a–b–c–...* in simple cases which are almost certainly his own (see for example above, p. 119) shows that this objection can be disregarded. A glance at the diagrams in the beginning of Fibonacci's *Pratica geometrie* [ed. Boncompagni 1862: 2, 5*f*] supports this inference: at first comes a diagram proving *Elements* I.28 (not identified but following a generic reference to Euclid); it is lettered *a-b-g-d-e-...* and is almost certainly borrowed from the version translated directly from the Greek [ed. Busard 1987: 42]. Somewhat later, when Fibonacci speaks about how to measure a "quadrilateral and equiangular field", an illustrating diagram is lettered *a-b-c-d-e-f-g*; going on with more complicated divisions of the square, *a-b-g-...* returns.

[175] No sophisticated test is needed to show that this distribution is statistically significant. Using a simple model (that the probability to choose *tetragonus* is $\frac{27}{32}$ and that to choose *quadratus* is $\frac{5}{32}$) we find the probability of the present distribution to be slightly below 10^{-6}. A model based on combinatorics gives a probability of $\frac{5! \times 27!}{32!}$, close to $5 \cdot 10^{-6}$.

We may add that a writer who avoided as carefully as Fibonacci to refer to any sources beyond Euclid (and once Ptolemy together with what can be regarded as an explanatory commentary, see above, note 86) would hardly try to intimate by his lettering of diagrams that his own inventions were borrowed.

On the other hand we should not be misled by the instances of faithful copying which we can identify (together with the many others that we may suspect, for instance on the basis of the lettering of diagrams) that Fibonacci did not understand what was in his book. Faithful copying was rather a strategy making sure that no unintended misunderstanding crept in. We may think of the explanation offered by Charles Homer Haskins [1924: 152] of the tendency to translate Greek texts *de verbo ad verbum* (in part paraphrased from a 12th-century translator's preface). It had nothing to do with ignorance. Instead,

> Who was Aristippus that he should omit any of the sacred words of Plato? Better carry over a word like *didascalia* than run any chance of altering the meaning of Aristotle. Burgundio might even be in danger of heresy if he put anything of his own instead of the very words of Chrysostom.

As also observed by Haskins, the translations he discusses are "so slavish that they are useful for establishing the Greek text". Once we recognize Fibonacci's way of working we also discover that he opens new vistas on forgotten mathematical schools and traditions.

15.3, introduction to algebra

Part 3 – almost 10% of the whole work – states in the heading [B406;G622] to deal with "the solution of certain questions according to the way of *algebra et almuchabala*, that is, of "apposition"[176] and restoration".

As we remember, *algebra et almuchabala* was translated differently in chapter 14, in connection with the presentation of the "keys", namely as *contentio* and *solidatio*. Apart from an inverted order and from the miswriting *contemptio*, this is an adequate translation: *algebra* (*al-jabr*) means "restoration", "bringing back to normal state", which may well be rendered *solidatio*; in mathematics, it refers to the restoration of what is lacking (which, in an equation, is accompanied by an addition to the other side – cf. above, note 97); *al-muchabala* (*al-muqābalẖ*) means "encounter", "comparison", etc., not far from *contentio*. We now find the same inversion in chapter 15, "restoration" corresponding to *al-*

[176] *Appositionis*, probably a miswriting for *oppositionis* but possibly an alternative translation, meaning "setting before". Boncompagni, following his manuscript, has *ad proportionem*, obviously an attempt to repair the impossible grammar of two other manuscripts having "a proportionis" [Giusti 2020: 808]. Two rather desperate attempts to make sense of Boncompagni's *ad proportionem* ([Hughes 2004: 324 n. 43] and [Høyrup 2011a: 94*f*]) can now be happily discarded.

jabr.[177] *Oppositio* for *al-muchabala* is also a reasonable translation.

The evident starting point for the discussion of how Fibonacci presents the discipline is al-Khwārizmī's *al-jabr*,[178] the core of which is 6 "cases", equation types – originally riddles about an a "possession" or amount of money (a *māl*) and its (square) root, all provided with numerical examples [ed. Hughes 1986: 233–236] (*C* stands for *census*, the standard Toledo translation of *māl*, below also for Tuscan *censo; r* stands for *root/radix/radice*, *N* for number, α for an undetermined coefficient signalled by the use of a plural):[179]

Kh1	$C = \alpha r$, first example $C = 5r$.
Kh2	$C = N$, first example $C = 9$.
Kh3	$\alpha r = N$, first example $r = 3$.
Kh4	$C+\alpha r = N$, first example $C+10r = 39$.
Kh5	$C+N = \alpha r$, first example $C+21 = 10r$.
Kh6	$\alpha r+N = C$, first example $3r+4 = C$.

We observe that all cases except Kh3 are presented in normalized form, in agreement with the respective first examples.[180]; the rule for Kh3 is not normalized (although the first example is). This shows that al-Khwārizmī thinks of *r* as the real unknown, since then the normalized equation *is* the solution, as indeed shown by the first example. Additional examples show how to normalize equations where *C* (or, in Kh3, *r*) carries

177 The inversion should make us doubt the level of Fibonacci's understanding of Arabic. (According to [Tangheroni 2002], Pisa merchqants would understand Arabic, but at what level is unknown.) Beyond that, it is striking that the same inversion is found in chapter 14 and chapter 15, in spite of differing translations. A 13th-century manuscript (Florence, Biblioteca Nazionale Conv. soppr. J.V.18) of Gerard of Cremona's translation of al-Khwārizmī's *algebra* expands "in computatione algebre et almuchabale" into "computatione oppositionis algebre et responsionis almuchabale", which seems to be a witness of the same misunderstanding, with a further enigmatic *responsio* (is the idea that of a university disputation, where an opposite opinion has to be answered and refuted?).

178 My references will be to Gerard of Cremona's Latin translation because it is closer to the Arabic original than the extant Arabic texts, all later by a small century or more [Høyrup 1998; Rashed 2007: 83, 86]. When referring to the Arabic text I shall use [Rashed 2007].

179 A mnemotechnic trick that may help to remember the order: in the first three, first the number, then the *root*, then the *māl* is lacking; in the last three, first the number, then the *root*, then the *māl* is isolated. Whether this has anything to do with al-Khwārizmī's thinking is undecidable but doubtful.

180 The extant Arabic texts define the cases in non-normalized form, but conserve the initial, non-normalized examples – see [Rashed 2007: 96–107]. The text has evidently developed over the centuries, but since this did not influence Fibonacci, there is no reason to trace this process. Some elements are presented in [Høyrup 1998].

either an integer or fractional coefficient.

For each case an algorithm for solving it is given. We may look at the one given for case Kh4, the first of three “composite” cases:[181]

> The rule is that you halve the roots, which in this question are 5. Then multiply them in themselves, and from them comes 25. To which add 39, and they will be 64. Whose root you take, which is 8. Then diminish from that the half of the roots, which is 5. Hence 3 remains, which is the root of the *census*. And the *census* is 9.

Originally, when this was a riddle about an amount of money and its square root, the *amount* was evidently the unknown. Seen in this way, the problem translates thus into modern symbols:

$$y+10\sqrt{y} = 39$$

with solution

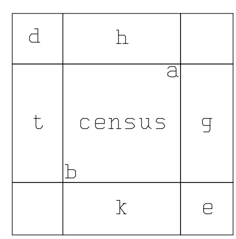
$$y = \left(\sqrt{\left(\frac{10}{2}\right)^2+39} - \frac{10}{2}\right)^2 .$$

As al-Khwārizmī presented the technique, the *census*/*māl* was still understood as an unknown to be found, as we observe. But it was no longer the fundamental unknown – as we shall soon see, the *root* was identified with the unknown *thing* which we have encountered in use in the *regula recta*. The corresponding reading of the equation is

$$x^2+10x = 39 ,$$

and the solution becomes

$$x= \sqrt{\left(\frac{10}{2}\right)^2+39} - \frac{10}{2} .$$



After the rules and the examples showing normalization, al-Khwārizmī gives geometric proofs for the three composite cases. For Kh4, two are given, of which this is the first one [ed. Hughes 1986: 237]: The *census* is represented by the square *ab*, each of whose sides is therefore the root. We distribute the 10 roots along the four sides, which gives us four rectangles *ghtk* with width 2½. Together with the square *ab* they have the area 39. Filling out in the corners the four lacking squares, each 2½×2½, we find the area of the larger square *de* to be $39+4\cdot 6^1/_4 = 64$. Therefore the large square has a side $\sqrt{64} = 8$; removing the

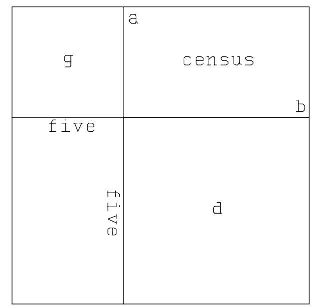
[181] Al-Khwārizmī writes all number in full words, and Gerard, always faithful, follows him faithfully. This canon was not taken over by Fibonacci nor in abbacus algebra. In order to make the argument stand out more clearly for the modern reader I shall therefore also disregard it.

width of two rectangles we get for the root 8–2·2½ = 8–5 = 3.

Strictly speaking, this proves the solution

$$x = \sqrt{4\cdot\left(\frac{10}{4}\right)^2+39} - 2\cdot\frac{10}{4} ,$$

and al-Khwārizmī needs to argue that it gives the same result as the one following from the algorithm.



The second proof [ed. Hughes 1986: 238] is much more adequate, and similar to the diagram used to prove *Elements* II.6; since the basic idea is the same as in the preceding proof, there should be no need to go through it in detail. One may ask why al-Khwārizmī first gave the less adequate proof since he knew the better one. There appear two possibilities; either the first one was what first came to his own mind, or he supposed it would speak more directly to the mind of his reader.[182]

As it turns out, there were reasons for this. Since Old Babylonian times (18th–17th c. BCE), the first diagram had been used to solve the riddle about "all four sides and the area" of a square, and that riddle was still circulating in al-Khwārizmī's (and Fibonacci's) time.[183] Al-Khwārizmī was likely to have known it, and so were his readers.

In the following sections, al-Khwārizmī teaches the multiplication, addition and subtraction of algebraic and arithmetical monomials and binomials like "*thing*" and "10 less a *thing*", "square root of 5", and "square root of 200 less 10". Here *thing* times *thing* turns out to be a *census*, which means that the *thing* and the *root* are identified (as also stated explicitly [ed. Hughes 1986: 242]). Numbers, moreover, are understood as numbers of *dragmae* (*dirham* in the Arabic original).[184]

The identification of *root* and *thing* turns out to be fundamental in the section of six problems illustrating the rules. We may look at the problem that serves as illustration of Kh5 [ed. Hughes 1986: 249]:

---

[182] Stylistic changes actually suggest that the second proof may have been added during a later revision of the text – cf. [Høyrup 1998: 169, 174].

[183] Since al-Khwārizmī is already two steps away from the abbacus school, I shall not offer documentation for this; but see [Høyrup 2001].

[184] Not consistently, however and with no thought about the dimension problem. A *dragma* times a *dragma* is stated to be a *dragma* [ed. Hughes 1986: 242] – the *dragma* thus functions exactly like Diophantos's *monas*, "unity", or a modern x^0.

> "Divide 10 in two parts, and multiply each of them in itself and aggregate them. And 58 results". Whose rule is that you multiply 10 less a *thing* in itself,[185] and 100 and a *census* less 20 *things* result. Next multiply the thing in itself, and it will be a *census*. Afterwards aggregate them, and they will be known as 100 and 2 *census* less 20 *things*, which are made equal to 58. Restore therefore the 100 and 2 *census* by the things that were diminished, and add them to 58. And say, "100 and 2 *census* are made equal to 58 and 20 *things*". Reduce therefore to one *census*. You therefore say, "50 and a *census* are made equal to 29 and 10 *things*". Oppose therefore by it.[186] Which is that you throw out 29 from 50. There remains therefore 21 and a *census* which is made equal to 10 *things*.

This is exactly the first example of the case Kh5, and it is solved accordingly.

The illustration of Kh6 [ed. Hughes 1986: 249*f*] is of interest by showing fluctuation in the use of *census* and *thing* (a phenomenon we shall encounter below in the *Liber abbaci* and elsewhere):

> "A third of a *census* is multiplied in its fourth, from which results a *census*. And let its augmentation be 24". Whose rule is, because you know that when you multiply $^1/_3$ of a *thing* in $^1/_4$ of a *thing*, results the $^1/_2$ of $^1/_6$ of a *census*, which is equal to a thing and 24 *dragmae*. Multiply therefore the $^1/_2$ of $^1/_6$ in 12 so that the *census* is completed,[187] and there will be a complete *census*. And multiply equally the *thing* and 24 in 12, and there result for you 288 and 12 *roots*. [...]

At first *census* appears in the original sense of a monetary possession understood as an unspecific quantity. Then, in order to allow its multiplication by itself, it is reinterpreted as a *thing*, whose square now becomes a (different, now algebraic) *census*, while the *thing* in the end appears as a *root*.

Further examples follow after the six initial illustrations. We shall return to them when needed, but there is no reason to discuss them systematically.

Instead we shall now turn to Fibonacci's presentation of *algebra et almuchabala*. Al-Khwārizmī [ed. Hughes 1986: 233] had stated

> to have found that three kinds of numbers were needed [in *al-jabr wal-muqābalah*]: *Roots*

---

[185] Already in the illustration of Kh2, where 10 is also divided into two parts, one of the parts was posited to be a *thing*, whence the other had to be 10 less a *thing*.

[186] This is the *muqābalah* operation, in later times understood as the subtraction from both side of the equation. As confirmed by the two more or less synonymous translations *contentio* and *oppositio*, what is intended is rather a *comparison*, which leads to the construction of the reduced equation.

[187] The verb used here (*reintegrare*, Arabic *kamala*) is distinct from the additive completion *restaurare*/*jabara*.

> and *census* and simple numbers neither related to a *root* nor to a *census*. The *root*, however, which is one of them, is something that is multiplied by one, and what is above that in numbers, or what is beyond that among fractions [that is, which carries a coefficient], while the *census* is something resulting from a *root* multiplied in itself".

Fibonacci [B406;G622] is obviously inspired, directly or indirectly.[188] But he changes the explanation according to his own understanding (the constructions are, mildly spoken, knotty):

> For the composition of algebra et almuchabala, three qualities [*proprietates*] that are in whatever number we consider, which are *root*, *square* [*quadratus*] and simple numbers. As when some number is multiplied in itself, and something results. A *square* is thus made from the multiplication of the multiplied [*multiplicati*]; and the multiplied is the *root* of its square. As when 3 is multiplied in itself, 9 results. 3 is namely the root of 9; and 9 are the square of the ternary. And when a number has no respect to square nor to root, then it is called a simple number. These, indeed, are made mutually equal in solutions of questions in six modes, of which three are simple and three composite. And the first mode is when a square, called a *census*, is made equal to roots. [...].

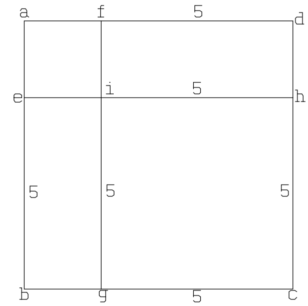
Fibonacci will certainly have known the normal Latin meaning of *census:* "wealth", "property", "estate valuation", etc. – quite adequate as a translation of *māl*. As we see, however, he chooses to present it as a synonym for *square* (of a number – in the geometric proofs he regularly uses *tetragon* about square configurations).

His list of six cases (F1, F2, ..., F6) is almost the same as that of al-Khwārizmī, the only difference being an inversion in the end, F5 = Kh6, F6 = Kh5; all cases are also defined in normalized shape, the reduction of non-normalized equations being taught separately. In the very first [B496*f*;G623], the term *census* is introduced as a replacement for *quadratus*, "when the square, which is called *census*, is made equal to roots".

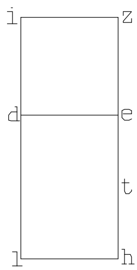
The numerical example used in connection with the rule and the proof for F4, "*census* and *roots* made equal to number", coincides with that of al-Khwārizmī; the others not. Nor are the proofs the same. We may look at that for F4. As in al-Khwārizmī's treatise, there are two, but they are different. The first builds on a diagram lettered *a–b–c–d–...*, thus apparently of Fibonacci's own making. Euclid is not mentioned, but the inspiration seems to be *Elements* II.4, not II.6, and the construction goes "the other way round": At first, a square *abcd* is made, where each side is requested to be larger than 5 ells (the

---

188 In the margin in Boncompagni's manuscript is also written "Maumeht", an obvious reference to Muḥammad ibn Mūsā al-Khwārizmī. However, this reference is absent from $\mathbf{V}_{\mathbf{F}}$, and can therefore be supposed to have been added by a later copyist or user. It corresponds to the beginning of Gerard's translation [ed. Hughes 1986: 233], *Liber Maumeti filii Moysi Alchoarismi de algebra et almuchabala incipit*.



style is thus that of a synthesis rather than an analysis). Next points *s*, *f*, *g* and *h* are made with the distances indicated in the diagram, and lines *eh* and *fg* are drawn, crossing at *i*. Then the *census* can be identified with the square *ef* – etc.



The second proof has some similarity with al-Khwārizmī's second proof, but there is no cutting and pasting. It is simply requested that the square *ei* be the *census*, that ten roots be applied to the side *de*, and that *t* be the midpoint of *he*. This has some family likeness to the proof given by Abū Kāmil, but both the Arabic [ed. trans. Rashed 2012: 254] text and the medieval Latin translation [ed. Sesiano 1993: 328] include a justified reference to *Elements* II. Given Fibonacci's propensity to cite Euclid when he knows it is warranted we may assume that he was inspired indirectly by Abū Kamil – but we may be sure that he did not use Abū Kāmil directly. Since even the first proof includes no reference to Euclid, we can be fairly certain that Fibonacci, though making his own version, was inspired not by Euclid but by some later source.

The proofs for the cases F5 and F6 are very similar – in particular that for F5, exemplified by the question "*census* made equal to 10 *roots* and 39 *denarii*". The close relationship between this and the exemplification and second proof of F4 could be what caused Fibonacci to change the order of cases (in other words, the idea to do so may well be his own).

99 questions with interspersed theory

Last in the third part, in chapter 15, and in the whole of the *Liber abbaci* [B410;G627] comes a collection of 99 questions,[189] with interspersed theoretical explanations. Many of them coincide with problems from al-Khwārizmī's *Algebra* in structure, often but far from always also in the choice of numerical parameters. Many (sometimes the same, sometimes other ones) coincide with problems from Abū Kāmil's *Algebra*. Both works were accessible in the Iberian Peninsula – at least two different manuscripts of al-Khwārizmī's algebra were translated there during the 12th century, and the *Liber*

[189] The precise number depends on to which extent variants are counted as independent questions. I follow the list in [Hughes 2004: 350–361], which along with the Boncompagni edition draws on the edition of chapter 15 in [Libri 1838: II, 307–479], based on a different manuscript of the *Liber abbaci*, and on Benedetto da Firenze's vernacular translation of the questions as rendered in [Salomone 1984]. A problem referred to as [H#*m*;G§*n*] is number *m* in Barnabas Hughes' list and §XV.*n* in [Giusti 2020], [G§*n*] refers to §XV.*n* in [Giusti 2020].

mahameleth refers repeatedly and correctly to Abū Kāmil. But the overlap and the occasional use of other numerical parameters shows that those same problems circulated, and that Fibonacci's inspiration may well have been indirect. The rather few agreements with al-Karajī's *Fakhrī*[190] are also not evidence that Fibonacci knew that work – some of them are also found in Abū Kāmil, and Fibonacci's numerical parameters often differ from those of al-Karajī.

An example suggesting that what "may well" be the case seems indeed to be so is offered by the problem [H#21;G§288]. It is one of 32 problems dealing with a "divided 10". Expressed in letter formalism:

$$10 = a+b\ ,\quad ({}^{a}/_{b}+10)\cdot({}^{b}/_{a}+10) = 122\,{}^{2}/_{3}$$

The same problem is solved by Abū Kāmil [ed. trans. Rashed 2012: 410*f*], while al-Karajī gives the sum as $143\,{}^{1}/_{2}$. According to the paraphrase in [Woepcke 1852: 94], al-Karajī appears to have posited a as a *thing*, and thus with a straightforward calculation to have reduced the problem to an instance of Kh5, namely $C+16 = 10r$. Abū Kāmil instead posits ${}^{a}/_{b}$ to be a "large *thing*" (presupposing $a>b$), and ${}^{b}/_{a}$ to be a "small thing" (as we see, al-Karajī was not the first to deviate from the practice of using coins as names for supplementary unknowns). Then

$$(R+10)\cdot(r+10) = 122\,{}^{2}/_{3}\ ,$$

and since $rR = 1$,

$$1+10\cdot(R+r)+100 = 122\,{}^{2}/_{3}\ ,$$

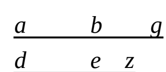
which is reduced to

$$R+r = 2\,{}^{1}/_{6}\ .$$

Thereby the problem is reduced to

$$10 = a+b\ ,\quad {}^{a}/_{b}+{}^{b}/_{a} = 2\,{}^{1}/_{6}\ ,$$

which has already been dealt with.



Fibonacci uses a line diagram, lettered *a–b–g–d–e–z*. Here, $ab = de = 10$, while $bg = {}^{a}/_{b}$, $ez = {}^{b}/_{a}$. That is, he replaces Abū Kāmil's two algebraic unknowns by line segments. The following procedure is parallel to that of Abū Kāmil, and also leads to the same reference to what has already been dealt with – actually, what has been dealt with by Fibonacci's source! Fibonacci himself [H#10,243;G§243] has treated the case where the sum of the two fractions is $3\,{}^{1}/_{3}$, not $2\,{}^{1}/_{6}$ (cf. below, note 192). An obvious trace of copying, though not from Abū Kāmil himself.

In the very end Fibonacci says that the reader should know that

[190] Some of those identified by Hughes turn out at inspection to be mistaken.

> when you have two numbers and divide the larger by the smaller and the smaller by the larger and multiply that which resulted from one division in that which resulted from the other, then from their multiplication always 1 is generated, and therefore I said 1 to come from *bg* in *ez*.

As shown by the lettering, we have a perfect parallel to the number problem from [B405; G622] (above, p. 140): A faithfully borrowed text, supplemented by a personal explanation coming afterwards, not integrated in what was taken over.

Comparing the three solutions, we notice that al-Karajī presents us with a typical *al-jabr* solution. Abū Kāmil's reduction makes use of a technique rather belonging with the *regula recta* (evidently, the problem to which he reduces the present one is then solved by *al-jabr*); Fibonacci, and his source, also removes anything that could make one think of *al-jabr* techniques (with the same proviso).

Line diagrams and geometric diagrams are used for other purposes too, a matter to which we shall return. First, however, we have to observe that there is no indication that Fibonacci tried to guide the reader systematically from simple to more advanced or difficult matters. Instead it is evident that groups of problems have been adopted together from the same source. In some cases, this source can be identified with approximation, in others not even that.

As the best example of the first category may serve the first eleven problems. Nine of them have a counterpart in the beginning of al-Khwārizmī's algebra, five in his list of six illustrations of the basic cases, four in his collection of varies problems. Internally in each of these groups, they follow al-Khwārizmī's order, but the two groups are mixed up.[191] By definition, the nine that have a counterpart have the same mathematical structure. Only two, however, have coinciding numerical parameters; and only one [H#10;G§243] has the same initial formulation as Gerard of Cremona's translation of al-Khwārizmī (though so simple that the coincidence might well be an accident); but in that case the numerical parameters are different, and the procedure is quite different.[192] According

---

[191] With Q referring to the six illustrating questions, V to the *varia*, and – indicating absence of a counterpart, Fibonacci's order is V1, –, Q2, Q3, –, Q4, Q5, V2, Q6, V4, V5. Using a simple combinatoric model we find that the odds that the order of borrowings from the two groups should be conserved by accident is $\frac{1}{4!\cdot 5!} = \frac{1}{2880}$.

[192] The problem is

$$10 = a+b\ ,\quad \frac{a}{b}+\frac{b}{a} = 3\tfrac{1}{3}\ ,$$

almost the same as the one to which the problem

$$10 = a+b\ ,\quad \left(\frac{a}{b}+10\right)\cdot\left(\frac{b}{a}+10\right) = 122\tfrac{2}{3}$$

[H#21;G§288] was reduced (above, p. 148), just with the sum being $2\frac{1}{6}$ (which is also the sum in al-Khwārizmī's version of the present problem). That problem (coming later in the *Liber abbaci*) was reduced, we remember, by means of line segments representing a, b, $\frac{a}{b}$ and $\frac{b}{a}$, respectively.

to what we have seen above in note 142, this should exclude that Fibonacci used al-Khwārizmī's *Algebra* (in Gerard's or any other version) directly for this sequence, but on the other hand show that he used an introductory work descending from that model (and made by a writer who was less faithful to his sources). The similarity of the demonstration of [H#10;G§243] to that of [H#21;G§288] observed in note 192 suggests origin in the same school of thought, while the stylistic difference seems to exclude inspiration by the same treatise.

Another cluster of problems is characterized by the appearance of an *avere*, a Romance (Italian, Catalan, Provençal or Castilian) loanword meaning "possession". Obviously it translates *māl* – but only when this term is used about an unknown quantity, literally an amount of money. The first time it appears is in [H#62;G§387]:

> Further, I multiplied the root of the sextuple of some *avere* in the root of its quintuple, and I added the decuple of the same *avere* and 20 *denarii*, and all this was as the multiplication of the same *avere* in itself. I shall posit for the same a thing, and I shall multiply the root of its sextuple in the root of its quintuple, that is, the root of 6 things in the root of 5 things. The root of 20 *census* results, since when a thing is multiplied in a thing it makes a *census*, whence when the root of a thing is multiplied in the root of a thing the root of a *census* results. Then I shall add above the root of 30 *census* the decuple of a thing and 20 *denarii*, and I shall have 10 things and the root of 30 *census* and 20 *denarii*, which is made equal to the multiplication of a thing in itself, that is, a *census*. In this falls the rule of roots and numbers which are made equal to a *census*.

The concluding statement presupposes that $\sqrt{(30C)}+10r$ is understood to be $(10+\sqrt{30})r$, but this would not be acceptable according to the canon that only integers and, in practice, rational fractions are accepted as numbers.[193] We shall return to this, but for the moment observe two things: throughout, in a number of problems that correspond to what is found in al-Khwārizmī and/or Abū Kāmil,[194] their initial, non-algebraic *mal* appears as "number" (*numerus*) or *avere*. This is not systematic, however, and we may suppose

A similar strategy is used here, just with the four segments being separate and each designated by a single letter (*a*, *b*, *g* and *d*). Al-Khwārizmī, in contrast, has no line representation (he never has).

193 The force of this canon, from al-Khwārizmī until the European 16th century, is dealt with in [Oaks 2017]. Since the difficulty was seen to be an obstacle that was to be, and was, circumvented, the avoidance of irrationals as coefficients *was* a canon, and not the result of failing understanding of possibilities, cf. [Høyrup 2004].

194 Unfortunately, Woepcke's paraphrase of al-Karajī [1853] translates into modern algebraic symbols, and therefore does not allow those who do not have access to the Arabic text to see what al-Karajī does.

that the choice depends on Fibonacci's source for the problem where it happens. Even to this we shall return.

Avere reappears in 13 further problems.[195] Sometimes the *avere* is posited to be a *thing*, sometimes to be a *census*. That obviously depends on what will yield a convenient equation, and does not tell us more than that.

More interesting is that all of these constitute a closed group, adopted from the same source. The apparent interruptions in the sequence all deal, either with a divided 10 (once 12) or with *two* numbers or quantities, and therefore would not allow the appearance of any substitute for *māl*, *avere* or otherwise. Since no problem after the last making use of an *avere* except the very last [H#99;G§682][196] would have allowed its appearance, the group may well have extended further (as we shall see on p. 155, there is more evidence for that).

This source, moreover, must already have used the term *avere*. There is no reason that Fibonacci should suddenly on his own choose a new translation – earlier problems use the standard translation *census* for *māl* in both roles, or replace an original Arabic initial *māl* by *numerus* or *quantitas*. We cannot exclude that this source was already written in a Romance vernacular (Italian, Catalan, Provençal or Castilian, though Italian seems even more unlikely than the others); more plausible, however, is a Latin translation prepared in a Romance-speaking (and thus Iberian) environment and, as Fibonacci does regularly, borrowing terms from the vernacular. Since we already encountered one source corresponding to this which also makes use of a non-standard terminology, namely in the introduction to chapter 14 (above, p. 118), we might suppose it to be the same – but only if we find it quite improbable that *two* such treatises should have disappeared, or at least disappeared from view.[197]

Other clusters can be suspected, but they are not as neatly delimited nor as informative,

---

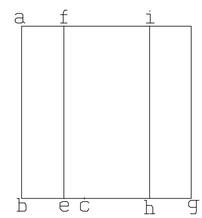
[195] [H#66;G§410], [B70;G439], [H#76;G§531], [H#77;G§539], [H#78;G§543], [H#79;G§546], [H#80;G§549], [H#81;G§551], [H#82;G§554], [H#83;G§557], [H#84;G§559], [H#85;G§561], [H#87;G§570]

[196] A simple first-degree problem, "I multiplied the 30-double of a *census* by 30 and what resulted was equal to the addition of 30 dragmas and the 30-double of the same *census*" – noteworthy at most (but hardly) for the use of *addition* in the sense of sum, which is unique in the algebra section though found in the last problem of chapter 15 section 2 and occasionally in chapter 14.

[197] Not the same thing! Remember that the *Liber mahameleth* was not known to have existed until Jacques Sesiano discovered it in 1974 – see [Sesiano 2014: v]. The best of the three manuscripts (Paris, BN, ms latin 7377A) had already been inspected by Michel Chasles [1841: 506], who mentions what has later been identified as the Latin translation of Abū Kāmil's *Algebra*. Louis Karpinski [1911], working on the manuscript, described Abū Kāmil's work. But none of these outstanding scholars noticed the *Liber mahameleth*.

and there is no reason to discuss them.[198] Let us instead return to [H#62;G§387] (above, p. 150) and look at how Fibonacci manages to circumvent the difficulty that he is not allowed to apply the rule he has seen to be pertinent:

> In order to show that, let there be placed hereby an equilateral and equiangular quadrangle *ag*, whose side is *bg*, and posit *bg* to be a *thing*. Therefore we cut off from the square *ag* a rectangular surface *ae*, which should be root of 30 *census*, and from the surface *fg* is removed the surface *fh*, which should be equal to 10 roots of the *census ag*, wherefore *eh* is 10. From the whole square *ag* remains the surface *ig*, which will be 20. And because the surface *ae* is the root of 30 *census* and comes from the multiplication of *ab* in *be*, and *ab* is a *thing*, it follows by necessity that *be* must be the root of 30, since from the multiplication of a *thing* in the root of 30 results 30 *census*. We add thus *be* with *eh*, and the whole *bh* will be 10 and root of 30, which is a fourth binomial; and we divide it in two equals at he point *c*, and each of the lines *bc* and *ch* will be 5 and the root of 7 $^1/_2$. And because the surface *ig* is 20, that which results from the multiplication of *ih* in *hg*, that is, from *bg* in *hg*, if above 20 we add the multiplication from *ch* in itself, which is 32 $^1/_2$ and the root of 750, we shall have 52 $^1/_2$ and the root of 750 for the square on the line *cg*. Then *cg* is the root of 52 $^1/_2$ and the root of 750, and if we add to it the line *cb* we shall have for the whole *bg*, that is, for the requested *avere*, the root of 52 $^1/_2$ and the root of 750 and 5δ and the root of 7 $^1/_2$ δ; all of which is according to approximation around 16 $^2/_3$.



The argument may be difficult to follow, but makes use of *Elements* II.6 or the corresponding "key" (none of which are mentioned), according to which $bh{\cdot}hg+bc{\cdot}bc = cg{\cdot}cg$. $bh{\cdot}hg$ indeed equals $ih{\cdot}hg$ and therefore 20. As we see, the rule for the case *census* equals roots and number does not turn up again, and the (correct but redundant) observation that $10+\sqrt{30}$ is a fourth binomial points back to the secondary layer of chapter 14. However that may be, we see that the difficulty of irrational coefficients is eschewed by the application of geometry.

The final approximation is worth observing. Approximation is used nowhere else in Fibonacci's algebra, nor anywhere I have noticed in abbacus algebra.

The lettering, including a *c* entering late in the argument, suggests that Fibonacci has intervened himself. This is confirmed by a comparison with the preceding problem [H#61; G§383], $\sqrt{(8n)}\cdot\sqrt{(3n)}$ (*n* being here a *numerus*, no *avere*). This *n* is directly identified with a line *bg* and n^2 with the corresponding square, here spoken of as a *tetragonus*. In

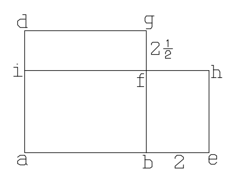
[198] A tempting but false trail is offered by the observation that some problems use *denarius* as the unit for pure numbers and some instead *dragma*. Since the text may alternate between the two within the same problem solution (e.g., [H#28;G§315] and [H#58;G§372]), no classification can be derived.

this case, the lettering is *b–g–d–f–h – h* taking the place of *c*, and *a* being left out because the corresponding corner of the square is not mentioned. Moreover, in this case a binomial (here the result) is identified as a *sixth binomial*, whereas the problem before that [H#60; G§381], $(8\sqrt{n})\cdot(3\sqrt{n})+20=n^2$ has a *fifth binomial* (and no diagram, since the problem reduces to $24n+20=n^2$, with no irrational coefficients occurring). Such references to the classes of *Elements* X occur nowhere else in the collection of algebraic questions. It appears that in [H#62;G§387] Fibonacci has borrowed a proof technique from [H#61; G§383], which does not belong to the *avere* group, but adapted the proof to the situation where the coefficient is $10+\sqrt{30}$ and no simple root, employing also his own terminology ("an equilateral and equiangular quadrangle" then becoming simply "square", *quadratus*). As observed on p. 140, Fibonacci shifts to his own language when he is creative, and avoids imitation.

In the following problem ([H#63;G§392] still belonging to the *avere*-cluster), a "divided 10" also leading to an irrational coefficient, a diagram lettered *a–b–c–d–g–e*, is made use of. Even here, and perhaps more radically, Fibonacci seems to work independently. Once more, the quadrate is spoken of as *quadratum equilaterum et equiangulum*.[199]

A final use of diagrams serves the translation of a question into an equation. That happens in a sequence of five problems, the first of which [H#12;G§252] runs like this:

> I divided 60 between some men, and something resulted for each; and I added two men above them, and between all these I divided 60, and for each resulted $2\,^1\!/_2$ less than resulted at first. Let the number of the first men be the line *ab*, and on it is erected at a right angle the line *bg*, which should be that which falls to each of them of the mentioned 60δ, and draw the line *gd* equal and parallel to the line *ba*, and the straight line *da* is connected. Then the space of the quadrangle *abgd* will be 60, as it is contained[200] by *ab* in *gb*. Then protract the line *ab* to the point *e*, and let *be* be 2, that is, the number of men to be added. And on the line *bg* the point *f* is marked, and let *gf* be $2\,^1\!/_2$, that is, that each one



199 However, the use of *tetragonus* versus "equilateral and equiangular quadrangle" is not quite systematically coupled to diagrams lettered *a–b–g–...* respectively *a–b–c–....* Evidently, Fibonacci may have sometimes have borrowed a diagram but written his own text, or *vice versa*. He may also have changed his preferred terminology over time. We do not know, indeed, how much of chapter 15 goes back to 1202, and how much was inserted in 1228, even though problems belonging to a particular cluster like the *avere*-cluster almost certainly entered the work at the same time.

200 *Colligatur*, not the standard terminology, which would be *continetur*. We observe, moreover, that this being contained is not formulated as a geometric fact but as a multiplication of "*ab* in *gb*".

got less by the addition of two men. And through the point *f* the line *hi* is protracted equal and parallel to the line *ea*, and the straight line *eh* is connected; the quadrangle *heai* will be 60, since it is contained by *ae* in *eh*, namely by *ae* in *bf*, where *bf* is that which resulted for each of the men *ae* from the 60δ. The surface *ei* is thus made equal to the surface *bd*. The multiplication of *gb* in *ba* is thus made equal to the multiplication of *ea* in *fb*. Whence these four lines are proportional. Therefore, the first *gb* is to the second *fb* as the third *ea* to the fourth *ba*, whence, by dividing,[201] as *gf* is to *fb*, so is *fb* to *ba*. But the ratio *gf* to *eb* is as 5 to 4. Thus *fb* contains once and one fourth the number *ba*.

So, posit for the number *ab* a *thing*. *bf* will thus be $1\,^1\!/_4$ *thing;* and multiply *ab* in *bf*, and $1\,^1\!/_4$ *census* results for the surface *bi* [...].

Nothing similar is to be found in the original version of al-Khwārizmī's algebra as we know it from Gerard of Cremona's translation, nor in the somewhat extended version translated by Robert of Chester [ed. Hughes 1989]. In the later Arabic manuscripts we have a version where the amount to be distributed is 1 dirham, only one man is added, and the difference is $^1\!/_6$ [ed. trans. Rashed 2007: 190*f*]. Even here, the solution consists of several parts: first a description in general terms, which appears to correspond to a diagram which however has disappeared; then the same with explicit numerical values; and finally, as in the *Liber abbaci*, the solution of the resulting equation. However, this may have crept into the tradition at any moment before 1222, the date of the earliest Arabic manuscript [Rashed 2007: 85], and there is no reason to believe it inspired Fibonacci, neither directly nor indirectly.

On the other hand, there is no reason to doubt a link to Abū Kāmil's algebra [ed. trans. Rashed 2012: 352–355], where 50 dirham are shared first among some men, them among 3 more, the difference between what each one gets in the two situations being $3\,^3\!/_4$ dirhams. The solution follows the same pattern as that of Fibonacci, but instead of using proportions the argument about the diagram is arithmetical all the way through.

Next in the *Liber abbaci* follows a problem [H#13;G§259] where first 20 is divided between some number of men, next 30 between 3 more, the difference between the shares in the two situations being 4. The solution is based on a diagram of the same character though slightly more complicated, lettered *a–b–g–d–e–...*, and on proportion techniques (followed by algebraic solution of the resulting equation). Once again, Abū Kāmil offers four problem of the same structure [ed. trans. Rashed 2012: 358–371], presenting solutions based on diagrams of similar structure and never referring to proportions.

The following problem in the *Liber abbaci* [H#14;G§271] has the same mathematical structure. This time, however, a diagram lettered *a–b–c–d–e–...* is used, and the algebraic

201 That is, we transform $\frac{gb}{fb} = \frac{ea}{ba}$ into $\frac{gb-fb}{fb} = \frac{ea-ba}{ba}$, whence $\frac{gf}{fb} = \frac{eb}{ba}$. That could, by the way, be seen directly in the diagram, just by removal of the shared surface *af* from both of the surfaces *ag* and *ah*. The ensuing "permutation" leads to $\frac{gf}{en} = \frac{fb}{ba}$.

entities (*thing* and *census*) enter directly in the discussion of the diagram, and proportions are not referred to. To judge from the lettering of the diagrams in these last two problems, the reformulation of Abū Kāmil's technique in terms of proportion techniques solution is borrowed, while Fibonacci's own (more straightforward) solution does not mention them.

In the last problem about changing numbers of men sharing money [H#15;G§276], when 10 are divided between a certain number of men and then 40 between 6 more, they get the same in the two cases. It would be obvious for anybody tending to think in terms of proportions to state this as $\frac{h-6}{h} = \frac{40}{10}$, from which would follow $\frac{6}{h} = \frac{40-10}{10}$, whence $6 \cdot 10 = 30 \cdot h$. But Fibonacci working on his own[202] does not appear to have such preferences on the present occasion. He just observes (thus not using algebra) that the 30 extra monetary units must be the share of the 6 extra men, each of whom therefore gets 5. Since the first men get the same, their number must be $10 \div 5 = 2$.

There can be no doubt that the sequence [H#12–16] is part of a cluster adopted from the same source (for the last one, however, Fibonacci seems to have presented a simpler solution of his own making). Since [H#11] belongs to the cluster borrowed indirectly from al-Khwārizmī's algebra, [H#12] is the first member of the present cluster; whether it extends beyond [H#16] seems undecidable (but rather unlikely according to internal criteria of style).

A final question to address is whether and how Fibonacci deals with higher-degree problems.

Some of the problems that have been considered biquadratic by earlier workers only become so because of failure to understand the distinction between the two roles of *census* – for instance [H#44;G§344]:

> I multiplied the third of a census and $1\,{}^{3}\!/_{4}$ in its fourth and $2\,{}^{3}\!/_{4}$, and a census augmented by $13\,{}^{3}\!/_{4}$ resulted. Posit a thing for the census. [...].

If it is not realized that the initial *census* is of the kind that elsewhere is sometimes spoken of as an *avere* or a *number*, this looks like a biquadratic solved by means of a substitution of variable. When no positing is needed – for instance, in [H#38;G§340] – the *census* in question is indeed considered the solution, its root is not found, which confirms its meaning as an "amount". There are more of these, and there is no reason to discuss them any further.

Others are properly biquadratic or lead to solvable third-degree equations. They are all found within the "extended *avere* group" (cf. above, p. 151), confirming the suspicion

---

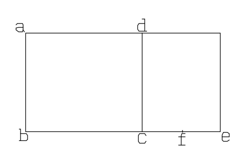
202 The same problem is in Abū Kāmil's algebra [ed. trans. Rashed 2012: 370–373]. At first Abū Kāmil gives an unexplained numerical prescription, stating only that "the reason of that is obvious"; next he actually formulates the proportion $\frac{h}{h-6} = \frac{10}{40}$, and then identifies the second ratio with the number ${}^{1}\!/_{4}$. Identifying h with a *thing* he obtains an algebraic equation.

that this really *is* a group.[203] Here, a corresponding terminology is also used (*cubus, cubus cubi, census census, census census census, census census census census,* for the third, the sixth, the fourth, the sixth and the eighth power), and scattered theoretical observations though no systematic presentation of higher-degree techniques can be found.

We shall look at a single example [H#88;G§575], in which Fibonacci appears to have intervened actively in a justification:

> Of three unequal quantities, when the major and the minor are multiplied it is as the middle in itself,[204] and when the major is multiplied in itself results as much as the minor in itself and the middle in itself joined, and from the multiplication of the minor in the middle results 10. Posit for the smaller a *thing* and for the middle 10 divided by a *thing*, and multiply 10 divided by a *thing* by itself, and 100 divided by a *census* results, which you divide by a *thing:* 100 divided by a *cube* result, and this will be the major quantity. Then multiply the minor quantity, namely a *thing*, in itself, and a *census* results; and multiply the middle in itself, namely 10 divided by a *thing*. 100 divided by a *census* results, which you shall add with the *census*, they will be a *census* and 100 divided by a *census*, which is made equal to the multiplication of the major quantity, namely 100 divided by a *cube* in itself, from which multiplication result 10000 divided by a *cube of cube*. Then multiply everything you have by *cube of cube;* and to multiply by *cube of cube* is as multiplying by *census of census of census*. Then if we multiply 10000 divided by *cube of cube* by *census of census of census*, 10000 result; and if we multiply a *census*, namely the square of the minor quantity, by *census of census of census*, we shall therefore have a *census of census of census of census;* and if we multiply the square of the middle quantity, namely 100 divided by *census*, by *census of census of census*, results *100 census of census*. Therefore a *census of census of census of census* and 100 *census of census* are made equal to 10000 dragmas.

At this point, we might perhaps have expected Fibonacci to posit a (new) *thing* for the *census of census*, or simply to have applied the standard rule for the case "*census* and *things* made equal to number". Instead, as when application of the standard algorithms would presuppose the explicit use of irrational coefficients in [H#62;G§387] and [H#63;G§387] (above, p. 151), Fibonacci uses the geometric configuration that serves to justify the rule in question, positing the square *ac* for the *census of census of census of census*. The lettering as well as the use of *quadratum* indicates



[203] Since this group is found close to the end on the chapter, we may get an impression of theoretical progress. This impression, however, is an artefact, and the very final trivial first-degree problem disproves it.

[204] In other words, if they are in continued proportion.

that the proof was inserted by Fibonacci himself (the two proofs circumventing irrational coefficients, as we remember, were also characterized by the lettering *a–b–c–...*, and also used *quadratum*). Together, Fibonacci's need to intervene actively in these three cases[205] suggests that his source for the *avere* group had fewer qualms with irrational coefficients than he had himself and handled higher powers more freely.

In both cases, Fibonacci's independent construction of proofs shows that he had no difficulty in understanding what his source was doing. That seems to hold throughout the algebra-part, with a single exception, an alternative solution to [H#71;G§448],

> I divided 10 into two parts, and divided the larger by the smaller, and the smaller by the larger; and aggregated that which resulted from the division, and they were 5 δ.

The alternative solution [G§461] starts that

> you posit one of the two parts a *thing*, and the other certainly 10 less a *thing*. And let from the division of 10 less a *thing* in a thing a *denarius* result.

Obviously, Fibonacci here adopts the Arabic use of coin names for supplementary algebraic unknowns (cf. above, note 104), unfortunately using the same term for the unit of pure numbers and for the extra unknown (for the former, Abū Kāmil would use *dirham*, for the latter *dinar*). If Fibonacci had understood the principle, he might perhaps have managed to keep the two functions of the term separate, but he does not (but since he copies, he evidently ends up with the correct result in spite of intermediate mistakes; cf. [Høyrup 2019b: 32–35]).

So, as final characterization of part 15.3, Fibonacci's algebra, we may say that it is no *treatise*, no new coherent and systematic presentation of the field. It is an anthology, a collection of excerpts from other texts (with an introduction and interspersed additional explanations) – with a single exception well-understood by Fibonacci. Being neither an elementary introduction nor an methodically progressing guide to advanced methods, it is no wonder that (as we shall see) it had no influence on the abbacus masters when some of them eventually took up algebra.

---

205 Supported by [H#63;G§392], [H#89;G§583], [H#71;G§448] and [H#73;G§497], which share these characteristics. The last of them has the letter sequence *c–d–e–f–g*, *a* and *b* having been used already as one-letter line-carried symbols for $^{10-r}/_{r}$ and $^{r}/_{10-r}$, respectively, similarly to Abū Kāmil's "large" and "small thing" (above, p. 148).

IV. The real story in select detail

I shall not object to those who see the preceding chapter as an “almost insupportably detailed description of the *Liber abbaci*”, to quote Cantor’s characterization [1892: 31] of his own much shorter analysis. Now, however, we shall turn to the history of abbacus mathematics proper, divided into periods.

“Generation 1”:, *Livero de l’abbecho*, Pisan *Libro di ragioni*, “Columbia Algorism” and *Liber habaci*

Three abbacus books have, for various reasons, been dated to the late 13th century: a *Livero de l’abbecho* [ed. Arrighi 1989; ed. Bocchi 2017],[206] known from an early-14th-century copy;[207] the Pisan *Libro di ragioni* [ed. Bocchi 2006; ed. Franci 2015]; and the “Columbia algorism” [ed. Vogel 1977]. Because of the character of the text, a *Liber habaci* from *ca* 1309 [ed. Arrighi 1987] should probably be counted to the “first generation” together with these – and perhaps even be seen as a reflection of “generation zero”.

Livero de l’abecho

The *Livero* (mentioned above, note 19), written in Umbria, plausibly in Perugia [Bocchi 2017: 7 and *passim*] offers what superficially looks like confirmation that the abbacus tradition is based on Fibonacci’s *Liber abbaci*.[208] Its initial lines represent it as “the book on the abbacus according to the opinion of master Leonardo from the house of the Bonacci sons of Pisa”.[209] This – apart from the very term *abbaco* – looks as the only positive evidence that the abbacus tradition really had Fibonacci and his *Liber abbaci* as its starting point. Later abbacus books, indeed, take next to nothing from the *Liber abbaci*,[210] but since they share much with the *Livero* – see, as an example, the similarity of the ways the rule of three is formulated in note 19 (Fibonacci, as we remember from p. 71, has nothing similar, nor any name for the rule). So, through this double coupling, it seems that the abbacus tradition can really be linked to Fibonacci and his *Liber abbaci*.

Precise analysis of the *Livero* reveals that this conclusion is fallacious – the details can be found in [Høyrup 2005]. Here I shall restrict myself to a summary.

It turns out that the *Livero* moves on two levels. On the basic level we find everything that was taught in the abbacus school; nothing on this level comes from Fibonacci (some complex problems about similar topics do, however).

---

206 Henceforth, references [A*m*;B*n*] to the text stand for [Arrighi 1989: *m*; Bocchi 2017: *n*].

207 Florence, Riccardiana, Ms. 2404, Fol. 1^r–136^v – a beautiful vellum manuscript indicating that the book was appreciated.

208 Since my earlier analysis of the *Livero* [Høyrup 2005] was built on Arrighi’s edition (and a visit to Florence where I could inspect the manuscript), this edition shall be the main basis for my references.

209 *Quisto ène lo livero de l’abbecho secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa* [A9;B163].

210 See below, p. 249 onward, as a late and very particular (thus partial) exception.

At first the rule of three is presented, in the formulation that with very small variations was to become the standard of the abbacus treatises (cf. above, note 19); the basal presentation is followed by rules for how to eliminate fractions and some examples. Several chapters present the use of metrological shortcuts, and four the exchange and barter of monies and the purchase of bullion. The initial parts of chapters on alloying and on simple and composite interest also belong on this level, as do the full chapters on discounting and partnership (details in [Høyrup 2005: 29–30, 44–53]).

On the other level we find sophisticated material. Most of this is translated from the *Liber abbaci* – often with errors revealing that the compiler of the *Livero* (and his source, if the translation is borrowed, as it appears[211]) did not understand what he translated – as Andrea Bocchi observes [2017: 16], in contrast to the exquisite quality of the manuscript, the mathematical substance is characterized by "an impressive series of errors and lacunae [...], in particular in the part derived from the *Liber abbaci*". We may restrict ourselves to a couple of instances.

A plausible though not fully certain example of this is that Fibonacci's composite fractions are understood as normal fractions, $\frac{3\ 5\ 14}{4\ 6\ 25}$ becoming simply $\frac{3514}{4625}$.[212] In principle that might be due to a misunderstanding on part of the 14th-century copyist of

---

211 "The vernacularization of Fibonacci does not derive directly from a Latin antigraph: the model of the Riccardiano [manuscript] was not directly the *Liber abbaci* but an already profoundly adapted vernacularization" [Bocchi 2017: 32]: a conclusion based on mistakes that are difficult to explain unless we presuppose the use of a vernacular intermediary – for example, Fibonacci's *minus* becoming *viene* through the intermediary *meno*, whose *m* is easily read as *vi*.

212 [B110;G190] respectively [A26;B191]. In general, Bocchi, knowing what these fractions *should* be, corrects them tacitly [Bocchi 2017: 108 n. 95], Arrighi not. Having inspected the manuscript I can confirm Arrighi's readings.

The present fractions results in the *Liber abbaci* when "uncie 13, et denarii 14, et carubbe 5, et grana 3" are expressed as a mixed number, namely as $\frac{3\ 5\ 14}{4\ 6\ 25}13$, in agreement with Fibonacci's explanation of the metrology [B84,107;G142,183], according to which 1 *uncia* consists of 25 *denarii* [*di cantera*], 1 *denarius* of 6 *carubbe*, 1 *carubba* of 4 *grani*. Fibonacci leaves to his reader to understand this.

In the *Livero*, an explanation is given, "This is its rule, that we shall bring to one stroke all the *denari* and the *carubbe* and the *grani*, like this, $\frac{3514}{4625}13$". "Stroke" (*verga*) is used elsewhere about the fraction line, but also [A132;B400] when the compiler takes over a graphically similar but mathematically different notation in the problem about seven old women go to Rome [B311;G489] – $7\ \frac{1\ 1\ 1\ 1\ 1}{7\ 7\ 7\ 7\ 7}\ 1$ standing for $7{\cdot}(1+7{\cdot}(1+7{\cdot}(1+7{\cdot}(1+7{\cdot}(1+7)))))$. Actually, the *Livero* writes $7\ \frac{1}{7}\frac{1}{7}\frac{1}{7}\frac{1}{7}\frac{1}{7}\frac{1}{7}\frac{1}{7}\ 1$, adding two extra 7s, using multiple strokes but still speaking of the *verga* in the singular, in agreement with Fibonacci's singular *virga*. Here, at least, the compiler seems to have copied without understanding (and without counting well). Whether he understands what was possibly added to the original text in the vernacular translation of the *Liber abbaci* which he uses cannot be decided.

a substantially correct original version of the *Livero*; Such misunderstanding are also found in the Boncompagni manuscript of the *Liber abbaci*. However, there they are not systematic as here, which suggests that the misunderstanding of the composite fractions must be imputed to the original compiler of the *Livero* (if not to the earlier translator into the vernacular). If this is the case, the compiler has not followed the calculations when copying; if the mistake goes back to an earlier translator, it would be impossible for him to do so, since the calculations are meaningless.

Indubitably the responsibility of the compiler (again, probably already of the earlier translator) is the omission of most of Fibonacci's alternative solutions by means of *regula recta*. In one case, however [A89;B7317], in a problem about travels with gain and expenses based on [B258;G418], only the beginning is skipped, where the unknown is posited to be a *res*; in the passage that is taken over, the compiler translates this *res* as a nonsensical non-algebraic "thing" (*cosa*); the method itself is called *per regola chorrecta*.

In one case, finally, where Fibonacci [B399;G613] solves a problem of the second degree (repeated commercial travels with constant profit rate; above, p. 137) by means of proportions in a lettered line diagram and *Elements* II.6 in key version, all letter-references disappear from the text [A93;B324*f*], as does the line diagram itself. The possession of the traveller after the second travel is misread consistently as 229£ (Fibonacci has 299£), blatantly contradicting the copied correct result (namely that each 100£ earn 30£ at each travel.

In general, much is left out from the problems that are copied from Fibonacci. However, *that* which the compiler copies, he tries to copy faithfully – often repeating Fibonacci's cross-references, even when they are no longer valid in the new context.[213]

As explained in note 75, Fibonacci writes mixed numbers in the Arabic way – that is, not $69\frac{1}{3}$ but $\frac{1}{3}69$. In the problems he borrows from Fibonacci, the compiler of the *Livero* does the same, only adding in many cases a unit, which Fibonacci leaves out.

Those problems that do not come from the *Liber abbaci* are different. The first pages follow the habits of the time and write, for example, *libra 1, soldo 1 e denari 10,* $^{6}/_{7}$ *de denaro*.[214] Then suddenly the compiler starts to write the fraction links, for example, *libre 68, soldi 3, denari* $^{7}/_{11}$ *7 de denaio* [A18;B177]. This grammatically impossible structure shows that he has used an original writing ... *denari 7,* $^{7}/_{11}$ *de denaio* and then (whether inspired by Fibonacci or not we cannot know) shifted the order of the mixed number according to Arabic habits. This interpretation is confirmed by a few slips:

[213] One example: When presenting Fibonacci's first house-renting problem the *Livero* [A48;B231] explains that "this one is similar to the other one about travels, that is, that somebody had 100 £, from which of 5 £ he made 6 in each travel ...", exactly as does Fibonacci [B267;G430]; but unfortunately the chapter on repeated travels only comes much later [A88;B315] in the *Livero*.

[214] [A16;B174], abbreviations resolved, punctuation modern.

occasionally the shift is forgotten. The material that does not come from Fibonacci must therefore come from an earlier abbacus-text (possibly several). That other, later abbacus texts turn out to be related to *this* part of the material of the *Livero* hence does not at all link them to Fibonacci.

This brings us to the question of dating. Loan contracts in the text, dated 1288–1290, were taken by Warren Van Egmond [1980: 156] as justification for the dating "c. 1290, *i*[*nternal evidence*]". Gino Arrighi [1989: 6] judged the text to belong to the second half of the 13th century because of the general character (*stesura*).

However, the same distorted way of writing amounts of money involving mixed numbers is found in the loan contracts contained in the *Livero*. In consequence, these cannot originally have been composed by the compiler; as the rest of his text, they must have been copied from a model – certainly an earlier teaching text, not real-life contracts.[215] The date 1288–1290 is therefore only *post quem*. The complete ignorance of the compiler of even the most basic algebraic terminology tells us, however, that he cannot have written much later than 1310.

The manuscript, apparently a *de luxe* copy on vellum, may well be later. That is evidently of no concern for the dating, but it tells us that a *Primo amastramento de l'arte de la geometria* which follows the *Livero* in the manuscript need not originally be from the same hand nor date as the latter. As pointed out by Bocchi [2017: 85], it was clearly thought to be an independent treatise; it shares with the *Livero* the general character of drawing in part on the Fibonacci, in part of other sources. However, its way to designate concrete mixed numbers where it does not depend on Fibonacci is sometimes similar to the inconsistent way of the *Livero*, sometimes different.[216] If not due to the same hand, the originals of the two works seem to have been compiled by closely connected writers. We need not undertake a detailed analysis of this *Primo amastramento*, but we shall return to it on p. 178.

The Pisa *Libro di ragioni*

A unfortunately incomplete *Libro di ragioni* ("Books of problems") is contained in the manuscript Siena, Biblioteca degl'Intronati, L.VI.47.[217] On the basis of her

---

215 As observed by Bocchi [2017: 37], all accounts are opened 1 January 1288.

216 For instance [ed. Arrighi 1991: 9], *br.* $^5\!/_8$ *9 de braçio* and, in the following line after a multiplication by 4, *br.* $^1\!/_2$ *38*. The latter type, close to Fibonacci, is by far the most common.

217 Two editions have been made, [Bocchi 2006] and [Franci 2015]; Franci appears not to have been aware of the existence of Bocchi's edition. Bocchi, being a philologist and palaeographer mainly interested in the language, leaves out the numerical schemes contained in the manuscript from his almost diplomatic transcription; they are included by Franci, who as a historian of mathematics understands these to be important for her purpose. Bocchi also points out in one of four cases only

impression of the language and the shape of numerals Franci [2015: 11] claims it to have been written in the late 13th century. Bocchi [2006: 19], supported by the authority of Armando Petrucci, dates it "certainly to the first half of the 14th century, probably to its first fourth (1301–1325)".[218] Ulivi, independently [2011: 258], also dates the treatise to the beginning of the 14th century.

According to an early foliation, three leaves are missing in the beginning. They may have contained multiplication tables, but will not have allowed a general introduction to the Hindu-Arabic numeral system as we know it from Jacopo. The first conserved leaf starts (p. 22) in the middle of a sequence of metro-numerical tables – the first line closes a list of divisions of amounts of bezants by 100,[219] after which come

– divisions of $^1\!/_3$ δ, $^2\!/_3$ δ, $1\,^1\!/_4$ δ, ..., 100 δ by 100;
– divisions of $^1\!/_4$ δ, $^1\!/_3$ δ, $^1\!/_2$ δ, ..., 12 δ by 12;
– divisions (pretendedly) of $^1\!/_{12}$ß, $^1\!/_9$ß, $^1\!/_8$ß, ... $^{11}\!/_{12}$ß by 12; actually, $^1\!/_{12}$ß etc. are just converted into δ;
– conversions of fractions of £ into ß and δ;
– conversions of fractions of a pound (a *libbra sottile*, cf. note 21) into ounces;
...

In the end come divisions of fractions
– $^1\!/_5$, $^2\!/_5$, $^3\!/_5$ and $^4\!/_5$ by 100.

After these tables come (p. 28) a number of rules for divisibility, apparently meant to facilitate the reduction of fractions and divisions:
– When the letter [i.e., digit] in the beginning (*a chapo*) is even, one may reduce

that a leaf is missing within the stretch of text he transcribes (mentioning the general phenomenon only in the introduction, p. 23), while Franci identifies them and reconstructs the statements of problems where only the final part of the calculation survives after such a lacuna. On the other hand, Franci omits from her transcription the final ten surviving leaves, which are too damaged to allow her to produce an understandable text, while Bocchi includes them in his edition, which indeed allows us to understand the topics dealt with if not the procedures. Bocchi also offers a very useful glossary.

When nothing else is indicated, references to the text point to the pagination of Franci's edition.

218 Bocchi [2006: 22] further points out that the geographical awareness reflected in the problems carries no trace of how the Pisan trading network had looked before Pisa's defeat to Genova in the battle of Meloria in 1284 – probably indicating that this was already decades in the past.

219 Bezants *tout court* can be seen from a problem where they are used for purchase in Tunis (p. 46) to be *garbi*, not Byzantine (cf. above, note 102); p. 66 speaks of a *biçantio di migliarese*, the migliarese being a silver coin, valued (according to the present treatise, for example the present division) $^1\!/_{10}$ bezant. This bezant appears regularly in the treatise. A *biçantio di carato* (Byzantine, Egyptian or from the Crusader states?) turns up more rarely.

[*schizzare*, the term used for the reduction of fractions] to half. When proof is taken for 9 and is 3 or 6 and the first letter is not even, one may reduce to $^1/_3$.

– When the two letters in the beginning have the $^1/_4$, all the others have too.

...

– When the letter in the beginning is *çefaro* (i.e., zero), one may reduce to $^1/_{10}$ or $^1/_5$. Beyond the spelling *çefaro* (closer to original arabic *sifr* than the *zevero* that is often found, and no obvious descendant from Fibonacci's *zephirum* [B2;G5]) we may take note that a multi-digit number is supposed to begin to the right, as in Arabic (but also in many other abbacus books). The rules are all reasonable and correct; I have not observed them elsewhere.

Another table follows (p. 28), with heading *hoc est lasalma*, "this is *lasalma*". *Lasalma* is related to Fibonacci's *hasam* (above, note 79), rendering the Maghreb Arabic term *aṣamm* designating a number that cannot be factorized. *Lasalma*, however, must come directly from something like *al-ṣammāʾ*, with article, double consonant and (what to a Pisan merchant looks like) a feminine ending, none of which Fibonacci indicates; it must come directly from the Arabic.[220] Another indication that the table is not drawn from Fibonacci (as supposed by Franci [2015: 13]) is that Fibonacci's table lists only the *prime* numbers from 11 to 97 (that is, the *aṣammāʾ* numbers), while the present table lists *all* numbers from 11 upwards (because of missing leaves we cannot know how far), indicating either a splitting in two or three factors or that the number in question "has no rule"[221] – all in fraction form, confirming that *division* by the number in question is intended. It appears that the author has not understood (as he would if he had read Fibonacci) that *lasalma* means the same as "without rule" (that is, prime) and instead takes it to mean "factorization".

For 12, the format is $\frac{1 \qquad 1}{2 \;\; \mathit{of} \;\; 6}$, obviously inspired by the ascending continued fraction $\frac{1\;\;0}{2\;\;6} = {}^1/_{12}$ and a functioning explanation of the reading direction of such fractions but none the less mistaken. The format for 14, 15 and 16 is the same, but from 18 upwards it changes to "$^1/_2$ of $^1/_9$", etc. – with the exception of 84, for which "$^1/_2$ of $\frac{1\;\;1}{6\;\;7}$" is given.

Later in the treatise, mistaken use of the notation for ascending continued fractions also occurs – in a problem on p. 46 $\frac{1\;\;1}{2\;\;4}$ is thus explained to stand for $^1/_8$, which is next used three times, excluding a writing error. However, the notation for ascending continued fractions is mostly used correctly, always written (as in Arabic and the *Liber abbaci*)

220 The omission of the initial vowel and the misrepresentation of the double consonant suggests spoken Arabic, which is anyhow what we should expect.

221 Once written *sine regula*, afterwards *non à regula*. The former expression is close to what is used in the *Liber abbaci* (see note 79 – Fibonacci uses the plural); but Fibonacci, as we have seen, speaks of something which was already said "by us", and the present words therefore need not come from the *Liber abbaci*.

right-to-left – as explained on p. 46 concerning $^{7}/_{11}$ and $^{7}/_{9}$ of $^{1}/_{11}$, "both fractions are to be written on one stroke and disposed thus $\frac{7\ \ 7}{9\ \ 11}$". Sometimes, they go until three levels (never more). As a rule, however, they are only used when resulting directly from the calculation, and so rarely that the writer feels obliged to repeat on p. 53 and again on p. 70, (and repeatedly in more rudimentary form afterwards) the instruction given on p. 46. Much more often, complex fractions are expressed as sums of fractions connected with *e*, "and".[222] Mixed numbers involving a fraction or an ascending continued fraction may have the fraction to the left (the Arabic and Fibonacci-way), or to the right (the normal local habit), with a tendency that the former type takes over completely toward the end of the treatise. Often, mixed numbers are also written with a connecting word (for example, p. 51) *22 e* $^{2}/_{13}$. This writing, often expanded, prevails when a unit is involved; on p. 66 we thus find both *634 ß e* $^{1}/_{2}$ and *32 biç. e* $^{10}/_{79}$ *di biç*. Complex fractions expressed as sums invariably stand to the right, as for example (p. 34) "δ 5 e $^{3}/_{4}$ e $^{1}/_{100}$ e $^{1}/_{400}$ di δ" – even when reappearing in schematic calculations deprived of unit.

All in all, even though ascending continued fractions as well as mixed numbers with the fraction written to the left are shared with Fibonacci, there is no reason to conclude from this eclectic treatment of fractions that the present *Libro di ragioni* was inspired by the *Liber abbaci*, and even less to find a deliberate attempt to emulate Fibonacci's ways. Both features simply reflect Maghreb habits; and as shown by the (badly understood) notion of *lasalma* numbers and by his copious reference to Maghreb bezants, the present writer had direct contact to the Maghreb.[223]

With the exception of two clearly delimited borrowed sequences (below, p. 166 onward), the matters that are taught after the tables are definitely oriented toward what is commercially useful (and to a large extent linked to Pisa trade). By far the larger part of what is conserved teaches the rule of three. Often the questions involve several metrological levels and/or fractions in at least two of the positions – for example (p. 47), "*rotuli* 19 and $^{1}/_{3}$ cost £ 4 ß 13 and δ 7, at what come *rotuli* 58 and $^{1}/_{2}$?" They may also ask for serial application of the rule, as in this problem (p. 48),

> I buy in Palermo the *cantare* of cheese, which is *rotuli* 100, at *teri* 23 and grains 12, I leave and return to Pisa with my merchandise, and each Palermo *cantare* in Pisa becomes pounds 240. In Pisa I sell the *centonaio*, that is, pounds 100, at £ 7 and ß 13, I want to know what I get per ounce.

---

[222] In the *Liber abbaci*, in contrast, such sums stand without a connector.

[223] Beyond the Maghreb bezant with its subdivisions and coins from the Italian mainland and Sicily, only tornesi (minted in Tours) and the *biçantio di carato* (see note 219) are referred to. The Maghreb can be seen to have remained essential for Pisa trade a good century after the city had sent Fibonacci's father there as a public official.

Obviously, the calculations can be quite extensive.

Between the tables and the problems, three leaves are missing. We therefore cannot exclude that the rule of three was introduced in abstract form, as in Jacopo's *Tractatus* and in the *Livero* (above, p. 16 and note 19), even though the absence of traces in the language further on makes it doubtful. So much seems certain, however, that an introduction, if any has been there, did not make use of a counterfactual statement, as done in Ibero-Provençal works (below, p. 180); as we shall see when analyzing the "Columbia algorism", this would have left traces in the formulations of the problems.

Within the long sequence of rule-of-three problems we find two partnership questions, both of which, however, merely indicate the total capital and the share of a single partner – which means that these *are* indeed nothing but rule-of-three problems. The reader will evidently have learned from this how to solve also problems where the shares of all partners are given, but the aim of the text is shown by this choice to be training for trade, not for capital management.

Related to partnership problems, however, are two earlier recreational problems (pp. 51*f*) about distribution according to fractions whose sum exceeds or falls short of 1; first this one,

> There are 3 men who should share 75£, one should have the $^1/_2$, the other the $^1/_3$, and the other should have $^1/_4$. I want to know how much each should have. You should do like this: knowing that these questions are called fallacious [*fallace*], why are they called fallacious? Because when the number is more than a whole part one should have less, and when the number is less than a whole part, more £, ß and δ result. Now you should do thus: that you should know in what $^1/_2$ and $^1/_3$ and $^1/_4$ can be found, it is found in 2 times 3 multiplied by 4, they make 24, which you can reduce [*schizzare*] to $^1/_2$, and they are found in 12. Now say, $^1/_2$ of 12 is 6, and $^1/_3$ of 12 is 4, and $^1/_4$ of 3, now join together, and you have 13. The sharing thus falls to 13. Now say, 13 parts have to share the 75£, what results for the half of 12 which is 6, now you shall multiply 6 by 75£, and divide in 13. [...].

Straightforward application of the partnership rule would have found the first share as $(^1/_2 \cdot 75)/(^1/_2+^1/_3+^1/_4)$; what is done here reminds more of a method used in Islamic inheritance law (Ulrich Rebstock, private communication).

These problems follow a group of recreational problems similar to those treated in part 12.3 of the *Liber abbaci* [B173;G296] (the "tree problems"). The first (p. 49) says about a lance that $^1/_3$ and $^1/_4$ of it are below ground and 32 palms above. It is solved by means of a single false position, that is, in what Fibonacci called the vernacular way (above, p. 25). Three other problems belonging to the group deal with pure numbers.[224]

[224] Between the first and these comes (p. 50) a problem about buying and selling eggs, of which only the initial line survives, but might be similar in structure (need not, in any case the numerical

The two “fallacious” problems are followed by three problems about a house which is owned collectively, the first of which (p. 52) ($^1/_3$ and $^3/_5$ of the house is worth 300 £, what is the value of $^1/_6$ of the house?) makes use of a technique similar to that of the “fallacious” problems, while the other two (both presupposing the house to be divided into 24 shares called “carats”) use a more regular rule-of-three procedure. All five consistently state the monetary unit *after* the number (e.g., “7 £ e 10 ß”, whereas the rest of the treatise follows the habit shared by Fibonacci and most other abbacus treatises, where this amount would have been written “£ 7 e ß 12”. They are thus certainly taken from a particular source.[225]

After the house problems follows (p. 53) an unusual pure-number problem (unique in the present work, and unusual in general in the way it is dealt with):

> Divide for me 10 into 2 parts, so that one part divided by 3 makes as much as the other by 4. You shall do like this: 3 and 4 make 7, thus if 7 is worth 10, what results from 3 and from 4? That is, and if you want to say 7, you have to divide 10 so that it comes to 3 and at 4. You should multiply 3, multiplied by 10, and divided in $^1/_7$, and $4\,^2/_7$ results. The other part you will make like this: 4 times 10 and divide in $^1/_7$, and $5\,^1/_7$ results, as so much is the other part. One part is thus $4\,^2/_7$ and the other part $5\,^5/_7$.

A proof follows – maybe because the writer is not certain the exposition is convincing. The idea hiding behind the procedure seems to be a variant of the single false position: let us assume that the numbers are 3 and 4 – obviously, these fulfil the second condition. Their sum, however, is 7, not 10, and therefore by the rule of three the first has to be $^{10\cdot 3}/_7$, the second $^{10\cdot 4}/_7$. If I do not err, the formulation “if 7 is worth 10” is the only occurrence of a counterfactual statement in the *Libro di ragioni;* it is therefore a reasonable assumption that the present problem belongs together with the “house” and “fallacious” problems that precede – since no money appears the criterion that kept these together (monetary unit preceding or following the number) does not apply here, where no money is spoken of.

This group is an insertion in the long stretch of rule-of-three problems, which continues from p. 54 to p. 80, ending by the two rule-of-three problems about shares in a partnership. They are followed by a section teaching how to multiply two mixed numbers (pp. 81–89).

---

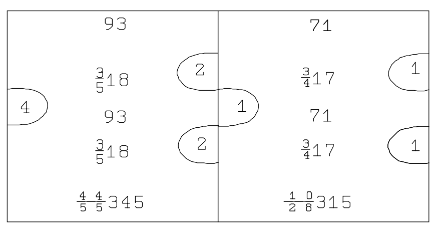
parameters are different) to a problem in the *Liber abbaci* [B179;G304].

[225] Both examples of *carato* designating $^1/_{24}$ of a partnership in [Edler 1934: 63] are from Florence. But for the corresponding Castilian *quilate*, [Corominas & Pacual 1980: IV, 727] reports the use about a tax (presumably of $^1/_{24}$) on the sale of fixed property in Murcia around 1300. Further, a Venetian *Libro dabaco* [Tagliente 1520x: 48^{v}] speaks of shares of $^1/_{24}$ of a ship-partnership as *carati*. The generalization of the carat, after all, cannot be used to determine the region from where these problems were adopted.

Mixed numbers have evidently been multiplied almost *ad nauseam* in the many rule-of-three problems, but by what Fibonacci calls the "vernacular" method (above, p. 62) – just as the method for serial application of the rule of three is his "vernacular" step-by-step calculation. Here, instead, the method is to "bring to fraction" both factors (in the terminology of manuscript **V** of Jacopo's *Tractatus* [ed., trans. Høyrup 2007: 241*f*]); the calculation is then controlled by casting our sevens, and the final result expressed by means of an ascending continued fraction. The products (12 in total) initially are squares; afterwards the factors are different:

$17\frac{3}{4}\cdot17\frac{3}{4}$	$41\frac{1}{4}\cdot54\frac{1}{5}$	$19\frac{1\ \ 3}{3\ \ 5}\cdot25\frac{1\ \ 2}{2\ \ 7}$
$23\frac{2}{7}\cdot23\frac{2}{7}$	$47\frac{5}{6}\cdot62\frac{2}{7}$	$29\frac{1\ \ 5}{3\ \ 10}\cdot37\frac{1\ \ 4}{4\ \ 11}$
$27\frac{5}{9}\cdot27\frac{5}{9}$	$56\frac{7}{9}\cdot72\frac{3}{10}$	$43\frac{3\ \ 7}{5\ \ 8}\cdot56\frac{1\ \ 5}{1\ \ 17}$
$32\frac{1}{11}\cdot32\frac{1}{11}$	$79\frac{2}{13}\cdot85\frac{5}{17}$	
$37\frac{1}{2}\cdot51\frac{0}{3}$		

After each calculation follows the same in schematic form – each time to the right of another schematic calculation with slightly higher numbers, whose appearance is not commented upon. At first this:



It seems certain that the whole sequence is borrowed – not least because a very similar sequence is found in manuscripts **M** and **F** of Jacopo's treatise [ed. Høyrup 2007: 405–407, cf. p. 55],[226] (schemes only, and no ascending continued fractions, but also there ordered right-to-left, and with the rare control by casting out sevens). There, there is no doubt that the sequence is a wholesale intrusion.[227]

226 The diagram just given indeed emulates the style of these two manuscripts (**M**, fols 12^{r}–13^{v}, **F** facsimile in [Simi 1995: 55–58].

227 Firstly, in contrast to **V**, **M**+**F** do not multiply mixed numbers in the way taught in these schemes. Secondly, in the two manuscripts the fractions of the two factors are written as far left as possible, while the integer part is pushed to the right; the fraction of the outcome is surrounded by a curved line similar to those enclosing the remainders modulo 7, showing that the compiler of this version did not understand what he copied as mixed numbers.

P. 89 returns to indubitable Pisa material:

> A man makes 3 travels, in the first he goes from Pisa to Lucca and doubles all his money and disburses δ 12. And then he goes from Lucca to Florence and doubles all his money and disburses δ 15. And then he goes from Florence to Siena and doubles the money and spends δ 21, and nothing remained for me. I want to know what my capital was.

This could be inspired by the first problem in the *Liber abbaci* about repeated travels with gain and expenses (above, p. 89). It is no less possible that Fibonacci took over a local Pisa variant of the recreational classic and solved it by means of his own sophisticated method. In any case there is no doubt that the two problems are linked though different (there is no mindless copying here, as in the *Livero* [A89;B315]). The method used in the *Libro di ragioni* also differs from that of Fibonacci (copied with or without understanding in the *Livero*), namely a scheme within which the backwards calculation can be arranged.

Three problems of the same kind but with varying numerical parameters follow. Then (p. 92) comes a short paradigmatic example showing how to subtract a fraction from a fraction ($^3/_5$ from $^6/_7$) as a continued ascending fration. The method proposed is to find a number of which both can be taken (in case $5 \cdot 7 = 35$); since $^3/_5$ of 35 is 21 and $^6/_7$ of 35 is 30, the difference between the fractions is $^9/_{35} = \frac{4\ \ 1}{5\ \ 7}$. This looks like a continuation of the borrowed sequence about the multiplication of mixed numbers that preceded the travel-problems.

After this comes a sequence of traditional recreational problems – first the beginning of one about the emptying of a cask through several holes, then after a missing leaf (from here onward only transcribed by Bocchi) two analogues of the apple-problem apple-picking problem we encountered in the *Liber abbaci* (above, p. 92), though with different parameters and solved in a different way.

After a single simple partnership problem follows [ed. Bocchi 2006: 70] a problem comparing two cylindrical volumes (the payment for a projected versus the realized well); the text is heavily damaged but at least shows that the calculations are wrong.[228] A last recreational problem [ed. Bocchi 2006: 71] deals with 4 men finding 4 purses. Even here something is wrong, probably because of ill-understood copying. The contents of the purses are given, and so are the ratios between the possessions of the men. Firstly, the ratios are given cyclically in such a way that the first man has 120 times as much as he has himself; secondly, the two sets of data are unconnected, and even with possible ratios therefore could not possibly lead to a solution (what remains from the solution also

[228] The idea of a badly used source appears to be that the volume of the cylinder is found at half-diameter times half-perimeter times height. The final result shows that the calculations of the source were correct, apart from a minor error in the fractions of a δ.

shows that the data are supposed to be connected).

The next section [ed. Bocchi 2006: 71–75] deals with alloying, said to be divided into five *differencie*, much the way the topic is divided in the *Liber abbaci* (which has seven *differentiae*, cf. above, p. 76). Most of the questions begin that “a man” has or wants to make a coin of specified fineness. We may take note, however, that one [ed. Bocchi 2006: 74] starts “I have coin, which coin is at 4 $^1/_4$ ounces per pound” – cf. above, p. 76.

The last two conserved leaves [ed. Bocchi 2006: 75–7] deal with area computation in Pisa metrology (also described by Fibonacci in his *Pratica geometrie* [ed. Boncompagni 1862: 3*f*][229]).

Area computation is likely to have been the last topic dealt with; all in all we probably have a good impression of the complete treatise. Summarizing, it seems to show traces of Maghreb influence not mediated by Fibonacci; the main objective can be seen to have been the training of the rule of three. Absent are metrological shortcuts allowing to dispense with the full rule with its multiplications and divisions (part of the 15th-century “Pisa curriculum”, and also known from Jacopo, see above, pp. 4 and 19). Barter is absent, and so is interest, and *a fortiori* discounting; the only commercially useful topic dealt with beyond the rule of three is indeed alloying. Recreational problems play a rather restricted role compared with many other abbacus treatises (not to speak of the *Liber abbaci*).

The “Columbia algorism”

Like the *Livero*, the “Columbia algorism” (henceforth *CA*) is known from a 14th-century vellum copy – once belonging to the Boncompagni collection, since 1902 in the possession of the Library of Columbia University, New York (Columbia X 511 A13) – cf. [Cowley 1923: 22*f*]. An edition was made in [1977] by Kurt Vogel; references to the text in what follows point to the pagination of this edition and its numbering of sections.

On the basis of some of the coins included in a coin list Vogel dated the *CA* to the second half of the 14th century, while admitting that the coin list *might* have been included when the copy and not the original was produced [Vogel 1977: 3*f*, 158]. Better identification of the coins in question allowed Travaini [2003: 92] to date the list to “later than 1278 and before 1284”.

That does not necessarily determine the date of the *CA* itself. Jacopo’s coin list can be dated to 1302 [Travaini 2003: 104], five years before Jacopo’s *Tractatus* was written; moreover, it was still copied in two abbacus books in the second half of the 15th century [Travaini 2020: lxi, lxiv]. Francesco Pegolotti, when putting together around 1340 material useful for international trade in his *Pratica de mercatura* [Evans 1936:xiv*f*], inserted a coin list from *ca* 1290, with additions to be dated *ca* 1320 [Travaini 2003: 86]. However,

[229] Cf. [Høyrup 2019a: 209].

another observation made by Vogel [1977: 12] supports a date close to that of the coin list: the shapes of the Hindu-Arabic numerals are not those that were current in the mid-14th century but seem to point to the 13th; apparently (as not quite uncommon) the copyist tried to emulate his original. As we shall see, the contents also speaks for an early date. Since most money-exchange problems involve the *cortonesi*, and since the coin list often evaluates other coin with reference to these, the likely place of origin is Cortona, close to the Tuscan border toward Umbria.

Like the Pisa *Libro di ragioni*, the *CA* is incomplete, missing the initial as well as several later leaves. How much is missing in the beginning cannot be known precisely, but hardly much, as argued by Elizabeth Buchanan Cowley [1923: 382*f*] after analysis of the binding (unless a complete quire has been lost). A full introduction to the Hindu-Arabic numerals is therefore not very likely to have been present, and hardly a detailed explanation of operations with fractions (they are trained in the first conserved problems). The *CA* was therefore *no* algorism,[230] as is Jacopo's *Tractatus*, an explanation of the Hindu-Arabic system, but another *libro di ragioni*, "book of problems".

In the *Livero*, we saw, ascending continued fractions were used consistently in the stratum more or less well borrowed from *Fibonacci*, and never in the basic stratum of the text. In the Pisa *Libro di ragioni*, they are used intermittently, and initially the notation apparently borrowed from the Maghreb is misunderstood. In the *CA* they are extremely rare, never go beyond two levels, and the writing direction changes: In problem #39, p. 64, they are to be read in the Arabic way, from right toward left, $\frac{1\ \ 1}{4\ \ 2}$ standing for $^{5}/_{8}$ and $\frac{3\ \ 1}{4\ \ 2}$ for $^{7}/_{8}$; in #60, p. 81, the reading goes is left-to-right, $\frac{1\ \ 1}{4\ \ 2}$ now standing for $^{3}/_{8}$;[231] mixed numbers, however, are always written left-to-right. Noteworthy is an emulation of the notation, where for instance (#40, p. 65) $\frac{1}{\mathit{grana}}\frac{1}{2}$ stands for "1 *grana* and $^{1}/_{2}$ [of a *grana*]", which in combination with the occasional use of (for instance) $^{1}/_{4}$ for "the fourth" meant as no. 4 in a sequence shows that the fraction line is understood, not as a division but as an indication of unit – in other words, what looks as a *denominator* is instead understood as a *denomination.* We shall encounter the pseudo-ascending fractions in Dardi da Pisa's algebraic notation on p. 220. Ordinal numbers written in the shape of fractions are widespread in the abbacus corpus.

The *Livero*, including no introduction to the number system, opened with an abstract formulation of the rule of three (the one that was going to be the standard of abbacus books). Whether the Pisa *Libro* did the same is doubtful (above, p. 166). In the *CA*, a

230 The line *Rascionei d'Algorsmo* on top of the first conserved leaf was obviously written when the initial leaves had already been lost. It is also written in a modern hand [Cowley 1923: 381].

231 Actually, the fraction line is discontinuous, $\frac{1}{4}\frac{1}{2}$, etc., as in the *Livero*, and the notation thus ambiguous; in #16, p. 45, $\frac{1\ \ 1}{10\ 20}$ is indeed meant as $^{1}/_{10}+^{1}/_{20}$, and $\frac{1\ \ 1}{5\ 10}$ as $^{1}/_{5}+^{1}/_{10}$.

general presentation is offered (#11, pp. 39*f*) after a sequence of problems and rules mostly teaching numerical techniques:[232]

> Remember, that you cannot state any computation where you do not mention three things; and it is fitting that one of these things must be mentioned by name two times; remember also that the first of the things that is mentioned two times by name must be the divisor, and the other two things must be multiplied together.

An example dealing with the exchange of money follows. Later this formulation is used a couple of times (#19, p. 48, and #21, p. 50) in examples that refer explicitly to the "rule of the three things". As we see, instead of referring to what is "similar" or "of the same kind" (cf. note 19) the *CA* here speaks about what is "mentioned by name two times". While the idea is the same, the words are different.[233]

Much more often, however, the problem is reduced to a counterfactual formulation (invariably so in cases where the rule serves inside a more complex calculation). A simple instance is a question (#114, p. 124) for two numbers the sum of whose squares is 64.

---

[232] However, the *CA* is far from systematic. Within the sequence in question we also find a problem about repeated travels with gain and expenses (#4, p. 33), solved step by step backwards (with no use of a scheme, as done in the Pisan *Libro* (above, p. 169), and in #5, p. 34, the grasping problem which we know from the *Liber abbaci* (above, p. 96) – with the numerical parameters which Fibonacci shares with al-Karajī but merely with an indication of the solution, not even hinting at a method, and thus teaching absolutely nothing. The story is told in words that are very far from those of Fibonacci.

[233] "mentioned two times" is not totally absent from the later Italian record, but it must have been rare. I know it from only three sources, two of them from 1478. One is Pacioli's Perugia manuscript [ed. Calzoni & Cavazzoni 1996: 19*f*], which gives this as an alternative to the normal "similar" formulation (repeated with minimal change in the *Summa* [1494: fol. 57[r]]):

> The same in other words. The rule of 3 says that the thing which is mentioned twice should be looked for, of which the first is the divisor, and the second is multiplied by the thing mentioned once, and this multiplication is divided by the said divisor, and that which results from the said division will be of the nature of the thing mentioned once, and so much will the thing be worth which we try to know.

The other occurrence is in Pierpaolo Muscharello's *Algorismus*, written in Nola (close to Naples, thus in a region under strong Spanish influence and outside the core abbacus region) [ed. Chiarini et al 1972: 59]; it is a simple nod to the "mentioned" formulation within the standard phrasing:

> This is the rule of 3, which is the fundament for all commercial computations. And in order to find the divisor, always look for the similar thing, which is mentioned twice, and one of these will be the divisor [...].

The third occurrence is much earlier, namely in an odd corner in a *Libro d'abaco* compiled in Lucca by several hands around 1330 (see below, p. 203).

We shall return to the appearance in the printed Larte de labbacho (below, p. 327).

> If you want to do this, find me a number that may be multiplied by two numbers [that is, which may be produced as the sum of two numbers each multiplied by itself], and which has a root, which is 25, and one number is 3 and the other is 4, and say thus: the root of 64 is 8, and the root of 25 is 5. Say, if 5 were 8, what would 4 be, and it would be 6 and $^{2}/_{5}$; and say, if 5 were 8, what would 3 be, and it would be 4 and $^{4}/_{5}$; and these are 2 such numbers which, each multiplied by itself, will make 64 precisely.

Such auxiliary use of the rule of three in counterfactual formulation is found in no less than 18 problems out of 141 sections (problems and rules).[234] Thrice the method is identified as "by the rule of three".[235] In 12 cases the multiplication and ensuing division are specified, in seven (as here) taken for granted. Sometimes the rule is appealed to without clear indication of which of the two approaches was thought of,[236] or it is simply applied without being named.[237]

As we shall see (below, p. 180), the approach by means of a counterfactual question points to the Ibero-Provençal world. There is more to such a connection – Iberian rather than Provençal. #111, p. 122, runs as follows:

> Somebody had δ in the purse and we do not know how many. The $^{1}/_{3}$ ⟨and the $^{1}/_{5}$⟩ were lost, and 10 δ remained for him. I ask, how many δ he had before the $^{1}/_{3}$ and the $^{1}/_{5}$ were lost for him. This is its right rule, that we shall say, in what are $^{1}/_{3}$ and $^{1}/_{5}$ found, and they are found in 3 times 5, that is, 15; and thus one shall say that he had 15 *denari* in the purse. Remove the $^{1}/_{3}$ and the $^{1}/_{5}$ of 15, 7 escape. Say thus; if 7 were 10, what would 15 be? Say, 10 times 15 make 150, to divide by 7, and from this comes 21 $^{3}/_{7}$, and so much did he have in the purse before the $^{1}/_{3}$ and the $^{1}/_{5}$ were lost for him.

In a Castilian *Libro de arismética que es dicho alguarismo* (henceforth *Alguarismo*), a closely related problem is found:[238] only 5 *denari* remain, and the

[234] #2, p. 32; #31, p. 57; #35, p. 61; #39, p, 64; #40, p, 65; #45, p. 70; #49, p. 73; #50, p. 74; #56, p. 78; #57, p. 79; #58, p. 80; #61, p.83; #64, p. 86; #68, pp. 90, 91; #69, pp. 90, 91; #93, p. 109; #111, p. 122; #112, p. 123; #114, p. 124. #96–98, pp. 110–112, further present three counterfactual calculations ("if $^{1}/_{3}$ and $^{1}/_{4}$ of 14 were $^{1}/_{2}$ and $^{1}/_{5}$ of 11, what would the $^{1}/_{3}$ and the $^{1}/_{5}$ of 23 be?", etc.). They are solved via reduction to simple counterfactual statements (the one just quoted to "if 8 $^{1}/_{6}$ were 7 $^{7}/_{10}$, what would 12 $^{4}/_{15}$ be?").

[235] #31, p. 57; #39, p. 64; #61, p. 83.

[236] Thus # 23, p. 51; #28, p. 54; #29, p. 55; #32, p. 58; #75, p. 96; #94, p. 110.

[237] Thus #25, p. 52; #75, p. 96; #95, p. 110.

[238] Ed. Caunedo del Potro, in [Caunedo del Potro & Córdoba de la Llave 2000: 167]); further page references to the *Alguarismo* point to this edition. The edition is based on a 16th-century copy of

story is told in the first and second, not the third grammatical person; apart from that, we find a quite faithful repetition (only the finding of 15 as 3·5 has been left out):

> The $^1/_3$ and the $^1/_5$ of my *dineros* were lost for me from the purse, and 5 *dineros* remained in it, I ask, how many *dineros* there were in it at first? This is its right rule and calculation, that you shall say, in what are found $^1/_3$ and $^1/_5$, which is in 15, let us then say that you had 15 *dineros* in your purse, $^1/_3$ and the $^1/_3$ were lost, 7 remained for you, say, if 7 were 5, what would 15 be? Say, 5 times 15 are 75, divide by 7, and from that come $10\,^5/_7$, and so many *dineros* were there at first in the purse.

The *CA* was not widely influential, and no other Italian abbacus treatise identifies the rule of three via a counterfactual structure. It is therefore next to certain that the influence went the other way, from Iberian vernacular practical arithmetic to the *CA*. That this could happen is also not implausible – we remember Fibonacci coping his treatment of barter from a "Castilian master" (above, p. 73).

In many cases, obvious similarities between the *CA* and the *Alguarismo* cannot be used to link the *CA* to the Iberian environment; that is the case if the problem type occurs elsewhere in the Italian corpus, if the phrasing is not characteristically similar, and if coinciding numerical parameters are either widespread or not sufficiently characteristic. A number of other instances are more telling, however, and confirm the partial Iberian inspiration for the *CA*.[239]

One example is #67 in the *CA* (p. 88):

> There is a tower which is 10 cubits high; and on this tower there is a dove which by day descends $^2/_3$ of a cubit, and by night returns upwards $^1/_3$ and $^1/_4$. I ask, in how many days the dove will come to the ground. This is its rule, how one should make all such computations, that you shall say how much $^2/_3$ is more than $^1/_3$ $^1/_4$, you see that it is $^1/_{12}$ of a cubit more. Hence it advances downwards each day $^1/_{12}$ of a cubit. If one wants to know in how many days it will be on the ground, one makes the 9 cubits and $^1/_3$ which it makes in 112 days, and remains to make $^2/_3$ of a cubit which it makes in the last day, and then it will find itself on the ground, and the computation is made in 113 days. And many masters say about this question that the dove will be on the ground in 112 days, not knowing about the hoax behind, that it makes $^2/_3$ on the day of return and finds itself

an manuscript from 1393, in itself a copy of an original going back at least to the early 14th century.

[239] Striking and almost certainly not accidental similarities are found in *CA*#67, *Alguarismo*#40; *CA*#123, *Alguarismo*#21; *CA*#61, *Alguarismo*#39; *CA*#106, *Alguarismo*#60; *CA*#56, *Alguarismo*#76; *CA*#126, *Alguarismo*#152.

Similarities that *could* be accidental are found in *CA*#108-109, *Alguarismo*#44,66-67; *CA*#85, *Alguarismo*#48; *CA*#93, *Alguarismo*#51; *CA*#118, *Alguarismo*#53; *CA*#108, *Alguarismo*#66.

on the ground.

Obviously, the errors of many masters (including Fibonacci, see above, p. 37) should be 120, namely $10 \div {}^{1}/_{12}$. We shall return to this mistake after having looked at how the same problem is dealt with in the *Alguarismo* (p. 162) – since the text is obviously corrupt, even to the point of being ungrammatical, I shall try to translate even more *verbatim* than usually:

> There is a dove on top of a tower, and the height of the tower is 10 *varas*.[240] And the dove mounts against upwards [*sube contra suso*] in a day ${}^{2}/_{3}$ of a *vara*, I ask you, in how many days will this dove be on the ground. And many say that it will be on the ground in 12 days, but it will be on the ground in 113 days, because it descends 9 *varas* in 18 days and ${}^{1}/_{3}$ of a *vara* in 4 days, which are 112 days and ${}^{1}/_{3}$ make in a day, and in this way it will be on the ground in 112 days.

There are obvious copying errors and misunderstandings – first, the nightly ascent is omitted, which may be behind the enigmatic "mounts against upwards"; what might hide behind "descends 9 *varas* in 18 days" is beyond my fantasy. But the reference to what "many say" is shared with the *CA;* most likely, both text descend from a text where the error attribute to "many" had been written "12" instead of "120". The *Alguarismo* has then copied that mistake better than the rest of the text, while the compiler of the *CA*, seeing that it cannot be correct, has misrepaired "12" as "112" instead of "120".

Also interesting is #61 (p. 82) of the *CA:*

> A merchant moved from France with his *denari*,[241] we do not know how many he carried. He moved from France to Pisa with these his ß invested and earned £ 15 of *denari* per the hundred. Then he left Pisa and when to Genova and earned £ 20 per hundred of £. He further left Genova and went to Sardinia and earned £ 25 per hundred of £, and turned back to Florence and earned £ 30 per the hundred, and then he counted his *denari* and found himself with precisely £ 1000 of *pisani*. This is its rule, how one should make all such computations, which can be made in two ways, one by false position to find it backwards; let us make it in the way of false position, and say that he moved first from France with £ 100, and in Pisa there were 115 and in Genova there were £ 138, and in Sardinia there were £ 172 ß 10, and returns to Florence and makes £ 224 ß 5. Now it can be made by the rule of 3, and say, if £ 1 ß 243 were £ 1000, what would £ 100 be,

[240] The *vara* is a length unit of around 80–85 centimetres. It corresponds to the Italian cubit (*braccio*) in abbacus texts in the sense that both are used to measure land as well as cloth.

[241] Here meaning "money", cf. above, note 21.

which would be £ 448 ß 3 δ 2 $^{420}/_{897}$.[242]

At first we observe that this appeal to a single false position corresponds to what Fibonacci does when dealing with the gains in the first problem about repeated travels with gain and expenses (and repeatedly afterwards; above, p. 90). More striking, however, is what we find in the *Alguarismo* (p. 162):

> A merchant moved from Lisbon with his *dineros* and we do not know how many, and came to Sevil and earned 15 £ per 100, and then came to Valencia and earned 20 £ per 100, and then he turned to Toledo and earned 25 per 100 and found as capital and gain thousand £, I ask you, how many *dineros* he had at first when he moved from Lisbon? Say that they were falsely 100£, and in Sevil he earned 15£, in Valencia he found himself with 138 and in Toledo he found himself with 172£ and $^1/_2$, say, if 172 $^1/_2$ were 100, what would 1000 be? Say, 100 times 1000 are 10000, and divide by 172 $^1/_2$, and from the division results 579 £ 19 ß and 2 δ, $^{10}/_{13}$ of δ.[243]

There is one travel less than in the *CA*, but apart from that the numbers are the same; in both cases the traveller is defined to be a merchant, in both cases he discovers how much he has after the last travel, and in both cases an explicit single false position is used, which is quite exceptional for this problem type. There can be little doubt that the two texts depend on the same source for this problem; moreover, we may notice that the three problems in the *Alguarismo* that were used for this comparison are immediate neighbours; at least the *Alguarismo* must therefore depend for them on a specific written source, not for something in general circulation; actually, the following 3 problems in the *Alguarismo* also have close counterparts in the *CA*, even though the similarities are less conclusive.

This does not exhaust the list of similarities between the *CA* and the *Alguarismo* which can hardly be accidental, but it should suffice to make the point: not only the main manner of the former to identify the rule of three but also a number of problems are borrowed from an Iberian – probably Castilian – environment.[244]

---

242 Should be £ 445 ß 18 δ 7 $^{369}/_{897}$.

243 Should be 579 £ 14 ß and 2 δ, $^{10}/_{13}$ of δ – most likely a simple copying error.

244 Links to the *Liber abbaci* beyond the use of a single false position for the tranvel problem can also be identified. Both the *CA* (#106, p. 118) and the *Alguarismo* (p. 167) present a two-participant "purchase of a horse", where the participants are defined as "companions", and the requests are $^1/_3+^1/_4$ respectively $^1/_4+^1/_5$ of what the other has. Both solve by means of an unexplained rule, which is based on this consideration: if $A+qb = B+pa$, than $(1–p)A = (1–q)B$, for which reason the any pair $(A,B) = (k(1–q),k(1–p))$ is a solution. Fibonacci's rule (above, p. 88) can easily be derived from the rule given in the *CA* and the *Alguarismo*, but since Fibonacci's rule only works when the fractions to be transferred are unit fractions, the *Liber abbaci* cannot be the source.

In abbacus books, the participants in this kind of deals are mostly just "men", "merchants"

Most of the problems in the *CA* are commercial or such widespread versions of traditional recreational problems that nothing precisely can be said about their affinities. Some are obviously related to problems from the *Liber abbaci*, but the similarities never go beyond the shared heritage.

Liber habaci

The last abbacus book that may belong to "generation one" or even to "generation zero" is the *Liber habaci*.[245] It is anonymous,[246] probably to be dated to *ca* 1309[247] (two examples in the computus chapter refer to this year, pp. 161 and 163), and apparently written in Provence.[248]

The reason to see this treatise as a possible reflection of "generation zero" is a unique feature: all integer numbers are expressed in Roman numerals, Hindu-Arabic numerals

---

or "the first, the second, ...". I have found "companions in a single problem with four participants in Paolo Gherardi's *Libro di ragioni* [ed. Arrighi 1987: 45] (written in Montpellier in Provence, cf. above, note 37), and a two-participant problem in the *Livero* [A67;B268], with the same fractions as in the *CA* and the *Alguarismo* but with given price of the horse and therefore with a different rule; even these two problems from the *CA* and the *Alguarismo* thus seem to be related, and Fibonacci's two-participant example might therefore (in spite of its bezants) be inspired from the same environment.

245 Florence, Biblioteca Magliabechiana XI.88, [ed. Arrighi 1987]; page numbers will refer to this edition.

246 Arrighi ascribes it to Paolo Gherardi, basing himself on a library catalogue note "Paolo / Gerardi / Arim" and on what is written on the spine of the binding, "XI./ Paolo / 418" [Arrighi 1987: 7]. The title on the spine must have been given by Giovanni Targioni Tozzetti when the manuscript was acquired for the Biblioteca Magliabechiana in 1752/53, at which occasion he assigned it to class XI ("418" is likely to have been its collocation in the Gaddi collection, from where it came); the catalogue note must also be due to Targioni Tozzetti, unless it is even later [McCuaigh 1990: 431]; both are probably inspired by the presence of another abbacus manuscript in the collection explicitly ascribed to Gherardi. None of this evidence has any weight, and the *Liber habaci* must be considered anonymous. We might observe that the orthographic habits of the two manuscripts differ, but even this is of no consequence, given that the Gherardi text is a copy.

247 Two examples in a computus chapter refer to this year, pp. 161 and 163. Van Egmond [1980: 115] gives *ca* 1310, and may combine these examples with material from a calendar that is not included in Arrighi's edition.

248 The calendar lists the days of saints that were important there but not in Italy [Arrighi 1987: 10]. Part of the mathematical contents points in the same direction. Only part, however – pp. 129*f* present Florentine area metrology (not the same as the Pisa metrology encountered in the Pisa *Libro di ragioni*).

do not appear; fractions are systematically expressed in words. Even the brief explanation of the place-value system on p. 155 speaks about *figure d'abacho* but does not show these but only speaks in the abstract about *figure*.

Evidently, Italian and Provençal merchants had to calculate also before they learned to use the Hindu-Arabic numerals. Once these had become commonly known, they certainly displaced Roman numerals; but the rule of three, and commercial calculation in general, could also be performed on the basis of the old notation or the spoken numbers to which they correspond – since Antiquity, saving intermediate results by means of finger reckoning had been standard.[249] The *Liber habaci* may therefore be a relatively late reflection of the ways of the earliest abbacus teachers.[250] However, the avoidance of the place value notation could also have resulted from a choice not to overburden an audience that did not yet know it.

Supplementary evidence that the *Liber habaci* is a reflection of an archaic stage is offered, however, by a list of square roots on p. 119. After the roots of perfect squares from 1 to 100 follows a list of approximate roots of non-square numbers – in modern number notation

$\sqrt{2} = 1\frac{3}{7}$, a little less	$\sqrt{12} = 3\frac{1}{2}$, a little less
$\sqrt{3} = 1\frac{3}{4}$, a little less	...
$\sqrt{7} = 2\frac{2}{3}$, a little less	$\sqrt{40} = 6\frac{1}{3}$, a little less
$\sqrt{10} = 3\frac{1}{6}$, a little less	$\sqrt{50} = 7\frac{1}{14}$

$\sqrt{10}$ and $\sqrt{40}$ *could* have been found as the usual "closest approximation" (above, p. 36), but the rest of the table (apart from $\sqrt{50}$) shows that the underlying idea (not stated in the text, and probably unknown to the compiler) is

$$\sqrt{2} = \sqrt{(2\cdot 49)}/7 \approx \sqrt{100}/7 = 10/7$$
$$\sqrt{3} = \sqrt{(3\cdot 16)}/4 \approx \sqrt{49}/4 = 7/4$$
$$\sqrt{7} = \sqrt{(7\cdot 9)}/3 \approx \sqrt{64}/3 = 8/3$$
$$\sqrt{10} = \sqrt{(10\cdot 36)}/6 \approx \sqrt{361}/6 = 19/6$$
etc.

Only $\sqrt{50}$, which is not indicated to be an approximation, is taken over from age-old tradition.

There are no approximated square roots in the *Livero*, nor in the Pisa *Libro di ragioni*.

---

[249] In this connection we should remember that not only the *Liber abbaci* but many abbacus books until Pacioli's *Summa* [1494: 26^v] presented these finger positions – and even Girolamo Tagliente's *Libro dabaco* from [1520x] – reprinted as late as [1579].

[250] "Relatively late" – but the Trento *Algorisimus*, printed only 70 kilometres north of Verona on the trade route toward southern Germany, was to do the same in *ca* 1475 – see below, p. 370.

The *CA* (p. 133) explains the "closest" approximation, yet without using this characterization. The *Primo amastramento*, probably close kin of the *Livero* (above, p. 162) extracts many square roots, never explaining how it is done but in ways that definitely excludes the "closest approximation". When extracting $\sqrt{14}$ [ed. Arrighi 1991: 10] it finds $3\frac{7}{9}$ as its *più sutile* root, which cannot come from the "closest approximation", neither from above not below; possibly, the basis is that $\sqrt{14} = \sqrt{(14 \cdot 81)}/9 = \sqrt{1134}/9 \approx \sqrt{1156}/9 = 34/9 = 3\frac{7}{9}$.The identification of $8\frac{2}{3}$ as the "most subtle" root of 75 [ed. Arrighi 1991: 13] can be explained both as a "closest approximation" *from above* or from the calculation $\sqrt{75} = \sqrt{(75 \cdot 9)}/3 = \sqrt{675}/3 \approx \sqrt{676}/3 = 26/3 = 8\frac{2}{3}$. Other radicands such as $194\frac{8}{11}$ (claimed to have the root $13\frac{19}{20}$ [ed. Arrighi 1991: 15]) will have required further approximation, and therefore defy explanation. All in all, however, the method of the *Liber habaci* may also have been the basic method of the *Primo amastramento*, confirming the archaic character of both.

However that may be: even though the *Liber habaci* is slightly later than Jacopo's *Tractatus algorismi* from 1307 (as we know it from the Vatican manuscript), it does not know the key innovation introduced by Jacopo, namely algebra. In spirit, it belongs together with the *Livero*, the Pisa *Libro di ragioni*, and the *CA*, if not to an even earlier stage.

As we have already seen (above, note 19), the *Liber habaci* and the *Livero* introduce the rule of three in what was going to be the standard way of the abbacus tradition; how (and whether) it was introduced in the missing sheets of the Pisa *Libro di ragioni* we cannot know (above, p. 166); the *CA* is clearly different on this account. When it comes to the contents, recreational problems play a much larger role in the *Liber habaci* than in the Pisa *Libro di ragioni* or in the basic stratum of the *Livero*, in this respect bringing it closer to the *CA*. If we consider the commercial aspect, on the other hand, the *Liber habaci* comes somewhat closer to the Pisa *Libro* than to the others; but it contains a single problem on barter, absent from the *Libro* (p. 147); in contrast to the *Libro*, it also considers simple interest. Geometrical computation, basic as well as recreational, interspersed among commercial and recreational-commercial matters, plays a larger role in the *Liber habaci* that in the others; its contents and methods, though not identical, is similar to what we saw in Jacopo's *Tractatus* (above, p. 34), seemingly reflecting the particular orientation of Tuscan abbacus authors writing in Provence.

All in all, the four surviving representatives of the first abbacus generation show no signs of descending from the *Liber abbaci*, apart for the easily separable and not necessarily well-digested sophisticated stratum of the *Livero*. Moreover, in their general character they differ so much from each other and suggest so many different contacts outside the Italian area that it is difficult to imagine that they should derive from an accurately defined common root. Their authors or compilers appear instead to have responded to a shared social need mediated and shaped by the newly arising abbacus school; this they did by drawing on shared commercial techniques and on inspiration from a variety of contacts in the Mediterranean world.

An Ibero-Provençal aside

The way of Ibero-Provençal writers to deal with the rule of three was referred to repeatedly in the preceding section. Which are the sources?

Oldest are two works freely translated from Arabic material into Latin somewhere around 1160: the *Liber mahameleth* and the so-called "Toledan *regule*" [ed. Burnett, Zhao & Lampe 2007]. The two are closely related, see [Burnett, Zhao & Lampe 2007: 145].

The next representatives of the area are all written in vernaculars, and all postdate the first Italian generation:

– The 14th-century Castilian *Alguarismo* (above, p. 173). Certain aspects call to mind the *Liber mahameleth*, enough to show it to be partially rooted in an Iberian tradition going back to the Arabic period.
– The anonymous "Pamiers Algorism" [ed. Sesiano 2018], according to monetary evidence written in the 1430s.
– The equally anonymous mid–fifteenth-century Franco-Provençal *Traicté de la praticque d'algorisme* [ed. Lamassé 2007], related to the "Pamiers algorism" but neither a descendant nor a source – see [Sesiano 2018: 9].
– Barthélemy de Romans' *Compendy de la praticque des nombres*, probably written around 1467 but only known from a revision prepared by Mathieu Préhoude in 1476 [ed. Spiesser 2003], somehow connected to the *Traicté*.
– Francesc Santcliment's Catalan *Suma de la art de arismetica* [ed. Malet 1998: 163], printed in Barcelona in 1482.
– Francés Pellos's *Compendion de l'abaco*, printed in Nice in 1492.

The *Liber mahameleth* [ed. Vlasschaert 2010: II, 185; ed. Sesiano 2014: 221] and the "Toledan *Regule*" [ed. Burnett, Zhao & Lampe 2007: 155] begin by an approach to the rule of three which I know from nowhere else. Of four numbers in proportion, the first and the fourth are declared "partners" (*socii*), and so are the second and the third. If one is unknown, then its partner shall divide any of the other two, and the outcome be multiplied by the third number; this, of course, is *not* the rule of three, where multiplication is performed first. Afterwards, both specify differently (without observing that there is a difference), namely in agreement with what we may call the "naked rule of three",

> thus, if three are proposed and the fourth is unknown, multiply the second in the third, and divide what results by the first, and what comes out will be the fourth.

No later Ibero-Provençal source contains anything similar to the *socii* explanation; being isolated from past as well as from future, this Latin explanation is thus likely to be a local invention in the learned Toledo environment which had no bearing on what merchants and their schools were doing.

The *Alguarismo* [ed. Caunedo del Potro & Córdoba de la Llave 2000: 147] explains the procedure as follows:

> This is the 6th species, which begins "if so much is worth so much, what will so much be worth".
>
> Know that according to what the art of algorism commands, to make any calculation which begins in this way, "if so much was so much, what would so much be?", the art of algorism commands that you multiply the second by the third and divide by the first, and that which comes out of the division, that is what you ask for. As if somebody said, "if 3 were 4, what would 5 be?", in order to do it, posit the figures of the letters[251] as I say here, the 3 first and the 4 second and the 5 third, 3, 4, 5, and now multiply the 4, which is the second letter, with the 5, which is the third, and say, 4 times 5 are 20, and divide this 20 by the 3, which stands first, and from the division comes $6\,^2\!/_3$, so that if they ask you, "if 3 were 4, what would 5 be?", you will say $6\,^2\!/_3$, and by this rule all calculations of the world are made which are asked in this way, whatever they be.

As we see, this combines the "naked" formulation of the Latin writings with use of the counterfactual calculation used as general model.

The "Pamiers algorism" [ed. Sesiano 2018: 285] says that the rule

> is called rule of 3 because there are always 3 things, 2 similar [*semblantz*] and one dissimilar. And if there are more, they are reduced to these 3. [...[252]].
>
> Multiply that which you want to know by its contrary, and then divide by its similar.

This is evidently the same rule as we know from the Italian material (including Italians writing in Provence), excepting the use of the term "contrary", which might refer to a rectangular scheme similar to that used by Fibonacci (above, p. 56); it has nothing to do with the alternative of the Latin treatises (nor with their primary formulation). The examples [ed. Sesiano 2018: 286*f*], on the other hand, differ in style from what we find in Italy:

> And first you ask, if ⟨so much is worth so much⟩, how much is so much worth, For example, if 4 are worth 7, what are 12 worth? [...].
> Further, $4\,^1\!/_2$ are worth 7, what are 13 worth? [...].
> Further, $4\,^1\!/_2$ are worth $7\,^2\!/_3$, what are 13 worth? [...].
> Further, $4\,^1\!/_2$ are worth $7\,^2\!/_3$, what are $13\,^3\!/_4$ worth? [...].

These are evidently not quite counterfactual, only so abstract that they become similar to that category. Only after these (and three more of the same kind) come the concrete examples where different monies are spoken of (mostly using the same numerical parameters), as we know it from Jacopo's *Tractatus* (above, p. 17), and as also habitual

[251] Elsewhere the author explains "the letters of algorism" to be the Hindu-Arabic numerals.

[252] Extensions to rule of 5 and rule of 7.

in other Italian abbacus books.

The presentation of the rule of three in the *Traicté* [ed. Lamassé 2007: 469] runs thus:

> This rule is called rule of three for the reason that in the problems that are made by this rule three numbers are always required, of which the first and the third should always be similar by counting one thing. And from these three numbers result another one, which is the problem and conclusion of that which one wants to know. And it is always similar to the second number of the three. By some this rule is called the golden rule and by others the rule of proportions. The problems and questions of this rule are formed in this way: "If so much is worth so much, how much will so much be worth?". As for example, "if 6 are worth 18, what would 9 be worth?". For the making of such problems there is such a rule:
>
> Multiply that which you want to know by its contrary and then divide by its similar. Or multiply the third number by the second and then divide by the first.

Here we see a combination of the second Toledan and the "Italian" way, leading to an abstract, quasi-counterfactual specification (the examples that follow are of the same kind).

Barthélemy de Romans' *Compendy de la praticque des nombres* says about the rule of three [ed. Spiesser 2003: 255–257] that it is "the most profitable of all", and gives two versions of the rule,

> Multiply that which you want to know by its contrary, and then divide by its similar,

and

> Multiply that which you know by that which is wholly dissimilar to it, and then divide by its similar,

after which it goes on with the composite rules. The first version, as we observe, is shared with the *Traicté:* the second, by using the term dissimilar (*dissemblant*) instead of contrary, is close to the Italian type. The exemplification is again of the abstract, quasi-counterfactual type.

Santcliment's *Suma de la art de arismetica* introduces the *regla de tres* in these words [ed. Malet 1998: 163]:

> It is called properly the rule of three, since within the said species 3 things are contained, of which two are similar and one is dissimilar. This said species is common to all sorts of merchandise. There is indeed no problem nor question, however tough it may be, which cannot be solved by it once it is well reduced.
>
> And in our vernacular [*nostre vulgar*] the said species begins: If so much is worth so much, what will so much be worth?
>
> The solution of this rule is commonly said: Multiply by its contrary and divide by its similar.

The first example offered after this explanation is abstract and quasi-counterfactual, “if 5 is worth 7, what is 13 worth?”.

Once again we encounter a combination of the Italian phrasing with an abstract, quasi-counterfactual explanation and the reference to the “contrary” instead of the dissimilar in the formulation of the rule; most interesting is the identification of the latter as the “vernacular”, very close to what Fibonacci says when he explains the finding of a fourth proportional by means of the (unnamed) rule of three, “in our vernacular usage” (above, p. 78). This leaves little doubt that the environment to which Fibonacci’s “we” refers on that occasion is Iberian of Provençal, since counterfactual formulations are seem to be absent from Arabic sources.[253]

Pellos’s *Compendion de l’abaco* starts by a general introduction to the theme [ed. Lafont & Tournerie 1967: 101–103] which does not look in detail like anything else we have seen except by speaking about the “contrary” and in the concluding *General rule to find every thing;* in its entirety is likely to be Pellos’s own description of the situation, yet still referring to familiar Ibero-Provençal parlance:

> In this chapter I want to give you a good mode and way in which you can always quickly and without great toil find all things that you want to buy or sell. And know that this chapter is called the chapter and rule of three things. In every computation of trade three numbers are indeed necessary.
>
> *The first number.*
>
> The first number is always the thing bought or sold, and you need to keep it well in memory.
>
> *The second number.*
>
> Know that the second number shall always be the value or the price of that which you have bought or sold.
>
> *The third example or number.*
>
> And the third number shall always be the thing that you want to know, that is to say, the thing that you want to buy.
>
> *Remember that the first and the third numbers are always the same thing* [*uns causa*].
>
> And know further that the first number and the third shall always be one thing. And if they are not certainly one thing, then you shall reduce them to a form where they speak of one thing, or matter, for in no way on earth they must not be different, as appears afterwards in the examples.
>
> *General rule to find every thing.*
>
> Always multiply the thing that you want to know by its contrary. And the outcome of this multiplication you divide by its similar, and that which comes out of such a division

---

[253] Actually, since Arabic does not use the copula, “if 5 were 7” would be a rather meaningless “if 5 7” in Arabic; the abstract “being worth” formulation is obviously not impossible and can also be found, but this is not what Fibonacci writes.

> will be the value of the thing that you want to know.
>
> [section on reduction of units]
>
> *This is the way how you should say in matters that ask: if so much is worth so much, how much is so much worth? In this way, you may understand more clearly in the following examples.*

The first examples that follow ask "if 4 are worth 9, what are 5 worth?", "if 3 and a half is worth 6, how much are 4 worth?", etc.

All in all we may conclude that the Tuscans who went to Montpellier and Avignon and wrote their treatises there may in general have gone to learn; but regarding the central piece of abbacus mathematics (if we do not count the Hindu-Arabic numerals), namely the rule of three, they brought it from home; that will have been something any future abbacus writer had learned long before going abroad.

On the other hand, the affinity of the *CA* with Iberian ways seems to be confirmed.

Before we leave this topic we should take note that the "Italian" formulation did not originate in Italy – see [Høyrup 2012] for precise references. From Bhāskara I onward Sanskrit mathematicians refer to the similar and the dissimilar in secondary formulations of the rule (which even for them is a "rule of three things") – apparently adopted from a vernacular, that is, mercantile environment. It is used (*pace* misunderstandings in the translations into Latin, English, French and Russian) by al-Khwārizmī in his presentation of the rule in his algebra, and also by other Arabic writers as a secondary formulation (the learned Arabs mostly prefer to begin with Euclidean concepts). So, it appears to have been shared by a mercantile community spread over the whole trading zone from India to the Mediterranean.

The "second generation": formation of a tradition, and the arrival of algebra

We still encounter non-Italian inspiration in the "second generation" – those abbacus writers who were active before 1340. Two figures, at least one of whom was influential, worked in Montpellier, and one almost certainly in the Papal city Avignon. Yet beginning in the early decades of the 14th century we can reasonably speak of the formation of an Italian abbacus *tradition*. More precisely, of a *North* Italian tradition – until the later 15th century we have no evidence at all south of Umbria.

Jacopo's *Tractatus*

The earliest representative of this generation is Jacopo da Firenze, writing in Montpellier in 1307. We already encountered his *Tractatus algorismi* above (p. 7) – but not Jacopo's original but a version, produced no later than *ca* 1410 (possibly well before that year). This version was adapted to the abbacus school curriculum and is thus a fitting representative of the abbacus tradition as it took form over the 14th century.

There are some (mostly minor) differences between the chapters contained in both **V** and **M+F**; some of them were identified above in the presentation of the latter version in chapter II. Beyond that, **V** contains several chapters that are not to be found in **M+F**:

– Algebra until the second degree, with rules and examples;
– Algebra of the third and fourth degree, rules only;
– a quasi-algebraic sequence of problems about wages in continued proportion;
– and a second collection of mixed arithmetical and geometric problems, not overlapping the earlier collection of mixed problems.

I have discussed the relation between the two versions in painfully pedantic detail in [Høyrup 2007: 12–25], and shall therefore only sum up the outcome:

Firstly, **V** is a meticulous copy of a meticulous copy of Jacopo's original, or at worst and not likely of an early stylistically homogeneous revised version.[254]

Secondly, **M+F** is a revised version descending either from Jacopo's original or from the hypothetical early stylistically homogeneous version from which **V** would also descend, making use of supplementary material circulating in Provence. For convenience I shall speak of the common archetype for all three manuscripts as the work of Jacopo: we cannot get behind it, and if not really his it must in any case be close to him in time.

Thirdly, the material in **V** with no counterpart in **M+F** goes back to Jacopo's original version or to the hypothetical early revision, and it has been eliminated in the preparation

[254] Fol. 46^{v} starts by stating that a section on silver coins has been omitted by error and is inserted *de rimpecto nel sequento foglio*, "opposite on the next sheet" (it follows indeed on fol. 47^{r}) – but the organization of the page shows that this passage was not inserted after the writing of the following section on "the alloys of small coins". It must hence have been present (together with a mark †† indicating the location of the omitted section) in the original used by the ultimate copyist, who will have preferred not to run the risk that attempts to repair would lead to extra errors.

of the archetype for **M+F**.[255]

Not mentioning the chapter on geometrically increasing wages, Van Egmond [2008: 313; 2009: 44] claims that the algebraic chapters in **V** descend from the algebra of *Tratato sopra l'arte arismetricha* (Florence, BNC, fondo princ. II.V.152, mentioned above, note 145, and described in some detail below, p. 240). Beyond glaring differences in level and style[256] he overlooks that a treatise which he himself dates to *ca* 1365 contains an algebra that indubitably descends from the one contained in **V**, which must therefore be earlier.

The algebra in **V** (of which I shall henceforth speak as "Jacopo's algebra") is the earliest abbacus algebra we possess. There is also evidence that Jacopo himself saw it as new, or at least as something new to his reader: The specific algebraic terminology is never abbreviated, in contrast to what happens elsewhere in the *Tractatus* – even *meno*, appearing as m̃ in the coin list, is written in full.

Though early, Jacopo's algebra is *not* the archetype from which all later abbacus algebras descend, but it is representative of their distinctive character and style and of the ways in which they differ from the algebras of al-Khwārizmī, Abū Kāmil and Fibonacci. So, for the general argument it is immaterial whether **V** really presents us with the earliest abbacus algebra – apart from details, the other early abbacus algebras collectively would allow us to draw the same conclusions. In any case, it is clear that the algebra in **V** descends from some source or cluster of sources which inspired the whole abbacus algebraic tradition; it is therefore merely a convenient window to that source cluster, not a historical milestone.

In one respect, unfortunately, **V** is not representative – but the reason is that something (probably a single sheet) has been lost in a precursor manuscript (not in **V** itself, since it would have belonged between fols 36^r and 36^v); this is shown by a later backward reference, see below, p. 191. When discussing Giovanni di Davizzo (below, p. 206), we shall see what kind of material this missing sheet may have contained. Before that, it must also be supposed to have carried some kind of announcement of what follows, similar to those in the beginning of other sections, for example the one dealing with the rule of

---

255 In his review, Van Egmond [2009] denies all of this. If I may be approximately as frank as Van Egmond, this review simply reveals his ignorance of what anybody able to read French (no need for Arabic) and interested in the matter (including in the innovations due to the abbacus masters) should know since [Woepcke 1853]; etc. I shall not persevere, anybody interested may look at [Høyrup 2009b], which is open access.

256 The only thing the two presentations of algebra have in common (apart from what *all* algebras have in common) is that they contain no false rules; but the *Tratato* shows how one type of irreducible cubic equation can be transformed into another one, which is far beyond the horizon of **V**. All of this is obvious in Raffaella Franci's and Marisa Pancanti's edition of the algebra of the *Tratato* [1988].

three (above, p. 16).

So, the algebra of **V** begins (p. 304) so to speak *in medias res*, with a sequence of rules with appurtenant examples for the first and second degree:[257]

> When the *things* are equal to the number, one shall divide the number in the *things*, and that which results from it is number. And as much is worth the *thing*.
>
> I propose to you an example to the said computation. And I want to say thus, make two parts of 10 for me, so that when the larger is divided in the smaller, 100 results from it. Do thus, posit that the larger part was a *thing.* Hence the smaller will be the remainder until 10, which will be 10 less a *thing.* [...].
>
> Again, I want to propose to you another example, and I want to say thus, there are three partners who have gained 30 £. The first partner put in 10 £. The second put in 20 £. The third put in so much that 15 £ of this gain was due to him. I want to know how much the third partner put in, and how much gain is due to (each) one of those two other partners. Do thus, if we want to know how much the third partner put in, posit that the third put in a *thing.* Next one shall aggregate that which the first and the second put in, that is, 10 £ and 20 £, which are 30. And you will get that there are three partners, and that the first puts in the partnership 10 £. The second puts in 20 £. The third puts in a *thing.* So that the principal of the partnership is 30 £ and a *thing.* And they have gained 30 £. Now if we want to know how much of this gain is due to the third partner, when we have posited that he put in a thing, then it suits you to multiply a *thing* times that which they have gained, and divide in the total principal of the partnership. And therefore we have to multiply 30 times a *thing.* It makes 30 *things,* which it suits you to divide in the principal of the partnership, that is, by 30 and a *thing,* and that which results from it, as much is due to the third partner. And this we do not need to divide, because we know that 15 £ of it is due to him. And therefore multiply 15 times 30 and a *thing.* It makes 450 and 15 *things.* Hence 450 numbers and 15 *things* equal 30 *things.* Restore each part, that is, you shall remove from each part 15 things. [...].
>
> When the *censi* are equal to the number, one shall divide the number by the *censi*. And the root of that which results from it is worth the *thing*.
>
> Example to the said rule. And I want to say thus, find me two numbers that are in proportion as is 2 of 3 and when each (of them) is multiplied by itself, and one multiplication is detracted from the other, 20 remains. I want to know which are these numbers. Do thus, and posit that one number was 2 *things* and the other was 3 *things.* And they are well in proportion as are 2 and 3. Next one shall multiply the numbers, each (one) by itself. And remove one multiplication from the other. And 20 shall remain. And therefore multiply each (one) by itself, and say, two *things* times 2 *things* make 4 *censi*. And three *things* times 3 *things* make 9 *censi.* Now remove one multiplication from the

[257] As above, page references to **V** point to the edition in [Høyrup 2007].

> other, that is, 4 from 9. 5 *censi* is left, which equal 20 numbers. And we say that one shall divide the numbers in the *censi*, so that one shall divide 20 numbers in 5 *censi*. From which results 4 numbers, and as much is worth the *thing*, that is, its root, which is 2. We said that the first number was 2 *things* and the second 3 *things*. Therefore you see clearly that 2 *things* are 4 numbers. And three *things* 6 numbers. And thus I say to you that these numbers are 4, one, and 6, the other. And such part is 4 of 6 as 2 of 3. Now if you want to verify it, multiply 6 times 6, it makes 36. And multiply 4 times 4, it makes 16. Detract 16 from 36. 20 is left, and it goes well. And thus all the similar computations are done, that is, according to this rule.

This beginning already shows us the most important distinguishing features of Jacopo's and later abbacus algebra:

First, all rules are presented in non-normalized form. The complete sequence of cases can be summarized as on p. 142:

Ja1	$\alpha t = N$	Ja4	$\alpha C+\beta t = N$
Ja2	$\alpha C = N$	Ja5	$\beta t = \alpha C+N$
Ja3	$\alpha C = \beta t$	Ja6	$\alpha C = \beta t+N$

C stands for *censo*, *t* stands for *cosa*, "thing", *N* for number, α and β for undetermined coefficients signalled by the use of a plural. As we observe, this differs from al-Khwārizmī's original sequence in three ways. Firstly, the *thing* has taken the place of the *root* even in the presentation of the cases. Secondly, the first and the third case have been switched; the new order will probably have been felt to be natural, once the *thing* is understood as *the* unknown. Thirdly, since all cases are now presented in non-normalized form, the first step in the corresponding rules is a normalization.

All six cases are provided with examples, sometimes one, sometimes two, the case Ja5 (the one allowing a double solution[258]) three:

1a. Make two parts of 10 for me, so that when the larger is divided in the smaller, 100 results from it.

1b. There are three partners, who have gained 30 *libre*. The first partner put in 10 *libre*. The second put in 20 *libre*. The third put in so much that 15 *libre* of this gain was due to him. I want to know how much the third partner put in, and how much gain is due to (each) one of those two other partners.

[258] It should be observed that these two solutions (when they exist) are regarded as *possibilities* – if one does not work, the other certainly will. The unknown is really seen as an unknown but already existing number, and *not as a variable* that may take on different values which fulfil the given condition.

2. Find me two numbers that are in proportion[259] as is 2 of 3: and when each (of them) is multiplied by itself, and one multiplication is detracted from the other, 20 remains. I want to know which are these numbers.
3. Find me 2 numbers that are in proportion as is 4 of 9. And when one is multiplied against the other, it makes as much as when they are joined together. I want to know which are these numbers.

4a. Someone lent to another 100 *libre* at the term of 2 years, to make (up at the) end of year. And when it came to the end of the two years, then that one gave back to him *libre* 150. I want to know at which rate the *libra* was lent a month.

4b. There are two men that have *denari*. The first says to the second, if you gave me 14 of your *denari*, and I threw them together with mine, I should have 4 times as much as you. The second says to the first: if you gave me the root of your *denari*, I should have 30 *denari*. I want to know how much each man had.

5a. Make two parts of 10 for me, so that when the larger is multiplied against the smaller, it shall make 20. I ask how much each part will be.[260]

5b. Somebody makes two voyages, and in the first voyage he gains 12. And in the second voyage he gains at that same rate as he did in the first. And when his voyages were completed, he found himself with 54, gains and capital together. I want to know with how much he set out.[261]

5c. Make two parts of 10 for me, so that when one is multiplied against the other and above the said multiplication is joined the difference which there is from one part to the other, it makes 22.[262]

6. Somebody has 40 gold *fiorini* and changed them to *venetiani*. And then from those *venetiani* he grasped 60 and changed them back into *fiorini* at one *venetiano* more per *fiorino* than he changed them at first for me. And when he has changed thus, that one found that the *venetiani* which remained with him when he detracted 60, and the *fiorini* he got for the 60 *venetiani*, joined together made 100. I want to know how much was

---

[259] *in propositione*. Most likely Jacopo copied from a text whose terminology he did not understand – the idea of proportions was generally unfamiliar to early abbacus authors; but we cannot fully exclude that the scribe of an intermediate or the final copy expanded an abbreviation wrongly. In any case, the mistake is systematic – in total, there are seven instances of *propositione*, whereas *proportione* is wholly absent.

[260] Choosing the *thing* to be the lesser part, Jacopo obtains that the smaller of the two solutions is valid; he does not try the other, which indeed does not work.

[261] Here, both solutions are shown to be valid.

[262] Because he chooses the *thing* to be the smaller part, Jacopo obtains that only the subtractive solution works; he tries the additive solution and shows that it is not valid.

worth the *fiorino* in *venetiani*.

As we see, five are pure-number problems, five deal with pretended commercial questions. Of the former, three are of the classical “divided 10” type, which we know from Fibonacci (above, p. 148) but which was already used many times by al-Khwārizmī as illustrations of the potency of his algebraic technique. No examples are formulated simply in terms of *censi* and *things* (corresponding to the initial *census-root-number* examples used by al-Khwārizmī, Abū Kāmil and Fibonacci). Instead there is an easy substitute – we may say a cheap way to create seeming complexity – namely numbers in given ratio. These were to become very popular in abbacus algebras, and when more than two number are involved the ratios are always given so as to fit nicely together, for example as 2:3, 3:4, ... In that way the number can be posited as 2*things*, 3*things*, 4*things* (Jacopo’s examples, being restricted to two numbers, do not demand this trick); the above example for the second case shows how this allows to construct an example corresponding to a given case.[263]

The first monetary problem (1b), the second illustration of the first case, is in itself very simple – no wonder, it illustrates the first-degree case. But we observe how Jacopo circumvents the difficulty that he is not allowed to divide by an algebraic binomial (“this we do not need to divide” – as we shall see, his successors would soon take for granted that this division was permissible and write it as a “formal fraction”. The operations would evidently be the same.

The second (4a) deals with composite interest (“making up the account at the end of year”). It shares its mathematical structure with Fibonacci’s first problem about repeated travels with constant profit rate, which Fibonacci solved simply by finding a mean proportional. Jacopo, wanting an algebraic problem and an illustration of the fourth case instead chooses the *thing* to be the interest in δ per month of 1 £, and thus obtains a nicely intricate problem.

The third (4b) is a give-and-take problem, but in contrast with those we know from Fibonacci it involves a square root, and it is therefore of the second degree; positing that the possession of the first man is a *censo* (adequate for taking a square root), Jacopo is led to the fourth case, and finds that the *thing* is $\sqrt{54}$–2. Wanting to calculate the possession of the first man, Jacopo needs to determine the *censo*, which must be $(\sqrt{54}-2)^2 = 58\text{-}4\sqrt{54}$. According to prevalent aesthetics, this should be expressed 58–$\sqrt{864}$, but being unable

263 It also shows that Jacopo uses “restore” to designate subtractive (and elsewhere, as Fibonacci and al-Khwārizmī, additive) operations on both sides of an equation. *Opporre* (corresponding to Latin *opponere* and Arabic *muqābalah*) is absent from Jacopo’s text; however, as mentioned above, note 186, the original meaning of *muqābalah*/*oppositio* was probably the confrontation leading to the construction of a simplified equations, and this is probably reflected in the term *raoguaglamento* used in example Ja5b (p. 316) about a simplified equation.

to calculate 16·54 mentally Jacopo leaves an open space – and forgets to return.[264]

The fourth (5b), about repeated travels with constant profit rate, belongs to a family we already know from Fibonacci. But the way to use this dress to produce a mixed second-degree problem differs from what we know from Fibonacci (above, p. 137); Fibonacci, moreover, makes use of proportion theory and *Elements* II in key version, not of algebra.

The dress of the fifth (6) brings to mind some of the more intricate exchange problems of the earlier abbacus books and of Jacopo's collections of mixed and geometric problems (above, p. 27), but its mathematical substance is wholly different, and indeed leads to a second-degree equation – positing 1 *fiorino* to be worth a *thing* of *venitiani*, Jacopo derives the reduced equation

$$40\textit{censi} = 120\textit{things}+100 \ .$$

These rules and examples for the first and second degree are followed (p. 320) by the announcement

> Here I end the six rules combined with various examples. And begins the other rules that follow the six told above, as you will see,

which confirms that there must have been a corresponding opening of the algebra section promising these six rules and thereby that a whole sheet (if not more) has been lost (cf. above, p. 186).

The "other rules" concern solvable cases of the third and fourth degree, which can be summarized thus (*K* standing for *cubo*, *CC* for *censo of censo*, that is, the fourth power of the *thing*):

Ja7	$\alpha K = N$	Ja14	$\alpha CC = \beta t$
Ja8	$\alpha K = \beta t$	Ja15	$\alpha CC = \beta C$
Ja9	$\alpha K = \beta C$	Ja16	$\alpha CC = \beta K$
Ja10	$\alpha K+\beta C = \gamma t$	Ja17	$\alpha CC+\beta K = \gamma C$
Ja11	$\beta C = \alpha K+\gamma t$	Ja18	$\beta K = \alpha CC+\gamma C$
Ja12	$\alpha K = \beta C+\gamma t$	Ja19	$\alpha CC = \beta K+\gamma C$
Ja13	$\alpha CC = N$	Ja20	$\alpha CC+\beta C = N$

[264] As explained in note 254, what we possess is not Jacopo's autograph but a copy of a copy; but this copy of a copy leaves open *ca* 2 cm and writes in the margin *così stava nel'originale spatii*, "thus it was in the original, spaces", which must have been transmitted through the whole chain. It cannot be excluded that this chain contained more than two steps, but all must then have strived at faithfulness.

The lacuna indicates that Jacopo calculated on his own, and did not just copy.

Jacopo cannot have been a brilliant calculator. After all, 16·54 can be found as 16·(50+4), and 16·50 = 800, 16·4 = 64.

No examples are given. The biquadratics corresponding to the cases Ja5 and Ja6 are missing, apart from that all cases that can be solved by means of division, pure root extraction or the substitution $C \rightarrow t$ are there – and only these. That the rules that are offered are correct is not revolutionary, at least since Abū Kāmil and al-Karajī such equations had been solved routinely in Arabic algebra, as also in the *avere* cluster of *Liber abbaci* 1.3. Neither Abū Kāmil nor Fibonacci had offered any similar systematic exposition, however; al-Karajī, on the other hand, had formulated general rules [Woepcke 1853: 71] for all mixed cases where the three powers involved are in continued proportion. Though presenting us with no new mathematical insights, Jacopo's approach (that is, the approach of his source) thus differs from how Abū Kāmil and Fibonacci had dealt with reducible higher-degree problems.

Even these rules, as we see, are defined for non-normalized cases.

Soon after these properly algebraic sections follows one about the manager of a *fondaco* (a warehouse located abroad, from Arab *funduq*), whose wages grow geometrically from year to year. In between, however, comes what thematically looks like an intruder – an alligation problem dealing with the mixing of two grain sorts with different prices. It makes use of a diagram that also serves in the later chapter about alloying (in **V** only, **M** and **F** have none), and is therefore likely to be original. It therefore may serve as a reminder that the following group would *not* have seen as belonging to algebra. Apart from that the problem tells us nothing new.

The *fondaco* section does. If a, b, d and (when needed) e designate the wages of the consecutive years, it contains the following problems (we remember that the wages grow geometrically):

F1 $a+d = 20$, $b = 8$	F3 $a+e = 90$, $b+d = 60$
F2 $a = 15$, $e = 60$	F4 $a+d = 20$, $b+e = 30$

F1 begins like this (p. 324):

> Somebody stays in a warehouse 3 years, and in the first and third year together he gets in salary 20 *fiorini*. The second year he gets 8 *fiorini*. I want to know what he received accurately the first year and the third year, each one by itself. Do thus, and let this always be in your mind, that the second year multiplied by itself will make as much as the first in the third. And do thus, multiply the second by itself, in which you say that he got 8 *fiorini*. Multiply 8 times 8, it makes 64 *fiorini*. Now it suits you to make of 20 *fiorini*, which you say he got in the first and third year together, two parts which when multipli⟨ed⟩ one against the other makes 64 *fiorini*. And you will do thus, that is that you always halve that which he got in the two years. That is, halve 20, 10 result. Multiply the one against the other, it makes 100. Remove from it the multiplication made from the second year which is 64, 36 is left. And of this find its root, and you will say that one part, that is,

> the first year, will be 10 less root of 36. And the other part, that is, the second year, will be 8 *fiorini*. And the third will be from 10 less root of 36 until 20 *fiorini*, which are *fiorini* 10 and added root of 36. And if you want to verify it, do thus and say: the first year he gets 10 *fiorini* less root of 36, which is 6. Detract 6 from 10, 4 *fiorini* is left. And 4 *fiorini* he got the first year. And the second year he got 8 *fiorini*. And the third he got *fiorini* 10 and added root of 36, which is 6. Now put 6 *fiorini* above 10 *fiorini*, you will get 16 *fiorini*. And so much he got the third year. And it goes well. And the first multiplied against the third makes as much as the second by itself. And such a part is the second of the third as the first of the second. And it is done.

As we observe, the notion of "proportion" is absent from the problem (though not from the formulation of F3, which has the usual mistake *propositione*); this indicates that we have to do with a standard problem type, in which the geometric increase is taken for granted.

None of the four problems refer to *thing* or *censo*, or to anything else that points toward algebra. Obviously, neither Jacopo nor his source made that connection. Already here we may notice a parallel to part 15.1 of the *Liber abbaci* (above, p. 132).

There, *Fibonacci* solved problems about proportions, drawing heavily on *Elements* II in "key" version (without referring to this term, found instead in the beginning of chapter 14 (p. 117). There is no such justification here, Jacopo merely explains the numerical steps leading to the solution (an "algorithm", but a trivial algorithm without branchings, that is, a formula).

The algorithm used to find a is the same as the one used by Fibonacci to solve the analogous problem #1 in part 15.1:

$$a = \frac{a+d}{2} - \sqrt{\left(\frac{a+d}{2}\right)^2 - ad} \ .$$

That is not too informative. The same procedure is used by Diophantos in *Arithmetica* I.27 (ed., trans. Tannery 1893: I, 60–62), and also used to find the sides of a rectangle from the sum of the sides and the area in Abū Bakr's *Liber mensurationum* [ed. Busard 1968: 91; ed. trans. Moyon 2017: 160*f*]. But alternatives exist, and one in indeed used by Jacopo himself in example 5a (above p. 189).

The remaining problems have no counterpart in part 1 of *Liber abbaci* chapter 15, but they make use of a trick that is known from the *Liber mahameleth*, going via the proportionality factor p between the wages of successive years – cf. [Høyrup 2021a: 46*f*]. In F2, this yields the solution

$$p = \sqrt[3]{\frac{d}{a}\cdot a^3}\ , \quad b = \sqrt[3]{\frac{d}{a}\cdot a^3}\ , \quad d = \sqrt[3]{\left(\frac{d}{a}\right)^2\cdot a^3}\ .$$

In the fourth, equally simple,

$$p = \frac{b+e}{a+d}\ ,\ \ a = \frac{a+d}{1+p^2}\ ,\ \ d = (a+d)–a\ ,\ \ b = \frac{b+e}{1+p^2}\ ,\ \ e = (b+e)–b\ .$$

The third is more intricate, and quite astonishing. It makes use of the insight that

$$a{\cdot}e = b{\cdot}d = \frac{(b+d)^3}{3(b+d)+(a+e)}.$$

after which *Elements* II.5 in “key” version can be applied. Even these formulas are most easily derived if we make use of the factor of proportionality ($b = pa$, $d = p_2{\cdot}a$, $e = p^3{\cdot}a$), since then

$$\frac{(b+d)^3}{3(b+d)+(a+e)} = \frac{a^3p^3(1+p)^3}{a\,(3p+3p^2+1+p^3)} = a^2p^3 = a{\cdot}ap^3 = ap{\cdot}ap^2\ .$$

It is not quite as easy in words, but it is still possible, and it does not go beyond principles of polynomial algebra that had been known at least since al-Karajī [Rashed 1984: 37]. However, to my knowledge (supported by that of all those whom I have asked) precisely this consequence of what was familiar had not been drawn in known works. Most likely, we have to do with another invention of the al-Andalus-environment which produced the *Liber mahameleth* and gave to Fibonacci the sophisticated version of the unknown heritage and *Liber abbaci*, part 15.1 (and bits of part 15.2).

The formula turns up in Pacioli’s *Summa* [1494: 87ᵛ], in a pure-number version that does not seem to descend from Jacopo [Høyrup 2009d: 100] but rather from a shared source – very likely together with his “keys” about numbers in continued proportion (above, p. 119), close to which they stand; it returns on fol. 96ᵛ, now indeed with a reference to these “keys”. From Pacioli it was borrowed by Tartaglia, who appears to have used it for his (claimed but undivulged) first solution of irreducible cubic equations (those involving *cube*, *censi* and number) – see [Kichenassamy 2015].

Beyond the algebraic sections and these four problems about numbers in continued proportion, **V** also contains a final chapter with mixed arithmetical and geometric problems that has no counterpart in **M+F**. Even though there are no repetitions of what has been dealt with in earlier chapters, they do not bring much fundamentally new. A few things may be mentioned.

As mentioned in passing in note 107, one problem (p. 360) is of type “unknown heritage”. It does not deal with a heritage, however, but with apples:

> I go to a garden, and come to the foot of an orange. And I pick one of them. And then I pick the tenth of the remainder. Then comes another after me, and picks two of them, and again the tenth of the remainder. Then comes another and picks 3 of them, 3, and again the tenth of the remainder. [...].

As we have already seen it in the first collection of mixed problems (above, p. 45), Jacopo here takes a familiar dress (thought with oranges, not apples) and then applies it to a new mathematical structure – this time more advanced, but since Jacopo offers a solution only but no argument, from his point of view it was probably simpler than the backward calculation.

The solution he offers is the usual one: the number of men, and the number of apples each one gets, equals the denominator of the fraction diminished by 1, that is, 9. Jacopo (or, as usual, his source) may have been aware that in the absence of argument a proof is needed, and he offers one.

The very first problem in the chapter (p. 347) is a partnership, where the partners do not enter at the same time. As reasonably, the gains are distributed proportionally to the investments of the single partner weighted by the time they stay.

On p. 350 comes a more convoluted variant of the "fish problem" (above, p. 24), here dressed as dealing with a goblet:

> A goblet of silver consists of three pieces, or three parts. That is, the stem, the cup, and the lid. The cup weighs $^1/_3$ and the $^1/_4$ of itself and of the stem. The lid weighs the $^1/_4$ and the $^1/_5$ of itself and of the cup. And the lid weighs ounces 6. I ask you what the stem weighs and what the cup weighs by itself, and what all the goblet weighs.

This gives Jacopo occasion to introduce the method of a single false position, only hinted at but not really used in the twin problem (above, p. 22) (he never mentions nor uses the method of the double false position).

Quite without parallel in earlier chapters is a problem

> Find a number which, when the $^1/_2$ and the $^1/_4$ and the $^1/_6$ are detracted from it, and the remainder multiplied by itself, makes this same number.

Similar problems are found in the *Liber abbaci* [B175;G298] (above, p. 80), the *Livero* [ed. Arrighi 1989: 124] [A124;B385]; and in the *CA* [ed. Vogel 1977: 31]. All of these make use of a single false position – sometimes mentioning it by name, sometimes not. Jacopo too uses it, and since he is only going to introduce the method slightly later, even he obviously does not refer to the method by name. Since the fractions are different, there is no reason to believe in any direct connection – the type was already widespread in Arabic algebras (see [Høyrup 2007: 133*f*]), and thus no invention of the abbacus environment (nor of Fibonacci).

Paolo Gherardi

The two next datable abbacus books from the second generation which we possess were also written in Provence – Paolo Gherardi's *Libro di ragioni* [ed. Arrighi 1987] (henceforth "Gherardi's *Libro*"; page references point to Arrighi's edition), and the anonymous *Trattato di tutta l'arte dell'abacho* (henceforth *Tutta l'arte*; above, pp. 11

and 35), existing in a number of manuscripts, two of which I have consulted.[265]

According to its colophon, Gherardi's *Libro* was written in Montpellier in 1327; what we have, however, is a not too conscientious copy [Van Egmond 1978: 162]; it is even plausible that the book was not written by Gherardi himself but by an assistant or a listener – the colophon says that this "book of problems [*ragioni*] will be written according to the rules and the abbacus course held by Paolo Gherardi of Florence"; which gives us the extra information that Gherardi was actually holding school. For simplicity, in the following I shall speak of the author of Gherardi. Occasional Provençal spellings (e.g., *nubre, valura*) confirm that the book was written in Provence.

After the colophon comes the rule of three in abstract formulation for integers and explanation of how to eliminate fractions. The words are the same as in the *Livero* and the *Liber habaci* (above, note 19) – and thus the same as those of Jacopo, apart from the latter's slight expansion of the final step, in **M+F** "divide in the other, that is, in the third thing" and in **V** "divide in the third thing, that is, in the other that remains". Gherardi thus does not copy from Jacopo but clearly belongs to the same tradition (and does not here follow the Catalan-Provençal habit).

After the rule of three with three examples follow multiplication and division of mixed numbers, and a sequence of number problems (n stands for a single unknown number, a, b, ... for sequences of numbers, for instance resulting from the splitting of a given number, $^{u}/_{v}$ for an unspecified fraction):

p. 16	$(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5})n+3 = 25$	(1)
p. 17	$[(1-\frac{1}{3}-\frac{1}{8})n] = 120$	(2)
p. 17	$(1-\frac{1}{2}-\frac{1}{4})n = \sqrt{n}$	(3)
p. 17	If 9 is $\frac{1}{2}$ of 16, what part is 12 of 25?	(4)
p. 17	$\frac{3}{5}a = \frac{5}{9}b\ ,\ \ ab = a+b$	(5)
p. 18	$12 = a+b\ ,\ \ 5a = 8b$	(6)
p. 18	$a^2+b^2 = 1$	(7)

[265] See the index listing in [Van Egmond 1980: 365] – not all are complete. Van Egmond ascribes it to Paolo Dagomari alias Paolo dell'Abbacho, but his only reason is apparently that a 15th-century manuscript announces a fragment as "some small rules drawn from Master Paolo's book, and various ancients measures and weights" [Van Egmond 1980: 145], which rather shows that the compiler of that manuscript thought of Paolo's *Regoluzze*, contained in the three preceding sheets. Comparison of how *Tutta l'arte* deals with the geometry of the circle and how that is done in the *Regoluzze* should exclude common authorship, see [Høyrup 2019a: 304 n. 29]. The two manuscripts I have used are Florence, BNC, fond. prin. II,IX.57 (henceforth $\mathbf{T_f}$); and Rome, Accademia Nazionale dei Lincei, Cors. 1875 (henceforth $\mathbf{T_r}$).

Arrighi [1980] also ascribes *Tutta l'arte* to Paolo, but giving no other reason than a reader's or librarian's note written "in a considerably later hand" than the 14th-century hand in which the manuscript is written.

p. 18	$6 = a+b\ ,\ \ (\frac{1}{4}+\frac{1}{5})a = (\frac{1}{5}+\frac{1}{6})b$	(8)
p. 18	$4 = a+b\ ,\ \ a = (\frac{1}{3}+\frac{1}{4})b$	(9)
p. 19	$(1+\frac{1}{3}+\frac{1}{4})n+5 = 6n$	(10)
p. 19	$n+5 = \square_1\ ,\ \ n-5 = \square_2$	(11)
p. 19	$n^2+\frac{1}{4}n = 19$	(12)
p. 19	$12 = a+b,\ ab = 31$	(13)
p. 20	$10\frac{1}{2} = a+b\ ,\ \ a+11\frac{1}{5} = 2b$	(14)
p. 20	$14 = a+b,\ a/b = 4$	(15)
p. 20	$a^2+b^2+c^2+d^2+e^2 = \frac{1}{7}$	(16)
p. 21	$n^2+24 = \square$	(17)
p. 21	$n^2 = 11+\frac{u}{v}$	(18)
p. 21	$(\frac{1}{3}n)\cdot(\frac{1}{4}n) = 20$	(19)
p. 21	$3\frac{1}{5}\cdot n = 17\frac{1}{3}$	(20)

Many are solved by means of the rule of three (4), other basic arithmetic (5, 6, 14, 10, 20) or a single false position (1, 2, 3, 7, 8, 9). Others make use of algebra (12, 13, 19), still others of what could be considered elementary number theory – (17), for instance, gives the solution $n = (\frac{24}{4}-1)^2$, which, for $p = 24$, corresponds to the identity

$$\left(\frac{p}{4}-1\right)^2+p = \left(\frac{p}{4}+1\right)^2 ,$$

related to the one that produces Pythagorean triples, and to what is used in (11)

This fundament for the solution, goes unexplained (as do most of the others, even when the reason for the numerical steps of the prescription are less than evident). None the less, while Gherardi does not borrow from the *Liber abbaci*, the environment on which he draws evidently shares interests beyond the commercially relevant with that which had once inspired Fibonacci. The writings from “generation 1” as well as Jacopo’s *Tractatus*, though containing some problems of the same kind, all treat the topic as less important.

These number problems are followed (pp. 21–26) by a presentation of the arithmetic of (square) roots and of binomials containing roots, ending by the rule for finding the “closest root” – the same approximation as the one Jacopo as well as other abbacus authors designate thus (and, like Jacopo, approximating only from below).

Commercially relevant matters begin on p. 26, though not in any convincing pedagogical order or progression. At first:

– Calculation of average price;
– rule of five (cf. above, p. 74)
– transformation between interest per year, month and day (pp. 27–29);
– alloying of silver and gold (pp. 29–32);
– exchange (pp. 33–37).

Before going on with partnership (pp. 38–41) Gherardi presents two recreational outsiders, the “twin problem” (above, p. 22), and the “unknown heritage” (above, p. 92); the former of these at least has the excuse that Gherardi (as Jacopo) solves it by means of an explicit fictitious partnership, but before partnerships proper are dealt with; the second is a *bona fide* outsider, a purely recreational teaser (and, since it does not explain the method, a teaser which teaches nothing, not even indirectly)

The partnership section is followed by a sundry collection of mixed, mostly recreational problems – and mostly of kinds we have already encountered. On p. 47, however, we find a puzzling variant of the pursuit problem. The two runners move along a circle, which one of them completes in 4 days, the other in 5 $^1/_2$ days. It is asked, firstly, when the faster runner will reach the slower for the first time, and secondly, when they will both be together at the starting point. What is puzzling is not the mathematics but the dress: before the hands of pendulum clocks and wristwatches and when the ancient circus was forgotten, what ran around in circles were planets rather than men. Could Gherardi’s variant of the pursuit problem somehow be an echo of astronomical calculation, perhaps of the Indian “pulverizer” (hardly of the technique), cf. [Plofker 2009: 134]?

Noteworthy is also Gherardi’s version of the grasping problem (above, p. 97): the shares ($^1/_2$, $^1/_3$ and $^1/_4$) differ from those of Fibonacci; moreover, the solution is obtained by means of first-degree algebra with unknown *thing* (the amount the three participants put back) – Fibonacci would probably have spoken about *regula recta*, but Gherardi does not know that expression. A four-participant “purchase of a horse” of the type where each asks his neighbour in the sequence (p. 45) is solved by means of a double false position – not called by name, instead the text speaks about “adjusting” one (position) with the other.

A problem on p. 49 combines a dress that is familiar from the the collection of arithmetical epigrams contained in book XIV of the *Greek Anthology* [ed. Paton 1918: 31, 101] with a counterfactual calculation: So large a part of the night has passed that “if $^1/_3$ had been $^1/_4$ of the part that has passed and $^1/_4$ were $^1/_5$ of what is to come, then it would be midnight”.

Two more make use of algebra: One (p. 49) is a number problem $a/b = 2/3$, $ab = a+b$, dressed up as dealing with the *denari* possessed by two men; the other is a three-participant give-and-take problem, where algebra is subordinated to a double false position.

A long section (pp. 61–83) deals with geometry. As pointed out above, note 37, Gherardi distinguishes between rules *di misure* and *di giomatria*, where the former could refer to Arabic *misaḥa* while latter could refer back to the Latin post-agrimensor tradition, which was probably more alive in Provence than in Italy;[266] but with no other evidence

[266] One problem named *giomatria* and certainly going back to the agrimensor tradition was mentioned above, note 47 and preceding text; it determines the area of a regular pentagon with side 6 as $(6^2–6)÷2$. This is the formula for the sixth pentagonal number, and believed to determine the area of a regular pentagon for example in the *Geometria incerti auctoris*, in ps-Varro, *Fragmentum geometriae*, and in Epaphroditas et Vitruvius Rufus [ed. Bubnov 1899: 346, 504, 534]. With an

for such a distinction, this remains hypothetical.

After another sequence of mixed (commercial, recreational-commercial and geometric) problems (pp. 83–97) Gherardi closes his book with a systematic presentation of "the rules of the thing", that is, algebra.[267] He deals with the following cases:

Gh1	$\alpha t = N$	Gh9	$\alpha K = \beta t$
Gh2	$\alpha C = N$	Gh10	$\alpha K = \beta C$
Gh3	$\alpha t = \beta C$	Gh11	$\alpha K = \beta C+\gamma t$
Gh4	$\alpha C+\beta t = N$	Gh12	$\alpha K = \beta t+N$
Gh5	$\beta t = \alpha C+N$	Gh13	$\alpha K = \beta C+N$
Gh6	$\alpha C = \beta t+N$	Gh14	$\alpha K = \beta t+\gamma C+N$
Gh7	$\alpha K = N$	Gh15	$\alpha K+\beta C = \gamma t$
Gh8	$\alpha K = \sqrt{N}$		

All are provided with examples – never more than one for the single case, even when (as in Gh5) the rule speaks of a double solution.

Two of the examples for the first six rules coincide with examples given by Jacopo; the data of Gh4 – a composite-interest problem – are those Ja4 divided by 5, otherwise the examples are identical (cf. above, p. 190); Gh3 is a slight numerical variation of Ja3 (numbers i ratio 2:3 instead of 3:4); the example for Gh2 is a new pure-number problem, $(n-{}^1\!/_3\,n-{}^1\!/_4\,n)^2 = 12$. Most interesting is the example for Gh6, the division of 100 by some quantity and then by 5 more, the sum of the two divisions being given:

$$100 \div q + 100 \div (q+5) = 20 \ .$$

With subtraction instead of addition, we know this problem type not only from the *Liber abbaci* but also from Abū Kāmil and the extended version of al-Khwārizmī's algebra (above, p. 153). Without being fully unfolded, however, Gherardi's way to solve the problem builds on an new idea: formal fractions. In its full form (as we encounter it in later texts), the "quantity" is posited to be a *thing*, the quotients written as fractions and "added as if they were fractions"

$$\frac{100}{t}+\frac{100}{t+5} = \frac{100\cdot(t+5)+100\cdot t}{t\cdot(t+5)} = \frac{200\,t+5}{C+5\,t} = 20 \ .$$

Gherardi does not mention fractions, but he performs all the requisite operations, and he

---

extra mistake, we remember (omission of the halving), the formula is also used in Jacopo's *Tractatus*, **V** as well as **M+F**, and with further repair of the dimensional nonsense in *Tutta l'arte* – more in [Høyrup 2007: 96]. A paving problem on p. 62 (above, note 36) is also categorized as *giometria*.

267 [Van Egmond 1978] is an edition of this section, with English translation and mathematical interpretation.

does refer to the scheme for cross-multiplication that produces the numerator (forgotten in the copy but easily reconstructible):

$$\begin{array}{lcl} 100 & & 1\ \textit{thing} \\ & \times & \\ 100 & & 1\ \textit{thing}\ \texttt{plus}\ 5 \end{array}$$

The full form is found often in later abbacus books, which cannot have derived it from Gherardi's obscure rudiment; Gherardi must have borrowed it from earlier writers to whom other abbacus authors had access, too. Similar formal calculations had been made in the Maghreb since the outgoing 12th century – see [Abdeljaouad 2005: 24–29]; a link is plausible, but details cannot be traced.

We find more innovations in the higher-degree rules. Those cases that can be solved by extraction of a cube root or reduced by a division to second-degree problems are already in Jacopo's *Tractatus;* the rules for the cases Gh12, Gh13 and Gh14 are false. The solutions given for Gh12 and Gh13 are identical,

$$t = \sqrt{\left(\frac{\beta}{2\alpha}\right)^2 + \frac{N}{\alpha}} + \frac{\beta}{2\alpha} \ ,$$

just copied from the solution to Gh6. Gh14 is no less preposterous,

$$t = \sqrt{\left(\frac{\beta}{2\alpha}\right)^2 + \frac{N-\gamma}{\alpha}} + \frac{\beta}{2\alpha} \ .$$

Anybody who understood algebra would have seen that Gh12 and Gh6 can only have the same solution if $K = C$, and hence if $t = 1$, that is, if $\alpha = \beta+N$ (or, if against the prevailing habits of the time, $t = 0$, which entails $N = 0$). One might believe that the examples would uncover the fraud, but since the solutions contain irreducible radicals and approximations were never made in abbacus algebra, that was less easy.

These fake solutions were a great success; more would be added over the next century, and Gherardi's three false solutions were still repeated by Bento Fernandes in 1555 [da Silva 2008: 200]. We shall return to the question why some abbacus masters (not all) would indulge in such pseudo-mathematics on p. 394.

The rule for the case Gh9 is correct, but a modern reader might wonder why it is listed separately from the preceding case. We should remember, however, that only positive integers and fractions were accepted as numbers, and think of the difficulty which the appearance of an irrational coefficient had caused Fibonacci (above, p. 152). $\sqrt{N}$ is not a coefficient, but its appearance here instead of the number term forebodes a practical widening of the number concept which was to take place over the following centuries, only maturing in the 16th century (cf. [Oaks 2017]).

All examples for the higher-degree cases, reducible as well as non-reducible, are of the easy type asking for numbers in given ratios – for example (Gh7) "find me three numbers where the first be such part of the second as 3 of 4, and the second be such part of the third as 4 of 5" – with the variation that Gh14 and Gh15 instead use the terminology "be in proportion".[268]

Tutta l'arte

As observed by Van Egmond [1980: 140], manuscript $\mathbf{T_f}$ of *Tutta l'arte* (above, note 265) is the author's draft. From internal evidence Jean Cassinet [2001] has shown that it was written in Provence and almost certainly in Avignon. It is dedicated ($\mathbf{T_f}$ fol. 17^r) to pope *Benedetto*, with space left open for his number; this, as further pointed out by Cassinet, implies that the dedication was written while it was still undecided whether the previous Benedetto had been a pope or an anti-pope – that is, in 1334.

Various aspects of *Tutta l'arte* were presented above (pp. 11, 35, and note 47). In order to illustrate the emergence of abbacus algebra we shall also have a look at how this topic is dealt with.

In $\mathbf{T_f}$, the draft manuscript, one page (fol. 171^v) contains the beginning of an introduction to "the rules of the thing, by means of which many beautiful and subtle problems can be solved". It is written in a different hand, and the date therefore uncertain. The cases are defined in the usual terms and solved by the usual rules; they may be abbreviated thus:

$$\alpha t = N \qquad\qquad \alpha K = N$$
$$\alpha C = N \qquad\qquad \alpha C+\beta t = N$$

As we see, the order is unusual (and not to be found in later treatises I know about). The examples are quite elementary (the example for the fourth rules is missing; on the next page (fol. 172^r) the text goes on in a different hand with medical advice):

– Find me a number which, multiplied by 3 and divided by 4 makes 20.
– Find me a number which, when $^1\!/_3$ and $^1\!/_4$ (of it) are subtracted and the remainder multiplied by itself, makes 12. This is the same as Gherardi's example for the case in question (one of the two where Gherardi deviates by more than a change of numerical parameters from Jacopo).
– Find me a number which, multiplied by itself and then multiplied by this number, makes 12.

In *Tutta l'arte* proper, there is no systematic presentation of algebra, but a number of problems are solved by means of *thing* and *censo* (all but one listed in [Cassinet 2001: 124–127]) – I indicate the folio numbers in $\mathbf{T_f}$:

157^r $\quad (1-{}^1\!/_3-{}^1\!/_4)n-5 = {}^1\!/_3(1-{}^1\!/_3-{}^1\!/_4)$

[268] The text writes *in positione*, certainly a mistake for "in proportione" – the formulation which Jacopo or his copyist had changed into *in propositione*, we remember from note 259. Whether Gherardi made the mistake (abbreviation perhaps assisting) when copying from his source or his compiler or a subsequent copyist miscopied is undecidable.

157^{v} (three men having money, with structure) $A+B+C = 104$, $A<B<C$, $A:B = B:C$,[269] $A = 8$

158^{v} $a+b = 16$, $(\frac{3}{4}a)^2 = (\frac{3}{4}b)^2-20$

160^{r} $a/b = 5/7$, $a/c = 5/9$,[270], $ab+c = c^2$

161^{v} $a+b = 10$, $ab/(a-b) = 2\frac{2}{3}$

162^{v} $(1+\frac{1}{3}+\frac{1}{4})n = \sqrt{n}$

166^{r} $(60\div q)\cdot 60\div(q+2) = 150$

166^{v} A rectangle with area 180 square cubits, length = $1\frac{1}{3}$ width

167^{v} Repeated travels with given profit rate and given costs and given net profit rate, equivalent to $\frac{8}{7}(\frac{6}{5}C-25) = \frac{6}{5}C$

Together with Gherardi's scattered use of algebra (above, pp. 197 and 198), this gives us an impression of the form in which algebra was disseminated in the Provençal abbacus environment around 1330.

It was not disseminated in Provence alone. That will follow when we look at three representatives of the second generation written in Tuscany.

The Lucca *Libro d'abaco*

The first of these is a *Libro d'abaco* written in Lucca by several hands – according to internal evidence around 1330.[271] We may guess that it was produced by an abbacus master and his assistants, or perhaps of the assistants alone – where else would we find a group that had occasion to engage in such a work?

There is no algorism, that is, no introduction of Hindu-Arabic computation. It opens with the rule of three (p. 17), in words that only differ slightly from what we have seen so far:

When you make some calculation [*ragioni*] by the three things, always take the thing

---

269 Expressed in terms of *proporzioni*.

270 Expressed as "such part as"; the text has $b/c = 5/9$, but the rest of the text shows this to be a slip caused by the habitual sequential formulation.

271 Lucca, Biblioteca Statale, ms, 1754. ed. [Arrighi 1973], cf. [Van Egmond 1980: 164*f*]; page references point to Arrighi's edition. A change of hand in the middle of fol. 23^{v} shows that it is the result of collaboration, not of discordant works or fragments put together – and also that what we possess is the original, not a copy.

The dating follows from (fictional) loan contracts (pp. 182–188) expiring in 1329–1333. Such dates may evidently have been thought of as being in the future, but hardly in distant future; in some cases they are indeed in similar fictional loan documents in *Tutta l'arte*, but no more than five years [Cassinet 2001: 107].

> you ask for or want to know, and that which is not of the same kind [*ragione*] or quality, and multiply one against the other, and divide that amount in the other thing, and that which results will be the effect of the question of the calculation.

This is followed by an example, first solved according to the rule just enunciated (where the missing "not similar" turns up),

> 8 cubits of cloth are worth 11 *fiorini*, what will 97 cubits be worth? You should do like this. The thing that we as for is what 97 cubits will be worth, the not similar thing to the said cubits is 11 *fiorini*, and therefore we should multiply 11 times 97, [...].

Then, rather unusually, come the two alternative methods where the intermediate result is meaningful; either that 97 cubits is $^{97}/_{8}$ times as much as 8 cubits, for which reason the price must be $^{97}/_{8}$ times 11 *fiorini;* or that the price of 1 cubit is $^{11}/_{8}$ *fiorini*, whence the price of 97 cubits will be 97 times $^{11}/_{8}$. These alternatives – in particular the first one, called "by relation" (*nisba*) – are well-known in Arabic mathematics. For instance, they are discussed by al-Karajī in his *Kāfī* [ed. trans. Hochheim 1878: II, 16*f*]; but even though they may occasionally be *used* by abbacus authors (cf. above, p. 21), they are rarely expounded directly as here.[272]

As elsewhere, the presentation of the rule of three invites the teaching of multiplication and division of fractions (called "elements of fractions", *Elementi de' rotti*, evidence of some kind of interaction with university mathematics).

In the very end (pp. 201–205) comes another version of the habitual beginning of abbacus treatises: First arithmetical tables (not reproduced by Arrighi); then metrological shortcuts; and then the rule of three in the unusual "mentioned"-formulation (above, p. 172), duly followed by teaching of how to deal with fractions and mixed numbers, and finally a case of proportional sharing said to be "strange" and called "oblique [*traverso*] sharing" – namely the sharing between partners of which one should have $^{1}/_{2}$ and the other $^{1}/_{3}$. The type is not rare – we have encountered it in the Pisa *Libro di ragioni* (above, p. 166), where it is spoken of as "fallacious"; but I have not noticed the present name elsewhere.

What we find between these two beginnings is a fairly full coverage of the usual abbacus topics and methods – sometimes going beyond the usual, for instance when teaching the rule of five and the rule of seven systematically, and when offering (p. 151) a rule for gauging the volume contained in a Florentine standard wine barrel.[273] Particularly noteworthy is a long account (pp. 153–175) of the weights, measures and customs of a number of trading places, and the relations between metrologies. The places

---

[272] Unfortunately a third method follows, where 97 cubits becomes 97 *fiorini*.

[273] Namely as $^{11}/_{25}$ of length×diameter^2. In the 16th century, German *Rechenmeister* were to integrate this topic ("doliometry") in their teaching – see for example [Ries 1550: 182–196].

spoken of reach from Accra, Alexandria and Constantinople in the East over Bejaïa, Tunis, Messina and Palermo in the South to Mallorca, Nîmes, Montpellier and Marseille in the West and North, and also encompass numerous cities from the Italian mainland. In the end of this *tariffa* comes a list of the fairs of France, Flanders and Apulia, with their calendars.

As Gherardi's *Libro* and *Tutta l'arte*, this *Libro d'abaco* solves scattered problems by means of algebra, thereby illustrating what was diffused in the environment; moreover, it contains not merely one but two systematic presentations, a *Regola della cosa* (pp. 108–113), and an *Aligibra amichabile*, pp. 194–197).

The *Regola della cosa* states 16 rules:

Lc1	$\alpha t = N$	Lc9	$\alpha K = \beta C$
Lc2	$\alpha C = N$	Lc10	$\alpha CC = \beta K$
Lc3	$\alpha C = \beta t$	Lc11	$\alpha CC = N$
Lc4	$\alpha C+\beta t = N$	Lc12	$\alpha CC = \beta t$
Lc5	$\beta t = \alpha C+N$	Lc13	$\alpha CC = \beta C$
Lc6	$\beta t+N = \alpha C$	Lc14	$\alpha K+\beta C = \gamma t$
Lc7	$\alpha K = N$	Lc15	$\beta C = \alpha K+\gamma t$
Lc8	$\alpha K = \beta t$	Lc16	$\alpha K = \beta C+\gamma t$

All cases are also dealt with in Jacopo's *Tractatus*, which implies that only reducible cubics and quartics appear. Moreover, in the likeness of Jacopo, the present compiler provides the first six cases with examples (only one for each), while the cubics and quartics have none. Apart from changed numerical parameters, the examples for Lc4 and Lc5 agree with problems offered by Jacopo, the others are different – sometimes even simpler than the "such part" problems, for instance (Lc6),

> find me a number which, when 30 is added to it, makes as much as when it is multiplied by itself.

The example for Lc2 coincides with what is proposed by Gherardi (above, p. 199), apart from a changed numerical parameter.

After the higher-degree rules come four divided-ten problems solved by means of algebra (all of the second degree), and two stated explicitly to be solved without the *thing*.

Given how close the rules are to what we know from Jacopo's *Tractatus*, it appears certain that the compiler draws (directly or indirectly) *either* on Jacopo *or* on a close precursor to his algebra chapter; the new examples he may have drawn from what was already circulating (they fit what we know from Gherardi and from *Tutta l'arte*), or he may have constructed them himself (the example for Lc6 being nothing but an instantiation of the rule, with specification of the numerical parameters).

The *Aligibra amichabile* (pp. 194–197) states 13 rules:

La1	$\alpha t = N$	La7	$\alpha K = N$
La2	$\alpha C = N$	La8	$\alpha K = \beta t$
La3	$\alpha C = \beta t$	La9	$\alpha K = \beta C$
La4	$\alpha C+\beta t = N$	La10	$\alpha K+\beta C = \gamma t$
La5	$\beta t = \alpha C+N$	La11	$\beta C = \alpha K+\gamma t$
La6	(omitted)	La12	$\alpha K = \beta C+\gamma t$
		La13	$\alpha CC = N$

These are simply Jacopo's first 13 cases in the same order[274]. The first five are provided with a single example – all but that for La2 drawn coinciding with examples we know from Jacopo, sometimes with changed numerical parameters; that for La2, however, coincides exactly with Gherardi's example (above, p. 199). The phrasing of the example for La4 is so similar to that of Lc4 (both numerical variants of the first example for Ja4, which however is phrased very differently) that they must clearly make use of a shared source ultimately depending on Jacopo or his source but already reformulated.

As mentioned above, the Lucca *Libro* also contains scattered problems solved by means of *thing* and *censo*. Most of them are similar in type to the scattered problems of Gherardi and *Tutta l'arte*, but one has close kin only in Jacopo's algebra (namely his second example of Ja4b – above, p. 190): a variant of the "purchase of a horse" which would hardly be imaginable without the prospect of solving it by means of algebra (p. 132):

> There are two men, and they want to buy a horse which is worth £ 10. The first says to the second, if you give me $^1/_3$ of your *denari*, I shall buy it. The second says to the first, if you give me the root of your *denari* and £ 5 more, I shall buy the horse. I want to know how many *denari* each one had. You should do like this: Posit that one had a *censo*, and then the other must have 30 less 3 *censi*. [...].

Over the next century and a half, similar problems turn up in many of the more advanced abbacus treatises.

Giovanni di Davizzo

My final representatives of the second generation are only known from being quoted in later works. One is the Florentine Giovanni di Davizzo (fl. 1339–1344), whose father, brother and two nephews were also abbacus masters [Ulivi 2002a: 39, 197, 200]. Within a sequence of number problems in the manuscript *Alchune ragione* from 1424[275],

---

274 That La6 has been omitted by mistake follows from the line following where it should have been (p. 196), "these are the six rules of *aliabra amichabile*".

275 Vatican, Vat. lat. 10488, see [Van Egmond 1980: 230]. It is written by several hands, often shifting in the middle of a page and thus a planned collaborative effort – once again we may think

starting on fol. 25^v and ending on fol. 38^v, on fols 28^v–32^r is inserted an extract announced as being

> extracted from a book from the hand of Giovanni di Davizzo dell'abacho from Florence, written the 15th september of year 1339, and this is 1424.

A later hand has added a heading *Algisbra*, which is indeed quite adequate. Fols 28^v to 29^r gives us a general idea (nothing more!) of what may have been lost in Jacopo's algebra (see above, p. 186).

At first come, mixed up with the four sign rules in §2, rules for the multiplication and division of powers:

¶[§1] Know that to multiply number by *cube* makes *cube*
and number by *censo* makes *censo*
and number by *thing* makes *thing*.
¶[§2] And plus times plus makes plus
and less times less makes plus
and plus times less makes less
and less times plus makes less.
¶[§3] And know that a *thing* times a *thing* makes 1 *censo*
and *censo* times *censo* makes censo of *censo*
and *thing* times *censo* makes *cube*
and *cube* times *cube* makes *cube of cube*
and *censo* times *cube* makes *censo of cube*.
¶[§4] And know that dividing number by *thing* gives number
and dividing number by *censo* gives *root*
and dividing *thing* by *censo* gives number
and dividing number by *cube* gives *cube root*
and dividing *thing* by *cube* gives *root*
and dividing *censo* by *cube* gives number
and dividing number by *censo of censo* gives *root of root*
and dividing *thing* by *censo of censo* gives *cube root*
and dividing *censo* by *censo of censo* gives *root*
and dividing *cube* by *censo of censo* gives number

---

of the assistants of an abbacus master with or without the participation of the latter. Occasional personal opinions (e.g., fol. 35^r, in the running text, not a marginal note) about procedures show that those who wrote were competent abbacists, not merely scribes. My references refer to the earliest foliation.

According to Van Egmond, the manuscript should be Venetian; he does not cite any evidence, and the language/orthography seems to fit Florence perfectly (and not at all Venice), which would make it less strange that Giovanni di Davizzo's manuscript was available.

and dividing number by *cube of cube* gives *cube root of cube root*
and dividing *thing* by *cube of cube* gives *root of cube root*
and dividing *censo* by *cube of cube* gives *root of root*
and dividing *cube* by *cube of cube* gives *cube root*
and dividing *censo of censo* by *cube of cube* gives *root*
and dividing *censo of cube* by *cube* gives ~~number~~ *censo*[276]
and dividing number by *censo of censo of censo of censo* gives *root of root of root of root*
and dividing number by *cube of cube of cube of cube* gives *cube root of cube root of cube root of cube root*.

¶[§5] If you want to multiply root by root, multiply root of 9 times root of 9, say, 9 times 9 makes 81, and it will make the root of 81, and it is done.

To divide root of 40 by root of 8, divide 40 by 8, it gives 5, and root of 5 let it be.

To divide root of 25 by root of 9, divide 25 by 9, it gives root of $2\,{}^7\!/_9$, done.

If you want to multiply 7 less root of 6 by itself, do 7 times 7, it makes 49, join 6 with ⟨49, it makes⟩ 55, and 7 times 6 makes 42, then multiply 7 times 42, it makes 294, and multiply then 4 times 294, it makes 1176, I say that 55 less root of 1176 will it make when 7 less root of 6 is multiplied by itself.

¶[§6] If you want to detract root of 8 from root of 18, do 8 times 18, it makes 144, its root is 12, and say, 8 and 18 makes 26, detract 24 from 26, and root of 2 will remain, done.

It you want to join root of 8 with root of 18, do 8 times 18, it makes 144, its root is 12, and say, 12 and 12 makes 24, and say, 8 and 18 makes 26, and join 24 and 26, it makes 50, and root of 50 will the number be.

If you want to multiply 5 and root of 4 times 5 less root of 4, do thus and say, 5 times 5 makes 25, and say, 5 times root of 4, do thus, bring 5 to root, it makes 25, and do root of 25 times root of 4, it makes root of 100, and make 5 times less root of 4, it makes less root of 100, 25 still remains, now detract 4 from 25, 21 remains, and 21 they make.

If you want to multiply 7 and root of 9 times 7 and root of 9, do 7 times 7, it

---

[276] From later versions it can be seen that this line was originally

"and dividing censo of cube by cube *of cube* gives number"

Somewhere in the process, this had become

"and dividing censo of cube by cube gives number"

Noticing the error, somebody – almost certainly the writer of the 15th-century manuscript, since the correction is made there – saw that this was wrong, and stated a correct result (but of a division Giovanni had not intended).

> makes 49, put (above) this 9, you have 58, and 9 times 49 makes 441, multiply by 4, it makes 1764, you have that it will make 58 and root of 1764, which is 42, done.
>
> If you want to divide 35 by root of 4 and by root of 9, do thus, from 4 to 9 there is 5, multiply 5 times 5, it makes 25, and say, bring 35 to root, it makes 1225, now say, 4 times 1225 makes 4900, divide by 25, it makes 196, and do 9 times 1225, it makes 11025, divide by 25, it gives 441. We have that dividing 35 by root of 4 and by root of 9 gives root of 441 less root of 196, and it is done.

The rules for the multiplication of powers (§3) show, firstly, that the names for higher powers are based on multiplication, not embedding: *cube of cube* stands for $t^3 \cdot t^3$, not for $(t^3)^3$; and secondly, that Giovanni has a full mastery of the sequence of successive powers.

Those for division (§4), on the other hand, demonstrate that here Giovanni's intuition fails. He appears to have nourished a vague idea that the inverse of taking a power is to take a corresponding root – raising to the third power and then taking the cube root evidently leads us back to the starting point. This is then mis-applied to negative powers (and we observe that all divisions in §4 apart from the one resulting from a copying error should give a negative power), which are identified with roots – with some difficulty corresponding to t^{-1}, which becomes *number*. The multiplicative composition of these "roots" confirms that they are nothing but postulates – when genuine roots are meant, also by abbacus writers, the cube root of 256 is 8, and the cube root of the cube root of 256 therefore 2, the ninth, not the sixth root.[277]

§5 and §6 deal with the arithmetic of monomials and binomials containing square roots; it is noteworthy that Giovanni often makes use of rational roots "as if they were irrational"; this would allow him to control the correctness of the calculations, but he does not do so, and the expression in quotes is not found in Giovanni's text but only in Dardi da Pisa's slightly later treatise (below, p. 218), which suggests Giovanni to have borrowed (which we would anyhow expect).

After these introductory matters come, as usual, a list of algebraic cases – rules only, no examples:

---

[277] This is the reason I have chosen to italicize these "roots" (but not "roots" in the normal sense), just as I italicize the algebraic powers *thing*, *censo*, *cube*, etc.

Gi1	$\alpha t = N$	Gi11	$\alpha K = \beta C+\gamma t$
Gi2	$\alpha C = N$	Gi12	$\alpha CC = N$
Gi3	$\alpha C = \beta \mathrm{t}$	Gi13	$\alpha CC = \beta t$
Gi4	$\alpha C+\beta t = N$	Gi14	$\alpha CC = \beta C$
Gi5	$\alpha C+N = \beta t$	Gi15	$\alpha CC = \beta K$
Gi6	$\alpha C = \beta t+N$	Gi16	$\alpha CC+\gamma C = \beta K$
Gi7	$\alpha K = N$	Gi17	$\alpha CC = \beta K+\gamma C$
Gi8	$\alpha K = \beta t$	Gi18	$\alpha CC+\beta C = N$
Gi9	$\alpha K = \beta C$	Gi19	$\gamma t+\alpha CC = \beta K$?
Gi10	$\alpha K+\gamma t = \beta C$		

All cases but the last are solvable; all agree, also in order, with what we know from Jacopo. Two of Jacopo's cases (Ja11 and Ja16) are skipped, however, and Gi10 is the mirror image of Ja11. It may not be significant that the rule Gi10 does not mention the possibility of a double solution, since it is added, apparently in the same hand, as having been omitted by mistake in the rule for Gi5, and since it is conserved in the rule for Gi16. All in all, however, Giovanni seems to share a source with Jacopo rather than copying from him.

The rule for Gi19 is almost illegible, just leaving enough traces to show that it cannot have been valid and to suggest that it was not one of Gherardi's false rules. A user of the manuscript appears to have discovered that the rule is wrong, and glued a slip of paper over it. The slip has disappeared, but the glue has made the paper as dark as the ink. In any case it appears that the fashion of inventing new false rules was spreading, as was algebra.

Biagio il vecchio

Our last representative of the second generation is Biagio *il vecchio*, "the old", a Florentine abbacus master who died around 1340.[278] Our source for his mathematics and for our scarce knowledge about Biagio himself is Benedetto da Firenze's *Trattato di Praticha d'arismetrica* (henceforth "Benedetto's *Praticha*"). This work, about which much more will be said in the following, was an "abbacus encyclopedia" written in 1463 and containing both Benedetto's own mathematics and extensive systematic extracts from

278 In spite of his fame, there is no reason to discuss Paolo dell'Abbacho, first a student and then apparently a partner of Biagio. Van Egmond [1977: 16] concludes that his fame was first of all due to his "cultivating friendship with prominent figures", and this even in spite of ascribing to him the *Tutta l'arte* on more than dubious grounds (cf. above, note 265). He also points out that Paolo's fame among contemporaries was due to his astrological activity and not to his practising of mathematics.

predecessors.[279]

Book XIV of the *Praticha* is announced as demonstrating "exemplary cases of the rule of algebra according to what master Biagio writes".[280] Benedetto explains that he reports what Biagio writes in his *Trattato di praticha* "not because others have not written rather copiously [about the topic] but because [Biagio] was, according to master Gratia de' Castellani the first who reduced this treatise to a good practice [*una buona praticha*]". There is no mentioning of Jacopo or Gherardi, perhaps because Benedetto does not know about them, perhaps because he restricts the perspective to precursors within his own school tradition, perhaps – and most likely – because only Biagio's extensive treatment of the topic has the depth that deserves the characterization as a *praticha*.[281]

Benedetto promises to follow Biagio's order, and starts by showing some editorial caution. The first problem asks for a "number", but Benedetto doubts Biagio would speak thus, and therefore corrects to a question about a "quantity"; Benedetto seems to have suspected the text he possessed had been tampered with.[282]

References within the problem solutions show that the problem collection comes from

---

279 The *Praticha* survives in three manuscripts [Van Egmond 1980: 356], of which Siena, Biblioteca Comunale L.IV.21 is Benedetto's working manuscript, as can be seen occasionally when computations have been made first and the text written afterwards in whatever space was left – an example is shown below, p. 297. The other two extant manuscripts are incomplete.

There is no full edition of the manuscript, but a number of partial editions have been made on the basis of the Siena manuscript. When referring to the manuscript and not to one of these editions, I shall use the foliation of the Siena manuscript.

280 Ed. [Pieraccini 1983: 1]. Further references in the format "p. *n*(#*m*) refer to page *n*, problem *m* in this edition (the problem numbering is due to Benedetto, and could go back to Biagio).

281 When Fibonacci wrote a *Pratica geometrie* in 1220, the meaning was probably (this would agree with the philosophical epistemology of the time) that geometry comprises two parts: a *theory* (*artificium* and a *practice* (*exercitatio*, practising of the theory) – cf. [Høyrup 2017: 209]. We have no certainty that this was still strictly meant when Benedetto wrote, but it was certainly still present as a connotation, remembered not least because of Fibonacci's work. However that may be, Biagio's algebraic *Praticha* certainly had no theoretical counterpart proper.

282 If anything, the tampering has probably gone the other way. Neither the Lucca *Libro* nor Gherardi – both roughly contemporary with Biagio – uses *quantità* when asking about a pure number. With one exception, the Lucca *Libro* only uses *quantità* when a concrete entity is referred to, as one of several possessions or a quantity of bullion. The exception to the rule (p. 195) is that if a quotient is multiplied by the divisor, then the result is "the quantity that is divided". The only time Gherardi speaks of a *quantità* that is not specified to be a quantity *of something* is when he divides 100 by "some quantity" and then by 5 more (above, p. 199). The use of *quantità* as a synonym for "number" may have originated in the second half of the 14th century (in note 314 we shall encounter it in Antonio de' Mazzinghi).

a larger algebraic work, which must then be Biagio's *Praticha*. It follows after a chapter stating the rules for solving 19 cases and promises (p. 27, #32) another chapter containing biquadratic mixed equations (and probably more). The order of rules in the preceding chapter (as they are referred to in the problem collection) is as follows:[283]

Bi1	$\alpha C = \beta t$	Bi10	$\alpha K+\beta C = \gamma t$
Bi2	$\alpha C = N$	Bi11	$\alpha K+\gamma t = \beta C$
Bi3	$[\alpha t = N]$	Bi12	$[\alpha K = \beta C+\gamma t]$
Bi4	$\alpha C+\beta t = N$	Bi13	$\alpha CC = \beta K$
Bi5	$\alpha C+N = \beta t$	Bi14	$\alpha CC = \beta C$
Bi6	$\alpha C = \beta t+N$	Bi15	$\alpha CC = \beta t$
Bi7	$\alpha K = \beta C$	Bi16	$\alpha CC = N$
Bi8	$\alpha K = \beta t$	Bi17	$\alpha CC+\beta K = \gamma C$
Bi9	$\alpha K = N$	Bi18	$\alpha CC+\gamma C = \beta K$
		Bi19	$\alpha CC = \beta K+\gamma C$

One might suspect the references to rule numbers to be due to Benedetto, but Benedetto's order (as given in chapter XIII of the *Praticha*) is different.[284] There can

283 No appeal is made to the third rule in the problems, and by mistake #76 identifies the rule for $\alpha k = \beta C+\gamma t$ as the 11th rule, which however has been explained in #75 to pertain to the case $\alpha K+\gamma t = \beta C$.

284 Benedetto's own list as contained in the *Praticha* (Book XIII, chapter 3 – fols. 374r–388v) is as follows (*R* stands for *cubo relato*, the fifth power of the thing):

be no doubt that Biagio's own numbering is quoted; there are certainly a few identifiable intrusions of Benedetto's pen, but these have the character of commentaries.

The order of the simple cases is the traditional Arabic order; for higher-order cases we discover a system inspired by Bi1–Bi2, Bi4–Bi6, for each power to the left, first decreasing powers to the right, and afterwards three cases that can be reduced to Bi4–Bi6 by division. Given how this system differs in part from what we find in Jacopo, Gherardi and the Lucca *Libro* it is likely to have been created by Biagio.

The majority of Biagio's 114 problems deal with abstract numbers (often spoken of as "quantities").[285] However, recreational commerce is not absent; we find repeated travels, interest, alloying, and several other mercantile dresses – all of course representing artificial questions that would never arise in real trade. The first problems are very similar

---

Be1	$\alpha C = \beta t$	Be19	$\alpha CC = \beta K+\gamma C$
Be2	$\alpha C = N$	Be20	$\alpha R = \beta CC$
Be3	$\alpha t = N$	Be21	$\alpha R = \beta K$
Be4	$\alpha C+\beta t = N$	Be22	$\alpha R = \beta C$
Be5	$\alpha C+N = \beta t$	Be23	$\alpha R = \beta t$
Be6	$\alpha C = \beta t+N$	Be24	$\alpha R = N$
Be7	$\alpha K = N$	Be25	$\alpha R+\beta CC = \gamma K$
Be8	$\alpha K = \beta C$	Be26	$\alpha R+\gamma K = \beta CC$
Be9	$\alpha K = \beta t$	Be27	$\alpha R = \beta CC+\gamma K$
Be10	$\alpha K+\beta C = \gamma t$	Be28	$\alpha KK = \beta R$
Be11	$\alpha K+\gamma t = \beta C$	Be29	$\alpha KK = \beta CC$
Be12	$\alpha K = \beta C+\gamma t$	Be30	$\alpha KK = \beta K$
Be13	$\alpha CC = \beta K$	Be31	$\alpha KK = \beta C$
Be14	$\alpha CC = \beta C$	Be32	$\alpha KK = \beta t$
Be15	$\alpha CC = \beta t$	Be33	$\alpha KK = \beta N$
Be17	$\alpha CC+\beta K = \gamma C$	Be34	$\alpha KK+\beta R = \gamma CC$
Be16	$\alpha CC = N$	Be35	$\alpha KK+\gamma CC = \beta R$
Be18	$\alpha CC+\gamma C = \beta K$	Be36	$\alpha KK = \beta R+\gamma CC$

We observe the absence of mixed biquadratics from Biagio's list; Biagio promises to deal with them in his next chapter, as also with cases to be solved by false rules. We also take note that the order of the basic six cases is that of al-Khwārizmī – perhaps because chapter 1 of the Biagio's *Praticha* drew on a translation of al-Khwārizmī's algebra (the three Florentine encyclopedias, including Benedetto's *Praticha*, all draw on al-Khwārizmī is Guglielmo de Lunis's translation, cf. below, p. 313, and they all belong within a tradition going back to Biagio).

Pieraccini [1983: iii] mistakenly exchanges Be5 and Be6, thereby producing Fibonacci's order.

[285] In the end of the book (p. 143) Benedetto tells us that "I could still write many more cases, but because I want to give space to others I shall finish this book". Biagio's original thus contained "many more" problems than the 114 copied by Benedetto.

to what we already know from the second generation and quite simple (n stands for "number", q for "quantity"):

p. 1 #1 $(\frac{1}{3}+\frac{1}{4})n = \sqrt{n}$
p. 2 #2 $(q-\frac{1}{3}q-\frac{1}{4}q) = \sqrt{q}$
p. 2 #3 $(q-\frac{1}{3}q-\frac{1}{4}q+) = \sqrt{q}$
P. 3 #4 $(\frac{1}{3}q)\cdot(\frac{1}{4}q) = q$
P. 4 #5 $(\frac{1}{3}q+3)\cdot(\frac{1}{4}q+4) = q$

However, the last of these, when the quantity is posited as a *thing*, leads to an equation having no solutions (for us, to two negative solutions), so Biagio points out that the question is not "reasonably set".

Further on, many illustrations of the higher-degree cases make use of the such-part structure – for instance (p. 68, #69), "find me three quantities so that the first be such part of the second as two of three, and the second be such part of the third as 3 of 4". If we look at the problems in mercantile dress it is sometimes glaringly clear that the dress is not taken seriously – thus in this problem (p. 134, #109):

> Somebody makes a certain number of travels, and as many travels as he makes, so many *denari* he brings. In each travel he earns 40 per 100, and after all the travels he has made in all 6 δ. It is asked with how many δ he set out.

At first it is shown that the number of travels is between 5 and 6, and it is posited that the merchant makes 5+t travels. t is then found to be $\sqrt{7\,{}^{1643489177}/_{4007902864}}-2\,{}^{26839}/_{63308}$, and the amount of *denari* he brought therefore to be $3\,{}^{24733}/_{63308}+\sqrt{7\,{}^{16434891177}/_{4007902864}}$. Discreetly Biagio does not state that this is also the number of travels (cautiously he also has not asked for that). In contrast, as we remember, when Fibonacci finds a non-integer number of travels he adjusts the parameters in order to get an integer solution.

We find a similar paradoxical acceptance of a non-integer result in a problem about a chess-board (p. 122, #101). In an ordinary chess-board with its 64 cases, as Biagio observe, 28 are at the edges and 36 internal. Now Biagio asks for a chess-board where the two numbers are equal. In brief, in an $n\times n$-chess-board, $4n-4$ cases are at the edges, and n^2-4n+4 therefore internal. Equating these, Biagio finds the solution to be $4+\sqrt{8}$ – tacitly omitting the other solution, $4-\sqrt{8}$, and (discreetly once more) not saying that this is the number of cases along each edge.

As we have seen (above, p. 199), Gherardi at one point refers to a cross-multiplication that only makes sense if connected to formal fractions involving algebraic binomials. Such formal fractions turn up in several of Biagio's problem solutions (#64, #92, #94, #97, #98, #99, #102). Mostly they make use of the abbreviation ρ (obviously not meant as the Greek letter *rho* but fairly similar) for *cosa*, "thing", and *mê* for *meno*, "less" (addition is indicated by juxtaposition). In #95 (p. 112) Biagio explains how to perform the addition

$$\frac{360}{1\rho}+\frac{360}{1\rho\ \textit{mê}\ 6} = \frac{1080\rho\ \textit{mê}\ 2160}{2\textit{censi}\ \textit{mê}\ 6\rho}$$

by means of cross-multiplication (similarly in #98, p. 117); next, when he has established that $\frac{1080\rho\ \textit{mê}\ 2160}{2\textit{censi}\ \textit{mê}\ 6\rho}$ equals 39 *fiorini*,

> in order not to have fractions, multiply both sides by two *censi* less 6 *things*, and you get that 1080 *things* and 2160 are equal to 78 *censi* less 234 *things* [...][286]

None of Biagio's problems correspond to any of Gherardi's false rules. One, however, presents us with another, related innovation: the introduction of rules that only function under specific (non-specified) circumstances. #31 (p. 25) asks this question:

> Find a number which, when multiplied by itself and over this is joined its root, makes 18.

To this Benedetto observes that

> in this our master gets lost, since he wants to compose a rule which does not apply to other similar questions. And the rule and way that he indicates is this. He says, you will posit that this number be a *censo;* multiplied in itself it becomes 1 *censo of censo;* put unto it the root of the said number, which is a *thing*, they make 1 *censo of censo* and a *thing*. Which you first bring to a *censo of censo*,[287] and you get the same, an then halve the *thing*, the half being $^1/_2$ *thing*, which multiplied in itself make $^1/_4$, and again multiplied in itself make $^1/_{16}$, which, added to 18, make $18\,^1/_{16}$, whose root is $4\,^1/_4$, from which quantity, when the square on the half is cut away, that is $^1/_4$, remain 4, and so much is the *censo* worth. And you posited that [the number] was a *censo*, thus it was 4, and as you see the rule is good for this case, but proposing it for other numbers does not serve, and therefore we call it the pronic root [*radice pronicha*]. And I think he did not look for other ways, or perhaps, since he did not intend so, it was written into his works.

It is not clear from this whether the pronic root of 18 is 4 or 2; elseswhere in his *Praticha* (fol. 361^{v}) Benedetto states that it is 2. Other sources do not all agree. Pacioli [1494: I, 115^{v}] has this:

> By *radice pronica* one normally understands a number multiplied in itself, and above it set the root of the said number, of this sum that number is called *radice pronica*. As 9 multiplied in itself makes 81, and above it set the root of 9, which is 3, it makes 84. The *radice pronica* is called 9 by practitioners,

---

[286] That this incipient use of symbolism is not added by Benedetto can be seen from an explanation given in Book XIII (fol. 374^{r}); Benedetto himself would also have used ρ for the *thing* but further have written *c* for *censo*. Cf. also note 316 below.

[287] This is evidently a reference to the normalization contained in a rule – a rule we may presume to have been contained in Biagio's next chapter.

according to which the pronic root of 18 should be 4.[288] The manuscript Florence, Bibl. Naz., Palatino 575 [ed. Simi 1992: 20*f*] agrees. On the other hand, Gilio da Siena [ed. Franci 1983: 18*f*], writing in 1384, as well as Pierpaolo Muscharello in his *Algorismus* from 1478 [ed. Chiarini et al 1972: 163], state that the pronic root of 84 equals 3. In any case, it is clear that the pronic root serves to solve equations $CC+t = N$. It also seems to follow from Benedetto's words ("we call it") that Biagio did not use the term, and in any case a "rule" would not serve if tabulated pronic roots were at hand. In the actual case, the rule only works because $\frac{1}{2}C = t$, that is, because $t = 2$.

Below, we shall encounter other special roots meant to solve irreducible equations.

Summing up, we may say that the second generation was strongly marked by an Italo-Provençal group, but that is also presents us with the establishment of a more homogeneous, properly Italian tradition than what we find in the first generation. The most conspicuous innovation of the second generation, however, it the introduction of *algebra*.

Abbacus algebra, already from this beginnings, differed in characteristic ways from the algebra that was known in Latin – the translations of al-Khwārizmī, Fibonacci, and (scarsely diffused) the *Liber mahameleth* and the translation of Abū Kāmil's algebra.

Firstly, there are no geometric proofs, and (perhaps connected to that) the first power of the unknown (really *the* unknown) is a *thing* (*cosa*), no *root* (*radice*); secondly, all cases are defined in non-normalized form, entailing that the first step of rules is a normalization; thirdly, there is *systematic* exploration of the possibilities to solve problems of higher than the second degree, as opposed to the solution of single cases we have encountered in the *avere* group of *Liber abbaci*, part 15.3 (some abbacus writers trying to show off by postulating false rules for higher-degree cases which only a modicum of algebraic understanding could unmask). Fourthly, algebra is applied to a range of sometimes complicated recreational business problems, reflecting that abbacus algebra was practised within and grew out of an environment fundamentally engaged in the teaching of commercial arithmetic. Finally, there is scattered evidence of incipient formal calculations and use of letter abbreviations within these, enough however to show that formal calculations existed and were made use of – which implies that these abbreviations served as *symbols* serving the mathematical argument directly and not through virtual expansion into a rhetorical argument.

[288] Piero della Francesca's abbacus collection [ed. Arrighi 1970b: 91*f*] agrees in different words.

Further into the 14th century

By 1340, abbacus schools were well established in many places, and the fundamental curriculum as we have seen it described above (p. 4) and as we have seen implemented in the revised version of Jacopo's *Tractatus* (above, chapter II) did not change perceptibly neither over the rest of the century nor before the advent of printing. There is no reason to go into details. The innovations that took place concerned the supra-utilitarian level, first of all the algebra.

Dardi da Pisa and the *Aliabraa argibra*

Biagio may have written the first extensive treatise (a *praticha*) on algebra, but we know it from Benedetto's extract only. The first *extant* treatise dedicated solely to algebra is the *Aliabraa argibra*, written by a certain Dardi da Pisa in 1344 – perhaps identical with the abbacus master Dardi Ziio (or Dardi de Zio) who is known to have taught in Venice in 1346 [Ulivi 2002b: 131]: beyond the date and the not very common name, a Venetian origin would also fit internal evidence.[289]

Dardi's treatise is extensive and falls in three parts, preceded by a preface opening with an echo of the "four causes" dear to contemporary university philosophy, explaining that the present book like others was made by four *rispetti*.[290] First comes the title, which we may assume is thought of as the formal cause – here *Aliaabra*, explained to be Arabic and to mean "explanation of subtle question"; second the author's intention (probably seen as efficient cause), namely to solve some questions by means of numbers and others by means of roots, namely those that have no discrete solution; third the matter which it deals with (the material cause), namely the "names" (in later terminology

---

[289] I have used the following versions:

– Vatican, Chigi M.VIII.170 (*ca* 1395, cf. [Van Egmond 1980: 211]); henceforth $\mathbf{D_1}$ (when referring, I use the recent foliation);
– Raffaella Franci's edition [2001] of Siena, Biblioteca Comunale I.VII.17 (*ca* 1470, cf. [Van Egmond 1981: 188]); henceforth $\mathbf{D_2}$;
– Warren Van Egmond's personal transcription of the Arizona manuscript, written in Mantua in 1429, for which I thank him heartily; henceforth $\mathbf{D_3}$.

A fourth manuscript is Florence, Biblioteca Mediceo-Laurenziana, Ash 1199, from c. 1495. I have only seen the extract in [Libri 1838: II, 349–356], according to which it is quite close to $\mathbf{D_2}$.

The Vatican manuscript is generally but not in all respects the best, cf. [Høyrup 2007: 170 n. 331, n. 332]. It follows Venetian orthography. Moreover, the abbreviation *ç* for *censo*, used in all manuscripts, corresponds to the northern writing *çenso* (in the 15th century changing into *zenso*); it therefore seems fairly certain that Dardi wrote in Venice or at least in north-eastern Italy (just like Fibonacci, in Pisa he would not have be identified as "from Pisa").

[290] The preface is in $\mathbf{D_2}$ only [ed. Franci 2001: 37*f*]. It is lost in $\mathbf{D_1}$ and replaced by a different one in $\mathbf{D_3}$ which speaks about the copying of that manuscript itself.

"powers") *things*, *censi*, *cubi*, *censi of censo* and *cubi of cubi*. Fourth (the final cause) the utility of the book, namely that the one who understands it well will be a good arithmetician and geometer – matters which when dealt with as theory (*spechulativa-mente*) belong to natural philosophy.

Even though abbacus masters did not belong to the university environment,[291] the intellectual separation was not absolute – and from the last observation it is clear that Dardi intends his present book to develop theory and thus to be counted among philosophers.

The preface goes on with an explanation of the meaning of the terminology for powers: the *thing* is a linear length and the root of the *censo*; the *censo*

> is a surface width and the square [*quadrato*] of the *thing*, and called *censo* from *cerno cernis*,[292] which stands for "to choose" because the *censo* chooses the mean proportional between the *thing* and the *cube*. The *cube* is a corporeal thickness, whose body includes the length of the *thing* and the surface of the *censo*, and is called *cube* according to the arithmetic of Boethius from this name *cubus cubi* which says as much as aggregation of numbers.

The final statement is inspired by *De institutione arithmetica* II.39 [ed. Friedlein 1867: 136; trans. Masi 1983: 163], according to which cubes are sums of subsequent odd numbers: $1^3 = 1$, $2^3 = 3+5$, $3^3 = 7+9+11$, etc. It has evidently nothing to do with "cubes of cubes", nor is it meant by Boethius as an explanation of the name; Dardi seems to write from approximate memory of what he has learned in Latin school (not that his etymologies are more fanciful than so many others from the epoch).[293]

The first of the three parts is introduced as a

> treatise about the rules that pertain to the multiplication, the divisions, the joinings and the subtractions of roots. And further to know to find the roots of square and cube numbers and other subtle rules that give to understand calculations magisterially.

In principle it thus corresponds to fols 28^v^ to 29^r^ of the extract from Giovanni di Davizzo and to the missing introduction to Jacopo's algebra. It is much longer, however (12 to

---

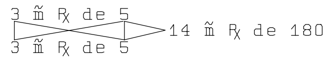
[291] Some of them taught astrology with the necessary astronomical underpinning at the medical faculties, where astrology served prognostication. One who did so was Giovanni di Bartolo, on whom below, p. 260 – see [Ulivi 2002a: 39 n. 141].

[292] "I distinguish, you distinguish" – a trace of how Latin verbs were taught in school. Apart from the other obvious flaws of the explanation, Dardi's proposed translation from the Latin is at best approximative.

[293] It is not to be excluded that Dardi builds on an intermediate source misquoting Boethius. Jacopo, with many others, certainly treats Boethius worse than Dardi, cf. above, note 8.

15 folio sheets in the various manuscripts). A full translation into modern symbolism of the contents can be found in [Franci 2001: 8–10]; here I shall point to some significant features, keeping closer to the text.

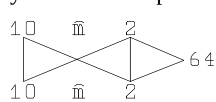
When teaching the multiplication of binomials, Dardi makes use of diagrams. For example (**D$_1$** fol. 6^r), for (3–√5)·(3–√5)



(℞ abbreviates *radice*, "root").

Instead of just stating the sign rules as Giovanni di Davizzo does (above, p. 206), Dardi uses a similar diagram (*m̃*, strictly $\widehat{m}$, stands for *meno*, "less") to support an argument for the most difficult of them (**D$_1$** fol. 4^v):

> Now I want to demonstrate by number how less times less makes plus, so that every times you have in a construction to multiply less times less you see with certainty that it makes plus, of which I shall give you an obvious example. 8 times 8 makes 64, and this 8 is 2 less than 10, and to multiply by the other 8, which is still 2 less than 10, it should similarly make 64. This is the proof. Multiply 10 by 10, it makes 100, and 10 times 2 less makes 20 less, and the other 10 times 2 less makes 40 less, which 40 less detract from 100, and there remains 60. Now it is left for the completion of the multiplication to multiply 2 less times 2 less, it amounts to 4 plus, which 4 plus join above 60, it amounts to 64. And if 2 less times two less had been 4 less, this 4 less should have been detracted from 60, and 56 would remain, and thus it would appear that 10 less 2 times 10 less two had been 56, which is not true. And so also if 2 less times 2 less had been nothing, then the multiplication of 10 less 2 times 10 less 2 would come to be 60, which is still false. Hence less times less by necessity comes to be plus.



As we see, we are well beyond the limit of established algebraic thought: instead of just arguing that (–2)×(–2) *must* be 64–60 since that is what is missing, Dardi has to make a double indirect proof[294] in order to rule out the possibilities (–2)×(–2) = 4 and (–2)×(–2) = 0.

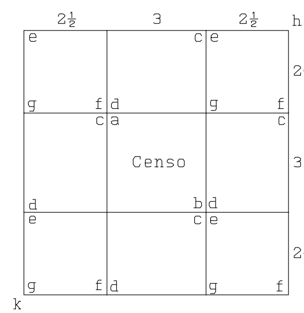
When explaining (**D$_1$** fol. 11^v) how to perform the division of 8 by 3+√4, Dardi makes use of the rule of three: Since (3+√4)·(3–√4) = 5, **5**/(3+√4) = **3–√4**; **8**/(3+√4) must

[294] Often, indirect proofs are believed to be much more difficult to grasp than direct proofs, and only introduced when mathematicians interacted with philosophers – most famously in [Szabó 1969: 341–346]. Dardi puts this assumption to rest (if need be, Aesopus's *salta hic* should suffice).

therefore be (8·[3–√4])/5. That √4 appears in the example is no accident. Dardi makes repeated use of rational roots "as if they were irrationals"; this should obviously make it possible to control the outcome, but Dardi mostly leaves it to the reader to do so.[295] Giovanni di Davizzo does the same, we remember (see above, p. 208), but never explains, nor does he take advantage. We may assume that both took over the idea from earlier writings but only Dardi understood.

Giovanni di Davizzo shows in two examples how to simplify the sum of two roots or their difference (say, $\sqrt{a}\pm\sqrt{b}$) – above, p. 207), but only as an unexplained numerical prescription, and does not specify that the method leads to a simplification if and only if *ab* is a square. Dardi (**D**$_1$, fol. 9^{r,v}) first explains this condition and then shows in the examples that the method builds on the squaring of $\sqrt{a}\pm\sqrt{b}$. Once again, Dardi certainly understands what he is doing, while Giovanni di Davizzo may simply have copied.

The second part of the *Aliabraa argibra* presents the six basic cases.[296] The order is the usual abbacus order, as we known it from Jacopo, both algebra presentations in the Lucca *Libro*, and from Giovanni di Davizzo; quite outside the beaten path, however, is the use of geometric demonstrations. Their principle is the same as we know from al-Khwārizmī's demonstrations, but the way of lettering is different. We may compare the demonstration for the fourth cases with al-Khwārizmī's corresponding demonstration (above, p. 143). Already al-Khwārizmī had deviated from the Greek canon by using letters to designate areas; Dardi deviates from this as well as from the canon of geometers (ancient Greek or of his own times) by designating identical areas by the same letters.



[295] At an early point (**D**$_1$ fol. 3^v), Dardi explains the multiplication of root by root on the example $\sqrt{4}\sqrt{\cdot 9} = \sqrt{(4\cdot 9)} = \sqrt{36}$, and as "prova manifesta" he explains that $\sqrt{4} = 2$, $\sqrt{9} = 3$, and $2\cdot 3 = 6 = \sqrt{36}$. Dardi may have expected his reader to have understood the principle and that repetition was superfluous.

[296] In **D**$_1$, a sheet has been lost, and it therefore starts in the very end of the third rule.

For equation, instead of Jacopo's *raoguaglamento* (also used by others) Dardi uses the term *adequation* (**D**$_1$, e.g., fol. 15^v) or *adequatione* (**D**$_3$, e.g., fol. 24^v). Seduced by the normal appearance of the word in the composite *l'adequation* (written without the apostrophe, not yet invented) and interpreting this as *la dequation*, the scribe of **D**$_2$ at least in the beginning believes the term to be *dequatione* (thus *nelle dequationi* [ed. Franci 2001: 64]); but *la ditta adequatione*, *overo de adequatione* and *la adequatione* [ed. Franci 2001: 73, 77, 88] show that the original term was *equation(e)*. Franci takes over the mistake except in a few cases where a preceding vowel differing from *a* prevents it. It may be noticed that *adequazione* (derived from *adeguare*) was in attested Tuscan use around 1350 [*Crusca*, p. 21], while no cognate of *dequation* can be found.

Once more it seems that Dardi works according to approximate memory of what was done in writings he had seen – this time producing a pedagogically efficient tool.

In this part, Dardi starts using the abbreviations *c* for *cosa*, “thing”, and *ç* for *censo* (referring to the Venetian spelling *çenso*, cf. note 289 – for relative ease of reading I shall use *Ç*); *m̃* is still used for *meno*. Most striking is a quasi-fractional notation for monomials, similar to the use of quasi-fractions for denominated numbers in the *CA* (above, p. 171): thus (**D₁** fols 15ʳ, 22ʳ, 46ᵛ), $\frac{1}{Ç}$ stands for 1 *censo*, $\frac{30}{c}$ for 30 *things*, $\frac{21}{n}$ for 21 “in numbers”, $\frac{36}{cu}$ for 36 *cubes*, $\frac{5}{c}\frac{1}{3}$ (emulating an ascending continued fraction) for $5\,{}^1\!/_3$ *things*. Nothing but a compressed linguistic expression and no operatory symbolism however rudimentary is intended. That can be seens from the way 1 *censo of censo* is expressed ($\mathbf{D_1}$ fol. 47ʳ) – namely as $\frac{1}{Ç}$ *de Ç*.

This algebraic notation is not Dardi’s invention – there is a single unexplained instance ($\frac{10}{cose}$) in *Tutta l’arte* ($\mathbf{T_f}$, fol. 159ʳ), showing that already the compiler of this work knew it in 1334, and that even he was not its inventor.

Using a notation that looks like a fraction but where the “denominator” is a *denomination*, a factor rather than a divisor, would obviously give rise to ambiguities *if* it were combined with the use of formal fractions involving algebraic polynomials. But it never is, by Dardi nor in any other Italian work I have inspected.[297]

On another account, Dardi is our first source for another strain in the development of algebraic symbolism, related to the algebraic *parenthesis*. Here, a possible misunderstanding should be cleared away. A parenthesis is not a bracket but an expression *enclosed*, for example, in a pair of brackets; in written language it can also be delimited between two dashes, and in spoken language by pauses. An *algebraic parenthesis* is a composite expression that is dealt with as a single entity – so, in $(a+b)^2$, $(a+b)$ *as a whole* is submitted to the squaring operation. Post-Cartesian algebra and analysis could not exist without the algebraic parenthesis. It is so pervasive that we tend to forget its crucial role.

We have already encountered one kind of algebraic parenthesis above, namely in the formal fractions. In Biagio’s $\frac{1080\rho\ me\ 2160}{2\,censi\ me\ 6\rho}$, the fraction line takes care that the numerator (1080ρ *me* 2160) as well as the denominator (2 *censi me* 6ρ) are algebraic parentheses. In modern notation, the extended root sign also delimits a parenthesis. The abbreviation ℞ cannot do that, and Dardi uses (invents?) a way to indicate that a root is to be taken of a composite expression – for example ($\mathbf{D_1}$ fol. 8ᵛ), he expresses

$$\sqrt{\frac{1}{4}+\sqrt{12}}$$

297 The “German algebra” (on which below, p. 363) does combine the two, but only because this late-15th-century compilaton is an eclectic combination of material drawn from a variety of sources. The very last problem [ed. Vogel 1981: 43] makes use of a formal fraction; but precisely this bit of text uses a different notation for the *cossa*, namely a superscript *c* known from other Italian writings.

as "℞ *de zonto* $^1/_4$ *con* ℞ *de 12"*, "root of, joined, $^1/_4$ with root of 12". Later abbacus writers instead speak of a *radice generale, radice universale* or *radice legata* ("universal" or "bound root"), or use Ⓡ, an encircled ℞. The notation is somewhat ambiguous, it is not always clear how far the composite expression goes; but normally it is restricted to two members, and it fulfilled its purpose within the ambit of abbacus algebra.

Part 3 of the *Aliabraa argibra* presents rules and examples for "194 regular and 4 irregular" cases – thus announced in the preface [ed. Franci 2001: 38]. The "regular" cases are those which can be solved by root extraction or by being reduced to one of the six fundamental cases; the irregular cases are solved correctly but only for particular parameters.

The reason Dardi reaches 194 regular cases is that he makes extensive use of radicals (square as well as cube roots, sometimes both within the same equation). Since the whole sequence is shared by no other writer, there is no need to recapitulate all of it.[298] After 16 cases where no radicals appear come these:

Da17	$N = \sqrt{(\alpha t)}$	Da22	$N = \sqrt{(\alpha K)}$
Da18	$\alpha t = \sqrt{N}$	Da23	$\alpha CC = \sqrt{N}$
Da19	$\alpha C = \sqrt{N}$	Da24	$N = \sqrt{(\alpha CC)}$
Da20	$N = \sqrt{(çC)}$	Da25	$\alpha t = \sqrt{(\beta t)}$
Da21	$\alpha K = \sqrt{N}$	Da26	$\alpha C = \sqrt{(\beta t)}$

We observe that Da21 coincides with Gherardi's Gh8; this tells us that that Dardi does not start completely from scratch. So far, everything looks quite simple from our perspective, in particular if we replace t, C and K by powers of x. However, as said in connection with Gherardi, "we should remember that only positive integers and fractions were accepted as numbers, and think of the difficulty which the appearance of an irrational coefficient had caused Fibonacci".

Later on things become more intricate, and more difficult to express. For instance, the rule for Da41, $N = \alpha K+\sqrt{(\beta K)}$ runs

> you shall divide the number [N] by the quantity of *cubes* [α], and serve what results separately. And then multiply the quantity of *cubes* that are not roots [α] in itself, and divide the quantity of *cubes* [β] that are said to be roots by this multiplication. And the fourth of that which results for you, join it above the division that you served, and the root of this sum, that is, the cube root, less the root of this fourth that you joined, that is, the division that results for the *cubes* called roots in the multiplication of *cubes* that

298 Full lists in modern symbolism can be found in [Van Egmond 1983: 402–417] as well as [Franci 2001: 26–33]; the former list also indicates the rule given by Dardi for each case in modern symbolic language.

> are not roots. And so much is worth the root of the *cube*, and this root multiplied in itself, so much is worth the *cube*, and the cube root of this multiplications comes to be worth the *thing*

– corresponding to the formula

$$t = \sqrt[3]{\left(\sqrt{\frac{1}{4}\frac{\beta}{\alpha^2}+\frac{N}{\alpha}}-\sqrt{\frac{1}{4}\frac{\beta}{a^2}}\right)^2}\ .$$

The formula makes heavy use of parentheses. As we see, Dardi instead calculates the single constituents separately; in the subsequent example, he does the same. We also observe that Dardi has an explicit concept of *coefficients*, "quantity of *cubes*" instead of the habitual "the *cubes*" ($\mathbf{D_2}$ mostly returns to the customary way).

In many cases, Dardi explicitly reduces a question to a another one which he has dealt with before – for example in Da82, $\alpha t+\beta\sqrt{C} = \gamma C$,

> You shall detract the *things* on each side [*parte*], and on one side will remain for you *censi* less *things*, and on the other root of *censo*, and then multiply that which remains, each part in itself, and you will have on one side *censi* and on the other *censi* and *censi of censo* less *cubes*. Then detract the smaller quantity of *censi* on both sides, and give the *cubes* that are missing on one side to both sides, and you will have that the equation will come to be that of the 70th or the 71st chapter,[299] and then proceed according to the way of the chapter already dealt with.

All 194 rules are correct, with two exceptions: for Da177, $\sqrt{(\alpha t)} = \sqrt[3]{(\beta CC)}$, the rule stated is $t = \sqrt[3]{(\beta^2/\alpha^3)}$ instead of $t = \sqrt[5]{(\beta^2/\alpha^3)}$, for Da179, $\sqrt{(\alpha K)} = \sqrt[3]{(\beta t)}$, the rule gives $t = \sqrt[4]{(\beta^2/\alpha^3)}$ instead of $t = \sqrt[7]{(\beta^2/\alpha^3)}$. As argued by Van Egmond [1983: 417], the likely reason is that no terminology was as yet available for the fifth and the seventh root.

All regular cases are provided with examples (as already Jacopo, Dardi offers three examples for the fifth rule). All are stated in terms of pure numbers – almost half ask for two or three numbers in given ratio (in "such part" formulation),[300] more than a fourth for a number which fulfils the conditions of the equation, around 15% are divided-ten problems.

Between Da182 and Da183[301] the four "irregular cases" with appurtenant rules

---

[299] Respectively (Da70) $\alpha CC+\gamma C = \beta K$ and (Da71) $\alpha CC = \beta K+\gamma C$.

[300] The "proportion" notion only appears when numbers in continued proportion are spoken of correctly ($\mathbf{D_1}$ fol. 44^r,) or misunderstood (fol. 23^r).

[301] In $\mathbf{D_3}$, they have been moved to the very end, and the order of the first two rules has been inverted.

are inserted,

D-i1	$\gamma t+\beta C+\alpha K = N,$	$t = \sqrt[3]{\left(\frac{\gamma/\alpha}{\beta/\alpha}\right)^3+\frac{N}{\alpha}}-\frac{\gamma/\alpha}{\beta/\alpha}$
D-i2	$\delta t+\gamma C+\beta K+\alpha CC = N$	$t = \sqrt[4]{\left(\frac{\delta/\alpha}{\beta/\alpha}\right)^2+\frac{N}{\alpha}}-\sqrt{\frac{\delta/\alpha}{\beta/\alpha}}$
D-i3	$\delta t+\gamma C+\alpha CC = N+\beta K$	$t = \sqrt[4]{\left(\frac{\gamma}{4\alpha}\right)^2+\frac{N}{\alpha}}+\frac{\beta}{4\alpha}-\sqrt{\frac{\delta/\alpha}{2\beta/\alpha}}$
D-i4	$\delta t+\alpha CC = N+\gamma C+\beta K$	$t = \sqrt[4]{\left(\frac{\gamma}{4\alpha}\right)^2+\frac{N}{\alpha}}+\frac{\beta}{4\alpha}-\sqrt{\frac{\delta/\alpha}{2\beta/\alpha}}$

These rules are said (**D$_1$** fol. 100^r) to be true only for the cases for which they have been arranged (*ordinati*) but included because “by some accident the said rules may turn up in certain problems”.

The first two examples deal with composite interest, the other two build on a divided ten. A look at the first will reveal how the rule is produced (**D$_1$**, fol. 100^r):

> Somebody lends to someone else £ 100, and in the end of three years he receives £ 150 in earning and capital, interest being made up at the end of year. I ask at what rate it was lent a month.

We have encountered a similar problem above (p. 190), namely Jacopo’s first illustration of Ja4, where the money was lent for two years. There as here, a simple way to solve the problem would be to take the value to which 1 £ (or 100 £) has grown after a year as the unknown, then the problem would be reduced to the extraction of a square or cube root; that is indeed the way Biagio (pp. 69, 84) solves problems about a capital growing over three respectively four years. Jacopo, as we remember from p. 190, instead produced a mixed second-degree problem by taking the monthly *interest* of 1 £ in δ as his unknown. Exactly the same trick works here. We may take the interest rate to be *t* ß per £ per month (ant thus $^{12}/_{20}t$ £ per year and £); then we have that $100\cdot(1+{}^{12}/_{20}t)^3 = 150$, whence

$$t = \sqrt[3]{\left(\frac{20}{12}\right)^3\cdot\frac{150}{100}}-\frac{20}{12}\ .$$

If instead we develop the equation we get

$$t^3+3\cdot\frac{20}{12}t^2+3\cdot\left(\frac{20}{12}\right)^2 t = \left(\frac{20}{12}\right)^3\cdot\frac{150}{100}-\left(\frac{20}{12}\right)^3\ .$$

We observe that $\frac{20}{12}$ can be found as the quotient between the coefficients of t and t^2, and that $(\frac{20}{12})^3\cdot\frac{150}{100}$ arises as the sum of the number term and $\left(\frac{20}{12}\right)^3$; and further, that t results if from this sum we extract the cube root and afterwards subtract $\frac{20}{12}$ – and that is exactly Dardi’s rule.

The second rule can be derived in a similar way. The examples for D-i3 and D-i4 both have the structure

$$10 = a+b \ , \quad \frac{ab}{a-b} = \sqrt{N} \ ,$$

N being respectively 18 and 28. If we posit $a = 5+t$, $b = 5–t$,[302] we get in both cases biquadratic problems (if we accept square roots as number terms, as done in Da18, quadratic equations). Positing instead $t = a$ or $t = b$ we get the equations of D-i3 and D-i4, and comparison once again allows us to derive the rule valid in these particular cases.

Whoever devised these rules was a very good algebraist, at the level of Biagio and certainly far above that of Gherardi. We can be confident that it was *not* Dardi. Firstly, there is no reason dardi should depart from his constant use of pure-number problems just for the first two irregular cases and nowhere else; secondly, the way he distances himself when introducing them ($\mathbf{D_1}$ fol. 100), "hereafter will be written certain chapters [...] though by some accident the said rules may turn up in certain problems" (when speaking about what he does himself, for instance in the preface, Dardi is not afraid of speaking in the first person singular).

Since (as we have seen) Biagio possessed the tools to derive the irregular rules, they *may* have come from his hand and have been among those that were not transcribed by Benedetto. As we have observed (p. 214), Biagio did not shy away from devising rules for higher-degree questions that only apply to specific cases. But we have no firm foundation for the belief that Biagio was the sole abbacus writer of his kind and level. In any case, it is not certain that Biagio or whoever it was knew that the rules derived by means of the ingenious method we have seen had no general validity; that may well have been Dardi's discovery.

Further evidence that the irregular cases and their rules did not originate in the *Aliabraa argibra* comes from their afterlife. A number of manuscripts contain sometimes only the two composite-interest problems, sometimes all four – sometimes with, sometimes without the examples.[303] A few also contain an extra case, $\gamma C+\beta K+\alpha CC = \sqrt{N}$, with

---

302 This is done by Biagio (pp. 39, 47, 49, 51) in five other divided-ten problems.

303 I know of the following instances;

– Florence, Bibl. Naz., Fond. princ. II.III.198 (see [Franci 2002: 96*f*]);
– Parma, Bibl. Palatina, Ms. Pal. 312 (the *Libro de conti e mercatanzie*, [ed. Gregori & Grugnetti 1998: 24*f*]);
– Palermo, Biblioteca Comunale, Ms. 2Qq E13; contains also the third irregular case (see [Franci 2002: 97*f*]);
– Vatican, Vat. lat. 4825 (Tomaso de Jachomo Lione), fol. 80^r–81^r; contains also the third case;
– probably also in Florence, Ricc. 2252, which according to [Franci 2002: 98] should be "quite similar" to the Palermo manuscript;
– Florence, Bibl. Naz., Palatino 567 (Raffaello Canacci, *Ragionamenti d'algebra* [ed. Procissi

the example (certainly as basis for the rule for solving the case) that 50 £ grow in 2 years to (50+$\sqrt{484}$) £.[304] None of these sources ever mentions the restricted validity of the rules, as one might have expected at least some of them to do if they had borrowed from Dardi.

Beyond reporting the first of Dardi's irregular cases, Vatican, Vat. lat. 10488, fol. 94v, also give fully valid rules for $\alpha t = \sqrt{N}$, $N = \alpha\sqrt{t}$, $\alpha C = \sqrt{N}$ and $N = \sqrt{C}$, while Florence, Bibl. Naz., Palatino 575 [ed. Simi 1992: 52–55] has rules for $\alpha C = M+\sqrt{N}$, $\alpha\sqrt{CC} = N$, $\alpha\sqrt{KK} = N$, $\alpha\sqrt{CC+M} = N$ and $\alpha\sqrt{CC-M} = N$.[305] Together, the only partial overlap with Dardi's material, the simple character of these cases and the closeness in style to Gh8, $\alpha K = \sqrt{N}$, suggest however that these rules are not borrowed from Dardi but instead (like Gh8) representatives of the type which had inspired Dardi for his thorough exploration of its possibilities.

So, however brilliant it is, and even though a number of copies of the treatise were made, Dardi's exploration was a dead end.

Alcibra amuchabile

Another compilation (rather than treatise) dedicated solely to algebra is a *Trattato dell'alcibra amuchabile* from c. 1365 (henceforth *Alcibra amuchabile*).[306]

In the likeness of the *Aliabraa argibra* (and many other algebras from al-Khwārizmī onward), the *Alcibra amuchabile* consists of three parts. The first of these teaches multiplication and division of roots or expressions containing roots. For the product of binomial by binomial, a diagram is introduced to illustrate the procedure – for instance (p. 18), for $(5+\sqrt{20})\cdot(5-\sqrt{20})$:

1954: 441]); as Tomaso di Jachomo di Lione, yet without examples;

– Vatican, Vat. lat. 10488, fol. 93v, brings Dardi's first irregular Rule without example;
– Florence, Bibl. Naz., Palatino 575 [ed. Simi 1992: 53], rules alone;
– finally, the first three rules with the usual examples are copied in Bento Fernandes's *Tratado da Arte de Arismética* from 1555 [da Silva 2008].

304 The example contains writing or copying errors as well as erroneous calculations, but the underlying idea is as good or as bad as in Dardi's irregular cases. Since we do not possess Biagio's treatment of the arithmetic of roots and arithmetical polynomials we do not know whether the appearance of $\sqrt{484}$ (= 22) is compatible with ascription of the irregular rules to Biagio.

Canacci has the rule without the example. Bento Fernandes copies both.

305 Rules only, no examples. The former group coincides with Da18–20, the first two in the latter group with Da38 and Da24, while the last three have no counterparts in the *Aliabraa argibra*.

306 Florence, Biblioteca Riccardiana ms. 2263, ed. [Simi 1994]. Page references will be to Annalisa Simi's edition. Dating according to watermarks. Written in two or three different hands [Van Egmond 1980: 151].

```
5 e  piu  ℞ di 20
via
5 e  meno ℞ di 20
```
.

Division of 100 by 10+√20 is accompanied by a similar diagram

```
10  piu  ℞ di 20
10  meno ℞ di 20
```
,

which serves to illustrate that both dividend and divisor are to be multiplied by 1–√20.

These look like reduced versions of Dardi's schemes, but since there is no hint that the compiler knew the *Aliabraa argibra*, either Dardi made a more elaborate version of what he knew from a shared background, or the present compiler reduced what was around; the existence of a shared source or source tradition is in any case beyond doubt.

The second part lists 24 algebraic cases with rules and examples:

AA1*	$\alpha t = N$	Ja1	
AA2*	$\alpha t = N$	Ja2	
AA3*	$\alpha C = \beta t$	Ja3	
AA4*	$\alpha C+\beta t = N$	Ja4	
AA5*	$\beta t = \alpha C+N$	Ja5	
AA6*	$\alpha C = \beta t+N$	Ja6	
AA7¤	$\alpha K = N$	Ja7;Gh7	Example differs
AA8¤	$\alpha K = \beta t$	Ja8;Gh9	Same example
AA9¤	$\alpha K = \beta C$	Ja9;Gh10	Same, with error
AA10¤	$\alpha K = \beta C+\gamma t$	Ja12;Gh11	Same example
AA11	$\alpha K = \sqrt{N}$	Gh8	Same example
AA12	$\alpha K = \beta t+N$	Gh12	Same example
AA13	$\alpha K = \beta C+N$	Gh13	Same example
AA14°	$\alpha K+\gamma t = \beta C$	(Ja11)	Ja11 inverted, with
AA15*	$\alpha K+\beta C = \gamma t$	Ja10;Gh15	Example in Gh15
AA16*	$\beta C = \alpha K+\gamma t$	Ja11	
AA17*	$\alpha CC = N$	Ja13	
AA18*	$\alpha CC = \beta t$	Ja14	
AA19*	$\alpha CC = \beta C$	Ja15	
AA20*	$\alpha CC = \beta K$	Ja16	
AA21*	$1\alpha CC+\beta K = \gamma C$	Ja17	
AA22*	$\beta K = \alpha CC+\gamma C$	Ja18	
AA23*	$\alpha CC = \beta K+\gamma C$	Ja19	
AA24	$\alpha CC+\beta C = N$	Ja20	

An asterisk * means that the rule and example (if such exists) are the same and use the same words as Jacopo's algebra. A pillow ¤ indicates that the rule coincides with Jacopo's corresponding rule, but that an example has been added – the last column states whether the example is shared with Gherardi or not. A degree symbol (°) indicates that the rule is worded differently than Jacopo's corresponding case.

As we see, the *Alcibra amuchabile* copies all of Jacopo's cases very faithfully. In four of the regular cases involving cubes an example is provided (as we remember, Jacopo supplied no examples after the first six cases). On one point, however, a telling correction is made. In the second example for Ja4 [ed. Høyrup 2007: 312*f*], Jacopo at one point has to compute $(\sqrt{54}-2)^2$, which should give him 58+ $4\sqrt{54}$, which according to prevailing norms should be expressed as $58+\sqrt{(16\cdot 54)} = 548+\sqrt{864}$. As pointed out above, note 264, Jacopo does not perform the computation $16\cdot 54$ but leaves the space open (five times in total); at least two consequent copyists reproduce this faithfully, the last of them (if not both) writing in the margin "così stava nel'originale spatii", "thus it was in the original, spaces".

This shows beyong all doubt that "Jacopo's algebra" (that is, the algebra contained in manuscript **V** of Jacopo's *Tractatus*) was older than *ca* 1365; in view of its almost certain influence on the two algebra sections of the Lucca *Libro*, we may say with reasonable confidence that if not already present in Jacopo's original from 1307, it cannot be much younger.

It *could* of course be older, in which case Jacopo would have copied it so faithfully from some source that he did not even take the trouble to perform the multiplication $16\cdot 54$ while none the less making it stylistically homogeneous with the rest of his treatise – see [Høyrup 2007: 23–25]. This seems quite implausible; so is a the stylistically harmonizing insertion of the algebra chapter in the *Tractatus* between 1307 and 1330.

As shown by the scheme, the *Alcibra amuchabile* also deals with a number of cases with a counterpart in Gherardi but none in Jacopo, and most of the new examples coincide with examples given by Gherardi. The example for AA7 does not, however, and the precise wording is never faithful as when the compiler copies Jacopo. Moreover, Gherardi's only four-term rule is absent. There is no reason that a compiler who copies one model verbatim should paraphrase another one; we must therefore conclude that Gherardi is no direct source, and therefore that the compiler drew on material that had also been at Gherardi's disposal in 1327.[307]

The third part of the *Alicibra amuchabile* consists of 41 solved problems. While those examples in part 2 that do not come from Jacopo are all pure-number problems of the

---

[307] As we remember from p. 196, "Gherardi" stands for a treatise "written according to the rules and the abbacus course held by Paolo Gherardi". The real Gherardi may this be somewhat earlier than 1328.

"part-of" type, such questions are totally absent from part 3. Problems 1–10 concern a divided 10, problem 11 a divided 20. Then follow 9 dealing with 100 divided by some quantity (we shall return to one of them), 3 about composite interest (one involving the square root of money already in the data) and 10 about exchange of money. One concerns the partial excavation of a well, which (since labour can be supposed to increase proportionally to depth) involves the formula for summation of an arithmetical series. In the end come three give-and-take problems, one of them involving a product and thus of the second degree; and amidst these, problems about 60 δ divided first between a number of men and then between one or two more.

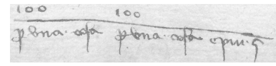
The most remarkable feature is the use of formal fractions. We may look at problem 13 (p. 41), which we have already encountered in Gherardi (above, p. 199):

> Somebody divides 100 in a quantity, and then he divides 100 in 5 more than at first, and these two results joined together made 20. I want to know in what 100 was divided at first and in what it was divided afterwards.

The method somehow presupposed by Gherardi is fully spelled out here with reference to a diagram

> Posit that you divided 10 in a *thing*, 100 divided in a thing results. And then say that you divide 100 in 5 more than at first, you shall thus divide 100 in a *thing* and 5, 100 divided in a thing and 5 results. Now you have to join 100 divided in a thing with 100 divided in a thing and 5. Now I will show you something similar so that you may be well advised about this joining and I will say thus: I will join 24 divided by 4 with 24 divided by 6, which you see should make 10. Thus posit 24 divided by 4 in the way of a fraction, from which results $^{24}/_{4}$. Also similarly posit 24 divided by 6 in the way of a fraction. Now multiply in cross, that is 6 times 24, they make 144, and now multiply 4 times 24 which is above 6, they make 96, join with 144, they make 244. Now multiply that which is below the strokes [*verghe*], that is 4 times 6, they make 24. Now you should divide 240 by 24, from which 10 should result. I say that if I multiply 10 which should result from it, against the divisor 24, it will make the multiplied, that is, 240,[308] and so it does precisely. Let us therefore return to our problem. Let us take 10 divided by a *thing* and therefore posit these two divisions as if it were a fraction, as you see it drawn hereby. And now multiply in cross, as you did before, that is, 100 times a *thing*, which makes 100 *things*. And now multiply the other way [*schisa*, literally "cleaving"], that is, 100 times a *thing* and 5, they make 100 *things* and 500 numbers; join to 100 *things*, you have 200 *things* and 500 numbers more. Now multiply what you have below the strokes, one against the other, that is, a *thing* times a *thing* and 5 more, they make a *censo* and 5 *things* more.

[308] Thus, correctly the manuscript. Simi writes 24, apparently taking the small final zero for a spot of ink.

> Now multiply the results, that is, 20 against a *censo* and 5 *things* more, they make 20 *censi* and 100 *things* more,
>
> $\frac{100}{\textit{per una cosa}}$ $\frac{100}{\textit{per una cosa e piu 5}}$
>
> 
>
> which quantity is equal to 200 and to 500 numbers.. Now take from each side 100 *things*, you will have that 20 *censi* are equal to 200 *things* and to 500 numbers. Bring to one *censo*, that is, that you divide each thing by the *censi*, you will have that one *censo* is equal to 5 *things* and to 25 numbers. [...].

Per se, the final step in the addition of the two genuine fractions seems superfluous. From the division of 240 by 24, not only 10 *should* result, evidently it *results*. It reflects that the use of the formal fractions is still a step "behind" how we would treat them. Instead of just multiplying an equation $p \div q = r$ by q without out thinking about why the step is valid, the text argues from the very definition of division, namely that the equation *means* that $p = qr$.[309]

For further elucidation we should remember a passage in Jacopo's second example for Ja1 [ed. trans. Høyrup 2007:305], a partnership problem (and as such involving a division) – quoted above, p. 187. There, no formal fractions were made use of, but we find the same reference to what the division means:

> [...] And therefore we have to multiply 30 times a thing. It makes 30 things, which it suits you to divide in the principal of the partnership, that is, by 30 and a *thing*, and that which results from it, as much is due to the third partner. And this we do not need to divide, because we know that 15 *libre* of it is due to him. And therefore multiply 15 times 30 and a *thing*. It makes 450 and 15 things. Hence 450 numbers and 15 things equal 30 things.

As we see, the use of formal fractions has not yet eliminated the need to keep in mind the underlying meaning of the operations that are performed on them. We may observe that the progress inherent in the above "behind" consists exactly in elimination of this need, freeing the mathematical mind for more creative task – in more recent times, say, solving integral equations without thinking about the definition of equations.

So, the author of this piece of text – whether the compiler of the *Alcibra amuchabile* or some predecessor – is on his way on "the royal road to *us*" – but he has still not advanced so far on it that the starting point has been lost from view.

Problem 22 (p. 48) illustrates how far away from us he is. It deals with a loan at composite interest over two years, and in this connection gives a general explanation that the solution

309 That Biagio had already multiplied by the denominator as a matter of course (above, p. 213) merely shows that we are not dealing with linear progress; nothing indicates that the present compiler knew Biagio's text.

> for one year follows from number, and in two years it comes by simple root, and in 3 years it comes by cube root, and in 4 years it comes by root of root, and in 5 years it comes by root of cube root, and in 6 years it comes by [cube?] root of [cube?] root.

As we see, at least the fifth root emulates the naming of the fifth power by multiplication; a copying error prevents us from seeing whether this was also the case for the sixth power. In any case, since the problem deals with two years, the compiler has no occasion to discover the absurdity.

Antonio de' Mazzinghi

Some of those abbacus books from the later 14th century that try to present the whole of abbacus mathematics leave out algebra; this we have seen exemplified by the redaction of Jacopo's *Tractatus*. Those that present the discipline mostly teach us little new; we shall return to an exception to this rule and bypass the others.

At first, however, we shall look at an outstanding figure, *Antonio de' Mazzinghi* from Florence, a representative of the school tradition spanning from Biagio *il vecchio* to Benedetto da Firenze.

According to Benedetto's *Praticha* (above, p. 209), fol. 451^{r} (transcribed [Arrighi 2004/1965: 157]), Antonio was a student of Paolo dell'Abbacho and began his career as an abbacus teacher when Paolo died (that is, in 1367); Benedetto further relates that Antonio was said to have died around the age of 30. On fol. 431^{v} this is said to have happened around 1390. Weighing this incongruous information (Antonio, however bright, cannot have started teaching at the age of 7!) against a number of documents from the Florence archives, Ulivi [1996] concludes that Antonio must have been born between 1350 and 1355, and probably have died in 1385–86 – thus essentially confirming what had been suggested by Van Egmond [1976: 354–356] on a narrower basis.

We know Antonio's mathematics from reports and excerpts in various later treatises. The encyclopedic manuscript Florence, BNC, Palatino 573 (above, note 145) relates (fol. 258^{r}, ed. [Arrighi 2004/1967: 183]) that Antonio is said to have produced the first tables of composite interest. The tables are reproduced on fols 262^{v}–277^{v}; they deal with the value, on one hand of 100 £, on the other of 1 £ (expressed in £, ß and δ) after 1, 2, 3, ... 20 terms, at the rate of 5, 5½, 6, 6½, ..., 20 percent per term (thus expressed, not as mostly done in δ per £ per month); as we remember from p. 19, 15 percent per year was in the upper end but still permissible.[310] The same treatise refers (fol. 397^{r}, ed. [Arrighi 2004/1967]) to a *Gran Trattato* from Antonio's hand in which he presupposes the reader

[310] Tables for the growth of 100 £ are also found in the manuscript Vatican, Ottobon. lat. 3307, fols 225^{r}–233^{r}, with an inversion on fol. 229^{v}. The announcing words on fol. 221^{v} state that such tables "were first composed" by Antonio, and thus do not promise to render Antonio's original faithfully.

to be familiar with part 15.1 of the *Liber abbaci*.

According to Benedetto's *Praticha* (fol. 451r, ed. [Arrighi 2004/1965: 158], Antonio "left many volumes about geometry and arithmetic, but the most sublime is the one entitled *Fioretti*, in which are written the cases that I shall show; so, be attentive".

Ulivi [1996: 123] suggests that this collection of "small flowers" should be identified with the *Gran trattato*. The title (in particular when used by somebody as familiar with Fibonacci as Antonio) seems rather intended to intimate a relation to the *Gran trattato* similar to that of Fibonacci's *Flos* to the *Liber abbaci*. There is also nothing in the *Fioretti* that relates to Fibonacci's part 15.1. In any case, what Benedetto copies corresponds well to the *Flos*, being a collection of often intricate and supposedly beautiful problems (whoever can be charmed by mathematics will agree).[311]

Starting just after Benedetto's "be attentive", Arrighi [1967a] contains an edition of the *Fioretti*, which I shall use in the following (checking the manuscript when there seems to be a reason to do so).[312] The text contains a number of editorial observations made by Benedetto (as does his extract from Biagio – cf. above, p. 212); but they are clearly separate, and what remains can (precisely because of Benedetto's care to make clear where he intervenes) be ascribed with fair certainty to Antonio; so can, as is particularly important, the order of the problems.

Quite apart from its appeal to the aesthetic feeling of mathematicians, the *Fioretti* as we know it from Benedetto is of the highest interest because it turns out at closer inspection to be a work in progress, or at least a work where Antonio has done nothing to eliminate the traces of his progressing insight. The most direct evidence for this (we shall come back to strong indirect evidence) is problem 29 (p. 63)

$$10 = a+b \ , \quad a^2+b^2+\sqrt{a}+\sqrt{b} = 86 \ ;$$

Antonio makes a position a = $5-t$; $b = 5+t$, which leads to

$$\sqrt{5-t}+\sqrt{5+t} = 36-t^2 \ .$$

At this point, Antonio exclaims "I do not like it, and therefore I do not complete it" – after which he goes on with a problem about three numbers in continued proportion.

Also unexplainable unless we assume a work in progress is the beginning of problem

311 It may be worth noticing that Antonio gave to Fibonacci's *Flos* the Italian title *Fioretto* – see the quotation in the manuscript Vatican, Ottobon. lat. 3307, fol. 348v [ed. Arrighi 2004/1968: 221]. In the same quotation, Antonio speaks of the *Liber abbaci* as Fibonacci's *Praticha d'aresmestricha* and about many Florentine citizens possessing Fibonacci's works.

312 Problem numbers and pages in the following refer to this edition. The problem numbering is too similar to what Benedetto does elsewhere to make us sure that it originated with Antonio; on the other hand, it agrees so nicely with what is found in other abbacus writings that nothing excludes Antonio's hand.

34 (p. 70) – a false start:

> Make two parts of 10 for me so that, when one is divided by the other and the other by the first and they are joined together, etc.
>
> Make two parts of 10 for me so that, when one is divided by the other and the other by the first and each division is multiplied in itself and they are joined together [...].

Benedetto's editorial intention is expressed on p. 47, where he says that something *could* be expressed in a particular way; "but since we speak like Master Antonio, we shall say" – and then the matter is formulated by means of formal fractions involving algebraic polynomials. There is thus no doubt than Benedetto tries to render notation as well as mathematical procedures faithfully.

The extract from the *Fioretti* ends (p. 94) with the words

> I would have many things to say; but for lack of time and because the volume would grow, we shall put an end to this chapter, and therefore to the book

– namely to book XV of Benedetto's *Praticha*. Comparison with the corresponding clause in the end of the extract from Biagio (above, note 285) seems to tell us that Benedetto's version of the *Fioretti* is complete.

The *Fioretti* consists of 45 problems[313] and a section "mirabile dictum" about properties of continued proportions. The most striking innovation in the treatise is the gradually developed use of two algebraic unknowns, wholly different from what we (and Antonio) have encountered in the *Liber abbaci*.

In problem 9 (p. 28) the beginning of the procedure suggests the use of two unknowns. It deals with two numbers (A and B), fulfilling the conditions that

$$AB = 8\ ,\ \ A^2+B^2 = 27\ .$$

At first, "though the case does not come in discrete quantity", Antonio solves it by means of *Elements* II.4, according to which (when it is read as dealing with "quantities" and not line segments)

$$A^2+B^2+2AB = (A+B)^2\ .$$

This leads to

$$A = \sqrt{10\tfrac{3}{4}}+\sqrt{2\tfrac{3}{4}}\ ,\ \ B = \sqrt{10\tfrac{3}{4}}-\sqrt{2\tfrac{3}{4}}\ .$$

Next Antonio states that

> we can also make it by the equations [*aguagliamenti*] of algebra; and that is that we posit

[313] The second-last and the last are both designated 44.

> that the first quantity[314] is a thing less the root of some quantity, and the other is a thing plus[315] the root of some quantity. Now you will multiply the first quantity [A] by itself and the second quantity [B] by itself, and you will join together, and you will have 2 *censi* and an unknown quantity, which unknown quantity is that which there is from 2 *censi* until 27, which is 27 less 2 *censi*, where the multiplication of these quantities [those of which the square root was taken] is $13\,^1\!/_2$ less a *censo*. The smaller part is thus a *thing* minus the root of $13\,^1\!/_2$ less a *censo*, and the other is a thing plus the root of $13\,^1\!/_2$ less 1 *censo*. [...].

If Antonio had worked with two algebraic unknowns, taking the "some quantity" as second unknown (say, q), he would have started with these steps (C stands for *censo*):

$$A = t+\sqrt{q}\ ,\quad B = t-\sqrt{q}$$
$$A^2+B^2 = 2C+2(\sqrt{q})^2 = 2C+2q$$

whence

$$q = 13\,^1\!/_2-C\ ,$$

which corresponds to the numerical steps in Antonio's argument, and obviously to his understanding. But what he does can instead be expressed

$$a = t+\sqrt{?}\ ,\quad b = t-\sqrt{?}$$
$$a^2+b^2 = 2C+??\ ,$$

and the fact that "??" equals two times "?" stays in his mind.

From this point onward, the method is algebraic, but with only one unknown.

The following problem 10 (p. 30) begins

> Find two numbers whose squares are 100, and the multiplication of one by the other is 5 less than the squared difference. Posit that the first number be a thing plus the root of some quantity, and the second be a thing less the root of some quantity, and multiply each number by itself and join the squares, they make two *censi* and something not known. And these squares should make up 100. Whence this unknown something is the difference

314 The two numbers of the statement have now become "quantities". There is nothing unusual in this, Antonio often replaces one word by the other. In the following lines that creates some confusion, only to be kept under control by keen unspoken awareness of what the various "quantities" refer to. Further on Antonio shows to be aware of the difficulty and to know how to eliminate it.

315 "Plus" translates *più*, literally "more" – but the expression "una chosa più la radice d'alchuna quantità" is ungrammatical if *più* is understood in this literal way. The word instead functions as a quasi-preposition, just like our "plus". Fortunately the English word "less" can serve as a quasi-preposition as well as in adjective function.

> there is from 100 to 2 *censi*, which is 100 less 2 *censi*. [...].

Antonio here gets even closer but still does not fully implement the possibility of working *algebraically* with two unknowns. But he is clearly be preparing mentally; then, in problem 18 (p. 41) the idea is unfolded:

> Find two numbers which, one multiplied with the other, make as much as the difference squared, and then, when one is divided by the other and the other by the one and these are joined together make as much as these numbers joined together. Posit the first number to be a *quantity* less a *thing*, and posit that the second be the same *quantity* plus a *thing*. Now it is up to us to find what this *quantity* may be, which we will do in this way. We say that one part in the other make as much as to multiply the difference there is from one part to the other in itself. And to multiply the difference there is from one part to the other in itself makes 4 *censi* because the difference there is from a *quantity* plus a *thing* to a *quantity* less a *thing* is 2 *things*, and 2 *things* multiplied in itself make 4 *censi*. Now if you multiply a *quantity* less a *thing* by a *quantity* plus a *thing* they make the square of this *quantity* less a *censo;* so the square of this *quantity* is 5 *censi*. And if the square of this *quantity* is 5 *censi*, then the *quantity* is the root of 5 *censi;* whence we have made clear that this *quantity* is the root of 5 *censi*. And therefore the first number was the root of 5 *censi* less a *thing* and the second number was the root of 5 *censi* plus a thing. We have thus found 2 numbers which, one multiplied in the other, make as much as to multiply the difference of the said numbers in itself; and one is the root of 5 *censi* less a *thing,* the other is the root of 5 *censi* plus a *thing*. Now remains for us to see whether one divided by the other and the other by the one and these two results joined together make as much as the said numbers. Where you will divide the root of 5 *censi* less a *thing* by the root of 5 *censi* plus a thing, this results, that is, $\frac{\textit{ra of } 5\ \textit{c less } 1\rho}{\textit{r of } 5\ \textit{c plus } 1\rho}$ [$\frac{\textit{ra di } 5\textit{c mê } 1\rho}{\textit{r di } 5\ \textit{c piu } 1\rho}$]. And then you will divide the root of 5 *censi* plus 1 *thing* by the root of 5 *censi* less a *thing*, $\frac{\textit{r. of } 5\ \textit{c plus } 1\rho}{\textit{r. of } 5\ \textit{c less } 1\rho}$ results.[316] And these two results should be joined together; where you will multiply the root of 5 *censi* plus a *thing* across,[317] that is, by the root of 5 *censi* plus *a thing*, they

---

316 We observe that Antonio, as Biagio (above, p. 213) uses ρ for the thing and *mê* for *meno*, "less"; but also that addition is not made by mere juxtaposition but indicated by a fully written *piu*, "plus", while *censo* is abbreviated *c*. The difference between the ways Biagio's and Antonio's texts are dealt with confirms that Benedetto does not impose his own ways on the texts he copies. To the same effect we may add that the manuscript Vatican, Ottobon. lat. 3307, also copying Antonio, uses the same notation when Benedetto does so – as seen for example on fol. 338[r] in the Ottoboniano manuscript confronted with fol. 456 in Benedetto's text.

317 The cross-multiplication is shown in a symbolic operation on the two formal fractions in the margin in the manuscript (fol. 458[v]) – Benedetto's autograph, but certainly copied from Antonio, as argued in [Høyrup 2010: 31–33]. A similar marginal calculation occurs when Biagio adds two formal fractions on fol. 403[v].

> make *censi* plus the root of 20 *censi of censo*; and further multiply root of 5 *censi* less a *thing* across, that is, by root of 5 *censi* less a thing, they make 6 *censi* less root of 20 *censi of censo*.[318] Which, joined with 6 *censi* and root of 20 *censi of censo*, make 12 *censi*. And this quantity we should divide in the multiplication of the root of 5 *censi* less a *thing* in root of 5 *censi* plus a *thing*, which multiplication is 4 *censi* because root of 5 *censi* in root of 5 *censi* make 5 *censi*, and a *thing* plus multiplied in a *thing* less[319] make a *censo* less, and when it is detracted from 5 *censi*, 4 *censi* remain, and multiplying 1 *thing* plus by root of 5 *censi* and 1 *thing* less by root of 5 *censi*, their joining makes 0. So the said multiplication, as I have said, is 4 *censi*, so these two results are 12 *censi* divided in 4 *censi*, from which comes 3. And we want they should make as much as the sum of the said numbers, whence it is needed to join the root of 5 *censi* less a *thing* with the root of 5 *censi* plus a thing, they make 2 times the root of 5 *censi*, which is the root of 20 *censi*. Whence the joining of the said numbers is the root of 20 *censi*, and we say that is should be 3; so 3 is equal to the root of 20 *censi*. Now multiply each part in itself, and you will have 9 to be equal to 20 *censi;* so that, when it is brought to one *censo*, you will have that the *censo* will be equal to $^{9}/_{20}$. So the *thing* is equal to the root of $^{9}/_{20}$, and if the *thing* is equal to the root of $^{9}/_{20}$, the *censo* will be worth its square, that is, $^{9}/_{20}$. So the first number, which was the root of 5 *censi* plus a *thing*, was $1\,^{1}/_{2}$ plus the root of $^{9}/_{20}$; and the second number, which was the root of 5 *censi* less a *thing*, was $1\,^{1}/_{2}$ less the root of $^{9}/_{20}$. And so is found the said two numbers [...].

This probably goes beyond what Antonio was able to do by mental implicit use of a second unknown, or at least beyond what he found it possible to convey to a reader in this way. This is the likely reason that he now makes the use of two unknowns explicit, and also chooses a more stringent language, pointing out that the *same* quantity is meant in the two positions. Awareness that something new and unfamiliar is presented to the reader is reflected in the explanation that now “it is up to us to find what this *quantity* may be” – it is never stated that the *thing* has to be found, neither here nor elsewhere in problems with a single algebraic unknown, that goes by itself.

It is also noteworthy that from this point onward, *quantity* in general use (cf. note 314) disappears from all problem solutions where that term is used to designate one of two algebraic unknowns (but not from other problems – in these *quantity* is still used profusely.[320]

---

[318] Arrighi has “20 *censi*”, but the manuscript (fol. 458^{v}), correctly, has “20 *censi di censo*”.

[319] We observe a distinction between additive and subtractive (not yet negative) numbers.

[320] There are two apparent exceptions, one in the present problem (“this quantity we should divide in the multiplication of the root of 5 *censi* less a *thing* in root of 5 *censi* plus a thing”), one in problem 28 (pp. 61*f*). Both, however, turn up after the *algebraic* quantity has been eliminated,

The procedure can be translated into familiar symbols as follows:

$$AB = (A-B)^2 \ , \quad {}^{A}/_{B}+{}^{B}/_{A} = A+B$$

with the algebraic positions

$$A = q-t \ , \quad B = q+t \ .$$

Then

$$(A-B)^2 = 4C \ , \quad \text{while} \quad AB = q^2-C \ ,$$

whence

$$q^2 = 5C \ ,$$

that is,

$$q = \sqrt{5C} \ .$$

In consequence we have the preliminary result

$$A = \sqrt{5C}-t \ , \quad B = \sqrt{5C}+t \ .$$

Inserting this in the other condition we get

$$\frac{A}{B}+\frac{B}{A} = \frac{\sqrt{(5C)}-t}{\sqrt{(5C)}-t}+\frac{\sqrt{(5C)}-t}{\sqrt{(5C)}-t}$$

which, after cross-multiplication, becomes

$$\frac{A}{B}+\frac{B}{S} = \frac{(\sqrt{(5C)}-t)^2-(\sqrt{(5C)}-t)^2}{5C-C} = \frac{6C-6C}{4C} = \frac{12C}{4C} = 3 \ .$$

Therefore, since

$$A+B = 2q = 2\sqrt{(5C)}$$
$$2\sqrt{5C} = \sqrt{20\,C} = 3 \ ,$$

whence

$$20C = 9 \ .$$

Tacitly interchanging “first” and “second” number, Antonio thereby obtains that

$$B = 1\,{}^{1}/_{2}+\sqrt{{}^{9}/_{20}}, \ A = 1\,{}^{1}/_{2}-\sqrt{{}^{9}/_{20}} \ .$$

This would probably have been very difficult even for a mathematician of Antonio’s calibre without the explicit use of two unknowns. Once Antonio had decided to make the step, things were easy. As we can see in the marginal calculations, Antonio routinely performed formal calculations involving ρ (standing for the *thing*, we remember) and *c* or c^o (both standing for *censo*) – his “multiplication across” refers to that.

Now, once the method has been invented and introduced, Antonio makes use of it even in problem 19 [ed. Arrighi 1967a: 43], which *could* have been solved according to

---

and the problem thus reduced to one with a single unknown *thing*.

the pattern we know from problems 9 and 10:

> Find two numbers so that the root of one multiplied by the root of the other be 20 less than the numbers joined together, and their squares joined together be 700. It is asked, which are the said numbers? You will make position that the first number be a *thing* less some quantity, and posit that the other number be a *thing* plus some quantity. And then you take the square of the first, which we said was one *thing* less one *quantity*, and its square is one *censo* and the square of this *quantity* less the multiplication of this *quantity* in a *thing*. And the square of the second number, which we say is a thing and some quantity, is a *censo* and the square of this quantity plus the multiplication of this quantity in a thing.[321] Which, joined together, make 2 *censi* and 2 squares of 2 *quantities*.[322] And we say that they should make 700, whence one of these squares is 350 less one *censo*. This quantity is thus the root of 350 less once *censo*. And we posited that the first number was one *thing* less one *quantity*, that is was hence one thing less the root of 350 less one *censo*. And the second number, which was posited to be a *thing* and a *quantity*, was one *thing* and root of 350 less one *censo*. And thus we have solved a part of our question, that is, to find two numbers whose squares joined together make 700. Now it remains for us to see what it makes to multiply the root of one by the root of the other. Therefore you thus have to multiply the general root of one *thing* less <u>root</u>[323] of 350 less one *censo* by the general root of one *thing* plus <u>root</u> of 350 less one *censo*, they make <u>root</u> of 2 *censi* less 350; and this is their multiplication. For these matters one has to keep the eye keen, I mean of the mind and the intellect, because even though they seem rather easy, none the less, who is not accustomed will err. Therefore we have thus found that this multiplication is the <u>root</u> of 2 *censi* less 350, and this we say is 20 less than the numbers joined together. And the said numbers joined together are 2 *things*, that is joining a *thing* less <u>root</u> of 350 less a *censo* with a *thing* plus <u>root</u> of 350 less a *censo*, which indeed make 2 things. Whence we have that 2 *things* less 20 are equal to the <u>root</u> of 2 *censi* less 350; whence, in order not to have the names[324] of roots, multiply each part

---

321 Obviously, the product of *quantity* and *thing* should be taken twice here as well as in the square of the first number. Antonio abbreviates, knowing that the two elliptical expressions cancel each other.

322 *2 quadrati di 2 quantità* – namely "the two squares coming from the two distinct quantities".

323 Underlined <u>root</u> renders ℞ or an encircled fully written *radice* (Arrighi does not indicate the encirclings, they have to be traced in the manuscript). Antonio may well be the one who introduced this notation for the "universal" or "bound" root, the root taken of a binomial (cf. above, p. 220). As we see, Antonio avoids the inherent ambiguity by using the further notion of a "general root", where the "general root of one *thing* less <u>root</u> of 350 less one *censo*" stands for $\sqrt{(t - \sqrt{[350 - C]})}$.

324 *Nomi*. At least from Dardi onwards (above, p. 216) the algebraic powers (*cosa*, *censo*, *cubo*,

in itself, and you will have that root of 2 *censi* less 350 multiplied in itself make 2 *censi* less 350, and 2 *things* less 20 multiplied in itself make 4 *censi* and 400 less 80 *things*. So 2 *censi* less 350 are equal to 4 *censi* and 400 less 80 *things*. Where you should make equal the parts giving to each part 80 *things* and removing 2 *censi;* and we shall have that 2 *censi* and 740 are equal to 80 *things*, which is the fifth rule. Where you bring to one *censo*, and you will have one *censo* and 375 equal to 40 *things*. Where you will halve the *things*, and let the half be 20, multiply in itself, they make 400, detract the number, they will make 25, that is, detracting 375 from 400, of which 25 take the root, which is 5, and detract it from 25, 15 remain. And you will say that the thing is worth 15, and the *censo* will be worth its square, which is 225. Whence the first number, which we posited that it was a *thing* less root of 350 less a *censo*, detract 225, which is worth the *censo*, from 350, 125 remain. And you will say, one part was 15 less root of 125, and the second number was 15 plus root of 125. [...].

In our usual translation:

$$\sqrt{A}\cdot\sqrt{B} = A+B-20\ ,\ \ A^2+B^2 = 700\ ,$$

with the position

$$A = t-q\ ,\ \ B = t+q\ ,$$

where Antonio no longer feels the need to point out that the two “some quantity” (*alchuna quantità*) refers to *the same* quantity. He does not quite return to the formulation of problems 9 and 10, $A = t-\sqrt{q}$, $B = t+\sqrt{q}$, since with the explicit position of q he can now operate freely with its square. Antonio calculates

$$A^2 = C+q^2-[2]qt\ ,\ \ B^2 = C+q^2+[2]qt\ ,$$

whence

$$2C+2q^2 = 700\ ,\ \ q^2 = 350-C\ ,\ \ q = \sqrt{(350-C)}\ .$$

Therefore

$$A = t-\sqrt{(350-C)}\ ,\ \ B = t+\sqrt{(350-C)}\ ,$$

which is seen as a partial answer, and is inserted in the other condition:

$$AB = \sqrt{t-\sqrt{(350-C)}}\cdot\sqrt{t+\sqrt{(350-C)}} = \sqrt{C-(350-C)} = \sqrt{2C-350}\ ,$$

a calculation which seems straightforward but where, according to Antonio, the untrained

etc.) were spoken of as “names”; as we see, Antonio sees the root as belonging to the same category.

will none the less err.[325] At all events, with the correct calculation we now have

$$\sqrt{2C-350} = A+B-20 = 2t-20$$

whence after squaring

$$2C-350 = 4C+400-80t ,$$

which can be reduced to

$$2C+750 = 80t .$$

Solving this equation by means of the standard rule or algorithm for the fifth algebraic case Antonio finds $t = 15$ – silently discarding the other solution $t = 25$.[326]

Several more problems are solved by means of two algebraic unknowns: number 20, number 21, number 22 (twice during the procedure), number 24, number 25 and number 28. All seven make the position

$$A = t-q \ , \ \ B = t+q \ ,$$

and all seven *could* instead have been solved in the same way as number 9 and number 10. They tell nothing new about the use of two unknowns, except that by now Antonio had taken full possession of the technique.[327]

Also noteworthy is the presence of no less than 11 problems dealing with numbers

---

[325] Those who doubt Antonio's words should be aware that near-contemporary algebraic writings might presume that $\sqrt{(a+\sqrt{b})} = \sqrt{a}+\sqrt{\sqrt{b}}$ – thus the *Libro di conti e mercatanzie* [ed. [Gregori & Grugnetti 1998: 116]. The somewhat cavalier use of language came at a cost for those who did not fully understand what was meant (from our point of view a cost – themselves they hardly had any occasion to discover).

[326] This alternative solution indeed leads indeed to a complex and thus impossible values for q, and hence also for A and B, which Antonio may have seen (not in our terms, of course). We should remember that abbacus algebra regarded the two solutions to the fifth case (when solutions exist) as *possibilities* of which at least one will be valid, cf. above, note 258.

[327] Only one detail is noteworthy. Number 20 (p. 44), begins

> Find two numbers so that their roots joined together make 6 and their squares be 60, that is, the joining of the squares be 60. Posit the first number to be a thing less the root of some quantity, that is, less some quantity; the other posit to be a thing plus the said quantity. [...].

Once more we see that Antonio as copied by Benedetto presents us with a work in progress: if the *Fioretti* had been polished, there would have no reason to leave a formulation "root of some quantity" then to be corrected. Antonio must at first have had in mind the method of problems 9 and 10; it is a plausible guess that he used an earlier solution of the problem – probably his own.

in continued proportion,[328] accompanied by two (numbers 14 and 15) about composite interest making use of Antonio's theoretical insight in the topic, solving a problem which had been considered impossible: to find the yearly interest equivalent to an interest of 20 percent over 9 months (#14); and to find the interest over 8 months that is equivalent to a yearly interest of 4 δ per £ and month (#15). These question are obviously related to Antonio's interest in tables of composite interest, mostly with terms of unspecified duration; in mathematical future perfect they are on the way towards Jost Bürgi's way to introduce logarithms.

Also related to the tables of composite interest is #16, to find the yearly interest of a loan which in five years grows from 1000 *fiorini* to 2500 *fiorini*, "which many ignorants have said cannot be resolved", but which Antonio solves via the extraction of a *radice relata*, a fifth root.[329]

These three problems have little if anything to do with practical commercial life. The problems about barter (#11, #13), partnership (#12) and exchange (#43, #44a, #44b) have even less – note 87, above, mentioned how Antonio manages to make the givens so abstruse that second-degree equations result. All of these, like the refined pure-number problems, *are* really *fioretti*, flowers picked on the field of abbacus mathematics, not matters to be taught to merchants *in spe* for use in their trade. These hopefuls were certainly also taught in Antonio's school, but not from this book.

---

328 Numbers 1, 2, 3, 4, 5, 8, 23, 24, 25, 26 and 33.

329 This gives Benedetto the occasion for a cross-reference to the last chapter of his book 12. Before wondering that Antonio's problem could be deemed impossible we should be aware of what Regiomontanus says about the analogous problem where 100 ducats grow to 900 over six years in a letter to Giovanni Bianchini [ed. Curtze 1902: 256]: namely that it "sent him onto a major rock" – namely because at first he had taken the yearly interest as his *thing* (cf. above, pp. 190 and 223). The solution he then gives shows him to have discovered Antonio's easier way (which, with fewer years, is also that of Fibonacci and Biagio).

The introduction of the *radice relata* solves the problem that made Dardi stumble: how to express roots that cannot be composed of square and cube roots (cf. above, p. 222).

The manuscript Florence, BNC, Palatino 573, fol. 258r [ed. Arrighi 2004/1967: 191] quotes Antonio for this explanation of the powers:

> *Thing* is here a hidden quantity; *censo* is the square of the said *thing; cube* is the multiplication of the *thing* in the *censo; censo of censo* is the square of the *censo* [*quadrato del censo*], or the multiplication of the *thing* in the *cube.* And observe that the terms of algebra are all in continued proportion; such as: *thing, censo, cube, censo of censo, cubo relato, cube of cube,* etc.

Since the sixth power is produced by multiplication, the fifth power could have been too, as *censo of cube* or *cube of censo*. It looks as if that the new name for the fifth root has called forth a corresponding naming of the fifth power, at the moment without general consequences being drawn.

The Florentine *Tratato sopra l'arte della arismetricha*

Final facets of the development of abbacus algebra in the 14th century show themselves in the manuscript Florence, BNC, fondo princ. II.V.152 (above, note 145), *Tratato sopra l'arte della arismetricha*.

Internal evidence (model loan contracts etc., and the contents of problems) tells us that the treatise was produced in Florence in the early 1390s, which fits watermarks dated to the years 1390–1399 [Van Egmond 1980: 138]. With this dating, the author almost certainly knew about Antonio as a recent colleague. As we shall see, however, he appears not to have known his mathematics too well; his innovations are his own, or in any case not borrowed from Antonio's lost writings.

The *Tratato* is written in a single hand. Its extensive algebra was edited by Franci and Marisa Pancanti [1988]; page references in the following will point to this edition.[330]

The algebra occupies fols 145^{r}–180^{v}. As we have seen in other cases, it begins with generalities – in the present case by explaining the sequence of algebraic powers, where explicit insight in the nature of this sequence as a continued proportion is combined with an astonishing terminological innovation.

The *thing* (*cosa*) is explained (p. 3) as

> nothing but a position that is made in many questions, and when it happens this position that has been made may represent [*portare*, literally "carry"] the quantity of a number at some occasion, or a quantity of time at another occasion, or a quantity of cubits [...].

Further,

> Having seen what a *thing* means, having shown that it is a position, we come to its multiplication: we should know that a *thing* multiplied in itself makes a root which is called a *censo*, so that it is the to say a *censo* as to say a quantity which has a root, engendered from a number multiplied by itself.

This turns out to be the beginning of a system. A *thing* multiplied against a *censo* gives

> a *cube*, that is a cube root, so that if you should say that if the *thing* should produce 6, then the *censo* will produce 36, that is, the square of the *thing*, the cube will produce 216 [...]. So it is the same to say a cube as to say a cube root of a given number.

Next (p. 4), *thing* times cube produces a *censo of censo*, "which is to say root of the root of a given quantity"; that of a *thing* and a *censo of censo* will make

[330] Inasfar as possible I have controlled critical points in a barely readable scan of a low-quality secondary microfilm.

> *cubo di censi*, which will be as much as saying a root which is engendered by a squared quantity multiplied against a cubed quantity; as it would be to say, if the *thing* were worth 6, the *censo* is worth 36, and the cube will be worth 216, and the multiplication that is engendered by the 36 against the 216 will be 7776, you will thus say that if the *thing* were worth 6, then the *cubo of the censo* will be worth 7776, and there are some that call this root the *radice relata*. So it is the same to say *cubo of censo* as to say *radice relata* of a quantity.

This passage shows us, firstly, that our anonyme can hardly have been too close to Antonio; either he has misunderstood his use of *radice relata* (namely that it refers to a proper fifth root and not to a fifth power), or he refers to an already current usage (by "some") without taking Antonio into account. Secondly we see that while Giovanni di Davizzo's impossible "multiplicative" composition *of roots* has no influence, multiplicative composition *of powers* was still in use, in spite of the unusual phrase *cubo of the censo* (*chubo del censo* , that is, including the definite article), even used about numbers where the grammatically proper reading should be "cube of 36" (the *censo* having just been said to be 36), that is, 46656.

If it had not been for what follows immediately, this might look as a pedantic imposition of modern thought; but the next step shows that we are in the midst of a "phase transition" of algebraic thought:

> If you want to multiply a *thing* against a *cube of censo*, it will be a *censo of cube* [*censo di chubo*], which means as much as to say, taken the root of some quantity, and of this quantity taken its cube root, as it would be if the *thing* were worth 3, the *censo* will be worth 9, the *cube* will be worth 27, the *censo of the censo* will be worth 81, the *cube of the censo* will be worth 243, the *censo of the cube* will be worth 729, because, taken the root of 729 it will be 27, whose cube root is 3, and that equals the value of the *thing*.

This is the preliminary concluding step of the explanation of powers. Whereas the fifth power, in the notation which is used so far, is KC, where the juxtaposition means multiplication (and K as well as C are thus understood as *entities*), the designation of the sixth power has to be expressed as $C(K)$, where C, in modern terms, is a *function*. We may assume that the transition, partial as it is, has been called forth by interaction with the taking of roots; to which extent it is also, at a different level, an outcome of challenges between abbacus masters is undecidable as long as we have not texts hinting at that.[331]

---

[331] A speculation is possible: a change due to explicit challenges would likely lead to explicit understanding and thus to a full transition involving *cube* as well as *censo*. My guess, which can be no more (beyond being based on psychological experiments which I performed half a century ago, never published), is that the full transition which occurs over the following century may well

Some further rules for multiplying powers follow on pp. 4*f*, pointing out that the powers are in continued proportion. Here, the sixth power – whether produced as *cube* times *cube* or as *thing* times *censo of cube* is named *cube of cube* (*chubo di chubo*). If these inconsistencies were produced by a bungler, they would tell us nothing of general importance; but as we shall see, the author was an eminent algebraist, and they are therefore evidence of a difficult birth.

The *Tratato* goes on with the multiplication of algebraic binomials and polynomials shown in schemes – for the binomials similar to what we find in the *Alcibra amuchabile* and in Dardi – for instance, for (6*t*–3)·(5*t*+4)

```
6 chose  meno  3
5 chose   più  4
```

When dealing with tri- or higher polynomials, the scheme is (by necessity) different. So, (6*t*+8+√9)·(6*t*+8+√9) is shown (p. 11) as

```
 6 chose e 8 e ℞9
 6 chose e 8 e ℞9
------------------
censi  ρ    n   ℞
       36   9
36     96   48
            64
------------------
36c  132ρ  121n
```

where advantage has been taken of the fact that √9 is rational. Such schemes would still be in full use in the printed algebras of the 16th century. Here it is (correctly) explained to emulate the multiplication of three-digit numbers *a casella* (not quite our algorithm, but based on the same principles).

As we have seen it before, this introduction about powers and polynomial arithmetic is followed from p. 44 to p. 97 by an enunciation of what in the closing words is called "the 22 rules". These are the same as those of Jacopo, supplemented by the two failing biquadratics. We may observe that there is no trace of Dardi's explicit notion of coefficients – to use the first rule as example, the initial normalization is spoken of as as "division of the number by the *things*".

All rules are provided with examples – 51 in total, since most are provided with at least two. In several cases (#10, #11, #12, #17[332]) it is also pointed out that they can

---

be the outcome of social interaction – discussion, challenge or both – but the present beginnings a result of private thought.

We may remember how Antonio's naming of the fifth root affected his naming of the fifth power (above, note 329). The present *Tratato* seems to have improvised like Antonio but not to have borrowed from him.

[332] The cases are not numbered in the edition, but in the manuscript they are.

be reduced to one of the basic 6 cases through division; in number 17 it is moreover claimed that *all* cases after the first six can be reduced to these, which is evidently only true if the root extraction of #2 is generalized to the extraction of any root. 15 of the examples are of the simple "such part" type, which leads directly to the corresponding equation type. Of the remaining six pure-number problems, two are of interest. The first example for rule 5 is

$$10 = a+b,\ a^2+b^2 = 82\ .$$

Taking a to be a *thing*, the problem is reduced to $2C+18 = 20t$, indeed the fifth case. The third example for rule 6 is strictly parallel,

$$10 = a+b,\ a^2+b^2 = 60\ .$$

Taking again a to be a *thing*, we would get $2C+40 = 20t$, again the fifth case. But instead the author chooses a to be a *thing*+4, whence b is 6–*thing*. This results in the equation $2C = 4t+8$. the sixth case. It appears that the author is full aware of the effects of a linear change of variable, which we shall see confirmed below.

The remaining 30 problems are in commercial or recreational dress, but as we have seen it in other algebras they deal with situations that could never present themselves in proper commerce. Often, traditional linear types like the give-and-take are twisted so as to become non-linear – for instance (p. 82) in the first example for rule 16 ($\alpha CC = \beta K$),

> Three men have *denari*, the first says to the second, if I multiplied my *denari* by themselves I shall have 2 times as much as you, the second says to the third, if I multiplied my *denari* by themselves I shall have three times as much as you; and the third says, and I have 4 times as many *denari* as when the *denari* which the first has are multiplied by those of the second, it is asked how many has each on his own.

Nothing absolutely new, of course, already Jacopo had a give-and-take problem involving a square root, though only with two participants (above, p. 189).

Seemingly commercial problems like barter (also by nature linear) are dealt with in similar ways, by augmenting the value of goods not by a fraction but, for instance, by its square root. As a result, the transformation of these warped problems into an equation is quite intricate – and solution without the use of algebra hard to imagine.

Two problems are worth mentioning not because they tell us something new about mathematical thought but as traces of connections over time.

One is the second example for rule 15 ($\alpha CC = \beta C$), which is simply Jacopo's fourth *fondaco* problem, with numbers doubled (first and third year together 40 *fiorini*, second and fourth year together 60 *fiorini*). In the present *Tratato*, the properties of continued proportions are discussed explicitly. The solution makes use of algebra, but – without saying so – first of the factor of proportionality which is also behind Jacopo's solution – namely by positing the salary of the first year to be 2 *censi*, and that of the second year 3 *censi*. The following algebraic calculation is much more complicated than it needed be. In particular, of course, the choice of the basic unknown as a *censo* seems strange. Admittedly, it serves to make a second-degree problem emerge as a biquadratic, but the author seems not to be have thought of that – once he has found the *censo* he feels obliged to find the *thing*, and then to return to the *censo* by squaring. It seems likely that he builds on a source where, in the original Arabic way, the *censo* stands for an amount of money.

More than likelihood is involved in the second problem. The fourth example for rule 13 starts like this (p. 76):

> Somebody lends to another one 1000 £ to make up at the end of year, and when he came to the end of the 4 years he gave him back in capital and interest £ 14641, it is asked at what were his *denari* per hundred. [...]

As can be guessed from the value after 4 years and as confirmed by the subsequent calculation, "1000 £" is a mistake for "10000". Exactly the same formulation, *including the mistaken 1000,* is found in Biagio's *Praticha* as copied by Benedetto. The rest of the calculation is also explained in almost the same words. It seems next to certain that the anonyme copied from – and thus, as later Benedetto, had access to – Biagio's work.

What follows on p. 98 after the 22 rules with examples builds on the tools that led to the discovery of Dardi's irregular rules, and were pointed out above (p. 224) to be *possibly* borrowed from Biagio. Here, they are used for a mathematically impeccable purpose – a mathematically valid approach to irregular equations:

> We have so far explained the 22 rules of algebra with examples, here we shall show how other rules can be made, by which are solved several questions which would not be solved by the 22 rules. And when we want to deal with this it is at first necessary to make clear that there are other roots than those of which one commonly speaks, that is, that there are other roots than square and cube roots, and among these there is one called cube root with a joined number [*radice chubica con l'aguagliamento d'alchuno numero*[333]], and about that one I want to show certain things.

On p. 214 we encountered the "pronic" root, which is connected to the equation $CC+t = N$. The cube root with addition α instead procures the solution to equations of the type $K = \alpha t+N$. So, as explained, the cube root of 44 with 5 added is 4 because $4^3 = 44+5\cdot4$. Similarly (still the text), the cube root of 65 with addition 12 is 5 because $5^3 = 65+12\cdot5$.

We might find this rather uninteresting, nothing but a synonym for "the solution to the equation $K = \alpha t+N$". Firstly, however, we should remember that as long as we make no approximations (and abbacus *algebra* never does), then the same can be said about

[333] As the further text shows, *aguagliamento* (meaning "equation") is a mistake for *agiugnimento*, which gives my translation. As we shall see (below, p. 281), a contemporary source speaking about the same type of root also speaks about "joining".

the square and cube roots, similarly synonyms for "the solution to the equation $C = N$" respectively "to the equation $K = N$".

Secondly, the author uses this particular root not to postulate solutions but to explore possibilities and connections. He shows that it is sometimes but not always possible for a given t and K to find a fitting (integer) value of α. For instance, for $N = 36$ we may choose $K = 64$ (whence $t = 4$), and then find $\alpha = (64–36)\div 4 = 7$. The cube root of 15 with added 2 cannot be found in this way, it is pointed out, whereas that with added 4 can.

After this explanation the author turns to such rules where this can be used – and these turn out to concern cases that can be transformed so as to have the shape $K = \alpha t+N$. One of them is $\alpha K+\beta C = N$ (p. 99). At first it is normalized, with an outcome that it is easier for us to deal with if we write it as

$$t^3+3at^2 = m\ .$$

Completion gives

$$t^3+3at^2+3a^2t+a^3 = m+a^3+3a^2t\ ,$$

that is,

$$(t+a)^3 = m+a^3+3a^2(t+a)–3a^2\cdot a\ ,$$

which is exactly what the rule of the text says, in this order and without reduction of the expression to the right. There can be no doubt that the transformation was derived in a way that corresponds closely to our use of polynomial algebra.

In this way it is shown that the notion of a cube root with added number can also solve problems of the type $\alpha K+\beta C = N$. After application to three examples it is shown on p. 102 and 104 to apply to the cases $\alpha K = \beta C+N$ and $\beta C = \alpha K+N$. The problem on p. 102 even shows that the new root can be taken of negative numbers: the cube root of "debt 80" with addition of 108 is indeed 10, since $10^3 = –80+10\cdot 108$.

In the end come 41 problems – some of them as difficult as those contained in Antonio's *Fioretti*, though not the same. Formal fractions are made use of when adequate, with the powers written in full words, without abbreviation.

Absent, however, are second-degree problems solved by means of two algebraic unknowns. The last four problems, on the other hand, throw oblique light on the gradual acceptance of a second unknown. They are all of the first degree, two of the type "purchase of a horse", while two deal with the "finding of a purse". All four are similar in their principles; we may take a closer look at the first of them (p. 145). Beyond the description the procedure, there are some metamathematical commentaries – here in spaced writing:

> Three have *denari* and they want to buy a goose, and none of them has so many *denari* that he is able to buy it on his own. Now the first says to the other two, if each of you would give me $^1/_3$ of his *denari*, I shall buy the goose. The second says to the other two, if you give me $^1/_4$ plus 4 of your *denari* I shall buy the goose. The third says to the other

two, if you give me $^1/_4$ less 5 of your *denari* I shall buy the goose. Then they joined together the *denari* all three had together and put on top the worth of the goose, and the sum will make 176, it is asked how much each one had for himself, and how much the goose was worth. Actually I believe to have stated similar questions about men in the treatise,[334] but wanting to solve certain questions in a new way I have found new cases which I do not believe to have [already] treated. [...]. Therefore I have made it in such way that in this one and those that follow it will have to be shown that the question examined by the thing will lead to new questions that cannot be decided without false position. [...]. I shall make this beginning, let us make the position that the first man alone had a *thing*, whence, made the position, you shall say thus, if the first who has a *thing* asks the other two so many of their *denari* that he says to be able to buy the goose, these two must give to the first that which a *goose* is worth less what a *thing* is worth, which the first has on his own. So that the first can say to ask from the other two a *goose* less a *thing*, and you know that the first when he asks for the help of the others asks for $^1/_3$ of their *denari*. So the two without the first must have so much that $^1/_3$ of their *denari* be a goose less a *thing*, and in this way you see clearly that the second and the third together have 3 *geese* less 3 *things*. Now it is to be seen what all the three have, and it is clear that the first by himself has a *thing* and the other two have 3 *geese* less 3 *things*, so that all three have 3 *geese* less 2 *things*. Now we must come to the second, who asks from the other two $^1/_4$ plus 4 of their *denari* and says to buy a *goose*. I say that when the second has had as help of the other two the part asked for, he shall find to have a *goose*[335]).

Further protracted arguments show that B is $^1/_3$ *goose* plus $^2/_3$ *things* less $5\,^1/_3$ in number (A, B and C being the three original possessions). Since $B+C$ has been seen to be 3 *geese* less 3 *things*, C is $2\,^2/_3$ *geese* and $5\,^1/_3$ in number less $3\,^1/_2$ *things*. Using then that $C+\,^1/_4(A+B)-5$ is a *goose*, it is found (I skip the intermediate steps) that $1\,^3/_4$ *geese* equals $3\,^1/_4$ *things* and 1 in number or, multiplying "in order to eliminate fractions",

$$7\mathit{geese} = 13\mathit{things}+4\ .$$

Moreover, since $A+B+C$ was seen to equal 3 *geese* less 2 *things*, and these together with the *goose* equalled 176, we have

$$4\mathit{geese}-2\mathit{things} = 176\ .$$

Now, for instance, the *thing* might have been found from the latter equation (namely,

[334] Namely in the sense that fols 97^v–110^r (coming before the algebra) contain a large number of "give and take", "purchase of a horse" and "finding a purse" problems.

[335] Thus the manuscript; the edition has a mistaken *thing*.

to be 2 *geese* less 88) and inserted in the former, which would immediately lead to the goal. Instead the author goes on,

> So, you have two equations [*aguagliamenti*], which are solved one by means of the other in this way: You have on one side that 7 *geese* must be worth as much as 13 *things* and 4 in number, on the other side you will have that 4 *geese* must be worth as much as two things and 176 in number, put the sides [*parti*[336]] together, now I shall make the position that the *goose* is worth 40, and take the first side, that is that 7 *geese* are worth as much as 13 *things* and 4, if the *goose* is worth 40, the 7 will be worth 280, thus 13 things and 4 are worth 280, and the *thing*, dividing the 276 by 13, the *thing* will be worth $21\,^3/_{13}$. With this go to the other side, and you will say, if the *goose* is worth 40 and the thing is worth $21\,^3/_{13}$ we shall see that 4 *geese* are worth as much as 2 *things* and 176, where we know that so much should be worth one as the other, from where it is manifest that the 4 *geese* are worth 160, and this is on one side, on the other side the 2 *things* and 176 in numbers will be worth $218\,^6/_{13}$, and we indeed said that they should be worth 160, there comes $58\,^6/_{13}$ more for us [than there should]. Thus save in this first position for 40 that you posited the *goose* to be worth there comes $58\,^6/_{13}$ more for us. Now make the other position and posit that the *goose* is worth 80 [...], so you shall say in the second position for 80 that you posited the *goose* to be worth, $58\,^6/_{13}$ are missing for me. No take the two positions that were made and follow the way to be made for positions that become plus and less, and you shall find that the price of the *goose* was 60. When the price of the *goose* is known you shall say, if the *goose* is worth 60, then 7 *geese*[337] are worth 420, and 13 *things* and 4 in number are worth 420, the *thing* is thus worth 32 [...].

This is far removed from the use of two algebraic unknowns as we have encountered it in the *Liber abbaci* or in the *Fioretti* – the author can hardly have been familiar with either. He seems to have been distantly acquainted with the use of a second unknown in *regula recta* computations but not to have known it well enough to apply the method; instead, drawing on the familiar technique of a double false position in a beautiful piece of *bricolage*, he invents a method of his own – a method which apparently was destined to remain his own, I do not remember having seen anything similar in later (or, for that matter, earlier) sources. One comes to think of Jean Paul's "little schoolmaster Maria Wutz" who, when informed about an interesting book which he evidently cannot afford, *writes it himself* [ed. Hecht 1987: I, 119].

Even the ingenious transformation of third-degree equation types seems to have been forgotten. Such transformations were to become a key element in Girolamo Cardano's

---

336 Here, the *parti* do not (as elsewhere, also in the present treatise) refer to the sides of an equation but to the two equations. Once more we see that the terminology was in a state of flux.

337 Thus the manuscript; the transcription by error has 3*geese*.

general solution of third-degree equations, but Cardano appears to have had to reinvent.

So, all in all, abbacus algebra unfolded impressively during its first brief century, if the compare Jacopo's beginning with Dardi, Antonio and the anonymous Florentine. But it does not allow us to discern cumulative progress, apart from the evidence we have that ideas shining through in Biagio's *Praticha* had matured and influenced both Antonio and our present anonym. Algebra, as a sophisticated outgrowth on the practically oriented teaching of the abbacus school, had insufficient social density to constitute a discipline – and also to wipe out the fake rules, which were far more likely to impress the mathematically incompetent judges in competitions for positions than the mathematically sound polynomial transformations of the *Tratato*.

The abbacus encyclopedias

The great innovation in 15th-century abbacus mathematics is the appearance of ambitious "abbacus encyclopedias" – three from around 1450–1465, and from 1494 Pacioli's *Summa*, to which we shall return in the next chapter. Here, we shall look at the three Florentine specimens, all of which carry the descriptive title *Praticha d'arismetricha*. As it turns out, here we see for the first time substantial borrowings from Fibonacci though only dominant in sections explicitly borrowed from him.

One of them we have drawn upon extensively, since it was our source for Biagio and Antonio: Benedetto da Firenze's *Praticha* (see above, note 279). The second, anonymous and contained in the manuscript Florence, BNC, Palatino 573 (henceforth the "Palatino *Praticha*"), was mentioned first in note 24; it is also our primary source for Antonio's tables of composite interest (above, p. 230) and for his explanation and naming of the algebraic powers (above, note 329). The third, equally anonymous, is in the manuscript Vatican, Ottobon. lat. 3307 (henceforth the "Ottoboniano *Praticha*"). Descriptions and extracts from all three were published by Arrighi in the 1960s, reprinted as [Arrighi 2004/1965], [Arrighi 2004/1967] and [Arrighi 2004/1968].

As said above (note 279), the principal manuscript of Benedetto's *Praticha* can be seen from marginal computations being made before the main text to be the author's working copy. The two other encyclopedias (each of which exists only in a single copy) can be seen in the same way to be authors' autographs.[338]

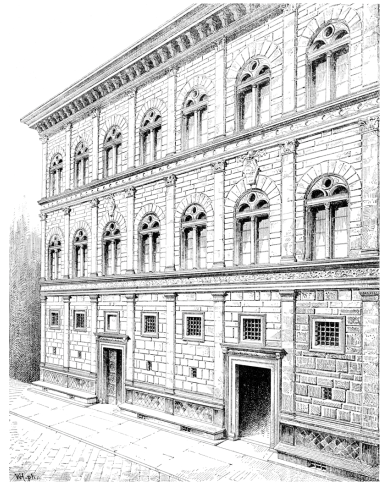
All three are in the Florentine tradition going back to Antonio de' Mazzinghi, Paolo dell'Abbacho and Biagio. The two anonymous writers both declare themselves to be students of Domenico d'Agostino Cegia, apparently a mathematical dilettante of standing and no abbacus teacher and known as *il Vaiaio* – "the fur dealer", which had been the profession of the family before protection by Lorenzo il Magnifico allowed it to improve its already considerable material conditions and social standing [Ristori 1979; Ulivi 2002a: 48*f*]. All three encyclopedias quote material from named earlier members of the tradition extensively; we have seen the lengthy extracts from Antonio in Benedetto's as well as the two anonymous *Pratiche*, and also Benedetto's extract from Biagio, but there are more.

The Palatino *Praticha* was prepared (as a gift) for a member of the distinguished Florentine Rucellai family, whom the author wants to "serve as a friend" [ed. Arrighi 2004/1967: 168]; the coat of arms of the Rucellai is depicted on the first page.[339] Girolamo di Piero di Cardinale Rucellai took possession of the manuscript on 22 April

---

338 See for instance Palat. 573, fols 64^{v} and 69^{r}, and Ottobon. lat. 3307 fols 48^{v}, 51^{r}–52^{r} and 53^{v}–54^{r}.

339 According to the same introduction, the treatise was a new version, with additions and deletions, of an earlier one, of which however we have no trace. Fictional loan documents on fols 288–291 refer to repayments to be made between 1450 and 1454; that might determine the date of the lost version.



Palazzo Rucellai, built around 1450–60 [Lübke & Semrau 1905:11].

1460;[340] we may safely assume that he was the intended recipient, and that the writing was not much earlier than this date. Benedetto says in the very beginning that his treatise was written for a "dear friend" in 1463, but since the same page carries a depiction of the coat of arms of the equally distinguished Marsuppini family [Arrighi 2004/1965: 130], we should probably think of a patron-friend – although abbacus masters could be counted as fairly wealthy when compared to master artisans, they were far below the immensely rich Rucellai and Marsuppini. The Ottoboniano *Praticha* carries no coat of arms, but the introduction ends [ed. Arrighi 2004/1967: 211] by asking "you, or whoever might get this work into his hand" to correct the errors that might be found. Exactly the same phrase concludes the introduction to the Palatino *Praticha;* even the Ottoboniano *Praticha* was thus meant as a gift to a particular person, most likely another patron-friend.[341]

Van Egmond [1980: 213] argues from watermarks that the Ottoboniano *Praticha* was written around 1465. However, the author writes on fol. 315^{r} about a certain problem that it was sent to Florence by a master from L'Aquila "already around 12 years ago". According to Benedetto [ed. Pieraccini 1983: 118*f*] this happened in 1445. Only one of the watermarks referred to by Van Egmond has not been identified in manuscripts dated before 1461, which after all is no strict criterion. The best dating thus seems to be 1457–59.

This writing for patron-friends, at least two of whom belonged to the absolute upper crust of Lorenzo ("il Magnifico") de' Medici's Florence, already puts the three *Pratiche*

340 [Arrighi 2004/1967: 1161]. In [Høyrup 2010: 39], repeated in [Høyrup 2019a: 858], an oversight and a misreading made me argue for a date around 1470.

341 Both also open the explanation of the addition of fractions (Ottoboniano fol. 9^{r}, Palatino fol. 12^{v}) addressing the recipient with a promise to be concise, being "convinced that you know these matters" – *intendo dire brievemente queste chose le quali certo sono che sai*, with the only difference that the Palatino manuscript inverts, *sono certo*. The two obviously shared more than a teacher. However, when arriving at the multiplication of fractions the Palatino *Praticha* repeats the promise (fol. 13^{r}), the Ottoboniano (fol. 9^{v}) not.

As we shall see in the following, the two treatises are largely drawn from the same archetype, though not throughout.

into a particular class of abbacus books. So does their size (all three between 850 and 1000 rather densely written folio pages) – and in particular their contents, which illustrates the development of abbacus mathematics at its mathematical best and its highest intellectual ambitions until the mid-15th-century.

The Ottoboniano *and Palatino Pratiche*

I shall first look at the Ottoboniano *Praticha*, to which least attention has been paid by earlier workers,[342] regularly confronting it with parallel passages in the Palatino *Praticha* and in the *Liber abbaci*. The reader who does not like to eat pedantic dust may skip the details.

The Ottoboniano *Praticha* is divided into 11 parts, subdivided into chapters. The Palatino *Praticha* is very similar in structure, with the same parts and grossly the same chapters; as we shall see, there is no doubt that this structure, and most of the materiali is taken over from a shared model.

The Ottoboniano *Praticha* presents itself as a *Libro di praticha d'arismetricha*, "that is, *fioretti* drawn from several books of Leonardo Pisano". This has to be taken with a grain of salt, Fibonacci may well be the most important single source, but the general abbacus tradition overweighs him, and long stretches are also borrowed from other named predecessors. There is little doubt, however, that the author (or compiler, according to his own words as well as internal evidence, as we shall see) had access to a copy of the *Liber abbaci,* plausibly a vernacular translation. This is quite possible – as we remember from note 311, possession of Fibonacci's work had not been uncommon in Antonio's time, and those citizens for whom it was a prestige object were not necessarily well trained in Latin (admittedly, the Latin originals might still serve them as prestige objects).

Part 1 (fols 1^r–8^r) presents the shapes of the numerals, the place-value system, and addition, subtraction, multiplication and the beginnings of division. The division of 574930 by the prime divisor 563 gives rise to the introduction of fractions. The last chapter deals with the factorization of non-prime numbers of more than 2 digits. Most, not all examples are drawn from the *Liber abbaci*. The term used for the factorization is *ripiegho*, "folding back" (current at the time, and also used by Benedetto), even though Fibonacci's *regula* occurs; at times it designates the single factor (thus fol. 24^r). Whereas Fibonacci gives the factorization of a number n as the sequence "under the stroke" in ${}^1/_n$ written as an ascending fraction for ${}^1/_n$ (that for 951, for instance [B39;G65] as $\frac{1\ \ \ 0}{13\ 113}$), the present *Trattato* (fol. 6^v) simply writes the sequence of factors, in the present case as "3 and 317"

[342] [Arrighi 2004/1968] transcribes the introduction; the final considerations listing predecessors from Euclid until the *Vaiaio;* the part- and chapter-headings; and a single long problem (fols 174^r–175^r) explaining and making use of formal fraction involving a *quantità*. [Simi 1999] transcribes the section containing a vernacular version of Fibonacci's *Pratica geometrie*.

or, sometimes, just separated by dots, as "4·10·389" for the *ripiegho* of 15560 (fol. 7^{v}).[343] In spite of the convenient borrowing of examples already worked out by Fibonacci and the nod to his terminology, the whole part 1 must be said to correspond to familiar abbacus ways.

The same can be said about *part 2* (fols 8^{r}–19v), with similar exceptions. It deals with the arithmetic of fractions. Noteworthy in chapter 7 – dealing with "extraordinary fractions", that is, fractions taken of numbers – is the observation (fol. 13^{r}) that "to take $^{3}/_{4}$ of 59" is "vernacular for multiplying $^{3}/_{4}$ times 59" – similar to what Fibonacci does (above, p. 62). This is the first time the author stakes his claim in the world of "magisterial" learning – so far modestly.

Not quite as modest, though this time implicit, is the beginning of chapter 8, which with a generic reference to Euclid, Boethius and Jordanus explains (fol. 14^{r}) "the way to bring to a known part the ratio which one quantity has to another one of the same kind". Firstly, the ratio concept (*proportione*) is peripheral in abbacus culture; secondly, when applied, the observation that ratios only exist between quantities of the same kind is rarely made (if ever). Here the author shows that he knows the scholarly way, and provides a bridge to abbacus habits. This, however, is little more than an aside. After two paragraphs, the second of which is closed by the observation that "it is not our habit to denominate proportions if not according to parts, like $^{1}/_{2}$ or $^{1}/_{3}$ or $^{2}/_{5}$ or $^{7}/_{8}$", the chapter goes on with the expression of amounts of *denari* as parts of a *soldo*, of numbers of *soldi* as parts of a *lira*, numbers of months as part of a year, etc. – all questions that were close to the interests of the abbacus school.[344]

Part 3 (fol. 21^{r}–33^{v}) promises to "demonstrate some rules about proportions and the nature of numbers", specified afterwards to concern "the rule of 4 proportional quantities, in the vernacular called the rule of three things", which also intimates a "magisterial" orientation.

Its first chapter presents the fundamentals of proportion theory, starting with the explanation of continued and non-continued proportionality and repeating the need that the "first two" quantities in a proportion should be of the same kind, as also the last two. This is evidently different from the usual abbacus introduction of the rule of three, but

---

343 These should *not* be taken as an early instance of the dot meaning multiplication; they correspond to the dots used to set out numbers in the text from surrounding words, as in the corresponding formulation *se'lla reghola di ·15560· vuoi trovare*, "if you want to find the rule for ·15560·".

344 The Palatino *Pratica* (fol. 17^{v}) has more precise initial references to Boethius and Jordanus, offering correct quotations of their respective definitions of a proportion (with equally correct references to book and chapter). It also gives more examples of how to denominate proportions. It closes the magisterial aside with the formulation, "we in the schools do not use such words" (*noi alle schuole non usiamo tali vocabuli*).

also from Fibonacci's introduction of the topic (above, p. 71). It is clearly a *new* "magisterial" rationalization of the rule of three – hardly invented by the present writer but perhaps introduced by the *Vaiaio*, since something rather similar though more elaborate is found on fol. 24r of the Palatino *Praticha* [ed. Arrighi 2004/1967: 177].

The second chapter first repeats the request of similarity of kinds and then gives examples of how to apply the rule that are not borrowed from Fibonacci. Chapter 3 turns to the "nature and properties of numbers", explaining the categories known from Boethius but opening with a reference to *Maestro Antonio:* the division into even and odd and the arithmetic of these classes; the division into abundant, deficient and perfect numbers; and figurate numbers (*numeri geometrici*), that is, linear, triangular, square, oblong (*parte altera longiore*, thus named in Latin), pentagonal, hexagonal and pyramidal, together with the rules for generating these from sums.[345]

The chapter includes two schemes. The first illustrates the Euclidean rule for producing even perfect numbers ($2^{n-1}\cdot[2^n-1]$, provided that the second factor is prime), and after the appurtenant calculations states the first six perfect numbers to be 6, 28, 496, 8128, 3550336 and 8589869056.[346] The interest and the idea for the scheme (and even the structure of the whole chapter) is also found in the Palatino *Praticha* (fol. 28r) as well as that of Benedetto (fol. 24v), both of which however stop at 3550336 (Benedetto performing a control for this number). While the idea and the extension until five numbers is thus likely to be derived from preceding tradition or from an earlier writing, the calculation of the sixth number can with some plausibility be ascribed to the author of the Ottoboniano *Praticha*.[347]

---

[345] Triangular numbers as Σn, square numbers as $\Sigma(2n-1)$, oblong numbers as $\Sigma(2n)$, cube numbers as 1, 3+5, 7+9+11, 13+15+17+19, ..., etc.

[346] 496 has been skipped in the scheme, but it is found in the text.

[347] As we shall see, Regiomontanus drew (at least in 1464) on the model on which the Ottoboniano *Praticha* was mainly based (below, p. 359). One might therefore try to elucidate the question by looking at Regiomontanus's list of perfect numbers in his *Rechenbuch* (Wien, Österreichisches Nationalbibliothek, cod. vindobon. 5203, fol. 167r), which also offers a stepwise computation and control but gives up before having certified that 131071 is a prime number, and therefore only gives the first five. However, according to the shapes of the numerals 4 and 7, this list may be written some years before 1464. Moreover, Regiomontanus's layout is quite different from what we find in the Ottoboniano and Benedetto *Pratiche* but on the other hand almost the same as that found in the manuscript Venice, Biblioteca Nazionale Marciana, fondo antico 332, fol. 36v – a Latin 13th-century manuscript owned by Bessarion at the time Regiomontanus met him. The only difference is that the Marciana manuscript has the higher numbers on top, while Regiomontanus starts with the lowest ones. Both show that 4096×8191 = 33550336 is perfect, while 8192×16383 is composite; Regiomontanus takes three steps more but fails to control whether 131071 is prime (which it is) and therefore does not discover that 8589869056 is perfect. In the parts they share the two texts

The second scheme shows the generation of the various polygonal numbers as sums. Even this is shared with the two other encyclopedias (though split up in that of Benedetto, corresponding to a much more detailed exposition).

The fourth chapter of part 3 is stated to deal with "cases solved by the principle of *elchatain*" (fol. 25^{v}). The same term is used in the Palatino *Praticha* (fol. 29^{v}) with the same meaning, and the new meaning given to it here can therefore not be an idiosyncrasy. In the *Liber abbaci* (as in Arabic), it stands for the double false position (above, p. 108). The present chapter applies simpler methods to a sequence of first-degree number problems.

The first (fol. 25^{v}) asks for a number whose $^{2}/_{7}$ is 20, and finds it by simple division, as $20\div{}^{2}/_{7} = 70$. A variant changes the formulation "of a number, seven parts are made, 2 of which amount to 20", and solves by the variant that the remaining 5 parts amount to $2\,^{1}/_{2}\cdot 20 = 50$, which added to the 20 gives the same 70.

A slight complication follows: $^{2}/_{7}$ of the number with 8 added amounts to 28, which is easily reduced to the first case. After several analogues of this follow divided-number problems – a small selection will suffice:

(fol. 26^{v}) $30 = a+b\ ,\ \ a = 2b–6$
(fol. 27^{r}) $10 = a+b\ ,\ \ a : b = 4 : 3$
(fol. 27^{v}) $10 = a+b+c+d\ ,\ \ a : b = 1 : 2\ ,\ \ b : c = 3:4\ ,\ \ c : d = 5 : 6$
(fol. 28^{r}) $10 = a+b+c\ ,\ \ a : b = 1 : 2\ ,\ \ a : c = 3 : 4$
(fol. 28^{v}) $10 = a+b+c\ ,\ \ a = {}^{2}/_{3}b\ ,\ \ a = {}^{2}/_{5}c$

Starting with the last of these, the direct arithmetical and proportion-based methods are replaced by use of the *regula recta* with unknown *quantità*, abbreviated *q* in marginal calculations. Since the corresponding single position (of something to be a *quantity*) is said on fol. 28^{r} to be "one of the simple modes of *elchatain*", the students of the *Vaiaio* (and probably their teacher) seem to have taken the word to refer to any kind of false or *regula-recta* positing (I have not noticed the usage in question elsewhere before Pacioli, see below, p. 345).

Before applying the *regula recta* for the first time, the author explains on fol. 28^{v} that the *modo recto* (as he calls it) is used by "Leonardo [Fibonacci] and all the others who understand". Obviously, the writer knows it from the *Liber abbaci*, and also takes over Fibonacci's statement that the method comes from the Arab. But the reference to "all the others who understand" (*tucti gl'altri intendenti*) shows that he also knows if from a general abbacus tradition, within which, as he says, "some say it is one of the

---

are identical word for word.

We may conclude that Regiomontanus did not draw for this on contemporary Florentine material, but also (since their layout is quite different) that the Florentine encyclopedias do not draw on this Latin manuscript or any close kin.

exemplary modes of algebra" (*alchuno lo dice uno d'esemplari modi dell'algebra*), which is nowhere found in Fibonacci's writings. The use of *quantità* as unknown and the naming differing from that of Fibonacci leaves no doubt that the main reference of our writer is the living abbacus tradition, not Fibonacci.

More problems solved by means of the *modo recto* follow, in some of which it is much more needed than in the first one. In one (fol. 321^{r}) the author refers to "my Leonardo P.", just as Ficino would speak of "our Plato" (*il nostro Platone*) 29 years later [ed. Figliucci 1563: II, 188^{r}] – the (somewhat preposterous) way to speak of honoured friends.

As we see, the problems solved in this last chapter of part 3 are not very different in kind from what we know from other abbacus books, though not linked in any way (whether straight or distorted) to commercial activity. *Part 4* (fols 33^{v}–116^{r}), instead, addresses commercial matters – often fitted to Florentine usage. Chapter 1 is dedicated to monetary questions, chapter 2 to selling (cloth) according to length (Florence owed its position as one of the five most populous European cities to its textile production and trade [Brucker 1969: 51*f*, 53*f*]). Chapters 3 and 4 teach alloying and connected topics, chapter 5 is dedicated to barter, chapter 6 to mixed cash- and barter trading and to delayed barter payment, chapter 7 to Florentine and comparative metrology (but also questions like "4 pears are worth 5 apples, 3 apples are worth 10 nuts, 8 nuts are worth 5 figs ..." (fol. 98^{v}), training the composite rule of three. Last, chapter 8 deals with partnerships.

Part 5 (fols 116^{v}–221^{r}) contains recreational riddles, announced in the preamble (fol. 1^{r}) as "cases called rambling,[348] or we shall say gentle and pleasant" (*chasi detti erratici o vogliamo dire gentili e dilettevoli*). Much is drawn from the *Liber abbaci*.

The first chapter (fols 116^{v}–118^{v}) deals with "cases of horses eating barley, which are written in the ninth chapter of Leonardo F." As we remember from p. 74, these are barter problems in a recreational dress but used as the basis for a theoretical inquiry. The present text follows Fibonacci rather closely, but adds some extra problems.

Chapter 2 (fols 119^{r}–121^{v}) corresponds to part 12.1 of the *Liber abbaci*, "collections of numbers" (above, p. 78); here, it is announced as dealing with "cases solved by the nature of numbers". At first it teaches the summation of various arithmetical series with upper limit 60; 1+2+..., 2+4+..., 3+6+..., 8+16+...+56 – then varied with other limits, after which follow sums of squares and cubes. The second half (as that of the corresponding part of the *Liber abbaci*) deals with problems of pursuit where the speed of one man is constant and that of the other increases arithmetically from day to day.

The beginning of chapter 3 (fols 121^{v}–139^{v}), "certain cases solved by means of proportions and given rules", corresponds to part 12.2 of the *Liber abbaci* (above, p. 78)

[348] Cf. above, pp. 78 and 92 about Fibonacci's ambiguous (sometimes wider, sometimes more restricted) use of the term. The present writer shares the wider interpretation.

and is rather faithful to it – also in the explanation that the request for a fourth proportional is expressed in "vernacular" usage by the counterfactual question, "if 3 were 5, what would 6 be". The counterfactual *calculation* and the two ways in which it can be understood is skipped, however, probably judged to be too pretentious and of little interest. Also skipped is the request to divide 10 into four parts in proportion (cf. above, p. 80).

After the construction of a continued proportion with an arbitrary number of steps chapter 3 jumps (fol. 122^{v}) to part 12.3 of the *Liber abbaci*, "questions of trees" (above, p. 80), thus skipping the rest of Fibonacci's part 12.2. It starts with the tree of which $^1/_3+{}^1/_4$ is underground, and presents the solution by means of a false position. The following problems, formulated around amounts of money, do not depend on Fibonacci. More interesting is a sequence of problems of type "lion in the pit" (see note 44 and surrounding text). Different than Jacopo and Fibonacci, the present author understands the prank. In the first problem (fol. 123^{v}), somebody goes from Florence to Pisa. The distance in 40 miles, and by day he makes 8 miles, going backwards 6 miles by night. So, in the first day he makes 8 miles. 32 miles remain, and in each night with following day he makes 2 miles more. The total duration of the trip is therefore $1+32\div2 = 17$ days. After two more questions with similar dress comes Fibonacci's lion-problem (fol. 124^{r}); the writer avoids to censure Fibonacci himself, mentioning solely that "some" solve it differently, after which he follows his own method.[349] After this group more problems with varying dress follow (purchase of cloth, pursuit with two constant velocities) – mostly numerical variants of problems from the *Liber abbaci*. Other problems of increasing complexity but belonging to usual abbacus types follow. On fol. 126^{v} we find this:

> The cubit of cloth is worth $2\,^1/_2$ *fiorini*. The cubit of velvet is worth $7\,^1/_2$ *fiorini*. The cubit of cramoisy silk [*chermisi*] is worth 10 *fiorini*. Somebody has 1000 *fiorini* and wants to buy cloth, velvet and cramoisy silk. And he wants 2 times as much of velvet as of cloth plus 4, and 2 times as much less 4 of cramoisy silk as of velvet.

Here (as already once before in a pursuit problem), the *modo retto* is appealed to, the amount of cloth being posited to be a *quantity*. In the text the calculations are expressed rhetorically, in the margin (with q standing for the *quantity*) they appear as piecemeal symbolic calculations that can be summarized in the equation

$$2\,^1/_2\,q+7\,^1/_2\cdot(2q+4)+10\cdot(2\cdot(2q+4)–4) = 1000$$

which yields $q = 16\,^4/_{23}$.

Other problems follow, some about the purchase of cloth, wool, wine, or eggs and oranges, others about exchange of monies, about barter, or about wages; some of them make appeal to the *modo retto*, others not, but all are accompanied by marginal calculations. One (fol. 132^{r}) is of particular interest both for its mathematics and because

[349] When solved in this way, the problem clearly does not really belong together with the tree problems.

it illustrates how this kind of problems served as challenges among the abbacus masters (and thereby also as a gauge for the level that was considered difficult – trivial matters could never serve as challenges):[350]

> 5 eggs and 4 oranges and 10δ are worth 8 eggs and 2 oranges and 6 δ. And 7 eggs and 6 oranges less 3 δ are worth 5 eggs, 4 oranges and 7 δ. It is asked, what is an egg worth, and what is an orange worth? This case has been given to me a few days ago to solve.

The marginal calculations are indubitably algebraic, just with no symbols for the two prices, which are replaced by positions in columns; already the question itself is obviously thought of as a set of two rhetorical equations. We may present the marginal calculations like this, providing the columns with the headings which the writer had in his head, and which are present in his rhetorical version in the text; the long stroke serves for confrontation, here as mostly the confrontation of the two sides of an equation:

E	*O*	δ		*E*	*O*	δ
5	4	10		8	2	5
		5				
5	4	5	————	8	2	–
	4	5		3	2	
	2					
	2	5		3		

In this way, the equation 5*e*+4*o*+10δ = 8*e*+2*o*+5δ has been reduced in steps to 2*o*+5δ = 3*e* – just without *e* and *o* being written. In the same way, 7*e*+6*o*–3δ = 5*e*+4*o*+7δ is reduced to 2*e*+2*o* = 0*o*+10. Thus it is summed up that the question has been reduced to one kind (namely, on each side of the equation 3 are worth 2 oranges and 5δ, while 2 eggs and 2 oranges are worth 10δ.

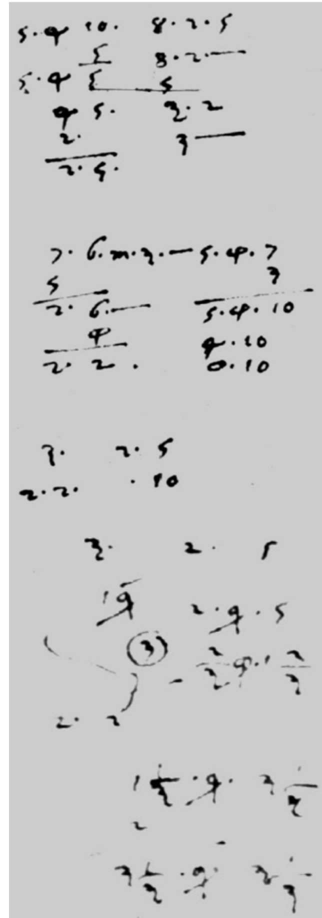
At this point, the *modo retto* is made use of, and the worth of the orange is posited to be a *quantity*. Therefore 3 eggs are worth 2 *quantities* and 5δ; 1 egg hence $^2/_3$ *quantity* and $1\,^2/_3$ δ; and 2 eggs and 2 oranges in consequence $3\,^1/_3$ *quantity* and $3\,^1/_3$ δ, but also 10δ. Therefore, $3\,^1/_3\,q$ are worth $6\,^2/_3$ δ, and hence the *quantity* equals 2δ, which is thus the worth of an orange, while the egg, being worth $^2/_3$ *quantity* and $1\,^2/_3$ δ, is worth 3δ.

Even though no explicit symbols but only loosely made columns are used for the

---

[350] We may compare with the give-and-take problem that had been proposed to Fibonacci in Constantinople some 250 years earlier (above, p. 83). The mathematical level was higher, but the problem types remained the same.

A number of challenges exchanged between abbacus masters are discussed in [Ulivi 2015].



unknowns, we see that the techniques of handling two linear equations with two unknowns could be implemented without being seen as a striking innovation.[351] Benedetto will present us with explicit use of single-letter symbols, which make the column organization redundant (below, p. 295).

A number of combined-work problems in familiar dresses follow from fol. 133^{v} onward: a lion, a leopard and a bear eating a sheep, two ants respectively two men meeting, a ship hoisting up two sails, the emptying of a tun respectively a cask through several outlets[352] – in middle the equally familiar fish in three parts, and afterwards other traditional question types, none of which tell us anything mathematically new or remarkable. However, one (fol. 136^{r}) about an abbacus school which turns out to have 1000 students, reminds us that the parameters of mathematical problems (whether in the 15th or any other century) may have little to tell about real circumstances and should only be used as evidence for social history with great circumspection.

Chapter 4 (fols 139^{v}–172^{r}) deals with "men who have *denari*". At first comes the problem for which Fibonacci (still in part 12.3 about trees) gives a *regula-recta* solution as an alternative (above, p. 81). The present writer gives only this method (as does the Palatino *Praticha*, fol. 160^{r}, with which the whole chapter is shared with minor deviations – details below). More problems from the *Liber abbaci* of increasing complexity follow until fol. 146^{r}, in order and at times with very

[351] They are used again in a similar though somewhat simpler problem on fols 133$^{r–v}$, "5 eggs and 8δ are worth 13δ more than a pair [of eggs]".

[352] The reasons behind the calculations are better explained than often. In the first question, where the lion is said to eat the sheep in 4 hours, the leopard in 5 and the bear in 6 hours, 60 hours is chosen as a common multiple, and in 60 hours the lion is seen to be able to eat 16 sheep, the leopard 12 and the bear 10, together thus 37 sheep – etc.

characteristic phrases that leave no doubt that Fibonacci is a direct source,[353] though sometimes with omissions and sometimes with somewhat changed formulations. Last (fol. 145^{v}, cf. [B202;G339]) comes this:

> Five men have *denari*, and the first and the second and the third ask the fourth and the fifth man for 7 δ, and say to have 2 times as much as they. The second and the third and the fourth ask the first and the fifth for 8 δ, and say to have 3 times as much as they. The third and the fourth and the fifth ask the second and the first for 9 δ, and say to have 4 times as much as they. The fourth and the fifth and the first say to the second and the third, if your give 10 δ, we shall have 5 times as much as you. The fifth and the first and the second say to the third and fourth, if you give 11 of your δ, we shall have seven times as much as you. It is asked what each one had. Because the first, second and third with 7 of the δ of the fourth and fifth have two times as much as they, it is necessary that the first and the second and the third had $^{2}/_{3}$ of the sum less the said δ. So the fourth and the fifth have the third of the said sum and 7 δ more. [...].

As we see, the arguments, though by necessity much more longwinded, are of the same type as those used in Fibonacci's first solution [B190;G324] to the problem copied by the present writer in the beginning of the chapter, *not* the second solution by *regula recta* which he copies for his own treatise. Actually, Fibonacci does not use the *regula recta* any more in this groups after having introduced it (in the 1228 version, as we remember), and after his own use of it in the first problem the Ottoboniano writer follows him faithfully.

The problems from fol. 146^{v} to fol. 155^{v} are independent of the *Liber abbaci* but present in the Palatino *Praticha* (fols 167^{v}–178^{r}) almost verbatim (and thus probably borrowed by both from the *Vaiaio*). They are of the same give-and-take type, with variations like this (fol. 152^{v}):

> Three have *denari*. The first has the half of the other 2 plus 10 δ; and the δ of the second are $^{3}/_{5}$ of the δ of the other two plus 24 δ. The third says to the others, if you gave me $^{1}/_{2}$ of your δ plus 20 δ, I should have twice as much as you plus 10 δ. [...]

– a few times with the further complication that what the participants have are quantities of silver, where the fineness of the metal also has to be taken into account. In all of these,

[353] The second problem in the sequence (fol. 140^{v}) starts "Two have *denari*. And the first having had from the second 7 δ ..." (*Due ànno denari. E'l primo avuto dal secondo 7* δ ...), corresponding to Fibonacci's [B192;G326] *primus, habitis 7 ex denariis secundi....* The Palatino manuscript (fol. 160^{r}) has the habitual "Two have *denari*. The first says to the second, , if you give me 7 δ of yours ...". There is probably no way to decide whether the shared source (probably the *Vaiaio*) was faithful to Fibonacci and the Palatino writer normalized the formulation, or instead the Ottoboniano writer controlled his vernacular source against Fibonacci's text.

the *regula recta* is used systematically,[354] illustrating how a convenient choice for the *quantity* can be made. At one point (fol. 153[r]) it is pointed out that "this type of questions is very fallible, that is, some cannot be solved".

We have already encountered non-linear give-and-take problems in Jacopo's algebra (above, p. 189) as well as in the Florentine *Tratato sopra l'arte della arismetricha* (above, p. 243). The problem collection which Benedetto takes over from Giovanni di Bartolo (a student of Antonio, and thus in the tradition between Antonio and the three encyclopedic *Pratiche*, also contains an appreciable number of non-linear give-and-take and other "men have *denari*" problems;[355] in the *Tratato* they served to illustrate algebraic cases, and Giovanni di Bartolo invariably solves them by means of *thing–censo* algebra. In the Ottoboniano *Praticha*, where they come long before the introduction of this technique, they are solved in other ways (being so mathematically simple that this is possible). Other abbacus treatises find it quite acceptable to use algebra before it is explained, but the present writer avoids it – a testimony of a not too common sense of mathematical order. In the Palatino *Praticha* the group is found in identical form (fols 177[v]–178[r]), so this testimony probably regards the *Vaiaio*, and not necessarily his students.

The first of these non-linear problems (fol. 156[r]) runs:

> Two have *denari*, and the *denari* of the first are more than the *denari* of the second, and the *denari* of the first multiplied by those of the second make as much as joined together. I ask what each one had. In this and similar ones you may posit that the first had as much as you please, taking an integer number, and the second will have one in number and a fraction which is denominated by a number that is one less than that which the first one has.

Expressed in familiar ways, the problem $ab = a+b$ can be transformed into $b(a–1) = a = (a–1)+1$, whence $b = 1+{}^{1}\!/_{a-1}$, which agrees with the solution that is offered, $a = n$, $b = 1+ {}^{1}\!/_{n-1}$. The text adds that if the first has 6 δ, then the second had $1\,{}^{1}\!/_{5}$ δ, if he has 5 δ, then the second has $1\,{}^{1}\!/_{4}$ δ; and if the first has 8, then the second has $1\,{}^{1}\!/_{7}$ δ, since 8 times $1\,{}^{1}\!/_{7}$ is as much as 8 and $1\,{}^{1}\!/_{7}$ (claimed – not verified but rather obvious).

The next problem adds the condition that ${}^{1}\!/_{5}\,a = {}^{1}\!/_{7}\,b$, and solves it by means of a tacit

---

[354] Never mentioned by name, the writer may expect the reader to recognize it by now after earlier references – or he copies from a source which uses the rule without identifying it by name. The later possibility would correspond to the virtual absence of marginal calculations in this section; in contrast, the preceding section, copied from Fibonacci, has its margins rather full of calculations, though apparently written after the main text.

[355] Ed. [Pancanti 1982: 18, 21, 28, 30, 31, 32, 36, 38, 40, 42, 44, 48, 51, 54, 57, 60, 63, 66, 99, 101, 105] – no less than 21 out of a total of 57 problems. The collection further contains a number of warped barter problems involving square roots, like those of the Florence *Tratato* (above, p. 244).

false position. The extra condition is evidently fulfilled if $a = 5$, $b = 7$, but then $a+b = 12$, $ab = 35$. Therefore, $a = 5\cdot{}^{12}/_{35} = 1\,{}^{5}/_{7}$, $b = 7\cdot{}^{12}/_{35} = 2\,{}^{2}/_{5}$. No explanatory arguments are given for the decisive step, but the solution is shown to be valid. Two further problems increase the complexity by means of coefficients without adding any new principles.

On fol. 156^{v} starts a group with a counterpart in the *Liber abbaci* (above, p. 96). First, however, comes a simple version that is not found in the *Liber abbaci* but which is present also in the Palatino *Praticha* (fol. 178^{r}):

> Three man have *denari*. The first and the second have 18 *fiorini*. And the second and the third have 20 *fiorini*, and the third and the first have 22 *fiorini*. [...]

As in the similar problems in the *Liber abbaci*, the trick is to add the three numbers, which must be twice the total possession of the three men. From the half (30), the total possession, 18 (the possession of the first and the second) is subtracted, whence the third man has 12 *fiorini*. Etc.

The next problem is a translation of Fibonacci's first problem (above, p. 96), about four numbers.

For the case where $a+b$, $b+c$, $c+d$ and $d+a$ are given, the example of an impossible question is taken over from Fibonacci, whereas the solvable example is independent.[356] Two similar questions about five men (fol. 157^{v}–158^{r}) are borrowed from Fibonacci.

The next problem (fol. 158^{r}) is not. It deals with three men having *denari*, and the structure is $a+b = 20$, $c+b = 22$, $c+a = 24+{}^{1}/_{4}b$. The same *question* is found in the corresponding place in the Palatino *Praticha* (fol. 179^{v}), but the two manuscripts solve it in different ways. The present text states explicitly that the solution will be according to the *modo retto*, and posits the first possession to be a *quantity*. The Palatino text does not mention the *regula recta* but posits the second possession to be the *quantity*. Moreover, it goes via the usual trick of adding all three sums, and then applies the partnership rule; the Ottoboniano text does neither. It seems most likely that the Palatino way, continuing the approach of the preceding problems, represents the shared source for the question, and that the Ottoboniano writer is the one who innovates here.

Fol. 158^{v} returns to the *Liber abbaci*, but whereas Fibonacci [B286;G456] has a question about the contents of three vessels, the present text (agreeing with the Palatino *Praticha*) normalizes the dress, and speak of the *denari* possessed by three men. The structure is $a = {}^{1}/_{18}b+{}^{1}/_{3}c$, $b = c-{}^{1}/_{3}a$, $c = b+{}^{1}/_{5}a$. Both follow Fibonacci's procedure, making no appeal to the *regula recta*.

The rest of the chapter goes on with problems that are mostly taken over (together with metamathematical observations) from the *Liber abbaci*. A few are only loosely inspired, or independent. On fol. 164^{r} an alternative way to solve an independent problem

356 Found identical, however, in the Palatino *Praticha* (fol. 178^{v}).

is ascribed to "Maestro Gratia".[357] Towards the end (fol. 165^{v}–170^{r}) come three- or four-participant versions of the grasping problem (above, p. 96), with the sterlings changed into *fiorini*.

Chapter 5 (fols. 172^{r}–176^{r}) is announced to be held short because the preceding one is so long, and to be "composed on the basis of cases about men making works extracted from the book of Master Gratia, perfect theologian" (see note 357).[358] The beginning (fol. 172^{v}) is simple:

> Two masters make a work. One would do it in 6 days, the other would do it in 8 days. I want to know, in how many days will they do it, that is, working together. [...]

which is explained in the same way as the common meal of the lion, the leopard and the bear (above, note 352).

Gradually matters become intricate, however – for example (fol. 173^{v}):

> There are two masters who undertake to make a house. The first says to the second, if you would help me for 10 days, I shall make the house in 12 days. The second says to the first, if you would help me for 6 days, I shall make the house in 15 days. I ask, in how many days each one would do it making alone. In this one should understand that 10 days of the second and 12 days of the first is one work. And thus 6 days of the first and 15 days of the second are one work. Then you collect the parts, taking away on each side 10 days of the second and 6 days of the first, and we shall have that 5 days of the second are for 6 days of the first. Thus in order to know in how many days the first would make the work you shall bring the 10 days which the second helped the first to days of the first, and you shall say that 5 days of the second are for 6 days of the first, and 10 days of the second, how many days they are of the first. You shall multiply 10 times in 6 days of the first, and divide in 5, 12 days of the first result from it. And you shall say that 10 days of the second are 12 days of the first, and thus the first would make the work, that is the house, in 24 days [...].

"Collecting the parts" by "taking away on each side" is clearly an algebraic operation on an equation. If we regard "day of the first" and "day of the second" (properly, the outcome of the working day of each) as algebraic unknowns, we thus encounter here

---

[357] Gratia de' Castellani, a contemporary of Giovanni di Bartolo – an Augustinian friar and teacher of theology at the university and within his order; according to Benedetto (*Praticha*, fol. 408^{v}) he was no abbacus teacher but a high-level amateur writer on the topic, from whom Benedetto also received some guidance for subtle questions.

[358] The Palatino *Praticha* (fol. 195^{v}) promises conciseness in the same words and for the same reason but says nothing about Gratia de' Castellani. In the beginning the problems are the same, but not throughout.

another instance of the use of two unknowns in a rhetorical algebraic problem solution.[359] As in earlier instances, it is not understood by the present writer as something noteworthy. The method almost certainly goes back to Gratia de' Castellani – not least because the same problem, with the same solution down to the details, is found in the Palatino *Praticha* (fol. 200^{r}). Here, it is followed by 5 more problems of the same kind, with two, three or four participants (sometimes each receiving the help of several of the others). In all cases, the basic method is the same, thus solution of simultaneous linear equations with up to four unknowns. This closes the chapter on works in the Palatino *Praticha*.

In the Ottoboniano *Praticha*, instead, the next problem (fol. 173^{v}) – the one which closes the chapter – makes use of the *regula recta*, with a single unknown:

> Three make a house in this way, that the first and the second, working together for 8 days, would make the work; and the second and the third, working together 14$^{2}/_{5}$ days, would make the work, that is, the house; and the third and the first would make the work, that is, the house, in 9 days. I ask in how many days each one of them would make the work on his own, that is, alone. You will take this way, that you make position that the first would make the work in a *quantity* of days. Where, in order to find in how many days the second (would make the work), you shall take this way: that you know, 8 days, which the first works with the second, which part are they of a *quantity*? Which is $\frac{8}{1\ \textit{quantity}^{\,th}}$.

As we see, the writer here makes use of a formal fraction; beyond the fraction line, however, he also uses the ordinal suffix (*esimi* in the original), supplementing the indication of the division by means of a fraction line with the way to speak of it rhetorically, similar to 8 *nth*s. Since this problem has been edited by Arrighi [2004/1968: 216–221], there is no urgent need to complete the translation, but note should be taken that the method leads to a second-degree equation (albeit with no degree-zero term, and thus reducible), where the square on the *quantity* is spoken of as "quantity of quantity" (*quantità di quantità*).

Chapter 6 (fols 176^{r}–189^{v}) deals with "men who find purses", corresponding to part 12.4 of the *Liber abbaci*, and following it rather closely. All problems are taken over from Fibonacci. Often the precise formulation deviates, but only three instances go beyond changes of words and details of pedagogical style.

First, on fol. 179^{v}, in a standard problem about three men finding a purse, the writer introduces the position that the first has a *quantity* (thus tacitly appealing to the *regula recta*); afterwards, Fibonacci's way [B216;G361] is given as an alternative (and Fibonacci's ensuing alternative discarded). In the *regula-recta* solution, the *purse* (*borsa*) is treated as a second algebraic unknown, and in the rudimentary marginal notes

---

359 We may even claim that the expression "one work" (*uno lavorio*) means that the texts works with three unknowns, of which one (the *work*) is eliminated straight away.

the two appear as single-letter abbreviations, *q* and *b*.[360]

Second, on fol. 181^{v}, in a problem about four men finding 4 purses the text observes that "you could solve it in the given way" (probably referring to that of the *Liber abbaci* [B220;G365]), but I will make it by the *modo retto* – and once again this leads to an argument where the *purse* becomes a second algebraic unknown (thus confirming that the *regula recta* was not restricted to a single unknown), supported by marginal calculations where *q* and *b* represent the two unknowns.[361]

Finally, on fol. 182^{r}, the text makes use of a position (and thus of an unidentified *regula recta* and not of Fibonacci's method [B222;G367].[362] This time, no second unknown is made use of.

In the end (fol. 188^{r}), after the observation that many other problems of the same kind can be made, the text adds a single example about three men having money, which is solved be means of the convenient position that the third man had 2 *quantities*. In the text the *purse* appears as a second unknown, and in the margin *quantity* and *purse* appear as *q* and *b*. The Palatino *Praticha* (fol. 217^{v}) is similar, but as usually has no marginal calculations.

Chapter 7 (fols 189^{v}–206^{r}) deals with "men buying horses", as does chapter 12.5 of the *Liber abbaci*, which the writer says to draw on.[363] It begins indeed not only with the same two-participant purchase (cf. above, p. 88) but also first finds the solution according to an opaque rule, only to explain its origin afterwards by means of what in agreement with Fibonacci is called "the rule of proportions" (that is, "the finding of proportion of the bezants of the one to the bezants of the other" – above, p. 88). As in the *Liber abbaci*, a solution by means of the *regula recta* follows (here called *modo retto*, Fibonacci's alternative name *per modum arabum* has disappeared).

On fols 190^{v}–196^{v} follow a number of problems, some borrowed from Fibonacci, others not, but all making use of the *modo retto* based on two algebraic unknowns *quantity* and *horse* (*quantità* and *chavallo*), abbreviated in the rudimentary marginal calculations *q* and *cha* (the latter kept together with an arch over the letters). Some of the questions are outside the beaten path, like this one (fol. 194^{r}), where the possibility to apply

---

[360] In the corresponding problem in the Palatino *Praticha* (fol. 208^{r}), the main text is the same (including the inclusion of Fibonacci's way as an alternative), but there are no marginal notes.

[361] The Palatino *Praticha*, fol. 210^{v}, has exactly the same words though not the marginal calculations. The "I" who wants to solve the problem by means of the *regula recta* is thus earlier that the two writers in question – maybe the *Vaiaio*, since neither refers to named predecessors for this chapter.

[362] The Palatino *Praticha* is identical (fol. 211^{r}).

[363] In my scan from the Palatino *Praticha*, part of the corresponding chapter is illegible. What can be read makes it clear that the substance is very similar, but some formulations different.

coefficients to the unknown *horse* seems to have influenced the question:

> Three men want to buy with their *denari* a horse. The first says to the second, if you give me $^1/_2$ of your *denari*, I shall buy one horse. The second says to the third, if you give me $^1/_3$ of your *denari*, I shall buy 2 horses. The third man says to the first, if you give me $^1/_4$ of your money, I shall buy 3 horses. I ask, how many *denari* each one had, and how many *denary* the horse had or rather was worth. Again by the *modo retto* you shall say, I make position that the first man had a *quantity*. [...].

Also unusual is what follows on fol. 194^v,

> Three have *denari* and want to buy a horse. The first says to the second, if I had $^1/_2$ of your *denari*, I should buy the horse, and 6 *fiorini* would be left over. The second says to the third man, if you gave me $^1/_3$ of your *denari*, together with mine I should buy the horse, and 8 *fiorini* would be left over. The third man says to the first, if you gave me $^1/_4$ of your *denari*, together with mine I should buy the horse, and 10 *fiorini* would be left over. [...].

Once again, one may suspect an inspirational offset from the solution technique (whether by the "rule of proportions" or the *regula recta*) to the question that is asked.

After another similar problem but now with deficits comes this warning (fol. 195^v):

> And observe that in these proposals one must keep the eye of the intellect open, because to every great calculator [*ragioniere*] they give shame when he believes to have an easy case, so, as said, take care![364]

From fol. 197^r onward comes a sequence of problems from the *Liber abbaci*, explicitly stated in the beginning of the sequence to be solved by Fibonacci's method. Three problems (199^r, 203^r, 205^v) do not come from this source, but still use Fibonacci's method (the first one saying it explicitly).

Fol. 202^r repeats Fibonacci's observation [B251;G408] that problems with an even number of participants in which more than two together make the requests sometimes possess solutions and sometimes not; also repeated are the two subsequent illustrating problems, one *impossible* and one possessing a solution (actually many, since the problem is indeterminate).[365]

While Fibonacci characterizes the whole collection of recreational problems as well as a particular subsection as "rambling problems" (above, pp. 78 and 92), part 5 chapter

---

364 All three anomalous questions as well as this warning are also present in the Palatino manuscript (fols 224^v–225^v). They are thus taken from the shared source – probably, since no name is given, the *Vaiaio*.

365 Both the observation and the examples are also in the Palatino *Praticha*, fols. 233^r–234^r.

8 of the present treatise (fols. 206^v–211^r) is told to contain "the rest of the rambling cases, and almost the end of the pleasing cases" (*el rimanente de' chasi erratici e quasi fine a' chasi da dare diletto*).[366]

Skipping Fibonacci's first problems, the chapter initially follows the *Liber abbaci* faithfully in its presentation of the simple as well as the sophisticated versions of the "unknown heritage" (above, p. 92) – only deviating by speaking of "sons" even when this leads to fractional sons.[367] Then (fol. 208^r) follows the "Chinese remainder theorem", also faithfully repeated, only with the difference that in one of the problems (fol. 2098^v) the number to be divided by successive divisors is replaced by eggs in a basket to be counted in twos, in threes, etc.[368] In agreement with the *Liber abbaci* then comes (fol. 209^r) the problem about a fallacious application of the partnership rule:[369]

> Two men have loaves, of whom the first has 3 loaves and the second has 2 loaves. And walking their way to a fountain they settle down to eat, and a friend of theirs passing by eats with them, and they eat equally. Their friend left, or should we say gave to them 5 ß for the bread he had eaten. I ask what each one should have. Even though some say that the first should have 3 ß and the second 2 ß, they do not speak the truth. Because it is a rather obvious matter that when the friend gave them 5 ß, all three eat as much as was worth 3 times 5 ß, that is, they eat as much as was worth 15 ß. And these 15 ß are the worth of 5 loaves. Therefore posit that 5 loaves are worth 15 ß, where one loaf is worth 3 ß. For which reason the first who had 3 loaves, which were worth 9 ß, but he eat for 5 ß. Therefore he should have 4 ß. And the one who had 2 loaves had 6 ß, and he eat for 5 ß, and he should therefore have 1 ß. And therefore the first ought to have 4 ß and the second ought to have 1 ß.

Three things have happened here to Fibonacci's text. His "soldier" has become a "friend"; the 5 gold bezants have become *soldi* (a slightly more realistic though still exorbitant

---

[366] The problems about composite travels, which in the *Liber abbaci* follow the horse trade, have indeed been moved from the recreational part 5 to the end of part 6.

[367] With minor changes of the words, the Palatino *Praticha* agrees with the present text in most of the chapter. So, even here, both draw on a shared source, presumably the *Vaiaio*.

[368] The eggs are broken by accident, and the owner does not know how many there were, only that when counting them by two's, three's, etc. until six's the remainder was always one below the measure, while 7 divided.

The same story, though with remainder 1 (thus the basic example of the Remainder Theorem) is told in similar words in a slightly earlier or roughly contemporary Byzantine problem collection [ed., trans. Hunger & Vogel 1963: 72*f*].

[369] Also in the Palatino *Praticha*, fol. 241^r; the following more complex additions to the single problem in the *Liber abbaci* are also there.

price). And the argument is different – Fibonacci calculates that each participant consumes $1\,^2\!/_3$ loaf, whence the guest gets $1\,^1\!/_3$ loaf from the first and $^1\!/_3$ loaf from the second.

Fibonacci has no more problems of this type, but the present writer adds three tangled variants. Then follows (fol. 210r) a mathematical riddle:[370]

> A father gives eggs to each of his three sons, and to the oldest he gives 50, to the second he gives 30, and to the youngest he gives 10. An he tells them to go sell them, and to sell them in the same way per *denaro*. And they go

– and when they come back it turns out they all have sold for the same amount. No calculation is given, but it is explained that they have sold at two markets, in the first 7 eggs per *denaro*, in the second 3 *denari* each. In the first market, the first sold 49 eggs, the second 28 eggs, the third 7 eggs; the rest they sell in the second market, and thus each gets 10 δ. So, this is no mathematical problem proper but a riddle, and has to be answered as such.[371] In Benedetto's *Praticha* (fol. 293r), the problem (formulated about "somebody" giving the eggs to three men) turns up in the vicinity of proper riddles – for instance, the classic about a wolf, a goat and a head of cabbage that are to be ferried over a river. Benedetto says about the problem that its kind is called *ragioni apostate*, "improper problems". Nothing similar is said neither here nor in the Palatino *Praticha*, where the problem is found on fol. 242r. As we shall see below (p. 308), there are fair though not compelling reasons to believe that the term was introduced by Benedetto.

Last in the chapter comes the twin problem (above, p. 22), differing from Jacopo's version only by stating in the beginning that the bequest is 3000 *fiorini*.[372]

---

370 Also in the Palatino *Praticha*, fol. 242r.

371 The *Liber abbaci* (above, p. 101) contains a similar problem about two men selling apples (and many variations). This problem is strongly underdetermined (even more because the gain of each is not given); Fibonacci offers a way to find one of many possible solutions.

The present story is found in a fifteenth-century problem collection (*Istratti di ragioni*) claiming to go back to Paolo dell'Abbacho [ed. Arrighi 1964: 85]. Paolo (?) comments that "this problem is without rule and of rather little utility", and then states the same solution as here.

372 Before the twin problem, the Palatino *Praticha* brings two other "rambling" problems which in the Ottoboniano manuscript are moved to chapter 10; one is borrowed from Fibonacci, the other not. First comes the "weight problem" (fol. 242v, cf. above, p. 99), then (fol. 242v) a problem about decanting between bottles containing 8, 5 and 3 ounces in such a way that the contents of the larger (which starts full) is distributed equally between the two larger bottles. This problem is known from the Columbia algorism (*CA*, #123, p. 131) but also presented in the problem collection purportedly going back to Paolo dell'Abbacho [ed. Arrighi 1964: 62] (where even this is characterized as being not very useful, "and made solely for orientation"). After the twin problem follow a pure-number problem and several give-and-take problems, none of which correspond well to the "rambling" category.

Chapter 9 (fols 211^{r}–217^{r}), *chasi d'indivinare*, follows Fibonacci's part 12.8 about divinations (above, p. 102), adding extra examples for the rules given and replacing others.[373] In the end is added the problem (the "Joseph game"), how to order two groups of 15 persons each in a circle in such a way that, by counting off by nines, one of the groups is separated out – here Christians and pagans.[374]

Most of chapter 10, "The doubling of the chess-board" (fols 217^{r}–221^{r}), follows *Liber abbaci* part 12.9 (above, p. 102). It even copies Fibonacci's blunder (above, p. 104) but changes the following calculation from Pisan to Florentine metrology; it also takes over the alternative interpretation of the doubling as well as the rest of Fibonacci's problems (described above, p. 106), only changing the bezants in the problem [B313;G491] into apples. In the end, as mentioned in note 372, the two problems are inserted which the Palatino *Praticha* brings before the twin problem in the chapter on "rambling" problems.[375] That closes part 5.

Part 6 (fols 221^{v}–252^{r}) returns to the commercial domain, more precisely to questions involving interest. The appearance of tables of composite interest derived from those of Antonio on fols 225^{r}–233^{r} was mentioned in note 310, but first comes an collection of problems borrowed from Gratia de' Castellani (fols. 221^{v}–224^{v}), dealing with matters like those we have already encountered in Jacopo's *Tractatus* (above, p. 19). After the tables come more of the same kind, together with simple discounting (fols 234^{r}–236^{r}) – for example (fol. 235^{v}), what is the interest rate if 140 £ to be paid in 18 months from now is reduced to a payment of 125 £ now, which obviously reduces to a question about the interest rate which makes 125 £ grow to 140 £ in 18 months?

Somewhat more interesting from a mathematical point of view and more demanding when it comes to the handling of numbers is the following chapter 3 (fols 236^{r}–240^{v}), concerning composite interest and corresponding discounting not necessarily linked to the year (cf. above, p. 230). At first the principle is explained – adding the interest (here of the year) to the capital, and taking this new value of the capital as basis for the interest to be paid the following year – and next what to do if the last term is only a fraction of a year. One of the examples (fol. 237^{v}) asks for the value of 125 £ 12 ß 6 δ after two years, 8 months and 10 days if the interest rate is 12 % per year. In the end come other terms than the year – for instance (fol. 238^{r}), to find the value of 100 £ after a year and 10 months at an interest rate of 4 δ per £ if accounts are made up every 6 months.

---

373 The Palatino *Praticha* (fols. 246^{v}–253^{v}) is very similar.

374 A short account of the history of this problem (referred to as *ludus Joseph* in [Cardano 1539: T $\mathrm{iii}^{\mathrm{v}}$]), with further references, can be found in [Tropfke/Vogel et al 1980: 652–655] – cf. also [Smith 1923: II, 541–544]. It appears to have been adopted by the abbacus tradition in the 15th century.

375 The Palatino *Praticha* does not repeat these, but apart from that it is similar.

Even more computationally demanding though not involving new principles are chapter 4 (fols 240^v–242^v), about the reduction to a single date of payments to be made at different moments; chapter 5 (fols. 243^r–245^v) about payments made in several tranches; and chapter 6 (fols 245^v–248^r) about making up debts due at different moments at a single date.[376] Chapter 7 (fols. 248^r–252^r) contains supplementary problems within the same area, at first five amortization problems about loans whose interest is repaid by the rent of a house, similar to what we have encountered in the *Liber abbaci* (above, p. 92), then others of recreational type – for example (fol. 249^v)

> Somebody lends to another one I do not know how much nor at how much the *lira* a month. And at the end of the year wanted to give him back 100 £ for interest and capital. And he says to him, keep them for another year at this same rate, and at the end of the second year he gave him back 120 £ for interest and capital. [...]

No difficult problem, evidently. The interest rate is seen directly to be 20%, that is, 4 ß per year per £ or 4 δ per month per £; discounting $^1/_6$ from 100 £ shows that the original loan was 83 £ 6 ß 8$^3/_4$ δ.

The *Liber abbaci*, we remember, added problems about amortization by rent due after those about repeated travels with gain and expenses in part 12.6, pointing out that the mathematical structure is the same. Here, the order is inverted, and chapter 8 of part 6 (fols 252^r–260^r) shows how to solve questions about "men making travels because these are linked with those on earning interest".

In general, the present text follows Fibonacci quite closely, with few deviations. In the first problem, the starting point for the first travel is Florence, not Pisa (cf. above, p. 89). Quite often, when Fibonacci introduces a variant, he merely states the parameter that is changed. Here, these variants are treated as independent problems, and the complete information given. Even the problems where Fibonacci makes use of two algebraic unknowns are taken over (*regula recta* as well as *regula versa*), the unknowns being designated *somma* and *quantità*.

In the end (fol. 260^r) of the chapter comes an apple-picking problem which in the *Liber abbaci* follows after the amortization problems [B278;G445]; in this case, Fibonacci offers an alternative solution by means of the *regula recta*. The present writer says that it can be done but "it is not to be explained" (*non est dicendum*, a Latin stock phrase showing that the writer had some basic Latin training[377]).

---

[376] The technical terms are *del meritare e schontare a fare chapi d'alchuno termine* (chapter 3); *il modo di rechare a termine denari dati in più partite et in diversi tenpi* (chapter 4); *del modo di fare resti e simili* (chapter 5); and *del modo del saldare* (chapter 6).

[377] In general, the travel chapter of the Palatino *Praticha* agrees with what we find here, but this point is made in Tuscan (fol. 310^r). The Latin may or may not go back to the shared model, but

Part 7 (fols 260v–267r) deals with the “rule of *chatain*”, this time to be understood as double false position. By far the larger part is drawn from the *Liber abbaci*.

As the *Liber abbaci*, the writer starts by stating that the name is Arabic, and gives the colourful interpretation as *due positioni bugiarde*, “two liar positions”. After this beginning he follows Fibonacci’s exposition and line-based proof of the rule (above, p. 108) over 7 folio pages (fols 260v–263v). From Fibonacci’s part 13.1, containing problems already dealt with earlier in the *Liber abbaci* (above, p. 113), it takes over only two problems, and then adds four problems that are not to be found in the *Liber abbaci* – the first of which, however, is simply the determinate version of the grasping problem (the indeterminate form borrowed from Fibonacci’s earlier treatment is on fol. 165v). None of them make use of Fibonacci’s nested structures, that is, of Fibonacci’s most advanced procedures (above, p. 116).[378]

Part 8 (fols 267v–298v) is dedicated to roots. An initial half-page describing the contents of the chapter is independent of Fibonacci while partially duplicated in the Palatino *Praticha* (fol. 317r).[379] After that, the text follows the *Liber abbaci*, beginning with the explanation of “certain necessary matters” from *Elements* II but here “demonstrated by means of numbers” (cf. above, p. 117), leaving out only the term “keys”, which however is conserved in the Palatino text (fol. 317r). The exposition is accompanied by lettered diagrams as well as numerical calculations in the margin; these are absent from Fibonacci’s margins as well as his text, and (with a single exception) from the Palatino *Praticha*. The diagrams in the Latin translations of the *Elements* are two-dimensional and quite similar to those of the Greek text, so the inspiration cannot have come from comparison with any of those. The single line diagram in the Palatino manuscript can make us confident that all of them were in the shared model – whether introduced by the *Vaiaio* or earlier in the tradition between him and Fibonacci is hardly to be known.[380]

Another innovation follows in the succeeding parallel to part 14.1 of the *Liber abbaci*, “extraction of square roots”. At first the writer, so to speak, renders allegiance to Fibonacci, ascribing to him (fol. 269v) the definition of a (square) root of a number that could be found everywhere, “a number which multiplied by itself gives the number”. Then he

in any case the Ottoboniano writer chose it.

378 The Palatino *Praticha* skips the demonstrations and gives the same problems as the Ottoboniano.

379 The Palatino version ends by referring to “what Euclid writes in the 10th book and some of the others”. This observation is absent from the Ottoboniano text; it seems most likely that the latter has left it out from what is copied from the shared model.

380 As we remember, the Palatino *Praticha* also drops the demonstrations in the *chatain* chapter, where similar line diagrams are used throughout in the Ottoboniano *Praticha*.

rearranges the material and describes the habitual approximation $n+{}^{a}/_{2n}$ to $\sqrt{(n^2+a)}$ in abstract terms. In the first example (the root of 10), the procedure is iterated. Fibonacci [B353;G548]stops at $3\,{}^{1}/_{6}-{}^{1}/_{228}$, while the present text completes the calculation, finding $3\,{}^{37}/_{228}$. Strikingly, this second approximation is called "the closest root",[381] a term which in the broad abbacus tradition is used about the first approximation. Another noteworthy deviation from the tradition is that the possibility of approximation from above as well as below is pointed out in the preceding general formulation of the algorithm (which is thus branched, yet another rarity in the abbacus tradition except in the algorithms for arithmetical operations on numbers[382]).

Approximations to the roots of fractions or mixed numbers follow – a theme not dealt with by Fibonacci in this way. The first example is $\sqrt{{}^{2}/_{3}}$, where ${}^{9}/_{16}$ is chosen as a fraction close to ${}^{2}/_{3}$ with square numerator and denominator. Since, more precisely, ${}^{2}/_{3}$ = ${}^{9}/_{16}+{}^{5}/_{48}$, the next approximation is

$$\sqrt{\frac{2}{3}} \approx \frac{3}{4}+\frac{5/48}{2\cdot 3/4} = \frac{54-5}{72} = \frac{59}{72}\ ,$$

which is then considered the "closest root" of ${}^{2}/_{3}$.

The next example is $\sqrt{13\,{}^{1}/_{2}}$, where the square number 16 is chosen as the starting point, which means that this time the approximation is from above. Even this time, the resulting first approximation is considered "the closest", but the first steps of the determination of the second approximation are offered.

An alternative method is then suggested – dealt with later by Fibonacci ([B355;G552], above, p. 120), but what is done here is independent. At first an abstract description is given, then as example once again $\sqrt{10}$, which is transformed into $\sqrt{{}^{10\cdot 900}/_{30}}$. The square number closest to 9000 being 9025 = 95^2, $\sqrt{9000}$ is approximated as $95-{}^{25}/_{(2\cdot 95)}$ = $94\,{}^{33}/_{38}$, whence $\sqrt{10}$ can be approximated as $94\,{}^{33}/_{38}\div 30$ = $3\,{}^{37}/_{228}$ – surprisingly the same "closest root" of 10, which suggests that the factor 30^2 has been chosen with the purpose of achieving this. Application of the procedure (still with factor 30^2) to $\sqrt{{}^{2}/_{3}}$ gives the

381 Using the high-style expression *la prossimana radice*, where for instance Jacopo has *la più pressa*. Beyond stylistic level, there is a slight semantic difference: *prossimano*, like the Latin superlative *proximus* from which it derives, can also mean "very close".

382 The rule of three and the double false position *could* both be formulated as branched algorithms, the former depending on whether fractions occur, the latter according to whether one guess leads to an excess and the other to a deficit, or both are of the same kind. But normally the different cases are explained as parallel branch-free algorithms.

Mostly, it must be emphasized, the problem solutions of the abbacus books are *not* meant as algorithms, that is, as fixed step-by-step prescriptions (whether with or without DO...UNTIL..., IF...ELSE and similar commands) but procedures that can be varied to the extent it is needed; cf. [Høyrup 2018].

approximation $^{49}/_{60}$, much better than the preceding one.[383]

Before repeating Fibonacci's circle-based geometric construction of a square root (above, p. 120), the present *Praticha* (fol. 270v) gives another one, closer to the spirit of abbacus geometry – namely to find it as the hypotenuse of a right-angled triangle – the examples used being a triangle with sides 1 and 3 containing the right angle.[384]

The rest of the chapter finds the roots of 743, 8754, 12345 and 927435, all taken from the *Liber abbaci*. The last part of Fibonacci's part 14.1, introducing the principle that $\sqrt{n} = \sqrt{(p^2 \cdot n)}/n$, is omitted – it has indeed been dealt with independently already.

Chapter 3, about "the multiplication of roots", starts by rendering Fibonacci's "part 14.2a", "the multiplication of roots and binomials" (above, p. 121) quite faithfully (fols 273r–274v), supplementing with marginal diagrams illustrating the various kinds of binomials. Next (fols 274v-282r) comes a partial counterpart of "part 14.2b", about the products of numbers, roots and roots of roots, at first following the *Liber abbaci* but soon giving different examples, and continuing with products involving also binomials and apotomes (above, pp. 123 and 125); this is the topic of Fibonacci's parts 14.3, but little of the present text is drawn from there (later, as we shall see imminently, the material from part 14.3 is used). Obviously, we are now moving within an area which had been explored in abbacus algebra, so here the writer draws on a different tradition,[385] returning to the *Liber abbaci* only occasionally – also when taking up roots of roots on fol. 276v.

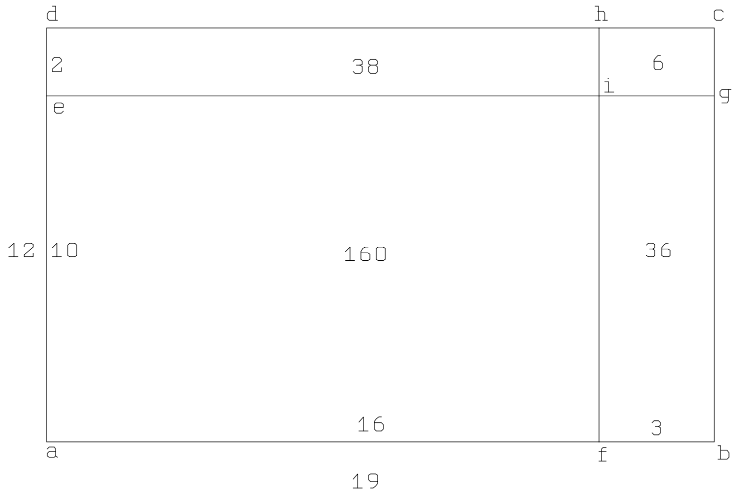
As we have seen, the arithmetic of monomials and binomials always went together with the sign rules (above, p. 206). So also here. Noteworthy (fol. 275v) is a demonstration borrowed from Fibonacci [B370;G571] of the rule for "less times less", based on next rectangle diagram on top of the next page, arguing first from the diagram-letters and then from the numerical example (the letters are those of Fibonacci, the numerical example independent). The argument runs like this: subtraction of the surfaces *dg* and *hb* from the surface *ac* leaves the surface *ai* less the surface *ic*, which therefore has to be added. As we see, the argument is structurally similar to that of Dardi (above, p. 218), but here the logic is impeccable.

Fols 277r–278v offer a surprise, a section dealing with the roots of the various types

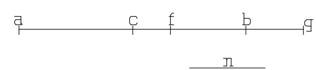
383 The text (as also the Palatino *Praticha*) finds how much $(^{49}/_{60})^2$ and $(^{59}/_{72})^2$ deviate from $^2/_3$, finding respectively $^1/_{3600}$ and $^{25}/_{5184}$, saying in both cases that this is "nothing much", not hinting at any comparison.

384 The Palatino *Praticha* also has these constructions (fol. 320r) but omits the diagrams.

385 Something, it is true, is taken over from the *Liber abbaci* – mostly matters drawing on *Elements* X. Thus (fol. 281r, cf. [B368;G569]) about the sum of the roots of binomials and corresponding apotomes, and fol. 276v, cf. [B360;G558], about finding two roots of roots whose product is a given rational number.



of binomials. It is clearly inspired by Fibonacci's statement about the root of a fourth binomial (above, p. 126), but it is independent, and supported by line diagrams similar to those used by Fibonacci but not found in the *Liber abbaci*. At first comes a theorem valid for all binomials – in paraphrase,



> Let *ag* be a binomial composed of *ab* and *bg*, and let *ab* be bisected at *c*. Let further the number *d* be the difference between the squares on the two components,[386] and *ef* be $\sqrt{(\frac{1}{4}d)}$. Then the root of *ag* equals the sum of the roots of *af* and of *fb*.

The proof, summarized and translated into symbols (replacing *ab* by $2p$ and *bg* by q, whence $af = p+\frac{1}{4}\sqrt{d}$, $fb = p-\frac{1}{4}\sqrt{d}$, $d = (2p)^2-q_2$), goes

$$\sqrt{p+\tfrac{1}{2}\sqrt{d}}+\sqrt{p-\tfrac{1}{2}\sqrt{d}} = \sqrt{2p+2\sqrt{(p+\tfrac{1}{2}\sqrt{d})(p-\tfrac{1}{2}\sqrt{d})}} =$$

$$\sqrt{2p+\sqrt{(2p)^2-d}} = \tfrac{1}{2}\sqrt{(2p)^2+\sqrt{(2p)^2-d}} = \sqrt{2p+q}\ .$$

The illustration in numbers (fol. 277^{v}) confirms that this further development of what Fibonacci had done by means of his technique was produced within the tradition of abbacus mathematics, even though we have no earlier traces: it considers the binomial $6+\sqrt{9}$, that

386 As we remember, this magnitude is fundamental for the definition of the different classes of binomials.

is, as Dardi and Giovanni di Davizzo it deals with the rational root as if it were irrational, and takes advantage of the control made possible by this computability.

On fols 278r–281r follow calculations of products mostly involving roots of binomials (no longer transformed as has just been taught). Even here, rational roots are often used – for example, $8\cdot(\sqrt{9}+13)$, $\sqrt{(6-\sqrt{4})}\cdot\sqrt{\sqrt{16}}$, $(\sqrt{\sqrt{343}}+\sqrt{7})\cdot(\sqrt{\sqrt{343}}+\sqrt{7})$ and $\sqrt{(\sqrt{9}+13)}\cdot\sqrt{(\sqrt{4}+7)}$ – also clearly made in continuation of the abbacus tradition.

Similarly, chapter 4 (fols 282r–286r), about "how to divide by roots", has more to do with the abbacus tradition than with Fibonacci's parts "14.4a" and "14.b" (above, pp. 126 and 128) although it depends to some extent on the latter. Initially it gives the sign rules for divisions among quantities of any kind (wholly alien to the *Liber abbaci*),[387] and then goes on with the subject-matter of "14.4a" proper, with explanations that do not come from the *Liber abbaci*, though the substance is evidently often the same – for example (fol. 282v),

> Divide root of 60 by 4. It is needed to divide the square of the root of 60, which is 60, by the square of 4, which is 16, and of that which results take its root. And from dividing 60 by 16 results $3\frac{3}{4}$, and the root of $3\frac{3}{4}$ is that which results from dividing root of 60 by 4.

Further (fol. 283r)

> Divide 30 and root of 3000 less root of root of 20000 by the root of root of 5. First you will divide 30 by the root of root of 5, where you multiply 30 two times in itself, and we shall have 810000. And you shall divide the root of root of 810000 by the root of root of 5, the root of root of 162000 results. And then you shall divide the root of 3000 by the root of root of 5, where you shall have to bring to or multiply 3000 by itself, they make 9000000, and you will have to divide root of root of 9000000 by root of root of 5. [...].

On fol. 283v begins the counterpart of Fibonacci's part "14.4b", about division by binomials and trinomials. At first the text follows this model, adopting also a line-based demonstration. Further, a number of examples are taken over. When coming to divisions by binomials involving roots of roots, however, the texts soon diverge. When computing $\frac{10}{2+\sqrt{\sqrt{3}}}$, the Ottoboniano text still takes over Fibonacci's line-based argument for the step $(2+\sqrt{\sqrt{3}})\cdot(2-\sqrt{\sqrt{3}}) = 4-\sqrt{3}$. However, when it comes to using this insight for the transformation

387 The rules given here speak of "added" and "diminished", not positive and negative numbers (in whatever terminology we might imagine).

The rules in question are absent from the Palatino *Praticha*. The two texts also differ in the following, even when the mathematical substance is the same.

$$\frac{10}{2-\sqrt{\sqrt{3}}} = \frac{10\cdot(2-\sqrt{\sqrt{3}})}{4-\sqrt{3}} = \frac{(20-\sqrt{\sqrt{30000}})\cdot(4-\sqrt{3}}{16-3} ,$$

Fibonacci feels the need to justify the operations by means of proportions, while the present writer, building on a century's familiarity with operations on formal fractions, does more or less as we do. After this, even the examples given differ from those offered by the *Liber abbaci*, until the very last topic – an abstract formulation followed by an example (fol. 285^{r}): the division of "some simple number or binomial or apotome", which again follows Fibonacci, adopting also his argument by means of a line diagram.

The short chapter 5 (fols 286^{r}–288^{v}), "about adding roots", starts by jumping back to the beginning of Fibonacci's part 14.3 (above, p. 125). It borrows an argument supported by a line diagram and a backward reference, that $\sqrt{7}+4$ is a binomial that cannot be expressed in a better way, the ratio between $(\sqrt{7})^2$ and 4^2 not being like that "between a square number and a square number" – it is indeed Fibonacci's example of a first binomial (above, p. 122), also expressible as $\sqrt{(23+\sqrt{448})}$.

So far this corresponds (with demonstration added) to what would also be done in abbacus algebra. Then (fol. 286^{v}) comes a stylistic rupture, and the text begins to discuss approximations, which in normal abbacus books only appear in solutions to geometric problems – first finding that $\sqrt{448} \approx 21\frac{1}{6}$, whence $\sqrt{(23+\sqrt{448})} \approx \sqrt{44\frac{1}{6}} \approx 6\frac{2}{3}$.[388]

More diagram-based considerations follow about the conditions for reduction being possible, for example (fol. 287^{v}) $\sqrt{18}+\sqrt{32} = \sqrt{98}$. Then (fol. 288^{r}), it is said concerning $4+\sqrt{\sqrt{20}}$ that it is expressed "in a vernacular way" (*sechondo un vulghare modo*) by means of the "closest approximations", opposing it to the transformation "according to the art" (*sechondo l'arte*); the example as well as the reference to the "vernacular way" is borrowed from the *Liber abbaci* [B364;G563]. The same double approach is next given to $\sqrt{12}+\sqrt{\sqrt{10}}$, for which Fibonacci only gives the exact solution. Except for the approximations, all calculations make use of the principle $a+b = \sqrt{(a^2+b^2+2ab)}$, where a and b can be numbers, irrational roots or roots of roots.

Chapter 6, "about the way to detract roots" (fols 288^{v}–290^{r}) also draws on part 14.3 of the *Liber abbaci*, beginning with a similar line-based demonstration and the example $4–\sqrt{7}$ – borrowed from Fibonacci [B363;G561] but simplified.[389] The rest of the chapter is independent – ending (fol. 290^{r}) with $\sqrt{\sqrt{32}}–\sqrt{\sqrt{18}}$ and $\sqrt{\sqrt{8}}–\sqrt{\sqrt{2}}$. The basic principle

---

388 The usual first approximation gives $\sqrt{44} \approx 6\frac{2}{3}$. The writer may have been satisfied with that. More likely, however, is that he approximated $\sqrt{7}+4$ directly as $2\frac{2}{3}+4$, a calulation which is presented as a check ($2\frac{2}{3}$ being miswritten as $6\frac{2}{3}$); the latter calculation is indeed offered as an alternative and not as a control in the Palatino *Praticha*, fol. 359^{v}.

389 Fibonacci speaks about "extracting a surd root from a rational number, or a number from an irrational root, or a root from a root that are commensurate in power only", the present text about "extracting a root from a rational number".

is that $a–b = \sqrt{(a^2+b^2–2ab)}$, where again a and b can be numbers, irrational roots or roots of roots.

Chapter 7 (fols. 290^r–294^v), "containing the way to find cube roots", corresponds to the beginning of part 14.5 of the *Liber abbaci* [B376–384;G582–590]. Apart from two alternative examples and an addition to a diagram (all on fol. 292^2), it follows the model, and leaves out nothing. There is no reason to add anything to what was said about Fibonacci's text on p. 129.

Chapters 8 (fols 294^v–295^v) and 9 (fol. 295^v–296^v), about "the way of multiplying cube roots of numbers" and "dividing by them", is parallel to the central section of part 14.5 of the *Liber abbaci* [B384–385;G590–591], yet with only interspersed borrowings. The corresponding pages of my scan of the Palatino manuscript are strongly overexposed and mostly illegible. The traces that remain show, however, that at least most is different from the Ottoboniano manuscript, not only in the choice of examples but also by not making use of rational roots. It also differs from the *Liber abbaci*. The explanation closest at hand seems to be that the shared model was too rudimentary in this section and the two compilers therefore decided to work independently.

At first in chapter 8 the rule for multiplying two cube roots is formulated in abstract terms (namely, to multiply the two radicands and to take the cube root of the outcome), illustrated by an example borrowed from Fibonacci ($\sqrt[3]{40} \cdot \sqrt[3]{60}$), and then an equally abstract rule for multiplying another "name" of root and a cube root (to bring both to the same "name" of root[390]), illustrated by examples not adopted from Fibonacci: $4 \cdot \sqrt[3]{27}$, $\sqrt{4} \cdot \sqrt[3]{10}$, $\sqrt{\sqrt{16}} \cdot \sqrt[3]{27}$. No advantage is taken of the possibility of control offered by the picking of rational roots. Only then comes Fibonacci's second example, $(2\sqrt{20})\cdot(3\sqrt[3]{40})$, and his first way to find two cube roots whose product is a rational number, with a different numerical illustration.

Chapter 9, about division, is wholly independent of Fibonacci, and makes extensive use of rational roots without ever taking advantage of the possibility of control. The first examples are $\sqrt[3]{120}÷2$, $100÷\sqrt[3]{20}$, and $\sqrt{16}÷\sqrt[3]{4096}$. Noteworthy is that the latter calculation is not finished, the result is given as $\sqrt{\sqrt[3]{4096}}÷\sqrt{\sqrt[3]{64}} = \sqrt{\sqrt[3]{64}}$. A final reduction to 2 is evidently only possible because of the friendly parameters; the compiler restricts himself to the steps that have general validity. Three more problems follow, of similar character and dealt with similarly – two involving also roots of roots and one including coefficients. We may observe that the solution of these problems makes more explicit the insights which Dardi must have possessed but apparently had no language to express.

---

390 As could be expected, it is taken for granted that root-taking is commutative, $\sqrt[3]{(\sqrt{n})} = \sqrt{(\sqrt[3]{n})}$, $\sqrt[3]{(\sqrt{\sqrt{n}})} = \sqrt{\sqrt{(\sqrt[3]{n})}}$. From the definition of what a root is this follows rather trivially from the associativity of multiplication, so there is no reason the abbacus writers should see this as a problem.

Chapter 10 and 11 (*sic*[391]) (fols. 297v–298v) translates the last part of part 14.5 of the *Liber abbaci* [B384–387;G591–594] (above, p. 129), deleting only an alternative solution on fol. 297r (for the good reason that the solution offered for the next question is very close to what has been left out), and adding in the end that the possibilities to speak about other roots in similar ways are infinite, and saying that this closes part 8.[392]

This closing demonstrates that the ensuing lacuna (fol. 200r begins in the middle of a sentence) belongs wholly within *part 9*, which is drawn from part 15.1 of the *Liber abbaci* (above, p. 132). A comparison with the folio numbers indicated in [Boncompagni 1857] suggests that 2 sheets have been lost.[393]

After the lacuna, the text follows that of Fibonacci's part 15.1 closely, leaving out nothing and adding nothing except the closing formula on fol. 303r,[394]

> And this will suffice concerning the 9th part. Next we shall speak about the 10th, so be attentive. And for finishing this part we shall say *deo gratias*.[395]

---

391 The Palatino *Praticha* (fol. 375r), having a slightly different organization of the preceding chapters, similarly speaks of the "ninth and tenth chapter".

392 The Palatino *Praticha* has almost the same text (with a proviso for the reduced legibility); it does not omit Fibonacci's alternative solution, and the final words added after the borrowings from Fibonacci are slightly different.

393 The foliation of the manuscript was made after the loss of these sheets, and is thus continuous.

394 There was almost certainly also an introduction not drawn from Fibonacci. The corresponding introduction in the Palatino *Praticha* [ed. Arrighi 2004/1967: 190*f*] runs as follows:

> Many strain themselves to show that this ninth part of this treatise is not necessary for the rule of algebra; and among these are some moderns whose names I shall at present not disclose. But (among) those who are demonstrators that not in vain work hard in algebra, the first is Leonardo Pisano since, in the first part of the 15th chapter, he gives names to the proportion of 3 and 4 quantities. And master Paolo says, in the second part of the treatise on continuous quantities, that without Leonardo's 15th chapter nothing can be done, I say its first chapter. And master Antonio, in the *Gran trattato*, says "I presuppose that the proportions of the first part of the 15th chapter are clear to you". And my noble master Domenico, in the memories he left to me, said, "don't depart with these" [...].

The last period may be taken as a warning that this is a personal testimony and not taken over from the model shared with the Ottoboniano writer.

395 This final invocation is also found in the closing formulas of parts 5 to 8. Part 10 instead has *laus Deo*. The Palatino *Praticha* closes its parts 3, 6, 7 and 9 with the former and parts 5 and 8 with the latter formula. The abbacus environment was decidedly pious in its written expression – whether as a matter of mere routine is difficult to know, and hardly the same for everybody.

Since the text follows the *Liber abbaci* faithfully, there is no reason to say more about this part than was said about Fibonacci's text above – except that this faithfulness[396] shows that the abbacus environment had not worked actively on the topic, in spite of the opinion of the Palatino writer quoted in note 394 (whose words, when closely read, indeed do not contradict this conclusion).

Part 10 (fols. 303r–343r) deals with algebra; the prestige of this topic within abbacus culture is reflected in the opening words:

> All that which has been said on this point would be in vain without the present, since here is shown the rule that solves all cases that can be solved, speaking in squares and cubes and in all the continuous quantities, as I shall show in the cases. And therefore you shall apply the intellect to this part [...].

The chapters are then listed:

> In the first we shall write the definitions of the said rule.
>
> In the second the way to multiply, divide, detract and join the names of that rule with each other, and with simple numbers.
>
> In the third are 6 equations of the said rule showing their demonstrations clearly.
>
> In the fourth examples of the said rule, that is, the said 6 modes.
>
> In the fifth many other equations.
>
> In the sixth cases solved by the said equations.

Chapter 1 (fols. 303v–304r) is quite short. It explains that some speak of the "rule of algebra" and others of the "art of the thing" (*reghola d'algebra* respectively *reghola della chosa*); that according to "those who understand" (*gl'intendenti*) it means "rule of opposition and of recuperation" (*d'oppositione e di ricuperatione*, with the inversion we already know from Fibonacci, cf. above, pp. 117 and 141, and an alternative translation which we shall encounter again on p. 314 in Benedetto's quotation from Guglielmo de Lunis); and that many have written broadly about it, in particular "Leonardo Pisano and master Antonio de' Mazzinghi". But because a book called *Algebra maumetti* sets things out very clearly, that will be the fundament.[397]

The following explanation of *censo*, *radice* and simple number is indeed close to al-Khwārizmī – and so close to what is offered by Benedetto [ed. Salomone 1982: 2*f*] that they must have used the same vernacular version (the obvious but wrong guess would be a vernacular translation of Gerard of Cremona's Latin version, cf. below, p. 313). After

[396] Shared by the Palatino manuscript everywhere I was able to read my scan.

[397] Since the Palatino manuscript (fol. 391r) does the same though with a somewhat different justification ("so that the work of the Arab Maumet, which was almost lost, may be restored, I shall begin by that"), the use of al-Khwārizmī's treatise must go back to the shared model.

this, however, and before the definition of the six cases, the excerption from al-Khwārizmī stops.

Chapter 2 (fols 304^r–305^v) leaves the classics and turns to the abbacus tradition in its presentation of the rules for multiplying and dividing algebraic "names" – in our sometimes inadequate language, "powers".[398] At first it states the products of powers until the sixth. This is named *cubo of cubo*, while the fifth power is either *cubo of censo* or *censo of cubo* (told to be the same). There is no trace of Antonio's *cubo* relato (above, note 329), the system is purely multiplicative. For all products, a second example with the coefficients 6 and 8 is given – for example, "6 *censi* times 8 *cubi* make 48 *censi of cubo*. Next follow divisions within the same limit, sometimes (in our terminology) leading to a positive, sometimes to a negative power, this time with corresponding examples with coefficients 48 and 6 – for example (fol. 305^r)

> From dividing *things* by *cubi* results a fraction denominated by *cubi*, as from dividing 48 *things* by 6 *cubi* results this fraction $\frac{8\ \textit{things}}{1\ \textit{cubo}}$.

Slightly later it is explained that this can be reduced (*schisato*, the same term as is used about the reduction of common fractions: $\frac{8\ \textit{things}}{1\ \textit{cubo}}$ is the same as $\frac{8\ \textit{dragmas}}{1\ \textit{censo}}$. In the end (fols 305$^{r–v}$) it is explained that addition and subtraction of different powers cannot be simplified, joining of 6 *censi* and 8 *cubi* and 3 *censi di censo* makes 6 *censi* and 8 *cubi* and 3 *censi di censo*.

Chapter 3 (fols. 305^v–307^r) presents the fundamental six cases – or so it says, the copying from the model (which is much better reflected in the Palatino manuscript) omits much.

In order to see the changes made by the Ottoboniano writer, we shall first have a look at the corresponding Palatino section (fols 391^r–399^r). It starts by stating the six fundamental cases in Arabic order, all normalized, and all provided with the same examples as in al-Khwārizmī's text[399]. It is noteworthy that the writer uses *dramme* (corresponding to Latin *dragmae*) much more consistently than Gerard's text for numbers; as we shall see (p. 316), it is also informative. It is explained regularly that it is the same as "simple numbers", as is the identity of *roots* and *things* (*chose*).

---

398 Sometimes inadequate – namely to the extent that *thing*, *censo*, *cubo*, ... are seen primarily as different entities, and only secondarily as members of a geometric series. The latter view, though always present (already Jacopo's rules for higher-degree cases demonstrates this) was becoming more outspoken from Antonio onwards, and was clearly that of higher-level abbacus algebraists in the outgoing 15th century.

399 Both the extant Arabic text [ed. Rashed 2007] and Gerard's translation [ed. Hughes 1986]. The first four cases are provided with three examples each, one normalized, one with an integer and one with a fractional coefficient of the highest-degree term. The last two are presented by a normalized example only.

As in al-Khwārizmī's text, geometric demonstrations follow. There are ten diagrams (al-Khwārizmī has four). Even in those that have a counterpart in Gerard's version, the lettering is wholly different.

The Palatino writer then presents the rules for multiplication and division of powers, saying that here he is going to follow Antonio. As we remember, that topic has already been dealt with earlier in the Ottoboniano *Praticha* (without reference to or use of Antonio).

Chapter 3 of the Ottoboniano *Praticha* starts with a listing of the six cases in normalized form but abbacus order, beginning by *thing* equal to number. For this case, the numerical example differs from that of al-Khwārizmī, for the others it is the same.

Then the three simple cases are repeated or summarized, still in abbacus order, and now in non-normalized abbacus style.

The first and second composite cases (*censo* and *roots* equal number respectively *censo* and number equal *roots*) are presented as in the Palatino text, still with al-Khwārizmī's examples. Then it skips the third composite case (*roots* and number equals *censo*) and shows a sample geometric demonstration, namely for that unexplained case, and closes the chapter (fol. 307^{r}) with the same words as the Palatino *Praticha* (fol. 399^{r}).

Chapter 4 (307^{v}–309^{r}) is another presentation of the fundamental six cases, now in full abbacus style: abbacus order as standard since Jacopo (above, p. 188), all cases non-normalized and the first step of the rules thus being a normalization, and the sixth case in the form *censi* equal *things* and number, not *things* and number equal *censo* as in al-Khwārizmī and the preceding chapter copied from him. All examples pretend to deal with commercial matters, though most are nothing but pure-number-problems said to concern the monetary possession of somebody.

The corresponding chapter in the Palatino *Praticha* (fols 402^{v}–406^{v}) presents different examples – still of types we know from earlier abbacus algebras – "find me two numbers so that ...", "find me a number which ...". In the end (fols 409^{v}–410^{r}) it reproduces a letter (transcribed in [Arrighi 2004/1967: 166]) from 1397, written by a certain Masolo da Perugia and addressed to Giovanni di Bicci de' Medici (initiator of the Medici rise to power in Florence).[400] It speaks of the solution of higher-degree equations by means of special roots:

> I understand that you find it marvellous that *cubi*, *censi* and *things* can be made equal to number, considering that the opinion of all predecessors is that this should not be possible. I say that every equation [*adeguagliamento*] can be determined, and, if the way had not been very long and difficult, I would have sent it herewith; although I shall send you what you ask for.

[400] Actually an answer to a request made by Giovanni, who thus still took time for such matters in spite of being fully occupied by the expansion of the Medici bank.

> First I say that, when *cubi*, *censi* and *things* are equal to number, there may be three necessary and diverse answers, all of which will be explained to you.
>
> The roots are infinite, which the geometer calls incommensurate lines, as it appears from the 107th of the 10th of Euclid.[401]
>
> The cube root of 84, with detraction [*abattimento*] of 5 *things*, is 4; because 5 things are 20 which, detracted from 84, leave 64, whose cube root is 4; as is said. Another example: the cube root of 33, with detraction of 2 *things*, is 3; because 3 times 2, which are the *things*, make 6 which, when detracted from 33, leave 27, whose cube root is 3, as is said.
>
> The cube root, with joining of the *things*, are done as I have said above; only that above is detracted, to these are joined. The example: the cube root of 44, with joining of 5 things, is 4 in number; because 4 times 5, which are the things, make 20, which, joined to 44, make 64, whose cube root is 4, as is said.

The Palatino writer goes on

> In his letter he shows these equations; but the reason they are not good I do not know.

It seems implausible that Masolo himself should have explained in the letter that the roots he speaks of are not good, so the present writer must refer to other objections.

However that may be, there is a close connection to what we have encountered above (p. 245) in the Florentine *Tratato;* even the example 44 is the same.[402] Masolo's opening words also indicate that is his environment (and that of Giovanni di Bicci) the kind of false solutions that start with Gherardi were not accepted.

Chapter 5 of the Ottoboniano *Praticha* begins (fols 309^r–310^r) by listing rules for higher-degree cases, with several peculiarities. The cases listed are (the numbering is in the manuscript, the preceding "Ot" obviously not – *dK* stands for *duplici chubi*):

---

[401] Campanus numbering, X.115 in modern editions, following [Heiberg 1883: III, 370]. The proposition is actually irrelevant to what follows, the unlimited number of roots of which is speaks are the lines produced from a medial line – corresponding arithmetically to the continued taking of square roots. We may take that as evidence that there was an ongoing interest in *Elements* X among certain abbacus teachers – but with the proviso that interest seems not to have entailed much understanding, at least not as a necessary consequence.

[402] As we see, instead of speaking of a cube root with added debt, as done by the Florentine *Tratato* (above, p. 246), Masolo operates with a distinct cube root with detraction.

Ot7	$K = N$	Ot16	$CC = K+C$
Ot8	$CC = N$	Ot17	$K+t = C$
Ot9	$dK = N$	Ot18	$K = C+t$
Ot10	$KK = N$	Ot19	$K+C = t$
Ot11	$CCC = N$	Ot20	$CC+N = C$
Ot12	$CC = t$	Ot21	$CC+K = C$
Ot13	$CC = C$	Ot**	$CC+t = N$
Ot14	$CC+K = C$	Ot22	$CC+C = N$
Ot15	$CC = K$	Ot23	$CC = C+N$

Double cube, it turns out, is the fifth power, and the solution is expressed as the *radice relata* of the number term after normalization. There is not much order in the list; since the naming of powers is elsewhere multiplicative, we must expect *CCC* in Ot11 to be the same as *KK* in Ot10 (the solutions are expressed as *root of root of root* respectively *cube root of cube root*, which allows no cross-check); if this is true and the writer has not discovered that the two cases are identical, we may doubt his understanding of the matter – as supported by his change of terminology for the powers, which suggests copying without much thought. Apart from that, we may observe that coefficients are spoken of explicitly, sometimes as "the number of" (Ot11), sometimes as "the number that is for" (Ot13), sometimes with Dardi's "quantity of" (Ot16, Ot19). Ot14 and Ot19 are followed by the explanation that they are like the case $C+t = N$, and the existence of a double solution is pointed out in Ot17 and Ot20. Finally, the non-numbered and evidently irreducible case Ot** is discarded after having been defined and normalized.[403]

The rest of chapter 5 (fols 309r–343r), "cases solved by the said equations", contains problems said to be borrowed from the *Liber abbaci* (fols 309r–326r), Luca di Matteo (fols 326r–331v), Giovanni di Bartolo (fols 331v–335r), and Antonio de' Mazzinghi (fols 335r–343r). In the section otherwise coming from the *Liber abbaci*, two problems are inserted – the first, about a purchase of wheat and barley (fol. 315r), is the problem that "was sent to Florence by a master from L'Aquila "already around 12 years ago" (above, p. 250), and which goes back at least to Biagio. The second, on fol. 320r, is quite simple – in symbols, $(4\sqrt{n})\cdot n = 7n$, solved by the observation that if a number multiplied by some other number makes 7 times as much, then this other number is 7; hence $4\sqrt{n} = 7$, etc.

On the whole, the section follows Fibonacci, sometimes forgetting diagrams in his margin even though the text refers to them, sometimes conserving them; a couple of times (fol. 311r, 312v), Fibonacci's verbal multiplications of algebraic binomials are repeated

403 The chapter has no counterpart in the Palatino manuscript, which instead (fols 410r–478v) goes directly to problems borrowed from the *Vaiaio*, Fibonacci, Luca di Matteo (a Florentine abbacus master, 1356 to 1433 or 1436) and Giovanni di Bartolo – the counterpart of chapter 5 of the Ottoboniano *Praticha*.

in symbolic form in the margin, and thrice (fols 324[r], 325[r] and 325[v]) Fibonacci's diagram [B431,439,440;G654,665,667] is replaced by a marginal symbolic calculation; the present writer (or rather, his model) did not share Benedetto's aspiration to "speak like" his source (cf. above, p. 232). Where Fibonacci uses *census*, *avere* or *quantitates* about one or more *numbers* (thus not where *census* designates the algebraic second power, cf. above, p. 155), they become *numero*/*numeri*.

We have no possibility to compare the next section with Luca di Matteo's original, but a number of the problems ascribed to him can be traced back to Biagio, others to Antonio's *Fioretti* – among these one on fol. 330[v] which pretends to solve the problem which Antonio did not like and therefore did not complete (above, p. 231). Evidently a fallacy is involved.[404] Many of the problems are contorted versions of normal recreational abbacus problems or genuine business questions – give-and-take problems involving roots (several, e.g. fol. 327[r]); partnerships where the partners contribute with different monies whose ratio is given indirectly (fol. 328[r]), or involving a squared contribution (fol. 328[v]), combined works with abstruse conditions (fols 329[r,v]), loans with composite interest and changing but unknown rates with known difference (fol. 329[v]), and barter involving a root (331[r]). We have encountered such problems in the Florentine *Tratato* (above, p. 244), and modestly already in Jacopo's algebra (above, p. 189). In spite of the fact that the *Tratato* was produced during the period where Luca di Matteo was active, there is no textual evidence suggesting it was written by Luca;[405] the two texts seem to reflect a fashion of the Florentine outgoing 14th and early 15th century.

As already mentioned in note 355, similar problems were also abundantly solved by Giovanni di Bartolo. This is confirmed by the much more restricted selection here, which once again contains further problems coming from Biagio and Antonio (one, fol. 333[r], solved differently than by Antonio). Most interesting – not *per se* but because Regiomontanus presents the same solution and marginal scheme, se below, p. 359 – is a problem on fol. 331[v] (the second of those ascribed to Giovanni di Bartolo). It is one of the problems about dividing a certain dividend first by one divisor and then by another one exceeding the former divisor by a known amount, the difference between or the sum

[404] Antonio, we remember, had arrived at

$$\sqrt{5-t}+\sqrt{5+t} = 36-t^2 \;.$$

Squaring this, Luca gets

$$(36-t^2)^2 = 5+t+5-t+2\sqrt{25-t^2} = 10+2\sqrt{25-t^2} \;.$$

Subtracting 10 on both sides – left, unfortunately, from the expression that is squared – Luca arrives at a biquadratic. Rhetorical algebra requires great skill when matters become complex.

[405] With the slight proviso that even an insightful mathematician may sometimes make blunders, the fallacy pointed out in note 404 also speaks against identification.

of the two quotients being given – familiar since Fibonacci, we remember, and amply present also in the abbacus tradition since Gherardi (above, pp. 153 and 199); within the abbacus tradition it was one of the earliest problem types calling for the application of formal fractions (cf. above, p. 228). Using like the manuscript the abbreviation ρ for the *thing* but indicating addition with a modern "+" and replacing the long stroke by a modern equation sign, we may express the problem as it appears in the margin of the manuscript as

$$\frac{100}{1\rho}+\frac{100}{1\rho-7} = 40$$

The solution (in the margin as well as in the text body) goes via the calculation

$$\frac{100}{1\rho}+\frac{100}{1\rho-7} = \frac{100\rho-100\cdot(\rho-7)}{(1\rho)\cdot(1\rho-7)} = \frac{100\rho-(100\rho-700)}{1\,\mathit{censo}-7\rho} = 40\ ,$$

for which reason 200ρ+700 = 40*censi*+280ρ; in the margin, where *censo* is abbreviated σ, we find

$$\begin{array}{llll}
100\rho & & & \\
\underline{100\rho\quad 700\qquad} & & & \\
\underline{200\rho\quad 700} & & & \\
\quad 1\sigma\quad \underline{7\rho\qquad\qquad} & 40 & & \\
 & & & \\
200\rho\quad 700\ \text{——}\ 40\sigma\ \langle 280\rho\rangle & & &
\end{array}$$

which for the moment we may take note of as representative of the level at which symbolic calculations are performed in the manuscript; the long stroke "———" here as mostly functions as an equation sign, that is, the confrontation of two equals (elsewhere in the manuscript it is used occasionally for other confrontations, for instance between the possessions of two partners).

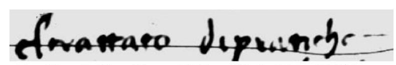
Part 11 (fols. 343^{v}–348^{r}) is a very traditional abbacus geometry; most problems are similar to what was already presented by Jacopo (above, p. 34 onward), the remainders build on similar principles. There is no need to go through the details, they would teach us nothing new. We may therefore restrict us to the first of two problems belonging to the same family as Jacopo's fallacious stone-in-a-well (above, p. 40, here fols 346^{v}), which illustrates that transmission had eliminated the fallacy as well as the intricacy of the *Liber mahameleth* and the *Liber abbaci* that had induced it – namely the use of different units for volume and hollow measure (above, p. 41). In the likeness of the *Liber mahameleth* and the *Liber abbaci* it speaks of a cistern, not a well:

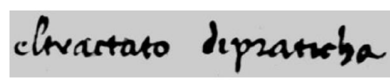
> There is a cistern, long 10 cubits, broad 8 cubits, high 6 cubits, and the water stands 5 cubits high. A stone falls in it which each way is 4 cubits. I ask how much water will remain. First bring the stone to square cubits, where you multiply 4 which it is long times 4 which it is broad times 4 which it is high, they make 64, and 64 square cubits is the

> said stone. Where you square the area of the area of the bottom of the said cistern, where you multiply 10 times 8, they make 80, and you divide 64 in 80, $^{4}/_{5}$ results, and $^{4}/_{5}$ of a cubit you shall say that the water raises in the said cistern, and thus you shall make the similar.

Apart the formulation of the question, which fits a situation where there is an overflow of water, this is impeccable. We observe that the unit for volumes is still spoken of as "square cubits".

Instead of this small collection of geometric problems, the Palatino *Praticha* offers a list of abstractly formulated rules, beginning with the computation of square and rectangular areas and of an equilateral triangle – the latter in a rather unorthodox formulation. Later come, among other things, transformations within Florence metrology and between volume measure and hollow measure for grain, and the contents of barrels and of a heap of grain (as it falls naturally). The compiler gives the reason that he had initially intended to deal with geometry at greater length; but since he has decided to write about the topic in a separate volume, he here gives only that which is needed in order not to leave the treatise incomplete.


eltrattato dipraticha (Palatino 573)


eltractato dipraticha (Palatino 577)

Judging from the style of its opening words, one might believe this separate volume to be the roughly contemporary manuscript Florence, BNC, Palatino 577, a *Praticha di geometria*; however, the hands are somewhat different, as can be observed.[406] On the other hand, the Ottoboniano manuscript contains on fols 355^r–429^v, after the *Praticha d'arismetricha*, another *Praticha de geometria*, which though not identical is clearly close kin of the Palatino 577. As we have seen, the Ottoboniano and Palatino *Pratiche di arismetricha* are in many places copied from the same source; even the *Pratiche di geometria* are apparently representatives of a whole family of geometric

[406] [Simi 1999] contains an edition of extensive extracts. [Simi & Toti Rigatelli 1993: 462*f*] argue from a reference on fol. 11^v–12^r to the "10 demonstrations and solutions" which the writer has given "in the first chapter of the eighth part" in his *Praticha d'arismetricha* that the writer must be the one who produced the Palatino *Praticha*. However, since this chapter of the Palatino and the Ottoboniano *Pratiche* coincide and both are copied from the same source (maybe from the hand of the *Vaiaio*), this reference only shows kinship, nothing more.

However, the reference excludes Ettore Picutti's identification [1989: 76] of the writer with Benedetto da Firenze.

encyclopedias.[407] Only the autographs and no copies of the Palatino and Ottoboniano *Pratiche d'arismetriche* have survived, while the earlier version of the Palatino *Praticha* (above, note 339) has been lost; other encyclopedias may therefore also have been produced and lost, including the geometric encyclopedia announced in the Palatino *Praticha*.

Due to their similarity with the two *Pratiche di arismetricha* I have spoken above of the two geometries as "encyclopedias". This might be considered a misnomer. Whereas the arithmetical encyclopedias are indeed encyclopedic, drawing on many sources (eclectically, of course, encyclopedias *are* eclectic and not monographs), the geometries depend if not exclusively then overwhelmingly on Fibonacci's *Pratica geometrie*. I shall therefore not discuss them further – a detailed discussion of their relation to the model is not very relevant to a portrait of abbacus mathematics.[408] At most they confirm that material drawn from Fibonacci did not enter into fruitful interaction with new

---

[407] Simi [1999: 68 and *passim*] asserts that the Ottoboniano geometry is a reduced copy of the Palatino geometry. A glance at the diagrams in the former and the latter on respectively fols 388^v–389^r and fol. 134^r and comparison with [Boncompagni 1862: 101] will suffice to show that this is impossible: the Ottoboniano writer knows all four diagrams from Fibonacci's *Pratica geometrie*, while only two of them (one of these moreover omitting one of the diagram letters) are in the Palatino geometry. The appurtenant texts confirm the conclusion. After teaching how to measure "a field or the figure of a fish" respectively "a figure similar to a fish" (an area delimited by two circular arcs), the Ottoboniano geometry goes on with the measurement of "a field having the figure called elana", some kind of oval shape; *elana* could possibly be a Latinization of *al-ʿayn*, "eye", which would fit the shape. This passage is not in the Palatino geometry, and Simi [1999: 60 n. 17] supposes it to be an addition; actually, it is translated from the *Pratica geometrie* [ed. Boncompagni 1862: 101] and thus instead omitted from the Palatino geometry.

Simi [1999: 44] also attributes "with almost full certainty" the Ottoboniano *Praticha* to the same hand as the Palatino *Praticha*, claiming it to be a reduced copy. Even here, the presence in the former of diagrams omitted in the latter show this not to be the case – a conclusion that is confirmed by the differences in algebraic terminology discussed above.

A similar consideration refutes Picutti's claim [1989: 76] that Pacioli copied from the Palatino geometry in his *Summa*. Pacioli, indeed, conserves many diagrams from Fibonacci's *Pratica* which are omitted in the Palatino geometry – cf. for instance [Pacioli 1494: II, fol. 16^r] with fol. 56^r of the latter and [Boncompagni 1862: 59] – the Palatino geometry omits the first of two diagrams here.

On the other hand, the similarities between the Ottoboniano geometry and Pacioli's *Summa* identified by Picutti leave no doubt that Pacioli used a pre-existing Tuscan translation of Fibonacci's text which was close to the Palatino and the Ottoboniano geometries; but Pacioli as well as the Ottoboniano writer must have used better versions than the Palatino geometry.

[408] For the same reason I shall also not take up other vernacular versions of Fibonacci's *Pratica*. See [Hughes 2010] and Arrighi's edition of one of them [1966a].

developments within the field of abbacus mathematics – not least, it has to be added, because geometry was peripheral to the curriculum of the abbacus school, for which reason there was no spur to direct creative work within abbacus mathematics toward geometry.

Benedetto

There is no need to say more about the Palatino *Praticha* than was done in the previous pages, in particular during the comparisons with the Ottoboniano namesake. Benedetto's *Praticha*, on the other hand, deserves extra attention.

We have already witnessed Benedetto's care for philological precision, which makes him a valuable and unique source for the development of abbacus mathematics. But there is much more than reliable copying in the treatise.

It opens[409]

> Begins [open space long $^1\!/_6$ of a line] of the treatise of practice of arithmetic drawn from the books of Leonardo Pisano and other authors. Compiled by B. for a dear friend in the year of Christ 1463.

A paragraph of almost half a folio page first extols that honest competition which makes everyone try without arrogance to do better than his neighbour, and then (addressing the dedicatee) speaks about a common friend Ł, who has communicated to Benedetto the desire of the two to be helped with certain difficult mathematical questions, for which reason he has now composed a treatise (of almost 500 densely written folio pages) – confirming that the dedicatee-"friend" was of such social standing that communication with Benedetto had to go via an intermediary.

A second part of the preamble approaches the subject-matter, beginning with two Latin quotations from Boethius's *Arithmetic* I.1 and I.2 – one explaining the primacy of arithmetic over the other mathematical disciplines, the other how everything is constructed on the basis of number. However, "since the treatise is rather for practical than for other uses", Leonardo Pisano and others who have written on practice will be excerpted and explained. Since arithmetic and geometry are connected and mutually supportive, along with the present treatise Benedetto intends to write another one about geometry – even this one lost if ever written.

The present *Praticha* is explained to consist of 15 books. In close paraphrase:

1. The representation, multiplication and subtraction of numbers [fols 1^v–20^v];
2. the nature and properties of numbers with respect to each other [fols 21^r–29^v];
3. division by integers [fols 30^r–45^r];
4. fractions [fols 45^v–64^r];
5. proportional numbers [fols 64^v–82^v];

409 The whole preamble (fols 1^{r-v}) is edited in [Arrighi 2004/1965: 138*f*].

6. everything that has to do with trade[410] [fols 83^{r}–171^{v}];
7. the principles having to do with the *chatain*[411] [fols 172^{r}–184^{v}];
8. what has to do with interest [fols 185^{r}–226^{r}];
9. problems solved by means of the double false position [fols 226^{v}–233^{r}];
10. recreational problems, corresponding to chapter 12 of the *Liber abbaci* [fols 233^{v}–299^{v}];
11. some proportions that have to do with continuous quantities [fols 300^{r}–310^{r}];
12. roots [fols 310^{v}–367^{v}];
13. the rule of algebra [fols 368^{r}–388^{r}];
14. cases exemplifying this rule [fols 388^{v}–408^{r}];
15. cases taken from several masters [fols 408^{v}–474^{v}];
16. cases having to do with square numbers [fols 475^{r}–506^{v}].

In the end of the preamble, Benedetto promises always to give due reference when reference is due, confirming thus his intended philological precision.[412]

Some of the books deserve closer description and analysis – in some cases summarily, in others in depth.

Book 2 (fols 21^{r}–29^{v}), "nature and properties of numbers with respect to each other", is a decent and well-ordered theoretical arithmetic, drawing on Euclid and Boethius, similar in topic to part 3, chapter 2 of the Ottoboniano *Praticha* (above, p. 253). It was edited separately by Arrighi [1967b]. It deals with the odd and the even, with the subdivisions into evenly even, etc.; with arithmetical series; with prime and composite numbers; with deficient, abundant and perfect numbers, including (fol. 24^{v}) determination of the first five of these[413] and a meticulous control of the last of them; and with the Boethian names for ratios. A second chapter presents figurate numbers: polygonal numbers until

---

410 The beginning of the book itself cautiously specifies that it deals with everything a merchant ought to know *which falls under number*.

411 As to the actual contents, see the description on p. 288.

412 As we shall see, Benedetto sometimes shares inspiration with the other two *Pratiche* without revealing his source; he appears to distinguish inspiration from philologically faithful copying.

413 As we remember from p. 253, the Ottoboniano *Praticha* also finds the sixth. Both organize the calculation in a scheme, Benedetto much clearer that the Ottoboniano writer:

Evenly even numbers	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
The sum		3	7	15	31	63	127	255	511	1023	2047	4095	8196
Perfect numbers		6	28		496		8128						33550336

heptagonals, as well as pyramidal and cube numbers; and in the end so-called circular or spherical numbers, those whose powers all end by the same digit, that is, 5 and 6.

The "principles having to do with the *chatain*" (*book 7*) is properly speaking a misnomer. The chapter itself specifies that is deals with "the way to solve cases by the simple mode of *chatain*", which turns out to be what Fibonacci calls the "rule of proportion" (cf. above, p. 88), demonstrated by means of line diagrams.

One of Fibonacci's examples of this method, we may remember from p. 84, speaks of selling three pearls in Constantinople, and an alternative solution by means of *regula recta* was given in the *Liber abbaci*. Benedetto repeats the problem (with a minor numerical change and as dealing with "precious stones") on fol. 182^{v} but gives the solution by means of *modo retto*, using the occasion to introduce this "almost universal" method.

Throughout this chapter there are many marginal calculations, and often these spread into the text column (line diagrams only appear on fols 172$^{r–v}$). In some of the latter cases it can be seen that space was prepared for them, meaning that the text was written first; such calculations are clearly meant to inform the reader, Benedetto did not need them himself (perhaps because the problem is copied). In other cases, as we shall see, it is obvious that complex calculations were made first and the text written afterwards, proving that Benedetto did not copy but worked on his own.[414]

Book 8, supposedly on interest, opens with moral deliberations similar to but more elaborate than those of the Palatino *Praticha* (above, note 24). Chapters 1–5 (fols 185^{r}–218^{v}) treat of simple and composite interest, discounting, making up of accounts at a specific time, etc. Here (fols 193^{v}–201^{r}) we also find tables of composite interest similar (also in layout) to those of Antonio as copied in the Palatino (above, p. 230), with the only difference that Benedetto starts with 6 percent per year instead of 5 percent – plausibly because Benedetto deemed this low rate to be deprived of practical interest.

There are many marginal calculations in these five chapters – the computations are indeed often quite intricate; but almost all seem to be of the kind that seems intended for the reader, not something needed by Benedetto; we may assume that Benedetto copies, from his own earlier or from foreign material (maybe updating such matters as dates of fictional payments due, which on fols 203^{v}–213^{r} run from 1458 to 1464, reaching into near future at the moment of writing). Only the text of a few problems in the end of chapter 5 dealing with loans with interest that is amortized by the rent of a house have calculations written first, showing them to be independently calculated by Benedetto in 1463.

Chapter 6 (fol. 218^{v}–223^{r}) deals with repeated commercial travels with expenses – a topic which Fibonacci deals with *before* the problems about amortization by rent, we

[414] That such calculations are made on the manuscript page itself seems to exclude that Benedetto used loose scrap paper or other external means for his calculations. As we shall see in note 430, however, he appears to have changed his practice at a certain moment.

remember from p. 92. Benedetto points out that these problems belong together with those about interest but concern gains that justly are legally accepted.

Almost all of the problems are borrowed from the *Liber abbaci*. Often even the way to solve them is taken over though in different words – Benedetto does not translate, except when explicitly reporting he is a teacher setting out matters in his own words. In many cases, however (in particular but not only when Fibonacci simply indicates the result corresponding to a variation of parameters) Benedetto makes his own calculations, which can then be seen to have been performed in the margin before he wrote the text. Here, his preferred method is the *modo recto*. In one case on fol. 221^{r} (in agreement with Fibonacci [B263;G425] but with new detailed calculations) Benedetto appeals to the *modo verso*. In the problem where Fibonacci appeals to two unknowns, *summa* and *res* ("amount" and "thing") (above, p. 90), Benedetto (fol. 222^{v}) uses *chosa* and *quantità* ("thing" and "quantity"), making an extensive marginal calculation spreading far into the text column, abbreviating *quantità* as *q* and writing *chosa* sometimes in full, sometimes as ρ.[415]

Chapter 7 (fols 223^{v}–226^{v}) is explained to deal with "the way of doubling, called doubling of the chess-board, which problem is fitting for this book". The opening of the chapter (fols 223^{v}–224^{r}) is missing from my scan,[416] and of the first interpretation of the doubling I have thus only seen the final part. This suffices, however, to show that Benedetto uses the Florentine metrology of his time, which means that he has had to do the calculations on his own; and further that he has replaced (or supplemented?) Fibonacci's pedagogical example (above, p. 104) with its sequence *grain – chest – house – city* and the step factor 65536 = 2^{16} by a sequence *grain – small bowl – sack – granary – house – castle – city – region – realm – world – sphere* with step factor 1024 = 2^{10}, which (supposedly) allows to imagine 100 doublings.

Fibonacci's alternative interpretation of the chess-board problem, where "the following square is the double of all its antecedents", is followed closely.

After this, Benedetto presents a selection from the further problems from Fibonacci's chess-board chapter – not in exactly the same order; not all of them; and without Fibonacci's extensive alternative solutions. Last in the chapter Benedetto copies Fibonacci's

---

415 It may be worth observing that at one point the text finds that at the end of the fourth travel (where nothing remains for him) the traveller has

$$5\textit{things}\text{ less }6\textit{quantity}\,{}^{5}\!/_{12}\text{ and }18\textit{fiorini}\,{}^{1}\!/_{4}\ .$$

In the margin this stands as the equation

$$5\rho\text{ equal to }\rule{6em}{0.4pt}\ 6q\,{}^{5}\!/_{12}\ \ 18\,{}^{1}\!/_{4}\ .$$

As mentioned above (p. 284), the long stroke is used regularly in the Ottoboniano *Praticha* as an equation sign (though not only); as we see Benedetto (here, not always) feels the need to explain it in words.

416 The introductory words are thus quoted from [Arrighi 2004/1965: 149].

rabbit problem (above, p. 95) – even here abbreviating.

Book 9 (fols 226^{v}–233^{r}) is presented as "containing the treatise on the rule of *chatain*, meaning 'of 2 false positions'", not only (as in the introduction) as "problems solved" by this method.

This time the term is given the normal explanation, copied from Fibonacci. As the Ottoboniano and the Palatino *Pratiche*, Benedetto goes on (fols 226v–228r) with Fibonacci's explanations and proofs (above, p. 108), once again omitting a number of alternatives. His examples (fols. 228^{v}–233^{r}), on the other hand, are all different from those found in the *Liber abbaci* and in the two other *Pratiche*, and accompanied by marginal calculations made before the text column was written – once again demonstrating that Benedetto is on his own. Most interesting is the last problem (fols 230^{r}–233^{r}), dealing with five men buying a horse. Here Benedetto applies Fibonacci's multiple nesting of double false positions (above, p. 115), calling the position for the price of the horse the "general" (not "universal") position; the subordinate positions are spoken of simply as "positions", not as (in the *Liber abbaci*) "particular positions"; none the less, inspiration from Fibonacci is hardly in doubt.[417] As shown by comparison with the other two

---

[417] I shall not go through the calculations – the six folio pages would become exorbitant even if exoressed in symbols – but only shoow the beginnings. If A, B, C, D and E are the possessions of the five men and H the price of the horse, the givens are (all numbers count *fiorini*):

$$A+{}^{2}\!/_{5}(B+C+D+E)-4 = H$$
$$B+{}^{1}\!/_{4}(A+C+D+E)+5 = H$$
$$C+{}^{1}\!/_{5}(A+B+D+E)+2 = H$$
$$D+{}^{1}\!/_{8}(A+B+C+E) = H$$
$$E+{}^{3}\!/_{5}(A+B+C+D)-14 = H$$
$$A+B+C+D+E+H = 190$$

Perspicaciously, Benedetto promises that "it will be very long to explain if made by position" (*molto lungho dire fia a farlo per positioni*). At first he makes the general position that $H = 50$, and the (secondary) position that $A = 20$. That gives

$${}^{2}\!/_{5}(B+C+D+E)\text{-}4 = 30\ ,$$

whence

$$(B+C+D+E) = 85\ .$$

Now the tertiary position is made that $B = 17$. Then

$$C+D+E = 68\ ,$$

whence

$$A+B+C+D = 88\ .$$

Inserting this in

$$B+{}^{1}\!/_{4}(A+C+D+E)+5 = H$$

gives $H = 44$, 6 less that the position.

Then the tertiary position is changed to $B = 21$; etc. This leads to $H = 47$, 3 less than needed. The true tertiary position would therefore be $B = 25$, whence $C+D+E = 85–25 = 60$ – etc. The outcome is that $A = 20$, $B = 30$, $C = 40$, $D = 50$, $E =$ debt 10, $H = 60$; as could be expected, the

Pratiche, what Benedetto takes over from his source is beyond the level of even the *Vaiaio* and his students).

Book 10 (fols 233^{v}–299^{v}) does not repeat the statement on fol. 1^{v} that it corresponds to chapter 12 of the *Liber abbaci*. However, inasfar as themes are concerned, the initial statement is not mistaken. In paraphrase, the chapters are introduced as follows:

1. the solution of certain cases about the nature and properties of numbers;
2. cases about men having *denari*;
3. cases about men who work;
4. cases about men who have *denari* and find *denari*;
5. about men who buy a horse, and similarly;
6. about rambling cases, whose rules vary;

Like Fibonacci's corresponding part 12.1, the present chapter 1 (fols 233^{v}–237^{r}) deals with summation of series, including application to pursuit and meeting problems; some problems are borrowed from the *Liber abbaci*, others not. Of some historical though not particular mathematical interest is a meeting problem on fol. 236^{v} (not borrowed from Fibonacci): Two men move around the earth, one towards east and one towards west. "According to the astrologers", the circumference of the earth is 20400 *miglia*.[418] Though the Portuguese had only started their travel eastward by going south in the previous decades and Columbus had not yet started dreaming about reaching the Indies by going west, the idea of travelling around the earth was becoming possible.

Chapter 2 (fols 237^{r}–256^{r}) begins with problems belonging to the easy type which in the *Liber abbaci* [B284–286;G454–456] follows immediately after the rabbit problem (above, p. 96); the problems and the discussion, however, do not depend on Fibonacci; then follow give-and-take problems. The first of these, though not in the *Liber abbaci*, appears to be borrowed (space is reserved within the text column for calculations, obviously to be borrowed – but then this space is not used). The next 17 can be seen to be based on Benedetto's own work: calculations are mostly written before the text column, and most are solved by means of the *regula recta*, which Benedetto from now on rarely identifies by name; mostly the possession of the first man is posited to be a *quantity*, but once (fol. 243 *v*) "in order to remove many fractions" instead 6 *quantità* – as we might expect, Benedetto does not act mechanically but chooses his algebraic unknown

---

value found for *B* in the first instance is not definitive.

As Benedetto observes, whoever insists on solving such cases by position will run into very great toil (*grandissima faticha*). We shall soon see that Benedetto himself has better methods to propose.

418 The value comes from al-Farghānī, see [al-Farghānī 1546: 28], and must thus refer to "miles" used in early ninth-century Iraq.

sagaciously.

Then, on fol. 244[r], we find a problem tacitly borrowed from the *Liber abbaci* [B194; G328], which may be expressed in symbols as

$$A+7 = 5\cdot(B-7)+1\ ,\quad B+5 = 7\cdot(A-5)+2\ .$$

Fibonacci argues in a way that is similar to what he had done with line diagrams; Benedetto instead proceeds by means of an unidentified *regula recta*, apparently calculating in the margin before writing the text.

The next problem (fol. 244[v]) is (correctly) identified as coming from chapter 7 part 3 of the *Liber abbaci*, but Benedetto points out that Fibonacci errs when claiming that the problem has no solution. The problem [B193;G328] can be expressed

$$A+7 = 5\cdot(B-7)+12\ ,\quad B+5 = 7\cdot(A-5)+12\ .$$

Fibonacci claims indeed that the problem when generalized as

$$A+7 = 5\cdot(B-7)+X\ ,\quad B+5 = 7\cdot(A-5)+\mathrm{X}$$

can be solved only as long as X (presupposed integer) does not exceed 11, and the problem with $X = 12$ is supposed to illustrate that. Unfortunately he makes a mistake – the limit is $X<41$, after which first A and then also B become negative.[419] Fibonacci, however, argues in a way that involves two auxiliary entities, the "major sum" $S = A+B$ and the "minor sum" $s = A+B-X$. When solving the case $X = 12$ Fibonacci finds that $(\frac{7}{8}+\frac{5}{6})\cdot s = s$, which he considers impossible – overlooking that it just means that $A+B = 12$, in which case $s = 0$. This is indeed the solution ($A = 5$, $B = 7$).

Benedetto relates Fibonacci's argument (slightly misrepresenting it as if it concerned the possessions A and B) and then shows (on a different example) how the *modo recto* can be applied; irrespective of his reverence for Fibonacci he is quite able to leave his trail (as also when warning against use of the nested double false position).

29 more problems follow – a few taken over from the *Liber abbaci*, but even then as a rule solved independently. Some are of the normal give-and-take type where transfers are defined in absolute terms, in some however they are defined as a fraction of the possession of the giver or the receiver; some belong to the same family as Fibonacci's grasping problem (above, p. 96), told in a way that involves a friend making peace; some state complex linear relations between the possessions. In four cases, *regula recta* with two unknowns is applied:

1. Fol. 249[v], the first part of the solution of problem about four possessions,
$$A = \tfrac{1}{2}B+\tfrac{1}{3}C+\tfrac{1}{6}D-8\ ,\quad B = \tfrac{1}{5}(A+C)+\tfrac{1}{4}D+2\ ,\qquad \ldots$$
is made by means of "the rule of *chatain*" – but as it turns out, what is meant is the "the simple mode of *chatain*" from book 7, that is, Fibonacci's "rule of proportion".

[419] Solution of the equation system leads to $A = (121-3X)/17$, $B = (167-4X)/17$.

However, that is not enough, so after almost one folio page of arithmetical arguments *C* is posited to be a *chosa* and *D* a *quantità*. In the marginal calculations they appear as ρ and *q*.

2. In a variant of the grasping-problem on fol. 254^v, *quantità* and *chosa* are used, still appearing as *q* and ρ in the margin; in the end of the marginal calculation its outcome is stated like this:

   the *quantità* is worth 13
   the *chosa* is worth 141

   Within the marginal calculations we also encounter the long stroke used as an equation sign.
3. Another variant of the same problem on fol. 255^r uses the same two unknowns, and is similar in all respects.
4. Yet another variant of the problem takes as its unknowns *quantità* and *somma*.[420] In the marginal calculations, *quantità* is abbreviated *q* but *somma* written in full. Once again we find a final summary in the margin and the long stroke used as equation sign.

The chapter closes by stating that its “problems are of great pleasure and great fantasy”.

Chapter 3, about “men who work” (fols 256^v–261^v), starts simply: “20 men plant 900 trees in $^1/_2$ day. It is asked, in 20 days, how many trees would 100 men plant”. Soon follow problems concerning combined works, of increasing complexity. The last problem (fol. 261^r) deals with a piece of work performed by three masters, “two without the first/second/third can finish in 6/8/9 days”. There are no marginal calculations, so everything seems to be borrowed without recalculation (from what Benedetto had already done elsewhere or from somebody else). Several of the problems are solved by means of the *regula recta* with a single unknown, involving formal fractions. These, as we know, had been in full usee for more than a century, and do not call for further analysis.

Chapter 4, “about men who have *denari* and find *denari*” (fols 261^v–272^v), contains problems of type “finding a purse”. At first comes this:

> Two have *denari* and find a purse in which there is *denari*. The first says to the second, if I had the money in the purse I would have 3 times as much as you. The second says

420 As we remember from p. 90, Fibonacci also uses *summa* (translated “amount”) in the travel problem. Since the problem is quite different, there is no reason to believe Benedetto copied this idea, in both cases the entity thus called *is* an amount of money. But they reflect a shared basis idea, to take the name of some entity appearing in the problem – be it “amount”, be it “goose” as on p. 247 or “purse” as on p. 86) as the second algebraic unknown.

> to the first, if you gave me the *denari* in the purse I should have 4 times as much as you.

The possession of the first man is posited to be a *quantità*. The contents of the purse (*borsa*) is not posited, but it is dealt with as a second algebraic unknown in the ensuing *regula recta* solution. In the marginal calculation (quite short, the problem is simple) the two appear as *q* and *b*. Even here, a long horizontal stroke serves as equation sign.

Soon the problems become more complex – sometimes becoming determinate, sometimes involving until five men, sometimes two, three or four purses with given linear relations between their contents; regularly, an algebraic solution involving the same unknowns is used,[421] at times with highly complex marginal calculations made before the text column was written.

On fol. 270^{v} something totally unexpected happens (to my knowledge and with a proviso to follow about Fibonacci, totally unprecedented):

> Four have *denari*, and walking on a road they found a purse with *denari*. The first and the second say to the third, if you give us the purse we shall have 2 times as much as you. The second and the third man say to the fourth, if we had the *denari* of the purse we should have 3 times as much as you. The third and the fourth say to the first, if we had the *denari* of the purse we should have 4 times as much as you. The fourth and the first say to the second, if you give us the *denari* of the purse we shall have 5 times as much as you. It is asked how much each had, and how many *denari* there were in the purse.

At this point (that can be seen from the organization of the page, redrawn below, p. 297) Benedetto starts making symbolic algebraic operations in the "margin" – in one place going more than 80% into what should be the text column. Using already familiar standard abbreviations for "the first", "the second", "the third" and "the fourth" (which I shall represent by α, β, γ and δ) and *b* for the purse he first writes the equations (juxtaposition as usually meaning addition, enlarged distance equality)

$$\begin{array}{ll}
\gamma & \quad {}^{1}\!/_{2}\alpha\ {}^{1}\!/_{2}\beta\ {}^{1}\!/_{2}b \\
\delta & \quad {}^{1}\!/_{3}\beta\ {}^{1}\!/_{3}\gamma\ {}^{1}\!/_{3}b \\
\alpha & \quad {}^{1}\!/_{4}\gamma\ {}^{1}\!/_{4}\delta\ {}^{1}\!/_{4}b \\
\beta & \quad {}^{1}\!/_{5}\alpha\ {}^{1}\!/_{5}\delta\ {}^{1}\!/_{5}b
\end{array}$$

[421] In the case of several purses, the contents of the first becomes the unknown *borsa*. In a problem on fol. 267^{v} where the contents of three purses is given, there is no algebraic role for the *borsa*. In a problem beginning on fol. 269^{r}, the *quantità* is the joint possession of the first two men – but Benedetto is so much in the habit that he writes "of the first" and then has to insert "and the second" above the line.

and then he starts operating algebraically on these. That is, Benedetto undertakes an algebraic calculation with five unknowns, apparently without thinking that this is something particular.

Actually, the four equations are already the outcome of a first algebraic operation, which is described in the text he writes afterwards, which goes on

> We shall do in this way, you shall say, the first and the second with the *denari* of the purse say to have 2 times as much as the third man. Whence the third man by himself had the $^1/_2$ of that which the first and the second and the purse have. And mark [*segnia*[422]] this. And then you shall say, the second and the third man with the purse have 3 times as much as the fourth, so the fourth man had the $^1/_3$ of that which the first and second have, and of the purse. And mark even this. And then you shall say, the third and the fourth man with the purse had 4 times as much as the first, and therefore the first man by himself had the $^1/_4$ of that which the third and the fourth man had, and of the purse. An mark this. And then you shall say, the fourth and first have with the purse 5 times as much as the second, so that the second will have the $^1/_5$ of that which the first and fourth man have, and of the purse. And this is marked. And you shall bring the *denari* of the third to a comparison.[423] And you shall say, the *denari* of the third man is as much as the $^1/_2$ of the *denari* of the first and the second and of the purse. From where it is to be known how much are the $^1/_2$ of the *denari* of the first, which we have ¿brought together?, that the *denari* of the first are the $^1/_4$ of the *denari* of the third and fourth man and of the purse, and let the $^1/_2$ of the *denari* of the first be $^1/_8$ of the *denari* of the third and fourth and of the purse. Therefore you shall say that the *denari* of the third should be as much as the $^1/_2$ of the *denari* of the second and of the purse and as much as $^1/_8$ of the *denari* of the third and fourth and of the purse. Therefore you shall take away $^1/_8$ of the *denari* of the third and join $^1/_8$ of purse to $^1/_2$ purse. And we shall have that $^7/_8$ of the *denari* of the third are $^1/_8$ of the *denari* of the fourth and $^1/_2$ of the *denari* of the second and $^5/_8$ of purse. [...].

This only takes up two lines (with two lines inserted indicating the factors $^1/_2$ and $^1/_8$) in the symbolic calculation. Without this incipient symbolic algebra it would probably not have been possible for Benedetto to plan the whole ensuing sequence of substitutions in a way easily leading to the goal.

With this tool at hand, he soon reaches the point where he can reduce the question to one with the usual two unknowns, the *quantità* (the same as β) and the *borsa*. Finding that $q = 2596b$ Benedetto chooses (the problem being indeterminate) the solution $b = 4897$,

---

422 Technically, this means that it is written down as a symbolic equation.

423 Technically, as we see in the following, this leads to an algebraic substitution.

q = 2596. This gives that the four possessions are 3717, 2596, 2596 and 4366 *denari* and the purse 4897 *denari*, which Benedetto reduces, dividing by the common divisor 59, to the solution "in smaller numbers" 63, 44, 95, 74 and 83 *denari*.

The marginal calculation is made within a number of areas delimited by curved lines – see the redrawing on the next page. Benedetto refers to the whole structure as a *castelluccio*, a castelet, and we may indeed see it as a building consisting of many chambers.[424]

It might be difficult to follow the path through the castelet without being guided by the text, but once one knows the order it can be seen to go through these steps, starting upper left:

(1) $$\gamma = {}^1\!/_2\alpha + {}^1\!/_2\beta + {}^1\!/_2 b\ ,$$

(2) $$\delta = {}^1\!/_3\beta + {}^1\!/_3\gamma + {}^1\!/_3 b\ ,$$

(3) $$\alpha = {}^1\!/_4\gamma + {}^1\!/_4\delta + {}^1\!/_4 b\ ,$$

(4) $$\beta = {}^1\!/_5\alpha + {}^1\!/_5\delta + {}^1\!/_5 \mathrm{b}\ ,$$

derived from the initial conditions of the problem.

As a first step, to the right in the same chamber, ${}^1\!/_2\alpha$ is found from (3) and substituted in (1), yielding

(6) $$\gamma = {}^1\!/_8\gamma + {}^1\!/_8\delta + {}^1\!/_8 b + {}^1\!/_2\beta + {}^1\!/_2 b\ ,$$

which is reduced to

(6) $${}^7\!/_8\gamma = {}^1\!/_8\delta + {}^1\!/_2\beta + {}^5\!/_8 b\ .$$

Next (2) is used to find ${}^1\!/_8\delta$, which is substituted into (6), leading to

(7) $${}^7\!/_8\gamma = {}^1\!/_{24}\beta + {}^1\!/_{24}\gamma + {}^1\!/_{24} b + {}^1\!/_2\beta + {}^5\!/_8 b\ ,$$

which is reduced to

(8) $${}^5\!/_6\gamma = {}^{13}\!/_{24}\beta + {}^2\!/_3 b\ .$$

Division by ${}^5\!/_6$ transforms this into

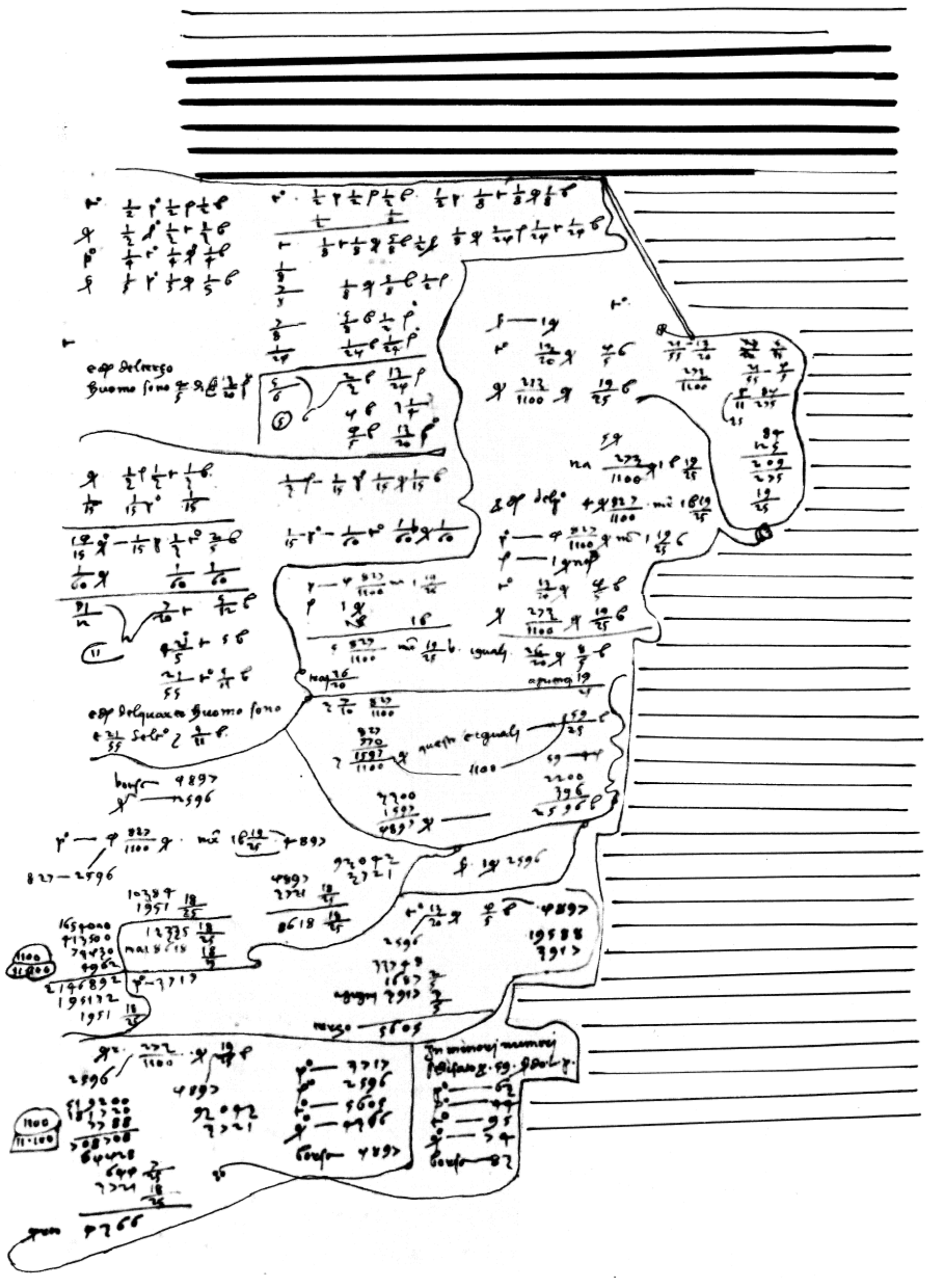
(9) $$\gamma = {}^{13}\!/_{20}\beta + {}^4\!/_5 b$$

In the next part of the calculation (next chamber downwards), ${}^1\!/_3\beta$ is found from (4) and inserted into (2), which leads to

(10) $$\delta = {}^1\!/_{15}\alpha + {}^1\!/_{15}\delta + {}^1\!/_{15} b + {}^1\!/_3\gamma + {}^1\!/_3 b\ .$$

---

[424] Elsewhere (also in Benedetto's *Tractato d'abbacho* [ed. Arrighi 1974: 55 and *passim*], the term designates a particular scheme for multiplication – in Benedetto's *Tractato* nicely contained within walls. For other occurrences, see [Smith 1923: 111].



Fol. 270v, redrawn. Thick lines represent the problem statement, thin lines the procedure description (the first two lines belong to the procedure of the previous problem).

This is reduced to

(11) $$^{14}/_{15}\,\delta = {}^{1}/_{15}\,\alpha + {}^{1}/_{3}\,\gamma + {}^{2}/_{5}\,b\ .$$

Now (3) is used to derive $^{1}/_{15}\,\alpha$, which is substituted into (11). That gives

(12) $$^{14}/_{15}\,\delta = {}^{1}/_{60}\,\gamma + {}^{1}/_{60}\,\delta + {}^{1}/_{60}\,b + {}^{1}/_{3}\,\gamma + {}^{2}/_{5}\,b\ ,$$

which reduces to

(13) $$^{11}/_{12}\,\delta = {}^{7}/_{20}\,\gamma + {}^{5}/_{12}\,b\ .$$

Division by $^{11}/_{12}$ reduces this to

(14) $$\delta = {}^{21}/_{55}\,\gamma + {}^{5}/_{11}\,b\ .$$

Now (the large chamber to the right) Benedetto shifts to the familiar set of two unknowns, q, identified with β, and b, already in service. From (9) we get that

(15) $$\gamma = {}^{13}/_{20}\,\beta + {}^{4}/_{5}\,b\ ,$$

and from (14) that

$$\delta = {}^{21}/_{55}\cdot({}^{13}/_{20}\,q + {}^{4}/_{5}\,b) + {}^{5}/_{11}\,b\ ,$$

whence

(16) $$\delta = {}^{273}/_{1100}\,q + {}^{19}/_{25}\,b\ .$$

Further, from the original condition behind (4) we know that

$$\delta + \alpha + b = 5\beta\ ,$$

whence

(17) $$\delta + \alpha + b = 5q\ .$$

This leads to

$$\alpha = 5q - ({}^{273}/_{1100}\,q + 1\,{}^{19}/_{25}\,b)\ ,$$

that is

(18) $$\alpha = 4\,{}^{827}/_{100}\,q - 1\,{}^{19}/_{25}\,b\ .$$

The values for α and β are now inserted into the original condition that gave rise to (1),

(19) $$2\gamma = \alpha + \beta + b\ ,$$

from which follows

(20) $$2\gamma = 5\,{}^{827}/_{1100}\,q - {}^{19}/_{25}\,b\ ,$$

that is,

(21) $$5\,{}^{827}/_{1100}\,q - {}^{19}/_{25}\,b = {}^{26}/_{20}\,\beta + {}^{8}/_{5}\,b$$

(even in the marginal calculation, $^{26}/_{20}$ appears without reduction). Addition and subtraction lead to

$$(22) \qquad 3\,^{1597}/_{1100}\,q = {}^{59}/_{25}\,b\ ,$$

and after multiplication by 1100 "so as to avoid fractions"

$$(23) \qquad 4897q = 2596b\ .$$

So (lower left chamber), if b is chosen to be 4897 (as Benedetto knows, the problem is indeterminate and allows this choice), q will be 2596. From (18) then follows that α = 3717; β is already known to be 2596, while γ is found for instance from (15) to be 5605, and δ (16) to be 4366; b is already known to be 4897.

"In smaller numbers" (bottom right chamber) this is reduced by the factor 59, which gives him $\alpha = 63$, $\beta = 44$, $\gamma = 95$, $\delta = 74$, $b = 83$. Since the coefficient 2596 in (23) arises as $59\cdot{}^{1100}/_{25} = 55\cdot 59$, it will have been obvious to try whether 59 is a common divisor. Benedetto does not explain from where he gets the number 59 but says to act "according to L[eonardo] P[isano]".

A confrontation with the problem from the *Liber abbaci* that was presented on p. 86 is illuminating. The *problem* is obviously the same, and in the beginning the procedures are parallel; but at the point where Benedetto brings "the *denari* of the third to a comparison" (6), they diverge, and the new symbolic tool helps him to make a much more transparent exposition that the one he could find in the *Liber abbaci*.

Nothing in the text suggests that Benedetto sees his new tool as revolutionary, and within his own horizon he is right.[425] At the background of Fibonacci's text, what Benedetto provides is for the moment a rather marginal improvement, no mathematical revolution.

Three simpler purse problems round off the chapter, two solved by means of *quantità* and *borsa* and one by arithmetical "consideration".

In general terms, chapter 5 (fols 272v–288v), which deals with "men who buy horses or similarly", does not differ from chapter 4. Many problems are taken over from Fibonacci, but as a rule Benedetto prefers his own method, mostly the *regola recta* with two unknowns. Noteworthy are four problems that gradually unfold symbolic algebra

[425] If he had done so, it would have been obvious to point it out in something meant as a gift to a patron, thus cautiously intimating the value of the gift – but he does not, neither here nor when the method returns in the next chapter.

involving more than two algebraic unknowns. The first of them (fol. 277^{r}) runs like this:[426]

> Four men have *denari* and want to buy a horse, and no one has so many *denari* that he can buy it. The first says to the second and the third, if you give me $^{1}/_{2}$ of your *denari*, with mine I shall buy the horse. The second says to the third and fourth man, if you give me the $^{1}/_{3}$ of your *denari*, with mine I shall buy the horse. The third man says to the fourth and the first, if you give me the $^{1}/_{4}$ of your *denari*, I shall buy the horse. Further, the fourth man asks the first and the second for the $^{1}/_{5}$ of their *denari* and says to buy the horse. It is asked, how many *denari* each one had, and what the horse was worth.
>
> Even though there are many ways to solve such cases I shall take the most convenient, or let us say the least tedious.|| That is that you shall say, we propose that the first with the half of the *denari* of the second and of the third man has a horse. And we say that the second with the third of the *denari* of the third and fourth man buy the horse. So the first with the $^{1}/_{2}$ of the *denari* of the second and third man has as much as the second has with $^{1}/_{3}$ of the third and fourth man. From there you will see confronting [*ragua-gliando*], that is, detracting first on each side [*parte*] $^{1}/_{2}$ of the *denari* of the second, and we shall have that the *denari* of the first man with $^{1}/_{2}$ of the *denari* of the third man are as much as $^{1}/_{2}$ of the *denari* of the second with $^{1}/_{3}$ of the third and fourth man. And then you remove from each side $^{1}/_{3}$ of the *denari* of the third man, and we shall have that the first man with $^{1}/_{6}$ of the *denari* of the third man is as much as $^{1}/_{2}$ of the *denari* of the second. And take note of this [*nota* [427]]. [...]

This time, the text until || was written first, after which Benedetto starts calculating in the margin. We observe, firstly, that Benedetto is aware that he is using a particular method, which furthermore is "most convenient", that is, "less tedious" than the alternatives. A modern reader will certainly agree, in particular when realizing that the marginal calculations and not the textual description is meant. Moreover we find a new technical term, "to confront" (*raguagliare*), meaning to construct the *reduced* equation, obtained by addition *and* subtraction on both sides of the equation. The marginal calculation is still organized in a "castelet", even though the word does not reappear (there is no occasion for that), and the symbolic operations are also quite similar to what we know from the purse problem. However, since the price of the horse does not enter until

---

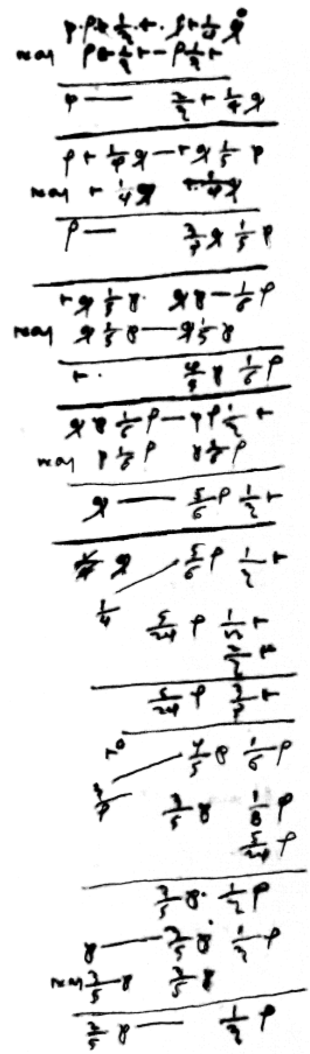
[426] A closely related problem is found in the *Liber abbaci* [B240;G393], only with fractions $^{1}/_{3}$, $^{1}/_{4}$, $^{1}/_{5}$ and $^{1}/_{6}$ – apart from that the structure of the question is identical. There too, Fibonacci's procedure is if not algebraic than at least quasi-algebraic, but even in the present case Benedetto proceed on his own.

[427] Technically, *notare*, translated "to take note" serves as a synonym for *segnare*, "to mark".

the very end, here we have symbolic algebra with *four* unknowns – towards the end once again reduced to algebra of two unknowns, here *quantità* and *chavallo*.

After a simpler three-participant problem solved algebraically by means of *quantità* and *chavallo* follows (fol. 278$^{\mathrm{v}}$) a further sharpening of the new tool in another problem taken over from Fibonacci, where it gets a characterizing name, "by equation".[428] Benedetto *might* have expressed Fibonacci's quasi-algebraic procedure within the new framework and does so in the initial steps – not necessarily copying, these are simply the obvious first steps. Then, once again, the two solutions diverge.

> Four have *denari* for which they want to buy a horse, and none of them has so many *denari* that he can buy it. The first and the second say to the third man, if you give us the $^1/_3$ of your *denari*, we shall buy the horse. The second and third man say to the fourth man, if you give us the $^1/_4$ of your *denari* we shall buy the horse. The third and fourth man say to the first, if you give us the $^1/_5$ of your *denari*, with ours we shall buy the horse. The third and first man say to the second, if you give us the $^1/_6$ of your *denari* we shall buy the horse. It is asked, how much each one had, and what the horse was worth. We shall do it by equation. Where you shall say, the first and second with $^1/_3$ of the third buy the horse. And the second and third man with $^1/_4$ of the fourth man buy the horse. Thus the *denari* of the first and second with $^1/_3$ of the *denari* of the third man are as much as are the *denari* of the second and third man with $^1/_4$ of the *denari* of the fourth. Where confronting the sides, taking away from each side the *denari* of the second and $^1/_3$ of the *denari* of the third, we shall have that the *denari* of the first are as much as $^2/_3$ of the *denari* of the third man and $^1/_4$ of the *denari* of the fourth man. And mark this. Then you shall say, the second and third man with $^1/_4$ of the *denari* of the fourth man buy the horse. And the third and fourth man with $^1/_5$ of the *denari* of the first buy a horse. So the *denari* of the second and third man with $^1/_4$ of the *denari* of the fourth man are as much as the *denari* of



428 *per aguagliatione* – as we remember, this term regularly refers to the *reduced* equation.

the third and fourth man with $^1/_5$ of the *denari* of the first. Therefore take away from each side the *denari* of the third and $^1/_4$ of the *denari* of the fourth man, and we shall have that the *denari* of the second are $^3/_4$ of the *denari* of the fourth and $^1/_5$ of the *denari* of the first. And then, going on, you shall say that the third and fourth man with $^1/_5$ of the *denari* of the first buy the horse. And the fourth and first with $^1/_6$ of the *denari* of the second buy the horse. It therefore follows that the *denari* of the third and fourth man with $^1/_5$ of the *denari* of the first are as much as the first and fourth with $^1/_6$ of the *denari* of the second. Where, confronting the sides, taking away on each side the *denari* of the fourth man and $^1/_5$ of the first, we shall have that the ~~fourth~~ third man ~~who~~ has the $^4/_5$ of the first and $^1/_6$ of the second. And mark this. And thus you shall do for the fourth man, saying, the first and fourth with the $^1/_6$ of the *denari* of the second buys the horse. The first and second with the $^1/_3$ of the *denari* of the third buy the horse. Therefore the fourth and first with the $^1/_6$ of the *denari* of the second have as much as the first and second with $^1/_3$ of the *denari* of the third man. Therefore confronting the sides, taking away on each side the *denari* of the first and $^1/_6$ of the *denari* of the second, we shall have that the *denari* of the fourth are $^5/_6$ of the *denari* of the second and $^1/_3$ of the *denari* of the third. And of that has been taken note. And you shall begin at the first equation,[429] saying, the *denari* of the first are the $^2/_3$ of the *denari* of the third man and $^1/_4$ of the fourth man. Therefore it has to be known what $^1/_4$ of the *denari* of the fourth are. From the others, however, we have found that the *denari* of the fourth man are the $^5/_6$ of the second and $^1/_3$ of the third man, where the $^1/_4$ of the *denari* of the fourth man are as much as the $^5/_{24}$ of the *denari* of the second and $^1/_{12}$ of the *denari* of the third. Where to the $^2/_3$ of the *denari* of the third man you join $^1/_{12}$ of the *denari* of

[429] *aguagliatione* – as can be seen in the marginal calculation, the first *reduced* equation.

> the third and $^5/_{24}$ of the second, they make $^3/_4$ of the third and $^5/_{24}$ of the *denari* of the second. And then bring the $^3/_4$ of the third ~~and~~ apart from the others, saying, the third man has the $^4/_5$ of the first and $^1/_6$ of the second, where the $^3/_4$ of the third man are the $^3/_5$ of the first and $^3/_{24}$ of the second. And you shall join to $^5/_{24}$ of the second $^3/_5$ of the first and $^3/_{24}$ of the second, they make $^3/_5$ of the first and $^1/_3$ of the second, and we shall have made that the *denari* of the first are as much as $^3/_5$ of the first and $^1/_3$ of the second. Therefore you shall detract on both sides the *denari* of the first, you shall have that $^2/_5$ of the *denari* of the first are $^1/_3$ of the *denari* of the second. That is, that the $^2/_5$ of the *denari* of the first are as much as the $^1/_3$ of the *denari* of the second. Thus, if the first should have 5, the second would have 6. Let us now try the others. You shall say that the third has as much as the $^4/_5$ of the first and the $^1/_6$ of the second. Therefore, the $^4/_5$ of the first and $^1/_6$ of the second are 5. And so much would he have. And the fourth has $^5/_6$ of the second and $^1/_3$ of the third, where the $^5/_6$ of the second are 5 and $^1/_3$, and the $^1/_3$ of the third man are $1\,^2/_3$, which all make $6\,^2/_3$. And thus it is done, the first has 5 and the second 6 and the third 5 and the fourth $6\,^2/_3$. Which, so as not to have fractions, multiply all by 3. And you shall have the first 15, the second 18, and the third man 15, and the fourth man 20. And so as to know what the horse is worth, you shall join 15 of the first and 18 of the ~~third~~ second, they make 33. To these joined the $^1/_3$ of the *denari* of the third man, that is, of 15, they make 38. And as much is worth the horse. And thus the first had 15, the second 18, and the third had 15, and the fourth man had 20. And the horse was worth 38.

Now the technique is mature. Firstly, Benedetto no longer falls for the temptation to shift to the traditional two unknowns, he uses the four unknowns (as observed above, the price of the horse does not enter the algebraic manipulations) until the end. Secondly, the marginal calculation (as redrawn here) is extremely neat. It fills only a narrow column in the margin and does not go into the text column, but the corrections in the describing text still suggest that the marginal calculations were made first.[430]

A related problem follows immediately on fol. 279r, dealing with five men who in groups of three ask the next one in the cycle for, respectively, $^1/_4$, $^1/_5$, $^1/_6$, $^1/_7$ and $^1/_8$ of their money so as to be able to buy the horse.[431] Even this case, Benedetto says, will be made "by equation", and he shows how the first reduced equation is to be constructed. For the others he refers to "the teaching made below", and a space large 11.5 centimetres and high 9 centimetres is indeed left blank there; unfortunately, it has not been filled by calculations, but none the less we see that Benedetto thought that his symbolic calculations would be preferable to a verbal description. At this point Benedetto may have felt that he had explained the technique sufficienty well. There are 24 more horse problems in the chapter, and most are solved by means of the familiar two unknowns *quantity* and *horse* (or, when a goose or a hare is bought, *quantity* and *goose* respectively *hare*.

Only one relatively simple problem (fol. 282r) about four men, of whom the first asks the second for $^1/_2$ of his *denari* and the third for $^1/_3$ of his (etc.) makes use of the method. No detailed calculations are presented, but when the first manipulations of the rhetorical equations gives the simplified equations that "$^4/_5$ of the first are as much as $^1/_2$ of the second" we find the characteristic phrase "and take note of this", and a corresponding note in the margin; the next two simplified equations are also written in the margin, but without the "take note". Given that the arguments are relatively simple Benedetto may well have made all arguments directly in words, in no need of symbolic calculations.

---

430 It is to be observed, however, that from this point onward, the manuscript contains no more invasive marginal calculations made before the text was written – those that intrude can be seen to be made in already prepared triangular or rectangular spaces. That in not very significant from fol. 300r onward: from then on most of the substance is taken over from Fibonacci, Campanus or earlier prestigious abbacus authors (due credit given), and whatever marginal material was to be inserted was known to Benedetto from the originals. Even fols 288v–299v, containing "rambling" recreational problems "for which the rules vary" may not be informative. But after the present problem there are still ten folios (twenty pages) with horse problems, where marginal calculations could be expected, and all we find in the margin are scattered brief notes extracted from the text, in the style "*t*o 12 *q* mê 6 *ca*" ("the third, 12 *quantità* less 6 horses – thus on fol. 282v)". It looks as if Benedetto has decided from this point onward to be a clean writer and make his draft work separately. There are a few inconsequential exceptions, on fols 314r, 315r, 325r and 335r. In all three instances, composite geometric diagrams (no calculation) seem to have been made first – most likely because it was difficult to predict how much space they would take up before they were effectively drawn.

431 The problem is once again borrowed from the *Liber abbaci* [B243;G398]. Fibonacci solves it by a quasi-algebraic procedure.

Beyond that, one problem dealing with five men and five horses (the prices of which differ by known amounts) (fol. 286^v) explains that

> where in the way given before, making and observing well, you shall have that the first man had 1589 *fiorini* [...].

There is no further explanation, nor any annotation in the margin. Here, given the complexity of the problem, the way referred to *might* be "by equation"; but the reference to "the way given before" also occurs in other situations where the method "by equation" is not meant, so we cannot be sure.

Neither the next chapter nor the following books offer any occasion to practise the method. No reader we know of seems to have noticed that Benedetto had introduced a new method; when Johannes Buteo (Jean Borrel) did something very similar in [1559] he did not even need to denigrate Benedetto, as he did with other sources so as to deny his debts;[432] as his contemporaries, he almost certainly neither knew nor knew about his Florentine predecessor.

The sixth and last chapter of book 10 (fols 288^v–299^v) deals with "cases called rambling"; different from Fibonacci and the Ottoboniano *Praticha* Benedetto explains that this means cases solved by varying rules (*chasi nominati erratici, de' quali le reghole si variano*) – thus the heading, the beginning of the text explains in more detail while adding that they are also *dilettevoli*.[433]

First come three problems of type *leo in puteo*, similar in structure but not coinciding with those which Fibonacci presents in his "tree" chapter (above, note 44 and p. 81). Since Benedetto solves them correctly, as we may say,[434] they are no longer homogeneous,

---

432 Among these probably Stifel, since like this predecessor he uses *A*, *B*, *C* and *D* for the unknowns – see below, p. 387.

433 The Palatino *Praticha* (fol. 237^v) also attempts an explanation, though less fortunately, claiming that they mostly have several answers – evidently a misunderstanding of what we find in the Ottoboniano *Praticha* (fol. 206^v), that they can be solved in several and different ways.

434 The first is about a man going from Florence to Pisa, a distance of 40 miles (it is said), walking 4 miles a day, and returning 2 miles each night. Benedetto prescribes to detract at first 4 miles, corresponding to what is reached after the first day; the remaining 36 miles are covered in 18 days, and it is immaterial for the answer whether he returns 2 miles in the following night – except that a following question asks for a return according to the same principle, which is then made in 18 days (Benedetto forgets that the initial 4 miles also correspond to a day and says 17).

The Ottoboniano *Praticha*, fol. 123^v has a very similar problem, we remember (above, p. 256), also with return, and solves it correctly.

and he has justly moved them here. Three slightly complex and badly formulated pursuit problems come next.[435]

Problems of type "unknown heritage" follow (fols 289^{v}–290^{v}) – the same as in the Ottoboniano *Praticha* (above, p. 266), with trivial changes (e.g., 1000 *fiorini* instead of 100 *fiorini*) and with characteristic shared phrases that show that they do not draw independently on Fibonacci even though there is no doubt that both as well as the Palatino *Praticha* are ultimately based on the *Liber abbaci*.[436] Benedetto also shows his better mathematical insight by explaining *why* the formula to be used for the sophisticated cases depends on whether $\alpha<\varepsilon$ or $\alpha>\varepsilon$ (in the sense explained on p. 94).

Next we find a number of problems depending on the "Chinese Remainder Theorem", all from the *Liber abbaci*. As the Ottoboniano and the Palatino *Pratiche*, one of the problems deals with the counting of eggs. However, Benedetto moves this dress to the first problem (as in the Byzantine problem collection, cf. note 368), probably as an appetizer. More interesting than this shared copying from Fibonacci are three problems on fols 292$^{r–v}$:

– To find the smallest number which when divided by 17 leaves 14 and when divided by 19 leaves 10;
– to find a number which when divided by 64 leaves 16 and when divided by 82 leaves 13;
– to find a number which when divided by 13 leaves 6 and when divided by 18 leaves 13.[437]

The first and third are solved semi-empirically – the first by adding so many times 17

---

435 For example, a dog is said (fol. 289^{r}) to follow a hare at a distance of 100 steps, and "each 5 steps of the dog are for 7 steps of the hare". It is not said whether the 100 steps are steps of the dog or of the hare – the solution shows them to be hare steps.

A similar problem in the Ottoboniano *Praticha* (fol. 125^{v}) specifies that the distance is measured in "steps of dog", and the following one that it is measured in "steps of hare" (there is no counterpart in the Palatino *Praticha*). Should we suspect Benedetto of sloppy copying here? Or is the Ottoboniano writer doing better than their shared source for the problem type?

436 For instance, the father wants to divide *la mia sustantia et mobile* (Ottoboniano fol. 211^{v}) respectively *la sostantia e il mobile mio* (Benedetto fol. 289^{v}). The Palatino *Praticha* (fol. 237^{v}) abbreviates to *la mia sustantia*. Evidently, Benedetto had access to the shared source of the two predecessors though he often goes beyond it. (That all three speak of "sons" when Fibonacci deals with the division of a number is immaterial – this change is too close at hand.)

437 Marginal insertions of forgotten passages of 15 and c. 6 words show us that Benedetto copies – probably all three questions.

to 14 that the outcome leaves 10 when divided by 19. The second is said to be one among several sent by the venerable Perugian [438] to master Giovanni [di Bartolo[439]]. It has no solutions, and the intention by the challenge was evidently to leave Giovanni dumbfounded; Benedetto shows instead (probably repeating Giovanni) that the question is impossible, since numbers leaving 16 when divided by 64 are even, while numbers leaving 13 when divided by 82 are odd. It thus illustrates the competitive nature of the ambience of the abbacus masters, and gives us supplementary insight in the types of problems that might serve as challenges.

A few related observations and questions follow, then comes a three+one variant of the loaf-sharing problem we know from Fibonacci and the Ottoboniano *Praticha* (above, p. 266), and a problem about cloth[440] with a similar mathematical structure.

After an inserted problem borrowed from Giovanni di Bartolo follows (fol. 293^{v}) the "apostate" problem about 50 eggs given to three sons (here "men") to sell – see above, p. 267. According to [Ghaligai 1521: 64^{v}] it appears that the term "apostate" was introduced by Benedetto about problems that cannot be solved by any specific rule but are presented so as to provide pleasure "in winter evenings when together around the fireplace", and the absence of the term from the Ottoboniano and Palatino *Pratiche* as well as from (ps.-)Paolo (above, note 371) would seem to support that hypothesis. However, Benedetto's words, "these are called apostate problems" (*queste si dichono ragioni apostate*) suggest that the term was already in use (but perhaps only in his own earlier practice?); also possible is that Ghaligai knows the term from Benedetto (whom he has read) and not from elsewhere, and supposes Benedetto to have introduced it.

As in the Ottoboniano *Praticha*, this is followed by the familiar twin-inheritance problem (above, p. 22), and then by the two problems which in the Palatino *Praticha* precede the twin problem and which the Ottoboniano *Praticha* presents in part 5 chapter

---

438 Space left open.

439 Benedetto relates on fol. 431^{v} that when Giovanni took over Antonio's school at the age of 19 after the death of the latter, invidious older masters tried to expose his incompetence by having their best students challenge him; instead Giovanni showed himself so much better than the competitors that their students left them for Giovanni.

Evidently, these students "of various subjects" (*di varie materie*) will not have been 11-year old boys following the normal curriculum but probably mathematical dilettantes and abbacus masters *in spe*.

440 Three men have, respectively, 14, 12 and 10 cubits of cloth and go together to a tailor to have a robe (*cioppa*) made for each. The tailor manages to make four, the last for himself, which he buys.

10 (above, note 372). Riddles characterized as “fables” follow: two versions of the river-crossing riddle (in wolf-sheep-cabbage and jealous-husbands versions) and the “Joseph game” (above, p. 268). Further come (among other things) a version of the question of how to share the stake of a game which is interrupted before it is finished (here not a game of chance but of cross-bow shooting, interrupted because the arc breaks[441]), and divination problems partly drawn from the *Liber abbaci*, with added explanations and exemplifications.

Book 11 (fols. 300r–310v) starts by explaining that so far all cases could be referred to numbers. For the six last books that will not suffice, even though Benedetto promises to explain as much as possible by means of discrete quantities. But “because the irrational continuous quantities are endless (*innunerabili*) as appears from 10th of Euclid” that is not always possible. Therefore Benedetto starts, with an unspecific reference to Campanus (that is, to his version of the *Elements*, no doubt to book II) how the product 6×6 can be represented by a rectangle. There is a further reference to the “last or indeed second-last book of Leonardo Pisano’s *Praticha*” (the chapter on roots, with the initial explanation of the “keys” – above, p. 117).

A first chapter of the book therefore goes through the same terrain as Fibonacci’s “keys” (actually, the complete *Elements* II in Campanus’s version, whereas Fibonacci stops at II.6), but with full geometric explanations, all obviously inspired by Campanus but with different numerical examples.[442] The chapter closes with an added discussion of the properties of the root of 72, and a reference to Boethius’s *Gran trattato*.

The second part is a translation of part 15.1 of the *Liber abbaci* (above, p. 132). It is very close to the counterpart in the Ottoboniano *Praticha* (part 9, see above, p. 277)

---

[441] 7 rounds are to be made, and the stake of 3 *fiorini* is to be divided proportionally to how many rounds each one has won. When the arc breaks, the first man has won 2 rounds, the second 1 round, the third none. Benedetto admits that many opinions in the matter exists, and presents a solution which he finds clearer than certain others. The solution given is that the first gets $^2/_7$ of the stake for his 2 victories, the second $^1/_7$, while the remaining $^4/_7$ are to be divided equally (the apparent difference in skill not being taken into account). Obviously a simpler situation than the usual question where the winner takes all, and some kind of justice has to be achieved. See select text excerpts in [Schneider 1989: 9–24].

[442] Or self-invented examples. It is not clear whether the Campanus manuscript used by Benedetto contained numerical illustrations. The Basel edition [*Euclidis megarensis ... Elementorum libri xv*, 1537] contains some, but they differ from those of Benedetto; neither the Ratdolt edition [*Preclarissimus liber elementorum Euclidis perspicacissimi*, 1482], [Lefèvre d’Étaples 1516] nor the modern critical edition [Busard 2005] contains any.

though not identical word for word; but since both are also close to Fibonacci's text nothing excludes independent translation. Further analysis is superfluous.

Book 12 (fols 310^v–367^v) is dedicated to roots. Having already presented an equivalent of Fibonacci's "keys", Benedetto starts chapter 1 (fols 310^v–312^v) by referring to the three basic categories considered in *la regola dell'algebra*, leading to the explanation of what a root and next a surd root are, with examples that differ from those of Fibonacci.[443] Next he inverts Fibonacci's order of numerical approximation and exact geometric construction (above, p. 120), giving the geometric construction first; he changes the diagram and the lettering, and adds the same triangle-based alternative as the Ottoboniano and Palatino *Pratiche* (above, p. 272). The method of the numerical approximation is spoken of as the *prossimana*, the term we encountered in the Ottoboniano *Praticha* (above, note 381), which must then have been the name that was locally current; Benedetto himself only speaks of the outcome as "rather close"; the traditional *la più pressa radice*, "the closest root" he uses instead about the closest *integer* approximation (for 10 thus 3). As the Ottoboniano *Praticha* he completes the second approximation ($3\,{}^{37}\!/_{228}$), and then continues by showing how to make a third approximation (though not going through the calculation). After this Benedetto, like the Ottoboniano *Praticha*, finds the approximate root of ${}^{2}\!/_{3}$, but in his own way: first making use of the procedure he has just taught, then the same procedure as the Ottoboniano *Praticha* but with higher precision – thus confirming that Benedetto knew the model of the Ottoboniano and Palatino *Pratiche* or some closely related work (which *could* reach as far back as to Antonio's *Gran trattato*) but deals with the inspiration independently, though forgotten long passages inserted in the margin suggest that sometimes he copies when he comes to the extraction of the square roots of multi-digit numbers.

Chapter 2 (fols 312^v–339^v) is presented as an explanation of the classes of irrational magnitudes from *Elements* X. It is independent of Fibonacci and instead a paraphrase of Campanus proposition by proposition, adding numerical exemplifications as done by Fibonacci. Many diagrams also come from the Campanus *Elements*, but quite a few do not. An amputated version can be found in the Palatino *Praticha*, fols 331^v–350^r, in the middle of a chapter about the multiplication of roots, which shows that Benedetto follows a pre-existing model; he may still have introduced pedagogical changes of his own but we cannot know. After the paraphrase, in what appears to be his personal style and thus

443 This introduction also differs from what is found in the *Palatino* and the Ottoboniano *Pratiche*, cf. above, p. 270.

almost certainly his own contribution, Benedetto admits that

> I certainly believe that many things because of the obscurity of the speaking will be badly understood, but if one puts himself at study he will certainly understand them with facility [...].

This serves as a preamble to a final one-page schematic summary, which (there a bit out of place) is also in the Palatino *Praticha* (fol. 350^r) and thus not original with Benedetto, in spite of his personal preamble.

With chapter 3 (fols 339^v–344^v) Benedetto returns to Fibonacci's footsteps (more precisely, to his part 14.2b), as he declares, requesting the preceding chapter to be well understood before the reader proceeds. In a pattern we have observed before, Benedetto at first follows Fibonacci but soon diverges, presenting many extra problems, following an abbacus-culture tradition which we also find reflected in the two other *Pratiche* but not their precise exposition. Obviously, replacing the numerical parameters in multiplications of monomials, binomials or trinomials is no great mathematical feat if only the principles are well understood. Towards the end, Fibonacci's problem about finding two roots of roots whose product is rational (above, p. 125) is taken over together with the letter-based proof.

Chapter 4 (fols 344^v–351^v) is announced to teach how to divide by the fourteen types of irrational lines (pointing out that division by rationals has already been treated in book 4); actually it starts by teaching how to divide by a root – exemplified by the calculation $30 \div \sqrt{20} = \sqrt{(900 \div 20)} = \sqrt{45}$; it then goes on with similar calculations involving also roots of roots. It thus corresponds to parts 14.4a and 14.4b in the *Liber abbaci*. However, it is independent of section 14.4a, while the counterpart of section 14.4b (like chapter 3) initially follows Fibonacci but then becomes largely independent, venturing into intricacies not dealt with in the *Liber abbaci* – for instance (fol. 350^v) $10 \div \sqrt{(1+\sqrt{9})}$; as we have seen repeatedly since Giovanni di Davizzo and Dardi (above, p. 208 and 218), Benedetto uses rational roots "as if they were irrationals" (and uses the trick for control). It is noteworthy that Benedetto when borrowing from Fibonacci uses the verb *dividere* but elsewhere the term *partire*, current in the abbacus tradition. He omits Fibonacci's discussion of the extraction of roots of binomials here, and moves it to the next chapter.

Of some interest is an observation made (fol. 346^v) at the intersection between the two parts of the chapter, namely that

> Having to divide a joined or movable quantity [*quantità agunto overo mobile*] by a diminished quantity, a diminished quantity results. As saying, you shall divide 100 in less 5, I say that 20 less results. Because to multiply 5 less by 20 less make 100, which are

movable. As you will show that multiplying diminished by movable makes diminished.

This is followed by rules for diminished divided by diminished and diminished divided by joined. "Movable" for a positive quantity almost certainly refers to the concept of "movable property", which (like "joined") seems to refer to something added (as movable property is additional to real property, cf. the testator's reference to *la sostantia e il mobile mio*, above, note 436). What is of interest is the idea of dividing by a "diminished quantity" – that goes beyond the notion of subtractive quantity, and thus implies that this idea was gradually giving way to a concept of negative quantity; still a concept rooted in practice, not yet theoretically based.

Also informative about changing patterns of thought and increasing precision of the mathematical language is the parallel between a passage from the *Liber abbaci* [B376; G579] and its counterpart in Benedetto's *Pratica* (fol. 349^{v}). Fibonacci's text runs

if you want to divide 10 by 2 and by the root of 3 and by the root of 5 [...]

while Benedetto has

if you want to divide 10 by 2 and root of 3 and root of 5 [...].[444]

Fibonacci's words are ambiguous, and would be the same if he had wanted to speak of the separate determination of the three divisions 10÷2, 10÷√3 and 10÷√5. Only the context (that division by a trinomial is taught) shows what is meant.[445] Benedetto instead, by avoiding repetition of the preposition creates an algebraic parenthesis, showing unambiguously (at least to the attentive reader) that a single division with divisor (2+√3+√5) is meant.

Chapter 5 (fols 351^{v}–356^{v}) is presented as dealing with "the joining and detraction of the said lines with certain other propositions needed in the whole book". It corresponds to and initially draws on part 14.3 of the *Liber abbaci* (above, p. 125). Once again, however, Benedetto's text diverges, changes the order of matters, and even when Benedetto treats the same topic he mostly does so in other words and with other numbers.

On fol. 355^{v} begins the section on the extraction of the root of binomials which Fibonacci deals with in the end of part 14.4b (above, p. 128); even here, Benedetto

---

444 Respectively

si vis dividere 10 per 2 et per radicem de 3 et per radicem de 5

and

se vuoi dividere 10 per 2 e radice di 3 e radice di 5.

445 As we remember from note 325, lesser mathematical minds than Fibonacci might fall into a similar trap and assume that $\sqrt{(a+\sqrt{b})} = \sqrt{a}+\sqrt{\sqrt{b}}$.

proceeds with proofs of his own (though based on the same principles) and different numerical examples. While Fibonacci deals only with examples of the first, second and fourth binomials (with a “do likewise” for the third, fifth and sixth), Benedetto deals in detail with all six.

Chapter 6 (fols 356^{v}–367^{v}) promises to say everything Fibonacci explains in the fifth part of his chapter 14, concerned with cube roots. It falls into three parts (not separated explicitly by Benedetto), only the first of which has approximate counterparts in the Ottoboniano and Palatino *Pratiche*. The first part (fols 356^{v}–361^{v}), on the approximate numerical determination of cube roots of numbers and on the conditions that the addition or subtraction of cube roots can be reduced to simpler form, follows Fibonacci’s text closely, omitting nothing substantial and adding nothing though often reformulating. The topic was obviously one where Benedetto had nothing to add and on which he can thus be assumed not to have worked himself.

The second part (fols 361^{v}–362^{v}) can be characterized as an addendum to Fibonacci’s work on roots from the perspective of the abbacus tradition. It leaves behind the specific topic of cube roots and begins by pointing out that there are many kinds of roots that “cannot be understood even though they are defined: *radice relata*, *radice pronicha* (cf. above, p. 214), and various other roots. The definitions of the two roots identified by name are explained and exemplified, and it is pointed out that the pronic root is not linked to a continued proportion as are the others; in order not to produce “major confusion” Benedetto says no more on the matter, and shifts attention to much simpler matters, namely operations with roots and fractions: ${}^2/_3$ of $\sqrt{20}$, ${}^1/_5$ of $\sqrt{80}$, ${}^3/_8$ of $\sqrt{128}$, ${}^2/_3$ of $\sqrt{36}$, $\sqrt{20}$ expressed as part of $\sqrt{30}$, $\sqrt{{}^5/_6}$ expressed as part of $\sqrt{{}^7/_8}$, $\sqrt{12} \div {}^1/_2$, and so forth until ${}^2/_3$ of ${}^3/_4$ of $\sqrt{50}$. All very elementary, but called for by the absence of fractions in the chapters related to Fibonacci’s chapter 14 and the important role of fractions in abbacus mathematics (algebra and otherwise).

The third part (fols 362^{v}–367^{v}) is meant to clarify the connection between continued proportions (starting with 1) and proper roots (that is, not such things as the pronic root, already pointed out not to enter this discussion). In the sequence 1–2–4–8–16–32–64, 2 is thus the root of 4 and the cube root of 8; and in general the product of any number with its root is a cube “as defined in the treatise of cube roots, that is, the present chapter”.

From here, in order that his treatise may have no imperfections, Benedetto moves on (fol. 363^{r}) to something not treated in the *Liber abbaci* but instead in Fibonacci’s *Pratica geometrie* [ed. Boncompagni 1862: 153–155], namely the finding of two mean proportionals. First Fibonacci’s Latin text is quoted, afterwards (fol. 364^{v}) a vernacular translation is given.

Three different constructions are described. Since the proofs are long and very challenging, wholly outside the normal abbacus tradition preceding and following after Benedetto (as also revealed by Benedetto's need to make the vernacular translation himself), there is no reason to include an analysis here. So, no more shall be said about Benedetto's book 12.

Book 13 (fols 368r–388r)[446] is announced as dealing with "how and in which way cases are solved by the rule of *algebra amuchabale*". As stated above (p. 278), the introductions to algebra in the three *Pratiche* share so much that they must draw on the same vernacular version. This version is not identified by the other two *Pratiche*, but Benedetto says it was made by Guglielmo de Lunis:

> Let us render thanks to the Almighty, thus begins the text of the Arabic Aghabar in the rule of Geber which we call algebra. Which rule of algebra, according to the translator Guglielmo de Lunis, embraces these 7 names, that is, *geber* [*al-jabr*], *elmelchel* [perhaps *al-muqābalah*[447]], *elchal* [*al-qabilah*[448]], *elchelif* [perhaps *al-khalās*, "liberation"/"riddance"[449]], *elfatiar* [?], *diffarelburam* [*difaʿ al-burhān*, "defense of the demonstration"[450]], *eltermen* [*al-tamām*[451]]. Which names according to Guglielmo are interpreted thus: *Geber* is as much as to say *recuperatione* because, as will be understood in the following, the case will be solved by the recuperation of two equal parts. *Elmelchel* is as much as to say *exenplo*, or *asomigliamento*, because the solution of the case is found by making similar [*asomigliare*] the quantity that is posited to the given case. *Elchal* is as much as to say *opositione*, because of two quantities that are found one is opposed [*oposta*] to the other, and when there are not two opposed quantities, then the case is insolvable. *Elchelif* is as much as to say *dispositione* because, even though there are two opposed quantities, if they are not disposed for the application of the rules, the case would be outside the rules and therefore there is a need that the quantities be

[446] The whole book 13 is transcribed in [Salomone 1982].

[447] Thus Ulrich Rebstock (personal communication).

[448] The disappearance of the "b" shows that an Iberian pronunciation of Arabic is rendered, as pointed out by Ulrich Rebstock (personal communication).

[449] Thus Ulrich Rebstock. Paul Kunitzsch (personal communication) proposes *al-taʾlīf* ("formation", "composition" etc.).

[450] Proposed by Ulrich Rebstock.

[451] "Completeness", "perfection", and similarly. Proposed by Paul Kunitzsch.

ordered [*disposte*].

Here, the "question Guglielmo de Lunis" needs to be considered, as being often maltreated. We know about Guglielmo's translation from two more sources. One is Raffaello Canacci's *Ragionamenti d'algebra* [ed. Procissi 1954: 302], where we find a similar but not identical passage:

> The role of algebra, which rule Guglielmo de Lunis has translated from the Arabic into our language. And the said Guglielmo and others say it was composed by an Arabic master of truly great insight, even though some others say it was one whose name is Geber, to which Leonardo Pisano says that *algebra muchalbile* is the interpretation of the rule in this language. The rest of the said rules begins, Let us give thanks[452] to the Almighty, and following the said Guglielmo the said rule in the said language contains seven names, that is, seven parts, called like this in the said language, *geber*, *el melchel*, *elchal*, *elchelis*, *elfatiar*, *diffarel buran*, *eltiemen* [...].

In what follows, Canacci's quotation from Guglielmo agrees with what we have seen in Benedetto, except that Canacci (like Jacopo as well as Biagio and Antonio as rendered by Benedetto) speaks of *aghuaglamento* instead of *asomigliamento*. By rendering *al-khalāṣ* as *elchelis* and *al-tamām* as *eltiemen* Canacci shows that he does not know Guglielmo's text from Benedetto (who writes *elchelif* and *eltermen*), or at least not through Benedetto alone.

In [1521: 71v], Francesco Ghaligai repeats Canacci's introduction, but with explicit reference to Benedetto. Like Benedetto he writes *elchelif* and *eltermen*, and he translates *elmelchel* as *assimigliamento*. He must thus know Benedetto's text as well as that of Canacci, or a precursor.

Beyond that, a gloss ascribes erroneously a manuscript of the Gerard-translation to Guglielmo [Hughes 1986: 223]). As Hughes observes, this is only evidence of awareness that Guglielmo had made a translation.

These are all the known references to Guglielmo's translation of al-Khwārizmī's algebra. Except for the last one, which betrays to know nothing beyond the existence of Guglielmo's version, all refer to the quoted list of translations of Arabic words. Whether Canacci's "our language" means Tuscan (or some other Italian) vernacular or Latin cannot be decided – when contrasted to Arabic by a Tuscan writer, Latin could also be "ours".

Now to the "question Guglielmo de Lunis". Since Pietro Cossali [1797: I, 7], historians of mathematics have been aware that Canacci ascribes a translation of Arabic algebra

---

452 Canacci has a meaningless "andano gratie" here, where Benedetto has the certainly correct "rendiamo gratie".

into “our language” to Guglielmo de Lunis, whom Cossali (reading Canacci’s words as a claim that Guglielmo was the *first* to import Arabic algebra) took to have worked before Fibonacci. When it turned out that the Latin algebra (*Liber restauracionis*) which Boncompagni edited in [1851: 28–51] on the basis of the manuscript Vatican, Vat. lat. 4606,[453] was not, as he had supposed, that of Gerard of Cremona, nor that of Robert of Chester, wielding of Occam’s razor led to the assumption that it had to be that of Guglielmo: three names, three translations, no need to multiply entities.

Unfortunately, this identification is impossible. All positive information which we possess about Guglielmo’s translation is that it contained the above list of transcribed Arabic terms together with explanations, and there is not the slightest trace of that in the *Liber restauracionis*. The difference is also revealed in the title of the latter, which corresponds to Gerard’s and Fibonacci’s translation of *jabr* as *restauratio*, while Guglielmo as quoted uses *recuperatio*. As any razor, that of Occam has to be handled with care – if not, if no blood flows, at least one ends up forcing square pegs into round holes.

Guglielmo himself seems to be identical with the Guglielmo de Luna who was connected to the *Studio* of Naples (instituted by Emperor Frederick II in 1224) and/or to the courts of Frederick and his son Manfred; and who translated Ibn Rušd (Averroës) and other philosophical writings into Latin.[454]

After the explanations quoted above, Benedetto goes on:

> And Leonardo Pisano says in the third part of the 15th chapter, the rule of *agebra amuchabale* means rule of opposition and restoration, that is, of restoring, as will be clearly seen in the cases. And, so that the rule will be well understood, I shall divide the said book in three chapters. In the first are the clear demonstrations by diagrams of the said rule. [...].

Which model Benedetto intends to imitate in what follow is not clear from these words. However, in the very end of the first chapter (fol. 371^v) he writes

> An even though Leonardo Pisano shows this same with clear demonstrations, none the less I have taken these as older and written in the said book of *aghabar*.

---

453 Since then, a better preserved manuscript has been discovered and edited by Wolfgang Kaunzner [1986]; and recently, Marc Moyon [2019] has discovered a third manuscript and produced a critical edition based on all three manuscripts.

454 See, lately, [Delle Donne 2007]. Roland Hissette [1997] raises tentative objections, but based on the impossible (and not fully endorsed) identification of Guglielmo’s translation with the *Liber restauracionis*.

That is, the first chapter follows or paraphrases Guglielmo's translation. While the above-mentioned *Liber restauracionis* is an evident redaction, comparison with Gerard's translation shows that Guglielmo was very close to al-Khwārizmī's original text (even in comparison with the extant Arabic manuscripts), both as concerns the order of case and by presenting cases in normalized form. On the basis of Benedetto's text alone it cannot be definitively ruled out that Guglielmo simply followed Gerard's Latin text, even though he must then also have known Arabic terminology. However, the consistent use of *dramme* in the presentation of the cases in the Palatino *Praticha* (above, p. 279) seems to exclude it; like Gerard he must have had an Arabic manuscript at his disposal that was close to al-Khwārizmī's original.[455]

In any case, from chapter 2 (fols 371ᵛ–374ʳ) onward Benedetto leaves Guglielmo and Fibonacci behind. Chapter 2 replaces al-Khwārizmī's discussions of sign rules and of the arithmetic of roots and binomials, which Benedetto has already dealt with extensively in earlier books. Here, instead, the sequence of algebraic powers is introduced, together with their products and quotients; the latter show kinship with what we find in the Ottoboniano *Praticha* (above, p. 279), for instance by explaining (372ʳ) that

> from dividing *things* by *censi* results a fraction denominated by *things*, as from dividing 48 *things* by 8 *censi* results $\frac{6}{1\mathit{thing}}$.

However, all formal fractions are reduced directly, as in the present instance. Also different from the Ottoboniano *Praticha*, the fifth power is called *cubo relato*. It is clear that the two draw on a shared tradition (an abbacus tradition, far beyond al-Khwārizmī and Fibonacci) but also obvious that they deal with it differently – not only is Benedetto more clear-minded mathematically, he is also closer to Antonio than the Ottoboniano writer. In the very end (fol. 374ʳ) comes an explanation of abbreviations;

> the *things* are written like this, ρ, and *censi* like this, *c*, and *cubes* like this, ♭, and *censo di censo* like this, *cc*, and *cubi relati* like this, ♭*r*, and *cubi di cubo* like this, ♭♭.

Chapter 3 (fols 374ʳ–388ʳ) replaces al-Khwārizmī's 6 problems illustrating the six

[455] Another misunderstanding should be cleared away. According to [Kaunzner 1985: 8], in the lettering of diagrams *z* is replaced by ζ or ξ in one manuscript of the *Liber restauracionis* as well as in Columbia University, Plimpton 189. This is taken as evidence of a connection to the Greek scholars in Frederick II's Sicily. Unfortunately for this conclusion, the letter in question is simply *ç* (at least in the Plimpton manuscript, a copy of Benedetto's *Praticha*), just as it is in Benedetto's autograph. It replaces *z*, in diagrams taken over from Fibonacci as well as in the running text – for example, in *terço*, *avança* and *sança* (*terzo*, *avanza* and *senza* in modern Italian).

basic case with a series of 36 cases and appurtenant problems (two for each case);[456] the first six cases are still in al-Khwārizmī's and not in normal abbacus order (above, p. 142), but the definitions and rules for the cases are now in abbacus style, non-normalized and hence having a normalization as their first step; the full list of cases can be found above (note 284). Most of the problems are of easily constructed types, not least asking for numbers in given ratios fulfilling conditions corresponding to the equation – in "such part" formulation, in spite of Benedetto's familiarity with ratio and proportion language.

First (fol. 387^{v}) this exemplification of the case Be1, $\alpha C = \beta t$:

> Find two quantities so that the first is such part of the second as 2 of 3, and when the first is multiplied by the second they make as much as when the total of the two is multiplied by 9.

As higher powers get involved, things become much more intricate – not by necessity but evidently because Benedetto (or his source for the problems[457]) chooses to exploit the possibility to construct something looking scary. The last question (fol. 387^{v}), illustrating Be36, $\alpha KK = \beta R+\gamma CC$, runs like this:

> Find two quantities so that the first is such part of the second as 1 of 2, and when the first is multiplied by the second and this multiplication is multiplied by the multiplication of the first in itself joined to the multiplication of the second in itself. And that which they make is multiplied by the multiplication of the square of the difference that there is from one quantity to the other multiplied in itself, makes as much as multiplying the first in itself, and this multiplication still in itself, and this multiplication by the first quantity, joined to it the multiplication of the second in itself, and the multiplication multiplied by the multiplication of half the difference that there is from one quantity to the other, and this multiplication multiplied by 6.

As a clue to the construction of complexity we may observe that with the particularly convenient choice of a ratio 1 : 2, "the difference that there is from one quantity to the other" is nothing but the first quantity, while "the multiplication of the first in itself joined to the multiplication of the second in itself" is nothing but 5 times the square on the first

---

456 As we remember (above, p. 279), the corresponding section of the Palatino *Praticha* instead returns to al-Khwārizmī – more precisely to Guglielmo's version, as revealed by the copious use of *dramme*.

457 A suggestion that Benedetto copies is found on fol. 379^{r}, where case Be17 ($\alpha CC+\beta K = \gamma C$) precedes case Be16 ($\alpha CC = N$); but that is no proof that Benedetto copies from somebody else, here and elsewhere when copying without indicating his source he may well have used some earlier writing from his own hand.

quantity.

In *book 14* (fols 388^{v}–408^{r}) “are shown cases that exemplify the rule of algebra according to what master Biagio writes”; it was already discussed above (see p. 209).

Book 15 (fols 408^{v}–474^{v}) “contains cases from various ancient masters”. At first, however, comes a kind of brief history of the abbacus tradition.[458] In doubt whether Paolo dell’Abbacho, Antonio or Giovanni di Bartolo had been the greatest, Benedetto is convinced that they are far beyond others who have written since 1300, “even though Leonardo Pisano was from around that time”. As we see, all three heroes are Florentine, as are the many others who are mentioned – and

> even though I do not deserve to be known as a teacher but still as a learner,[459] I may be counted together with the others. And if I were to write who has said and what he has said, the volume would certainly be fastidious; but I shall quote some problems from Leonardo Pisano, some from Master Giovanni and some from master Antonio. And because their volumes are at hand, I believe it is permissible to write these. And I am convinced that master Paolo composed rather copious works; but they cannot be found except in fragments [*ispeçata*].

So, at first (fols. 408^{v}–431^{r}) Benedetto presents faithful extracts from part 15.3 of the *Liber abbaci*, conserving also Fibonacci’s marginal diagrams, drawn within already prepared spaces (never delimited by curved lines as it happened earlier when Benedetto has calculated first in the margin and written the text afterwards). As introduction he quotes the autobiographical note from the beginning of the *Liber abbaci* in Latin (a marginal note in a later hand on fol. 409^{r} translates into Tuscan).

Next, as chapter 2 (fols 431^{v}–450^{v}), follow extracts from Giovanni di Bartolo.[460] They were already spoken about above (p. 260). So was (much more extensively) chapter 3 (fols 451^{r}–474^{v}), Antonio’s *Fioretti*.

“The sixteenth and last book of this treatise [...] contains cases about numbers and squares” – thus the heading of *book 16* (fols 475–504^{r}). The text itself begins

> The treatise about square numbers is the most difficult and in need of deeper thought

458 Edited in [Arrighi 2004/1965: 155].

459 Probably false modesty – Benedetto was born in 1429 and thus 33–34 years of age when writing this.

460 These extracts are transcribed in [Pancanti 1982].

> [*maggiore chonsideratione*] than any other. And I have found none who speaks of it in more elevated manner than Leonardo Pisano. Therefore I intend to write that in this volume; and I shall divide this book into two chapters: in the first giving the text of Leonardo; in the second giving certain cases that have been written by many others. [...].

Chapter 1 (fols 475^{r}–501^{r}) begins by quoting Fibonacci's introduction to the *Liber quadratorum*, once again in Latin (and even this time translated into Tuscan in the margin, but by another hand than the translation in the margin of fol. 409^{r})

The *Liber quadratorum* [ed. Boncompagni 1862: 253–279] begins by using the production of square numbers as the sum of successive odd numbers starting from 1 as a way to find Pythagorean triples (a name Fibonacci evidently does not give them). For example, 1+3+5+7+9 = (1+3+5+7)+9, whence $5^2 = 4^2+3^2$. This is illustrated by a line diagram [ed. Boncompagni 1862: 254] lettered *a–b–c–d–e–f*. After that, almost all diagram letterings start *a–b–g* (when not using later alphabetic sequences because they continue an earlier argument using *a–b–g*); it appears that Fibonacci draws the bulk of the work from an Arabic source – perhaps, once again, from a preceding Latin translation of an Arabic treatise.[461] A number of theorems follow that are either related to similar sums (including sums of squares) or to Pythagorean triples. However, the central interest (beginning on p. 265) is the topic of *congruous numbers* to which these theorems lead: integer or rational numbers which, when added to and subtracted from a specific square integer or rational number (the *congruent square*), produces other square integer or rational numbers – such as, for instance, 24, which added to and subtracted from 5^2, yields 7^2 and 1^2, while 120 added to and subtracted from 13^2 yields 17^2 and 7^2. Since this topic, though not quite ignored, remained a marginal concern in the abbacus environment, I shall not go into the details but restrict myself to generalities.

---

461 In the end of the treatise [ed. Boncompagni 1862: 279] comes "a question asked me by master Theodoros, philosopher of the Imperial Lord [Frederick II]":

> I want to find three numbers which collected together with the square on the first number make a square number. And if to this square the square of the second is added, comes out a square number; and when to this square the square of the third is added, similarly a square number comes from it.

Here, the letter sequence in the diagram used in the first transformation is *a–b–c–d–e*. In a lemma which is then proved, however, it is *a–b–g–d–e–f–i*, suggesting that the lemma was borrowed but its application to Theodoros's question independently developed by Fibonacci – documenting, if need should be, that Fibonacci had perfect understanding of what he borrowed and was able to use it creatively. Further on, when *a–b–g* returns in diagrams it is together with the unit *dragma* for numbers and ascending continued fractions – further suggestions of a direct or indirect loan from an Arabic source.

As to the *mathematical topic*, one may consult Laurence Sigler's matematically annotated translation [1987]; a translation of the results into symbols (but no further mathematical study) can be found in [Picutti 1979: 276–281].[462]

The earliest known evidence of *surviving interest* in the topic after Fibonacci appears to be a manuscript written by Gilio da Siena in 1384, which lists 35 problems with solutions but offers no indication of how the solutions were obtained [ed. Franci 1984: 12–17; ed. Franci & Toti Rigatelli 1983]. Evidently Gilio must have drawn on sources that escape us but which existed; Franci gives reasons to believe that Gilio may have been a student of Antonio, and if not at least knew his works well. Then, with an imperfect rule, a somewhat longer list of congruo-congruent numbers turn up in 1424 in the *Alchune ragione* (see above, note 275), fols 32^{v}–33^{r}.[463]

The next witnesses I know about are Benedetto's *Praticha* and the manuscript Florence, BNC, Palatino 577 (discussed above, p. 285). They are discussed together with a transcription of the latter version in [Picutti 1979]. This is where the erroneous claim is made that this manuscript was due to Benedetto (see note 407) – another accident following from clumsy handling of Occam's razor. The two texts also differ more from each other (and from that of Fibonacci) than Picutti acknowledges. From the way Theodoros's problem is presented it is obvious that Benedetto cannot have copied from Palatino manuscript but the latter possibly from the Benedetto: Benedetto (fol. 499^{v}) translates Fibonacci verbatim, "This was proposed to me", while Palat. 577 (fol. 279^{v}) relates, "A case, written for L. P. by Master Theodoros").

Then, to my knowledge, the next surviving appearance is in Pacioli's *Summa* [1494: 13^{v}–14^{v}] (the folio numbers are misprinted, "13" as "51", "14" as "15").

In chapter 1, Benedetto follows Fibonacci's work to the end – sometimes paraphrasing

---

462 Historiographically it may be added (since none of those historians or mathematicians who have written about the topic seem to be aware of it) that congruous numbers play an important role in Old Babylonian and even earlier Mesopotamian mathematics. They turn up because the square on the parallel transversal bisecting a trapezium (a favourite topic) is equidistant from the squares on the parallel sides – see [Høyrup 2002a: 237*f*]; whether there is any connection to what is found later in Diophantos and Arabic authors seems highly doubtful.

463 Already the manuscript Palermo 2Qq-E13, dated 1395, contains two problems of the kind on fol. 22^{r}, with congruous numbers 12 and 20. The latter is also in Gilio and *Alchune ragione* but with a different solution, the former is not even there. Its solution is that for given a (the congruous number)

$$\left(\frac{a^2}{4}+1\right)\pm a$$

will be square – namely $(a\pm1)^2$. It is obviously independent of the tradition which we know from Gilio.

rather than translating; but that he intends to follow is revealed by an observation made on fol. 501r: until this point the presentation has followed what was written to the illustrious Emperor [Frederick II]; now it follows what was addressed to Cardinal R[aniero Capocci, dedicatee of the prologue of the *Flos*].[464]

Chapter 2 (fols 501v–506v) is introduced by the remark that "the cases about square numbers are among the most difficult that we have", and that many of them are apostate (that is, allowing only solution by reason-guided experiment[465]), but they are very pleasant and ask for much thought. This is followed by a list of 130 congruous-congruent couples, followed by 4 pages of commentaries (fols 502v–504r), after which follow another two pages with 215 further couples.[466] Another page (fol. 506v) with quickly drafted numerical observations finishes the chapter – the book – and Benedetto's whole *Praticha*. Whereas the other books close formally, often with thanks to or praise of God, there is nothing similar here. After almost a thousand densely written pages and under pressure to finish his marvellous gift in time Benedetto seems to be tired.

Summary observations concerning the Florentine encyclopedias

After this long and admittedly fastidiously detailed analysis of the three encyclopedias it may be time to turn to general considerations.

---

464 The *Liber quadratorum* thus also existed as a master copy (perhaps an evolving master copy), from which Fibonacci made copies dedicated to different persons; cf. above, p. 59, on the *Liber abbaci*.

Ranieri is indeed the dedicatee *of the prologue* of the *Flos*, whereas the ensuing text of that work is directed to the Emperor; even here we thus have traces of the same process.

This observation explains a curious passage in the *Flos* [ed. Boncompagni 1862: 234]. Here, addressing the Emperor, Fibonacci speaks of the *Liber abbaci* as *your book*, which suggests that a copy had been dedicated to Frederick, even though the version we know has Michael Scot as its dedicatee (with the possible exception of manuscripts where a dedication is lacking).

465 This remained true at least less than two decades ago: according to [Guy 2004: 306], "it is only for the last twenty years, since the work of [Jerrold B.] Tunnell, that we have a reasonably complete understanding" of congruent numbers, yet "i[n] spite of improved computing techniques and machines, it may still be some time before some other of the more recalcitrant examples are discovered". If machines are needed, the field appears to remain apostate.

466 The numbers may be approximate. I might count the densely written lines a second and a third time hoping to get the same result but prefer to use the opportunity to quote what I learned from my high school physics teacher Kjeld Jensen some 60 years ago: "the only measurement with no uncertainty is a counting – and that is only in principle!"

Firstly, it has to be remembered that these encyclopedias, though apparently merely the surviving representatives of a larger group of similar works, do not represent general abbacus culture. Socially, they stand apart by being gifts to top members of the patriciate. This, furthermore, is reflected in distinctive attitudes.

One of these reflects the ways of the Humanist movement, which was also close to the heart of the Florentine patriciate – namely that the introduction of algebra is based on al-Khwārizmī "so that the work of the Arab Maumet, which was almost lost, may be restored" (the Palatino *Praticha*, see above, note 397) or because his demonstrations are older (Benedetto, above, p. 316). The authors are certainly not Humanists themselves – their style cannot be compared to that of, say, their contemporary Leon Battista Alberti.

Another distinctive attitude is the aspiration to integrate the mathematics of their own tradition with that of "magisterial" mathematics. In part this also corresponds to Humanists' ways, namely when Boethius is used (and not merely for empty namesdropping, as in Latin algorisms and in Jacopo's and other early abbacus books). Drawing on Fibonacci, on the other hand, goes beyond Humanism. It also goes beyond appeals to the prestige of Fibonacci as a culture hero (as we remember from note 311, many – obviously wealthy – Florentine citizens possessed his books in the late 14th century) – for that purpose, it would have sufficed to draw on the easier parts of the *Liber abbaci*, there would have been no need to copy for instance the work on roots, and even less (as done by Benedetto) to rewrite and expand his "keys" drawing directly on *Elements* II (above, p. 309) as introduction to the copy of Fibonacci's part 15.1. In particular, there would be no need to produce a complete paraphrase of *Elements* X, the notoriously difficult *crux mathematicorum*, as done by Benedetto but apparently also by the model from which the Palatino writer miscopies it – amputated and out of place.

As we have seen, Benedetto, though not copying the shared model of the two other authors as directly as they do, knows either this model or its close kin. We do not know how far back the distinctive characteristics of the three encyclopedias reach back in time, though we may guess that the presentation of the Boethian names for ratios is relatively new – *vide* the half-protest in the Palatino *Praticha*, "we in the schools do not use such words" (above, note 344), whereas interest in Fibonacci's part 15.1 may go back at least to An tonio among "demonstrators" (*dimostratori* – above, note 394).[467] Exactly what

---

[467] The claim of the Palatino writer that Paolo dell'Abbacho wrote a treatise about continued proportions (a subject wholly foreign to early abbacus books) could suggest that already Paolo had started looking into the advanced chapters of the *Liber abbaci*, not just copying with little understanding like the *Livero*. However, the lack of precise information concerning Paolo's mathematics beyond his elementary *Regoluzze* [ed. Arrighi 1966b] might call for scepticism – these

(if anything) is meant precisely by this term is not clear – that is, whether some particular current is referred to or the term just characterizes single actors. But we may take note that the Palatino as well as the Ottoboniano writers had been students of the *Vaiaio*, a mathematical dilettante, and that much of the advanced material from the *Liber abbaci* is shared by them and hence probably goes back to this teacher.

Apart from what Benedetto adds to Fibonacci's *Liber quadratorum* and the addition of the sixth perfect number by the Ottoboniano writer, however, the advanced magisterial material in the three encyclopedias is on the whole rendered as received – there are few traces of active *creative use*. Creative development is found instead within traditional abbacus-mathematics domains.

Most striking in the perspetive of later mathematics – though apparently not understood by others and therefore without consequences – is Benedetto's transformation of Fibonacci's probably line-based rhetorical quasi-algebra into a genuine symbolic linear algebra with four to five unknowns. Admittedly, most of the *problems* which serve Benedetto for the purpose are borrowed from Fibonacci, but the problems are of widespread recreational types, traditionally used for challenges in their most intricate variants.

More consequential, or at least part of an ongoing trend, is the expression of *regula-recta* solutions (thus still first-degree algebra) with two unknowns in symbolic marginal calculations. Abbreviations for algebraic powers and arithmetical operations (plus, less, root, universal root, fraction line) had been in use since Antonio (if we trust Benedetto's rendering on fol. 464[r]) and the Florentine *Tratato sopra l'arte della arismetricha* (above, p. 244), in the text column as well as in marginal calculations.

Also representative of a general tendency and active work within the broader abbacus environment is the interest in names for higher powers and for their products and quotients. The Ottoboniano division of "48 *things* by 6 *cubi*" (above, p. 279) and Benedetto's of "48 *things* by 8 *censi*" (above, p. 317) point to a shared background, while the different ways the outcomes are stated as well as the different namings of higher powers reflect a situation in flux and thus ongoing exploration.

All in all we may conclude that this early attempt to merge the abbacus tradition with Humanist culture and with magisterial mathematics led to juxtaposition but not to a fruitful synthesis transforming either of the three. Apart from the inclusion of al-Khwārizmī's algebra among the classics, all noteworthy innovations took place within the first

"small rules" are prescriptions absolutely devoid not only of demonstration but of the slightest hint of an explanation.

component. In consequence the encyclopedias – even the most innovative of the three, that of Benedetto – seem to have remained uninfluential.

Eventually, as we shall see, synthesis did take place; but that was under very different circumstances.

V. Abbacus goes into print and abroad

We shall now turn our attention to the afterlife of abbacus culture: on one hand to what happened as abbacus books went went into print; on the other to the emergence and unfolding of *Rechenmeister* mathematics in German lands after the mid-fifteenth century – an obvious descendant, but none the less a new and different mathematical culture.

Early prints in Italy

All levels of abbacus- and abbacus-related books were printed in Italy before 1530 – most of them close to the abbacus school curriculum, one at the level where also the algebraic prestige topic was well presented, and – in a class of its own – Pacioli's *Summa*.[468]

The basic to intermediate levels

The first abbacus book to be printed was [*Larte de labbacho*] – namely in 1478 in Treviso, close to Venice, and therefore known also as the "Treviso arithmetic".[469] Its first lines state that

> A *Practica* begins, very good and useful for anybody who wants to exercise the art of merchantry, in vernacular called art of the abbacus.

The anonymous author goes on that he responds to a request made by some much cherished youngsters who want to dedicate themselves to merchantry.

At first comes an algorism, presenting the writing of Hindu-Arabic numerals, and afterwards algorithms for addition and subtraction (also of monetary amounts consisting of *lire*, *soldi* and *piccioli*) which classical abbacus books often had taken for granted, followed by multiplications (according to different methods, and including control by casting out nines) and by division. This takes up 29 of 62 sheets. Fractions are not treated – they only turn up in the following pages, where the rule of three is dealt with. This rule is explained at first in almost the usual Italian way,

> The rule of three is this, that you should multiply the thing you want to know by that which has no similarity [*che non ha somiglia*] and divide by the other. And that which

---

468 Occasionally this class is denied to Pacioli, and his *Summa* is claimed – by scholars who have evidently read neither text – to plagiarize Piero della Francesca's abbacus treatise. That this is an impossible idea is made clear from Giusti's summary [1991: 64] of his analysis of the latter work. Its author (supposedly Piero) turns out to be

> at least confused, and a copyist who does not even [...] discover that what he is writing has already been copied one or two pages earlier,

also repeating uncritically the false algebraic solutions that has circulated since Gherardi's time.

A large number of sources for Piero is traced in [Giusti 1993]. There is no reason to add to Giusti's exhaustive analysis, but all the more to give space to Pacioli.

469 David Eugene Smith's English translation has been published in [Swetz 1987].

As we shall see below (p. 370), it may be the first *abbacus* book to go into print, by a much more modest introduction to commercial arithmetic in German had been printed in 1475 in Trento.

is engendered will be of the nature that has no similarity.

This is followed by a supposed clarification that states the same thing, just in "mentioned" terminology (above, p. 172), and next by an example confronting merchandise (saffron) and price. Then, as further explanation, counterfactual examples (here identified as "calculations by nameless numbers", *raxone de numeri senza nome*), and further a large number of mercantile examples often involving for example *lire* and *soldi* and deduction for tare weight (*tara*) – mathematically analogous to the problems about the loss of weight entailed by the washing of wool (above, p. 31). This takes up some 17 sheets, being followed by partnership. More or less an innovation is the introduction of *names* for the partners – in the first examples Piero, Polo (~Paolo) and Çuanna (~Giovanni). Tuscan-Umbrian abbacus books only speak of "the first", "the second" and "the third", but there is one Italian parallel, incidentally from the same year: Muscharello's *Algorismus*, written in Campania, which [ed. Chiarini et al 1972: II, 154–158, 193] in three problems dealing with the settling of accounts identifies the creditor and debtor, respectively, as Piero+Martino, Rinaldi+Simoni and Roberto+Martino, and in one about four gamblers, as Piero, Martino, Antonio and Francischo.[470] As said in note 233, Muscharello also knows the "mentioned" formulation of the rule of three.

Partnership takes up seven sheets, and is followed by a single page dealing with barter (not taking into account the different value of merchandise in cash and in barter), after which alloying takes up 2½ sheets. Three sheets contain very simple recreational problems, about couriers meeting, about pursuit, about master masons building a house together, and a problem of Fibonacci's "tree" type dressed as a finding of a purse. Two sheets deal with (lunar) calendar reckoning and two present metrological conversions.

Interest and loans are thus absent, as are the single and double false position as well as *regula recta* and algebra. The book is definitely meant for beginners, and perhaps for self-study; whether the author is an experienced abbacus master can perhaps be doubted. Measured by the number of further editions (pirated or otherwise) it did not command great respect in its time: none are known.

Measured by the same criterion, Piero Borgi's[471] *Opera de arithmetica*, printed

---

470 As we shall see, the first attested use of *letters* designating persons is slightly earlier in German area – see below, p. 358.

471 In the 1540 and 1550 editions, "Piero Borghi", in the 1561 edition "Pietro Borgo". The 1488 and 1491 editions still have "Piero Borgi". The others I have not seen.

Library catalogues list the work randomly under all three names, irrespective of how the author is identified in the edition they list.

in Venice in [1484], was a success; according to [Smith 1908], it was reprinted in 1488, 1491, 1501, 1505, 1509, 1517, 1528, 1534, 1540, 1550, 1551, 1560, 1561, 1567, and finally in 1572[472] – from [1540] onward under the title *Libro de abacho*. The printer was Erhard Ratdolt, who had brought out the first printed edition of Euclid's *Elements* (the Campanus version) in 1482. Counting pages, it is twice as long as *Larte de labbacho* – but each page contains *ca* 30%–40% more typographical units. Grossly speaking it covers the same matters as *Larte* (calendar matters being left out), but evidently in greater depth; fractions, in particular, are explained in all details.[473] Piero also teaches some extra topics – thus the proof by casting out sevens and the single and double false positions. Addition and subtraction are taught after multiplication and division (multiplication and division of multi-level amounts of money after addition and subtraction, however, but even for monetary operations multiplication and division come before addition and subtraction). This may not seem very pedagogical but probably reflects the habits of earlier abbacus books that did not explain addition and subtraction at all. Fractions and their arithmetic follow.

The rule of three is introduced (fol. 41^{v}) as the way to "deal with all merchantry calculations", and at first explains that the three things involved "contain two natures", two of them "being of the same nature" and the third of the other; that is, it does not use any of the formulations with which we are familiar, neither "similar–dissimilar" nor "names mentioned". It also reinvents the "secondary Toledo reference" to first-second-third (above, p. 180) – "reinvents", since the explanation is after all close at hand, and nothing suggests a link. After that, the explanation is made in terms of the quasi-counterfactual structure, "if 2 are worth 3, what should 4 be worth?"

Partnerships go well beyond the simple situations dealt with by *Larte*, presenting reversed cases (fol. 73^{v}, the contribution of one partner is unknown) as well as cases where the durations of investments differ (e.g., fol. 74^{v}); in the last partnership problem (fol. 81^{r}) the interest rates asked by the three partners differ.

The barter problems, starting on fol. 81^{v}, are also more complex, distinguishing between cash and barter value of merchandise.

Interest (obviously simple interest) is dealt with a single time (fol. 107^{v}) in the section presenting mixed problems (fols 101^{v}–116^{v}). That section, among other things, also contains a number of problems built on the principle of "combined works", explained in the way that appears to be behind Jacopo's calculation (above, p. 23). One of them is a two-

---

472 [Van Egmond 1980: 293–297] gives description of all but the last.

473 [Smith 1926] describes the way basic arithmetic is treated in greater detail than here.

participant variant of the "lion in the pit" (fol. 110r), which neglects that once the dove coming from the top of the tower encounters the sparrow hawk coming from the ground, none of them retreat.

In the end of this section (fol. 112r) come three problems introducing the false position, which turns out to be the double false; the first two are of the traditional type, and easy. The third, a give and take problem involving three men (fol. 113v), is nested with three levels of double false positions, and runs over six pages. After this Piero wisely stops, observing that

> Many other almost infinite problems could be brought together with the preceding ones, which it would be too prolix to couple to these and for the learned superfluous, since all ways in all problems belonging to merchantry have already been demonstrated.[...] *Laus Deo*.

The 1540, 1550 and 1561 editions do not agree that no more needs to be said, and add 10 problems after the give-and-take problem before declaring (in the same words) that enough is enough.

Another great publishing success was Girolamo Tagliente's *Libro dabaco che insegnia a fare ogni raxone marcadantile, et a pertegare le terre con larte della Geometria*, "Abbacus Book That Teaches to Make All kinds of Merchantry Calculations and to Measure Terrains by the Art of Geometry". Van Egmond [1980: 334–344] lists no less that 31 editions, the first from 1515, the last from 1586.[474]

Tagliente's volume is less than half as long as that of Borgi, and slightly longer than *Larte de labbacho*.[475] It confirms the impression given by these two predecessors that

---

474 Nine of these are undated but claim to have been produced by the same printer as the first three (Luca Antonio de Uberti); they may in reality be pirate editions, since a 10-year privilege was given in 1515 and renewed in 1520. The edition I mainly use (controlling with the [1530] and [1532] editions) is supposed by the Linda Hall library to be from 1520, though with a doubt (I shall refer to it as [1520x]); it appears to be the undated edition listed by Van Egmond [1980: 335] as number 4.

Van Egmond [1980: 344*f*] further lists four editions of a rather similar book, at first printed in 1524 under the title *Componimenti di arithmetica*, from 1525 until 1527 onward as *Opera che insegna a fare ogni ragione di mercantia* or *Opera nova che* .... However, the [1527] edition claims to be written by Giovanni Antonio Taglienti, Girolamo's brother, who had also assisted with the latter's *Libro de abaco*.

475 For this comparison, counting typographical units on select pages used for text only, I have not taken into account the many woodcuts in Tagliente's book. If I had done so Tagliente's book would probably go below 40% of that of Borgi, and somewhat below *Larte*.

the genre growing out of the basic abbacus books was in certain respects – first of all intended audience – moving away from what we find before printing; with the proviso, however, that all three are from Venice (the cradle of Italian high-quality printing), while most abbacus manuscripts – in particular those we have looked at – are Tuscan or in Tuscan tradition.

Manuscripts, unless disseminated through something like the bookseller system of late medieval universities, were not addressed to and did not reach a wide audience; we have seen that the Florentine encyclopedias were addressed to single “patron-friends”. Tagliente, instead, in his preface (fol. 2^{r}),[476] addresses “magnificent noblemen, noble citizens, enlightened artisans”, and later speaks about the utility of the book to “friars, priests, students, doctors, gentlemen, artisans”.

On the whole, the contents of Tagliente’s book is similar to that of Borgi. At first it explains the place-value system, and then the proof by casting out sevens before teaching the various multiplication algorithms. Even here, multiplication is followed by division; the arithmetic of pluri-level amounts of money, however, is dealt with very briefly, and fractions are left out – later it is stated (fol. 26^{v}), when a rule-of-three calculation leads to a fraction, that “this could be reduced. But in order not to occupy your mind I shall not explain such subtleties to you”.

The rule of three is explained (fol. 22^{r}) in terms of the similar and the not similar; as Borgi, Tagliente takes into account the complications arising from tare weight. After that (fol. 39^{v}), alligation (not only alloying) is dealt with, and then (fol. 47^{v}) partnership, which also encompasses a reverse case and different durations of the investments. One case (fol. 50^{v}) runs like this:

> Two men want to share 120 ducats. The first wants twice as many as the second, I ask what is due to each. Do thus, posit that to the second was due one. Then to the first is due two, join together, they make 3. Then multiply 2 times 120, it makes 240, divide by 3, from which results 80 ducats, and as much is due to the first. And rob [*bari*, properly “cheat”] these 80 ducats from the 120 ducats, remains 40 ducats, and as much is due to the second, and it will be made.

We notice, firstly, that an unexplained single false position is made use of as a way to transform the problem into one for which partnership may serve as a model; second, that the partnership rule is not followed to the end, as one would expect if the purpose had been to train that rule systematically.

The single false position is used tacitly once more in a similar problem on fol. 50^{v},

[476] The volume is not foliated; my counting begins with the title page.

and on fol. 55^{r} in a “tree problem” – and nowhere else, if I am not mistaken.

After partnerships and analogous problems follows barter on fol. 51^{r}, taking into account the difference between cash and barter value, and then from fol. 56^{v} until fol. 63^{v} a collection of mixed problems, mostly traditional and including also divinations.

The last two sections have no counterpart in Borgi’s book. First, on fols 54^{r}–68^{v}, comes a simple geometry; it goes beyond that of Jacopo (above, p. 34) in only one problem, which asks for the circumference of a wax sphere made from “small spheres” (*ballote*) with implausible circumferences 2, 3 and 6 cubits – supposed to be $\sqrt{(2^2+3^2+6^2)}$; Tagliente’s stereometric intuition is no better than that of Jacopo. In spite of the promise of the title to teach how to measure terrains (*pertegare*, literally to measure with the *pertica* rod), all that is taught is how to calculate areas (etc.) once the measurements have been made; even in this, the chapter does not go beyond traditional abbacus geometry.

Last comes (fols 68^{v}–78^{v}) a *tariffa* similar to the one that was included by Gherardi in his *Libro di ragioni*, but ending with a description of how various kinds of merchandise should be if of good quality (ginger and other spices, wax, sugar, rice – and many more); whether this is counted as part of the *tariffa* is not clear, since the *tariffa* proper is separated from this list by a description of the volume (fols 76^{v}–77^{r}).

This description ends by claiming that with these characterizations of good quality merchandise the reader will be ready to “stay in and go to all parts of the world”. One may entertain some doubts, just as one may doubt a teaching of commercial arithmetic without fractions. All in all, the final overselling of the product, the avoidance of subtleties that might overburden the mind, and the copious woodcuts might suggest that we classify Tagliente’s bestseller as a multi-media coffee table book – “abbacus mathematics made easy, beautiful and entertaining”.[477]

Not all descendants of the abbacus tradition obviously went the coffee-table-book way. Over the following centuries many texts were produced at the modest level which really served basic commercial teaching, similar to that of Borgi or simpler.

A early example of this simple category is Filippo Calandri’s *De arithmetricha*

[477] At least Girolamo’s assisting author Giovanni Antonio Tagliente was indeed the author of such books [Rivali 2019]: he published on calligraphy – [Tagliente, G. A. 1524] ran through 35 editions; on letter writing with models to be copied – love letters as well as official correspondence; on embroidery; on book-keeping; finally, a book offering to teach to read within a couple of months.

opuscula from [1491].[478] Calandri belonged to a renowned Florentine family of abbacus teachers; so, even though his book aims at being simple, he covers not only simple and composite interest but also discounting (simple and composite) and other more complex questions related to loans; barter is also dealt with more broadly than by the preceding books. A fair number of recreational problems are included, but none of them are so intricate that they call for a single or double false position (not to speak of *regula recta* or algebra). The geometry section does not go much beyond what we know from Jacopo (above, p. 34) – in style not at all.

The outstanding higher-level text is Francesco Ghaligai's *Summa de arithmetica*, printed in Florence in [1521][479] and dedicated to Cardinal Giulio de' Medici by his "humble servant". Already the title shows the higher ambitions, which are confirmed by the contents. Ghaligai had been a student in a school held by the distinguished abbacus master Giovanni del Sodo, probably in the late 1490s, who on his part is likely also to have been connected to the *Vaiaio* [Ulivi 2017: 16, 22].

I shall not make a detailed description but concentrate on the last four (of 13) books, which take up algebra. A few observations of what precedes are in order, however, since they make the same point. Book 1, about "what number is", starts by giving the Euclidean definition as "multitude composed of units", and also presents other matters from basic number theory: even/odd, composite/prime, perfect/abundant/deficient, and a few more. It is thus similar to part 3 chapter 3 of the Ottoboniano *Praticha* (above, p. 253) and to book 2 of Benedetto's *Praticha* (both about the "nature and properties of numbers") though much shorter than either. It also contains the same scheme for finding perfect numbers as Benedetto, though restricted to the first four perfect numbers (perhaps because the typesetting allowed no more).

After the arithmetic of fractions, book III takes up square roots (fols $21^{r–v}$) and proportions (fols 22^{r}–27^{v}). Certain aspects are similar to the three Florentine encyclopedias – for instance, the ascription of the definition of a square root to Fibonacci, and the determination of the square root of $^{2}/_{3}$ by the method used in the Ottoboniano *Praticha* (and secondarily by Benedetto) (above, p. 271). All in all, however, it is clear that Ghaligai did not draw directly on these works nor on the shared model of the Ottoboniana and the Palatino *Pratiche*. Instead, the similarities inform us about more widespread Florentine

478 Apparently a printed version of what had once been the school-book of Giuliano de' Medici (1453–1478), characterized on the title page as *nobilis et studiosus*. The original date may thus be around 1465. It was reprinted in [1518].

479 Reprinted in 1548 and 1552 as *Pratica d'arithmetica*.

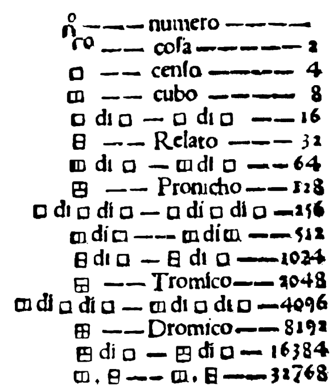
15th-century habits – seemingly also that material from the *Liber abbaci* was adopted more broadly than earlier and elsewhere in the abbacus tradition.

Book VIII (fols 57^{v}–61^{r}) contains pure-number problems, beginning with the relatively simple – for example

> make of 6 two parts so that one multiplied by the other make 8, and their squares joined together make 20. Divide the said 20 in half, 10 results from it, multiply in itself, it makes 100, and from this detract the square of 8, that is 64, 36 remains, whose root is 6; put above the half of the squares, that is 10, it makes 16, and its root will be the major part; and the minor the rest until 6, that is, 2.

This could of course be argued for instance from Fibonacci's "keys", as presented by the three encyclopedias (above, pp. 270 and 309); here, however, no arguments are given. As we shall see when discussing Pacioli's *Summa* (below, p. 342 and onward), however, a large number of similar rules were in circulation. Afterwards come (fols 59^{v}–61^{r}) extracts from Fibonacci's *Liber quadratorum*.

Book IX, about the double false position and "many rambling problems", is the one where Ghaligai refers to Benedetto as the originator of the notion of the notion of "apostate" problems (above, p. 308), showing that he knows Benedetto's work; all the more noteworthy is that his own text avoids the problems where Benedetto (or the Ottoboniano writer) make use of two or more algebraic unknowns.



Books X–XIV deal with algebra. Book X (fols 71^{r}–89^{v}) is told in the table of contents "to be the first of our algebra, drawn from the tenth of Euclid and Leonardo Pisano and Giovanni del Sodo". It begins by quoting Guglielmo de Lunis (cf. above, p. 315), but only the passage explaining Arabic terms. On fol. $71^{r–v}$ Ghaligai then presents and explains a set of abbreviations for the algebraic powers – with idiosyncratic geometric symbols and with particular names for the prime powers beyond the *relato*, the seventh power being *pronicho*, the 11th *tromico*, the 13th *dromico*. Other powers are produced by multiplication, and the whole system illustrated by the corresponding powers of 2. The whole system is taken over from Ghaligai's one-time master del Sodo, and is also used in the following to express powers of numbers – for example (fol. 73^{r}),

> When a line is divided into 2 parts, then the ◫ of each part with 3 times the multiplication of the square of each part in the other equals the ◫ of the whole line.

These signs (functioning only as abbreviations, never as symbols on which can be operated) are also used to express roots, and when the rules for multiplying powers and roots are set out on the following pages.[480] Then (fols 76^{r}–88^{r}) "because Benedetto has spoken broadly about it", the classes of Euclidean binomials and their relations are presented "following his style and way".

Last in the book (fols 88^{r}–89^{v}) come rules for algebraic cases with examples, and a half-page returns to matters connected to *Elements* X.

The short book XI (fols 90^{r}–92^{r}) is identified in the table of contents as "the second of algebra and drawn from the second of Euclid". Its actual contents is a sequence of problems implicitly referring to propositions from *Elements* II, and solved by means of algebra.

Book XII (fols 92^{v}–97^{v}) contains select problems taken from the part 15.3 of the *Liber abbaci*. Book XIII (fols 98^{r}–109^{v}) is a collection of problems taken over from Giovanni del Sodo, beginning with linear give-and-take problems solved algebraically.[481] Gradually matters become more complex, with give-and-take and purse-problems involving roots and products, which leads to second- and third-degree equations. Even higher degrees (until the *dromico*, the 13th) are called for in the solution of pure-number problems (none of the cheap type asking for numbers in given ratio, but many dealing with numbers in continued proportion).

Luca Pacioli

Pacioli was an abbacus teacher – or, better, a teacher of abbacus mathematics, but at universities; but not only.[482] Born in humble circumstances between 1446 and 1448 in Borgo Sansepolcro close to Arezzo (thus Tuscany), he appears to have been taught by or at least to have become familiar with Piero della Francesca while a boy. Around 1465 he went to Venice, becoming a private tutor in the Rompiasi merchant family while also serving in its commercial enterprises. During this stay in Venice he followed the teaching of Domenico Bragadin at the Scuola di Rialto – a municipal school at the level of a university arts faculty, concentrated on Aristotelian philosophy but under Bragadin also teaching algebra and geometry [Stabile 1971]. In 1470 he entered the Franciscan

---

[480] The same system was reported by Raffaello Canacci, another former student of del Sodo, in his *Ragionamento d'algebra* from *ca* 1495 – but with misunderstandings and almost never used elsewhere in the manuscript, see [Høyrup 2019a: 861, 895].

[481] The *regula recta* is no longer spoken about; as we have already seen in the Ottoboniano *Praticha* (above, p. 254), others had given up the distinction between this technique and algebra since long.

[482] Biographical details as given here can be found in [Di Teodoro 2014] and [Ulivi 1994].

order, wrote a first treatise on arithmetic and algebra, and shortly afterwards visited Rome, for several months the guest of Leon Battista Leone Alberti. That is, already before being called to the Perugia municipal university in 1477–1480 to teach there (thus not in an abbacus school), he had familiarity with philosophy as well as theoretical geometry and Humanism. From this period exists a huge manuscript known from the dedication as *Suis carissimis discipulis*, “To his very dear students”.[483] He was invited again for the year 1487–88, at which occasion he started to work at his *Summa de Arithmetica Geometria Proportioni et Proportionalita*, which was printed in Venice in [1494]; leaving aside his other publications and unpublished work[484] as not belonging to the afterlife of the abbacus school we shall now turn to this book.

Ghaligai’s *Summa* may be said to have justly changed its name into the more modest *Pratica d’arithmetica* in the second and third edition. In contrast, Pacioli’s “Total of arithmetic, geometry, ratios and proportions” carries its name with honour.

Like the other grand treatises we have looked at it is dedicated to a representative of political power – namely to the Duke of Urbino. The florid dedicatory letter (first in the vernacular, then in Latin) is stuffed with references to classical Antiquity, without forgetting famous contemporary painters – Humanism and courtly culture are much closer than in the Florentine encyclopedias written three decades before.

After the dedication follows a summary of the single parts.[485] That of the first part offers a list of alleged sources: most of the volume is asserted to have been taken from Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco and Prosdocimo de’ Beldomandi. We should be aware that these acknowledgments are strategic; much in the book comes from the abbacus tradition, including its higher levels represented by Antonio and others, and not from these Latin writers. Historians would be happy to be better informed, but we must accept that Pacioli follows his own agenda, not ours and still be grateful that he gives much information about the earlier abbacus tradition for which other sources have gone lost.

---

483 Vatican, Vat. lat. 3129, edited in [Calzoni & Cavazzoni 1996].

484 In particular the *Divina proportione* about regular polyhedra, printed in [1509a] but written almost a decade earlier; a vernacular translation of the *Elements* that has been lost (but see [Folkerts 2006: article XI, 222*f*]; an edition of Campanus’s Latin version of the *Elements* [Pacioli 1509b]; a manuscript *De viribus quantitatis* (“About the Powers of Quantities”), see

http://www.uriland.it/matematica/DeViribus/Pagine/index.html

(accessed 29 September 2021).

485 All these introductory matters are unpaginated.

Also strategic is the claim that Pacioli chooses to express himself in the vernacular in order to be more useful for the subjects of the dedicatee, the technical terminology of mathematics being no longer well understood because of the lack of good teachers. After all, most of the material that is used in the *Summa* comes from a vernacular tradition and vernacular sources, and Pacioli evidently repeats their terminology. Even when using an identifiable Latin source extensively, namely Fibonacci's *Pratica geometrie*, Pacioli prefers to take over what he can from an already existing vernacular version. Of the two parts of the *Summa*, the second, on geometry, is indeed largely a copy of an earlier vernacular translation of Fibonacci's *Pratica geometrie*, though *not* copied from the Palatino geometry (as oft repeated by those who never controlled Ettore Picutti's mistake, cf. above, note 407[486]). We shall leave it aside, as we have left Fibonacci's work and the other vernacular translations untouched.

The first part deals with "arithmetic, ratios and proportions"; it does so in nine "distinctions" divided into "parts".[487]

The first distinction (fols 1^r–18^v) deals with *quantity*, starting with the philosophical division into discrete and continuous. It further presents the gamut of concepts belonging to theoretical arithmetic, including perfect and congruous-congruent numbers; but Pacioli cannot abstain from speaking also about Platonic bodies, which occupied him in many other works, and at which he arrives via the consideration of how many equal components can make up a solid angle. He also delves at length in sacred and not so sacred numerology; for instance (fol. 5^r), the number 3 is found in:

– matter, form and privation (Aristotelian metaphysics);
– vegetative, sensitive and intellective soul (Aristotelian psychology);
– Asia, Africa and Europe;
– gold, silver and copper (coin metals);
– intellect, memory and will (the powers of our soul);
– [Aaron's] rod, manna and the Law of Moses (contents of the Ark of the Covenant, representing Christ's humanity, soul and divinity);
– Hell, Purgatory and Paradise;
– lasciviousness, pride and avarice (the three principal sins);

---

486 Even the reproach that Pacioli plagiarized, oft repeated in Italy, is glaringly strategic – namely anti-clerical strategy (Pacioli being a friar) – nobody scolds Fibonacci for plagiarizing just as much (rather, care has been taken not to discover). Every age and situation has its own strategies and underlying aims.

487 A detailed description can be found in [Rankin 1992: 367–370].

– the leopard, the lion and the wolf (the three animals encountered by Dante in the opening of the *Divine Comedy*);
– fast, alms, and prayer (three roads to Salvation);
– God, oneself, the neighbour (those offended by sin);
– the Father, the Word and the Holy Spirit;

and still more of the same kind. As we see, Pacioli is loquacious, entertaining – and a loyal member of his flock, probably a sincere believer.[488]

In the very end (fol. 18^v) Pacioli says not go on with cube, pentagonal, hexagonal and octagonal numbers, and so on "because of these one may rather speak to show off (*a ostentazione*) than for operating";[489] those who are interested, he adds, may find them fully explained in the second book of Boethius's *Arithmetic* – and then, true to himself, he does not stop but gives some hints.

The second distinction (fols. 19^r–47^v) is an algorism, presenting the place-value system and the various algorithms for addition, subtraction, multiplication and division, including casting out sevens and nines. It includes progressions and the extraction of square and cube roots,[490] and even the use of extensive tables of squares to find congruous-congruent numbers.

The third distinction (fols 47^v–53^r) deals with fractions, linking them initially to *Elements* VII ("basing them" on Euclidean theory would be an exaggeration), while the fourth (fols 53^r–56^v) goes on with the arithmetic of mixed numbers.

The fifth distinction (fols 57^r– 67^v) is dedicated to the rule of three, and various applications. As in the Perugia manuscript it is introduced in two ways (cf. above, notes

---

488 Hardly a profound theologian – but the theologians of his time were generally well below the intellectual level of their 13th- and early 14th-century predecessors.

489 Similarly, a longwinded explanation of the Boethian names for ratios is closed on fol. 72^v by the observation that "this has no other importance (for you, practitioner) than to assert solemnly the results you have found".

490 Latin algorisms do as much, and their inclusion thus justifies the reference to Sacrobosco as a source; however, the various algorisms that are shown are those of the abbacus masters.

The geometric way to find the square root of 10 is close to that of the Ottoboniano *Praticha* (above, p. 272; and also to Benedetto, fol. 311^r), but sufficiently different to exclude copying from the shared model of the Ottoboniano and the Palatino *Pratiche:* instead of a right triangle, Pacioli uses a rectangle. The method going back to Fibonacci is shown on a diagram closer to Benedetto than to Fibonacci or to the Ottoboniano *Praticha*, but again sufficiently different to exclude copying; it may have been influence by the diagram of the Campanus *Elements* II.14. The former must go back to the broader abbacus tradition.

19 and 233), first with the standard reference to the similar and the not-similar, and alternatively in “mentioned”-formulation. After some illustrative examples a foundation in proportion theory is provided on fol. 57ᵛ. This gives occasion to a digression on continued and discontinuous proportions, which includes the observation that ratios have to be taken between magnitudes of the same kind. There is also space for warnings about the variations of metrology and monetary values, for sequential use of the rule,[491] and for an autobiographic note (fol. 67ᵛ)[492] inserted within a listing of abbreviations for algebraic powers (until the 29th power of the *cosa*) and for roots.

This list combines a variant of the “root names” for powers which we encountered in the *Tratato sopra l’arte della arismetricha* (above, p. 241) with the more habitual names and their abbreviations, observing that *tante terre, tante usanze* “as many regions, so many usages”, and *tot capita: tot sensus*, “as many heads, so many opinions”:

℞ 1ª	nº	*numero*
℞ 2ª	co	*cosa*
℞ 3ª	ce	*censo*
℞ 4ª	cu	*cubo*
℞ 5ª	ce.ce	*censo de censo*
℞ 6ª	pº rº	*primo relato*
℞ 7ª	ce.cu	*censo de cubo e anche cubo de censo*
℞ 8ª	2ºrº	*secundo relato*
℞ 9ª	ce.ce.ce.	*censo de censo de censo*
[...]	[...]	*[...]*
℞ 29ª	ce.ce.2ºrº	*censo de censo de secundo relato*
℞ 30ª	[9º] rº	*nono relato*

We observe that powers have now systematically become *functions*, not entities (cf. above, p. 242). For instance, the sixth power, *censo* of *cubo*, corresponds to our $x^6 = (x^3)^2$. Unfortunately for practical use, the numbering does not correspond to exponents.

The root names are claimed by Pacioli to be “according to the Arabs, first inventors of these operative practices” – perhaps an extrapolation from the identification of the *cosa*/“thing” with the *root* of the *censo* (above, p. 144) (but as we see, Pacioli identifies the *cosa* with the *second* root).

---

[491] The rule of five (not identified by name) only comes on fol. 194ᵛ, in connection with a problem about grain-eating horses, similar to the one which gives Fibonacci the occasion to discuss the theory of composition of ratios (above, p. 74) but with different parameters.

[492] This note shows that the fifth distinction was written in 1487.

The sixth distinction (fols 67^v–98^v) deals with "proportions" (even though Pacioli sometimes uses *proportionalità* when speaking about a proportion, he mostly uses *proportio* about ratios as well as proportions); in several ways it goes beyond what we know from other writers – which probably does not mean that Pacioli innovates but rather that surviving sources give a very incomplete picture.

As the other distinctions, this one is divided into a number of "treatises" – here six. The first of them lists the authorities for the topic, from Plato, Aristotle, Euclid and Boethius to Jordanus of Nemore, Albert of Saxomy and Blasius of Parma, and explains the necessity to know about proportions in the fields of law; in medicine; when it comes to knowledge about nature and mechanical devices; painters' perspective; etc.

Proportions are, quite traditionally but in an abstruse formulation,[493] said to be the similarity of the mutual relation between two couples (fol.69^v). This opens the way to the "arithmetical proportion" ($a–b = c–d$) along with the normal "geometric proportion"; inspired by the three "means" (arithmetical, geometric and harmonic) Pacioli then further claims a "harmonic proportion". The geometric proportion "will be when comparison is made comparing one continuum and another, as one line to another line, one surface to another surface, one body to another body, one [duration of] time to another time"; the arithmetical "will be when comparison is made from one number to another one, whether they are equal or more properly between the excesses or differences between the numbers"; the harmonic, finally, "will be when comparison is made from one voice to another one and from one sound to another one" – and more badly understood metamathematics involving continuous and discrete quantities.

I shall not go through the whole distinction[494] but restrict myself to passages that inform us about developments within the abbacus tradition about which we have no other information.

The sixth treatise (fols 84^r–98^v) is stated to deal with "seven marvels [*mirabiles*] from the proportions between two quantities". As a matter of fact it *begins* with seven "marvels" involving two quantities, but afterwards considers others which concern three or more. The first marvel is that

[493] *Proportione communiter dicta e habitudine de doi cose asiemi comparate una a l'altra e l'altra a l'una, in alcun termino a loro univoco* – something like "proportion is commonly said to be a condition of two things compared together, one to the other and the other to the one, in some respect that is the same for them".

[494] Part of what I leave out is discussed either in [Bartolozzi & Franci 1990: 17–27] or in [Høyrup 2009d: 92–104].

> any two quantities you want in any ratio joined together, and then the sum divided by each of the said quantities; the results then joined together, and then the sum of the said results equally divided by each of the said results; and again these latter two results joined together, will always be the sum of the first two results, and it never fails.

In symbols,[495]

$$\frac{\frac{a-b}{a}+\frac{a-b}{b}}{\frac{a-b}{a}}+\frac{\frac{a-b}{a}+\frac{a-b}{b}}{\frac{a-b}{b}}=\frac{a-b}{a}+\frac{a-b}{b}$$

I shall not go through all seven marvels (all are rendered in symbols in [Bartolozzi & Franci 1990: 23–24][496]), but two are noteworthy – in symbols, respectively,

$$\frac{a-b}{a}\times\frac{a-b}{b}=\frac{a-b}{a}+\frac{a-b}{b}$$

and[497]

$$\frac{a-b}{a}+\frac{a-b}{b}=2+\mathrm{denom}(a:b)+\mathrm{denom}(b:a)$$

The marvels appear to have grown out from problems about the splitting of 10 into two parts a and b, where ${}^{a}/_{b}+{}^{b}/_{a}$ respectively ${}^{10}/_{a}+{}^{10}/_{b}$ is given. Such problems are known since the beginning of the algebra tradition.[498] It seems likely that Pacioli has borrowed at least the stock of his marvels, possibly adding some.

On fol. 85$^{r–v}$, other marvels about three, four or five numbers in continued proportion follow. The first states that if three numbers are in continued proportion, then division of their sum by the single numbers produces another continued proportion. This (but with an arbitrary dividend) had been considered "rather clear and obvious" by Antonio in the

---

495 The fraction lines stand for the operation which Pacioli speaks of as "division".

496 There is a (mathematical as well as translational) error in the fourth, which should be

$$\frac{\frac{a-b}{a}+\frac{a-b}{b}}{\frac{a-b}{a}}=\frac{a-b}{b}\quad\text{and}\quad\frac{\frac{a-b}{a}+\frac{a-b}{b}}{\frac{a-b}{b}}=\frac{a-b}{a}\,.$$

The authors have overlooked that the equality between the first and second results are said to be *e converso*. As Pacioli points out, the first marvel follows from this.

497 "denom" stands for "denomination of" the ensuing ratio, that is, the corresponding fraction or mixed number.

498 See [Rashed 2007: 167–165] for al-Khwārizmī, [Rashed 2012: 336–349] for Abū Kāmil; and [Woepcke 1853: 91*f*] for al-Karajī. All three give general rules for the behaviour of the quotients, e.g., ${}^{a}/_{b}\cdot{}^{b}/_{a}=1$ and $({}^{a}/_{b}+{}^{b}/_{a})\cdot ab=a^2+b^2$.

Fioretti [ed. Arrighi 1967a: 54], and is in fact a theorem which is useful for certain the problems about the splitting of a number into a sum of numbers in continuous proportion. We may take it for granted that Pacioli borrowed it – possibly directly or indirectly from Antonio, since he goes on (fols 85^{v}–86^{r}) to apply the rules to binomials in the way Antonio had done in his *mirabile dictum* (above, p. 232). As pointed out by Bartolozzi and Franci [1990: 24], Pacioli generalizes Antonio's method further than Antonio himself had done without controlling – and errs (so at least it seems – the text is not quite clear).

Next (fols 86^{v}–87^{v}) we find a number of rules about three, four or more numbers in continued or (at times) non-continued proportion. Most, as Pacioli states, follow from *Elements* VI.15–16 and VII.20 (Campanus's numbering, our VI.16–17 and VII.19 – the product rule for three or four segments or numbers in proportion/continued proportion): how, if two (or, an overdetermined case, three) neighbouring quantities in a continued proportion are known, to find the remaining one(s).

Slightly more intricate are the cases where the first and the last of four or five quantities in continued proportion are known. In the case of four quantities, this coincides mathematically with Jacopo's second *fondaco* problem (above, p. 192), but whereas Jacopo merely prescribes the extraction of the cube root of the ratio between the fourth and the first quantity without explaining why, Pacioli uses algebra, without which he finds it difficult to solve the problem. In the case of five quantities, the middle quantity is found first from the product rule.[499]

Between these two cases, Pacioli gives the abstract analogue of Jacopo's third *fondaco* problem. Without explanation Pacioli gives the same rule as Jacopo; he certainly does not know how it comes about (if so, the algebraic solution of the preceding problem shows that he would have explained). However, the last step of his procedure (how to find two numbers from their sum and their product) suggests that Jacopo is not his direct or indirect source: it contains a hint of an underlying geometric procedure (a reference to operation with two *different* halves of a quantity) which is absent from Jacopo's text, and which neither Pacioli not any intermediate abbacus writer is likely to have introduced on his own.[500]

---

499 In generic terms, Pacioli says that the same method can be used for "6, 7, 8, etc." terms, but he abstains (maybe wisely) from implementing this insight – fols 182^{r} he speaks of the sixth root as the "cube root of the cube root" and of the seventh root as the "root of the root of the cube root". These composite expressions might indicate that Pacioli believed they could be found by stepwise calculation; they may also be traces of copying from a source still expressing higher powers multiplicatively and emulating this system for *the naming* of roots.

500 When solving in the geometric part of the *Summa* the corresponding geometric problem, Pacioli [1494: II, fol. 18^{r}] merely refers to the contents of *Elements* II.5, as does his ultimate source (Fibonacci's *Pratica geometrie* [ed. Boncompagni 1862: 63]). Similarly also (here with explicit citation of Euclid's proposition) in the arithmetical part, fol. 93^{v}.

On fol. 88[r] begins a number of “keys [*claves*] or evidences concerning quantities in continued proportion”, likened to the two spiritual keys of gold and silver by which “in our Catholic Militant Church the first shepherd Saint Peter” opens and closes the doors of Paradise and Hell for us. While we recognize here Pacioli’s affection for his Church, the term “keys” is likely to go back to that of Fibonacci.

In their mathematical substance, however, Pacioli’s 15 keys differ from those of Fibonacci (which were derived from *Elements* II and had nothing to do with continued proportions). Like these, they are theorems (by Pacioli called “evidences”); in part they are near- or full repetitions of what he has already explained before or easy corollaries of familiar stuff, in part new to the book and not easily guessed without symbolic manipulation. Since Pacioli does not distinguish, he is likely to have borrowed the group as a whole (restrictions and further arguments below). All are illustrated by numerical examples. In symbolic translation they are

(1) If $a : b : c : d$, then $\frac{b-c}{a-b-c-d} : \frac{b}{a-c}$.

(2) If $a : b : c : d$, then $\frac{a-b}{c-d} : \frac{a}{c}$.

(3) If $a : b : c : d$, then $\frac{a-c}{b-d} : \frac{a}{b}$.

(4) If $a : b : c : d$ and $S = a+b+c+d$, then ${}^{S}/_{a} : {}^{S}/_{b} : {}^{S}/_{c} : {}^{S}/_{d}$; with three members, this was the first three-number “marvel” on fol. 85[r].

(5) If $\frac{a}{b} : \frac{c}{d}$, then $ad = bc$; the product rule, amply used before.

(6) If $\frac{a}{b} : \frac{c}{d}$ and if $c^2+d^2 = a{\cdot}b$, then $\sqrt{(a^2+b^2)(cd)}$ has the same value. Actually, given only the proportion, $(a^2+b^2){\cdot}cd = ab{\cdot}(c^2+d^2)$.

(7) If $\frac{a}{b} : \frac{c}{d}$, then $([a{\cdot}b]{\cdot}c){\cdot}d = (a{\cdot}d){\cdot}(b{\cdot}c)$; evidently, this does not depend on the proportionality.

(8) If $a{:}b{:}c{:}d$, then $(a+b+c+d)^2 = a{\cdot}(b+c+d)+b{\cdot}(a+c+d)+d{\cdot}(a+b+c)+c{\cdot}(a+b+d)+a^2+b^2+c^2+d^2$; this time, Pacioli himself points out that the rule does not depend on the proportionality.

(9) If $a : b : c$, then $(a{\cdot}b){\cdot}c = b^3$.

(10) If $a : b : c$, and if, for some quantity Q, ${}^{Q}/_{a}+{}^{Q}/_{b}+{}^{Q}/_{c} = a+b+c$, then $b = \sqrt{Q}$.

(11) If $a : b : c$, then ${}^{(a\cdot b)\cdot c}/_{a} = b{\cdot}c$, ${}^{(a\cdot b)\cdot c}/_{b} = a{\cdot}c$, ${}^{(a\cdot b)\cdot c}/_{c} = a{\cdot}b$, and ${}^{(a\cdot b)\cdot c}/_{a\cdot b} = c$, ${}^{(a\cdot b)\cdot c}/_{a\cdot c} = b$, ${}^{(a\cdot b)\cdot c}/_{b\cdot c} = a$; Pacioli points out that this does not depend on the proportionality.

(12) If $a : b : c$ and further $\frac{a}{b} : \frac{p}{q}$, then $p{\cdot}(b+c) = q{\cdot}(a+b)$.

(13) If $a : b : c$, then $2{\cdot}(a{\cdot}c+b{\cdot}[a+c]) = a(b+c)+b(a+c)+c(a+b)$. With references to *Elements* II.2 and the formulations “in other words” in *Elements* VI and IX Pacioli points out that this does not depend on the proportionality.

(14) If $a : b : c$, then ${}^{a(b+c)+b(a+c)+c(a+b)}/_{2\cdot(a+b+c)} = b$.

(15) If $a : b : c$, then $\frac{a^2}{b^2} : \frac{a}{c}$.

Under the heading “to find mean proportionals between two quantities”, two peculiar counterfactual calculations follow on fol. 89^{v} which could be Pacioli's own inventions: If 2 is the arithmetical respectively geometric mean between 5 and 11, what is then the corresponding mean between 7 and 13? In both cases, the true means between 5 and 11 and 7 and 13 are found (8 and 10 respectively $\sqrt{55}$ and $\sqrt{91}$), and the rule of three is applied. In the arithmetical case, a proof is performed, consisting in corresponding proportional change of the limits, after which the true means between these limits are shown to coincide with what was found before; in the geometrical case, a similar proof is sketched but not performed.

The "second case" under the same heading is a traditional question "Three is [too] little and 4 is [too] much". The "just or due" amount is said to be $\sqrt{12}$, the geometric mean; this – not the arithmetical mean – is then stated to be what is used in all commercial matters (*in omnibus mercantiis*). Primarily, this probably extrapolates from the observation that the rule of three is based on geometric *proportionality*. But Pacioli may also think of the use of the geometric mean in certain mathematical *problems* in commercial disguise.

In any case, a problem of this kind about three pearls, follows as the "third case". The first pearl weighs 1 carat and is worth 200 *ducati*, the second weighs 2 carats and is worth 1000 *ducati*, the third weighs 3 carats. What is its just price?

Pacioli posits a fourth pearl with weight 4 carats. To the weights 1:2:4 in continued proportion must correspond prices in continued proportion, i.e., 200:1000:5000. Therefore the price of the 4-carat pearl must be 5000 *ducati*. 3 carats being the (arithmetical) mean between 2 and 4, the price of the 3-carat pearl must be $\sqrt{(1000 \cdot 5000)}$.

A fourth case is also about justice. The Holy Father, Innocent VIII, orders that 10000 *ducati* be distributed justly between the citizens of Perugia for service rendered. This gives rise to a long discourse (more than 500 words) about Aristotle's two kinds of justice from the [Nicomachean] *Ethics* V.2–5 ($1130^{\mathrm{a}}14$–$1134^{\mathrm{a}}16$, [trans. Barnes 1984: II, 69–76]): "commutative"[501], applicable to commercial exchange, and distributive. Both, according to Pacioli, "can, broadly speaking, be understood in two ways, geometrically and arithmetically, though, strictly and properly speaking, the maximal distributive sort can only be geometrical".[502] After the digression into ethical theory it is then explained that the money is justly distributed if given in geometric proportion to the "quality" (*bontà*) of each.

The sixth distinction ends (fols 90^{v}–98^{r}) with 35 problems[503] and an epilogue (fol. $98^{\mathrm{r–v}}$). The last two problems have nothing to do with proportions – #34 is "Bachet's

---

501 Nowadays normally translated "rectificatory", but Pacioli follows his fellow friar Thomas Aquinas (*Summa theologiae* 1^{a} q. 21 a. 1 s 1 co, see [*Corpus thomisticum*]), whom he cites.

502 This point comes from Aristotle, whose Chapter 3 also contains as discourse on proportion theory. Mathematical proportions (represented by lines and letters) are used further in Chapters 4 and 5.

503 Pacioli also counts until 35, but has two #18, skips #19 and #28 and has two #29.

weight problem" (above, p. 99), and #35 belongs to the same family; parallels in the wording suggest that they are borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 297*f*]. In all the others, "proportions" play a role.

First come 23 problems about three numbers in continued proportion. In seven of them, a number (19, 19, 14, 10, "a number",[504] 10, 10) is split into such constituents; towards the end of the sequence, four are dressed as dealing with economic life.[505] In #1–6, specified "keys" are used as a first step in the procedures, which in these and the other cases often makes use of algebra or (in #5, #6 and #18) of *Elements* II.

Algebra is thus *used* by Pacioli well before he presents it systematically. Often, this algebra is quite complex. In #4, for instance, Pacioli has to operate with two unknowns in the same way as Antonio, that is, with "a *thing* less a quantity" and "a *thing* plus a quantity". The problem in which this is used is a numerical variant of #24 of Antonio's *Fioretti* [ed. Arrighi 1967a: 53][506] – in symbols, to find three numbers $a:b:c$ in continued proportion such that $abc = a+b+c$, $36/a+36/b+36/c = a+b+c$ (Antonio has 20 instead of 36). As may be remembered, all the problems solved by means of several algebraic unknowns we have encountered after those of Antonio were linear, so either Pacioli used the *Fioretti* directly, or there has been a transmission of the technique which escapes us. Since Pacioli's "keys" (essential for the formulation of the solutions) are hardly his own invention (see imminently) yet not present in the *Fioretti*, I opt for the latter possibility.

Next follows a sequence of ten problems about four magnitudes in continued proportion, none of them in concrete dress. Once again, the first ones make use of specified "keys" (#24–27 – and also #31–32). Most interesting are probably #31–33: #31 and #33 are pure-number versions of Jacopo's third and fourth *fondaco* problems; #32 is a similar problem where the sums of the first two and of the last two numbers are given. In #31, key (1) is used to reduce the problem; then the second number is taken as the *thing* and found by second-degree algebra to be $12\,{}^{1}/_{2}+\sqrt{7\,{}^{37}/_{84}}$ – at which point Pacioli cautiously leaves it to the reader to continue.[507] Since his present method does not lead

---

504 This problem (#15) is indeterminate. Afterwards, the number is chosen to be 10, whereby it is made determinate.

505 #18^bis deals with a gambler's gains, where the product rule is explained once again, suggesting perhaps the text to be borrowed (but Pacioli is too fond of repeating to make that inference certain); #21 is a challenge, "proposed to me in Florence in 1480, the 22nd of June", and deals with a purchase of saffron, cinnamon and mastic; #22–23 are about alloys.

506 Several other problems are also close to Antonio.

507 The solution is correct, but corresponds to a decreasing sequence, which is certainly not what Pacioli intended; in order to have an increasing sequence, he should have chosen the other root

easily[508] to the formula used by Jacopo and by Pacioli in the first presentation of the abstract problem just before the "keys" (above, text before note 70), Pacioli appears not to have noticed the connection.

The use of the "keys" in problem reductions leaves little doubt that these *new* theorems about the behaviour of proportions were created as tools for the solution of problems. Pacioli's way to add observations about (8), (11) and (13) strongly suggests that the basic set was not his own. It is likely to have been created during the fifteenth century, after Antonio's time and inspired by him; they seem to reflect a more intimate integration between algebra and proportions than other sources would make us expect.

In the very end of the sixth distinction (fol. 98v) Pacioli points out that his primary intention is always practice. "But if none the less higher theory [*speculare*] pleases you, have recourse to it" – and that is the purpose of what has been said about ratios and proportions.

The seventh distinction (fols 98v–111v) is claimed to present the rule of *El cataym*,

> which (according to some) is an Arabic word, which in our language is as much as saying the rule of two false positions. By which almost all questions can be solved: in particular those that have to do with trading; where normally one has no need to insert roots of any kind.

In spite of knowing the meaning of the term, Pacioli at first presents "simple *helcataym*", that is, the simple false position.[509] Most of the examples are slightly intricate, for instance asking for sequential use of the rule of three. One is a counterfactual calculation: "If 3 were the half of 7, what part would it be of 11", for which two different

---

of the equation, $12\,\frac{1}{2}-\sqrt{7\,\frac{37}{84}}$.

When applying later the same method to an analogous wage problem with rational solutions, Pacioli makes the complete calculation and chooses the correct solution – see presently.

508 Of course it *can* lead to it, but only if one is able to express the double root (the second and the third number, respectively) as

$$\frac{P}{2} \pm \sqrt{\frac{P^2}{4} - \frac{P^3}{3P-Q}}$$

(*P* being the sum of the second and the third number, *Q* that of the first and the fourth). The product of these is indeed

$$\frac{P^3}{3P-Q}$$

as required; but this, done without modern symbolism, will have been far too complicated for Pacioli (as it would be for us).

509 Benedetto too presents a "simple mode of *chatain*", but as we remember from note 411, Benedetto's rule is Fibonacci's "rule of proportion" (above, note 411). What Pacioli does is closer to the Ottoboniano and the Palatino *Pratiche* (above, p. 254).

interpretations are offered. Some deal with combined works, where neither a single nor a double false position is made.

On fol. 99^{v} begins the double false position proper. Initially it offers a quasi-algorithmic set of four small rules (followed by explanations) for what to do when either of the two positions leads to an excess or a deficit (and justly stated to be in reality only three):

more	and more	always	take away
less	and less	always	take away
more	and less	always	join
less	and more	always	join

Arithmetical explanations of the rules are given, which in their mathematical principle are similar to the one given first by Fibonacci (above, p. 108), but without his line diagrams. From fol. 102^{r} onward geometric proofs follow, whose diagrams are lettered almost as the corresponding rule in the *Liber abbaci*; while this classic is not copied, direct or indirect inspiration seems almost certain.

The illustrative problems that follow are only sketched, in the sense that they explain the reduction to questions where the two false positions can be applied but leave the rest to the reader.

One example is the "nightmare problem" about repeated travels with gain and expenses leading to bankruptcy (above, p. 89). After the hinted solution by double false Pacioli explains how the solution can be solved step by step backwards, adding that the same can be done in the problems about gardens with 3, 4 or more gates. Fol. 106^{v} comes a give-and-take problem which asks for nested double false positions, a technique which *is* explained in some detail; the problem, but not the positions and thus not the calculations, are shared with Piero Borgi, whose calculations are much more extensive (cf. above, p. 329). The structure is somewhat unusual – the first, second and third with, respectively, $^1/_2$, $^1/_3$ and $^1/_4$ of what the others possess will all have 20 ducats (Pacioli) or *denari* (Borgi). A borrowing is next to certain – but not necessarily from Borgi.

The last part of the distinction (fols 106^{r}–111^{v}) has nothing to do with false positions of any kind. It is rather a transition to the algebra of the eighth distinction, and consist of another collection of "evidences" (66 in total), preceded by some general observations, including that Euclid, in spite of all the qualities of his book, has not said everything.

The first ten evidences correspond to *Elements* II.1–10, and in so far also to Benedetto's substitute for Fibonacci's "keys" (above, p. 309). There are no indications of any link, however, what Pacioli offers are simply the statements (valid for any kind of quantities) with numerical examples, no geometric proofs.

The 11th is similarly a reflection of *Elements* II.11, and observes that the division in question[510] is a division into mean and extreme ratio, and that the ratio that results is irrational.

What follows are in part identities (in a few cases inequalities), in part it has some similarity to what Jordanus does in the *De numeris datis*, expressing solvability of certain problems – as in Jordanus's work illustrated by numerical identities. Many have to do with the division of a number into two parts. We may look at some extracts in symbolic translation:

(12) $(a+b)/a = 1+{}^{b}/_{a}$;

(13) ${}^{a}/_{b} \cdot {}^{b}/_{a} = 1$;

(14) $ab \leq ({}^{a+b}/_{2})^2$;

(15) $a^2+b^2 < (a+b)^2$ if $a<b$, the former difference increasing with the latter ($a+b$ is given);

(18) given $a+b$, a and b are determined unequivocally from $(a^2+b^2)/({}^{a}/_{b}+{}^{b}/_{a})$.

In order to express these, Pacioli has to develop some rhetorical strategies. We may look at (18):

> If a quantity be divided into 2 parts, which are mutually divided; and the two results are joined together; and save the sum. And then, if you square each of the said parts; and the squares joined together; and this sum divided in the saved sum; from which shall come a determined number. I say that who makes of the first quantity two parts; where the surface of one in the other makes the said number; will always have the said parts.

In symbols indeed

$$\frac{a^2+b^2}{\frac{a}{b}+\frac{b}{a}} = \frac{ab\cdot(a^2+b^2)}{a^2+b^2} = ab \ .$$

More interesting than this formula is the trick used to keep the ${}^{a}/_{b}+{}^{b}/_{a}$ together as one number that can be used as a divisor: it is saved, and then retrieved, exactly as we do when making a calculation on a pocket calculator that requires a parenthesis. The request that a "determined number" shall result corresponds to Jordanus's statement that the outcome is "given"; the discordant expressions makes it more than doubtful that Jordanus should have inspired Pacioli.

A number of cases are closed by a Latin phrase – *et sic habemus intentum* (fol. 110^r), etc. Apart from that, there is no Latin; my stylistic feeling (for what it is worth) suggests no borrowing but to the contrary that this set of evidences has been collected by Pacioli himself and not borrowed as a totality, as was probably the case with the evidences contained in the sixth distinction (above, p. 342) – also because there is nothing which looks like an added commentary to borrowed material.

---

[510] In symbols: a quantity is divided as $a+b$, so that $a^2 = b\cdot(a+b)$.

The ninth distinction (fols 111^{v}–150^{r}) opens with the declaration

> I find that I shall no longer defer the part which is most necessary to the practice of arithmetic and also of geometry, in the vernacular commonly called “the Major Art” or “the Art of the Thing” or “Algebra and Almucabala”, by us called “theoretical practice” [*pratica speculativa*]. Because in it are contained higher matters than in the minor art or mercantile practice.

At first (fols 111^{v}–115^{r}) the powers and their arithmetic are presented (including the impossibility to reduce expressions involving different powers), together with the sign rules for multiplication as well as division. A proof similar to that of Dardi (above, p. 218) is given for less times less – more loquacious than that of Dardi and referring to the general multiplication of binomials, but still including a double indirect proof. On fol. 113^{v} follows sign rules for division, similar to those presented by Benedetto, but in Pacioli’s version ordered with a scheme. New are (fol. 114^{r}) sign rules for additive procedures:

plus	with plus	joined	always makes	plus
less	with less	joined	still makes	less
plus	with less	joined	always one	subtracts
	and will be the major denomination			
less	with plus	the same as plus with less		

I would guess that this is Pacioli’s personal contribution. Evidently, Pacioli has come to consider “numbers less” as negative, not merely subtractive numbers (still with a two-class system, not a single number line going from negative numbers over 0 to positive numbers).

In the end of the section come corresponding rules for subtracting “numbers less”.

Next (fol. 115^{v}) follow roots. First the kinds, square, cube, related, pronic (as quoted above, p. 214); then (until fol. 119^{v}) the arithmetic of square and cube roots. The classes of Euclidean binomials and apotomes and their arithmetic are presented on fols 119^{v}–143^{r} (together with a little bit about trinomials), apparently independently of Fibonacci, see note 164.

Fols $143^{r–v}$ continues the listing of algebraic powers from fol. 67^{v} (above, p. 338) with an explicit backward reference, now tabulating their products, using the root names and illustrating their meaning by means of corresponding powers of 2.

Afterwards the abbreviations *n*^{o}, *co* and *ce* are explained to stand, respectively, for the first, second and third “root”; these are then what it used in the following explanation of algebra (where the highest powers play no role).

This explanation starts (fol. 144^{r}) with another tribute to the discipline:

> Having with God’s assistance come to the much desired place: that is, to the mother of all the cases by common people [*il vulgo*] called “The Rule of the Thing” or “the Major

> Art", that is, theoretical practice, also called *Algebra et almucabala* in Arabic language, or according to some in Chaldeic, which in our language is as much as to say "of restoration and opposition, *algebra, id est restauratio, almucabala, id est oppositio vel contemptio, et solidatio*. Because in the said way infinite questions are solved. And those which still cannot be solved I shall point out.

The Latin quotation leaves little doubt that Pacioli here draws upon chapter 15 as well as chapter 14 of the *Liber abbaci* (cf. above, pp. 141 and 117) – directly or perhaps indirectly, the latter suggested by his correct identification of *algebra* with *restauratio*.

This is followed by explanations of what number, *thing* and *census* are, and by the rules for the six simple cases (in Fibonacci's idiosyncratic order (above, p. 146). Versified versions of the first three rules are given in the margin.

Geometric proofs are given also for the simple cases (fol. 145^{v}), probably of Pacioli's own making; those for the composite cases are in line with the tradition but include explicit references to *Elements* II.

Four "essential things to note" follow, of which the first three teach how to simplify equations by the elimination of integral or fractional coefficients and by the operations of restoration and opposition. The fourth (fol. 148^{v}) steps outside the trodden path, pointing out that sometimes one has to posit two unknowns – second unknowns being called "deaf quantities". Pacioli takes as example a question for two numbers whose squares added together are 20 and whose product is 8. He posits the first to be a *thing* plus a *quantity*, the second to be a *thing* less a *quantity* (we recognize Antonio's trick – above, p. 233). Pacioli does not solve the problem, but since the resulting *thing* is 3 and the *quantity* is 1, "deaf" [*sordo*] must here mean "unknown".

Pacioli further explains that this quantity is called "second thing" (*cosa seconda*) in the ancient practical books (*ne li libri pratichi antichi*), while "the moderns call it simply $\tilde{\varphi}^{a}$" – that is, there must have been much more use of two unknowns than can be seen from extant sources.[511]

The last part of the distinction (fols 148^{v}–150^{r}) shows how to reduce certain higher-degree cases while others are impossible; in the end Pacioli explains (considering only three-member equations) that it has so far been impossible to create general rules if the intervals between the powers are not pairwise equal; they can only in certain cases be solved *a tastoni*, "by groping". As is the case with the squaring of the circle, also knowable but so far not found,[512] even though these equations may be possible, they have so

[511] None the less, the pair is not mentioned in the Perugia manuscript (with the proviso that the chapter presenting algebra systematically and thus corresponding to the present distinction has been lost). Instead, two "horse" problems make use of the algebraic unknowns *thing* and *horse*.

[512] An echo of Aristotle's *Categories* $7^{b}31$–33, almost certainly mediated by Boethius's commentary to the work [PL 64, col. 231]).

far not been solved.

The ninth distinction (fols 150^{r}–224^{v}) is, globally, concerned with mercantile matters. “Globally”, however, allows several local aberrations. Since most of the mercantile material is traditional, I shall concentrate on the aberrations and on the innovative mercantile substance.

The first treatise (fols 150^{r}–159^{r}) deals with partnerships, with the usual application of the principles to mathematically similar questions (for example, the twin problem). The second (fols 159^{r}–161^{r}) takes up a particular type of shepherding contract (*soccida*), warning against the many frauds that it may conceal (wholly outside abbacus habits); and next the traditional amortization of a loan by the rent of a house.

Barter is dealt with on fols 161^{r}–168^{r}, and exchange together with letters of exchange on fols 168^{r}–173^{v} – the latter also new in format in the abbacus context, even though the substance is familiar, for which reason the principle is explained in detail and a paradigm is shown in the margin.[513] In the end come recreational problems.

All matters regarding interest, discount, etc., are discussed on fols 173^{v}–182^{v}. Even here, toward the end we find recreational problems asking for the use of algebra. Alloying (and a single problem about the mixing of sugar, cloves, sandalwood, mace, nutmeg, cinnamon and ginger[514]) is the topic of fols 182^{v}–186^{r}).

From these mostly genuinely commercial concerns Pacioli turns to (mostly advanced) variants of the classical recreational problem types:

– repeated travels (fols 186^{r}–188^{r}), first- as well as (mostly) higher-degree problems, no less than six problems dealing with an unknown number of travels;
– complicated give-and-take and similar problems (fols 188^{r}–190^{v}), many solved by means of second- or higher-degree algebra;
– horse-buying and purse-finding “by way of ratios” (fols 190^{v}–194^{r}) and using algebra with one or two unknowns;
– problems about salaries of servants (fol. 194^{r}). Two coincide with Jacopo’s second and third *fondaco*-problem (above, p. 192); the former is said to be solved by means of what has been taught about (continued) proportions, the latter with a reference to the appropriate “key”;
– *varia* (fols 194^{r}–197^{v}).

---

513 Dated 9 August 1494, and involving the printer Paganino de Paganini.

514 An observation of general historical interest can be made concerning the prices: sugar is the most costly (32 *soldi* the pound, against 12 *soldi* for pepper and 11 for ginger). There was a strong economic incentive behind the Portuguese establishment of sugar-producing slave plantations in the Madeiras and the Azores, and for Columbus’s similar undertaking in Haiti in 1493 [Verlinden 1970: 21].

One of the give-and-take problems (fol. 191^{v}) gives Pacioli the occasion to return to the "deaf quantity" (I use Pacioli's notation, where *co* stands for the *thing*):

> Three have *denari*. The first says to the other 2, if you give me half of yours I shall have 90. The second says to the other 2, if you give me $^1/_3$ of yours, I shall have 84. The third says to the other 2, if you give me $^1/_4$ of yours plus 6, I shall have 87. I give you this solely to show how one operates by a deaf quantity which the ancients call second thing to differentiate it from the first positions. Posit that the first has 1 *co*, remove it from 90, 90 less 1 *co* remains, and this should be $^1/_2$ of the other 2, these then have 180 less 2 *things*, and all 3 have 180 less 1 *co*. Now do for the 2nd and posit that he has a *quantity*, which I depict thus, one $\tilde{\Phi}^{a}$, and to the 2 remain 180 less a *co* less a $\tilde{\Phi}^{a}$. Take $^1/_3$, from which results 60 less $^1/_3$*co* less $^1/_3\,\tilde{\Phi}^{a}$.

If A, B and C designate the three possessions, the conditions are thus

$$A+{}^1/_2(B+C) = 90\ ,\quad B+{}^1/_3(A+C) = 84\ ,\quad C+{}^1/_4(A+B)+6 = 87\ ,$$

and with A posited to be a *co*, B to be a $\tilde{\Phi}^{a}$, Pacioli has found that

$$^1/_3(A+C) = 60-{}^1/_3 co - {}^1/_3\ \tilde{\Phi}^{a}\ .$$

Inserting this in $B+{}^1/_3(A+C) = 84$ and using that $B = 1\tilde{\Phi}^{a}$ Pacioli derives the equation

$$1\tilde{\Phi}^{a} = 36+{}^1/_2 co\ ,$$

which is the second possession.

Now comes something new:

> Now for the 3d do similarly: Posit that he has a *quantity*, remove it from 180 less 1 *co*, that is, still from the amount of all three. [...].

Pacioli thus operates with three algebraic unknowns, but only with two at a time, which allows him to recycle the name *quantity*. This second position allows Pacioli to derive the equation

$$1\tilde{\Phi}^{a} = 48+{}^1/_3 co\ ,$$

which is the third possession. That brings him back to a single unknown,

$$A+B+C = 1co+36+{}^1/_2 co+48+{}^1/_3\ co\ .$$

But we know that $A+B+C = 180-1co$. This solves the problem. In the end Pacioli specifies that one shall always with this method isolate the *quantity*, and explains that

> by means of these deaf quantities which the ancients called second things a great many strong problems can be solved by the one who handles the equations well.[515]

---

[515] Nicolas Chuquet had used the same method of a recycled second unknown repeatedly in the appendix to his *Triparty*, written in 1484 – see [Heeffer 2012: 134*f*]. Since the *Triparty* was a

After these 24 pages dedicated to mathematical bravery Pacioli returns to what is really useful for the merchant – though, like the advanced recreational and algebraic material, well beyond what was taught in the abbacus school.

First (fols 197^{v}–198^{v}) Pacioli speaks (in headlines, but in much detail) about what should be in the *quaderno,* a book of inventory, transactions and, not least, of accounting (as we remember, in his early years in Venice Pacioli had participated in the mercantile activity of the Rompiasi family).

Next (fols 198^{v}–210^{v}) follows the exposition proper, whose main constituent is what won Pacioli the honour to have his portrait on an Italian 500 *lira* coin in 1994 and to be the founding hero in histories of accounting: the first (or at least the first surviving) description of the double-entry bookkeeping system that was in use in Venice.

Last in the ninth distinction and in the arithmetical part of the *Summa* (fols 210^{v}–224^{v}) comes a very detailed *tariffa* copied from one that had been printed in Florence in 1481 (*Libro che tracta di mercatantie et usanze de' paesi*), originally copied around 1450 by Giorgio di Lorenzo Chiarini from what was in use by Tuscan merchants in Ragusa (now Dubrovnik) and circulating in many manuscripts before being printed [Travaini 2003: 73, 164].

As said above, I shall not discuss the geometry part of the *Summa*, just recall that it is largely a vernacular version of Fibonacci's *Pratica geometrie*, mostly drawn from an earlier vernacular version but more complete than earlier surviving manuscripts in *volgare*. In this respect it can be seen to go beyond even the mathematically most advanced representatives of the abbacus tradition (not to speak, obviously, of abbacus geometry proper). On the other hand if reflects the lacking interest of the abbacus environment to develop the field of theoretical geometry beyond what was inherited.

The arithmetical part of the *Summa*, on the other hand, though building upon the abbacus tradition, goes much beyond it, while even the sophisticated recreational level and the algebra of the abbacus tradition in itself had gone well beyond what we find in such early representatives as the *Livero*, the Columbia algorism, and Jacopo – and, if we look at the 15th century where Fibonacci was to some extent re-adopted, often beyond the *Liber abbaci*. Those parts of the *Summa*, moreover, which do present genuine abbacus matters tend to present much fewer examples than had been the custom even in the Florentine encyclopedias, supposing that a single example suffices to teach a principle.

---

manuscript and apparently did not circulate (apart from Etienne de la Roche's use of it in [1520]), Chuquet was certainly not Pacioli's source. On the other hand, the shared principle supports Pacioli's claim that he presents existing ideas.

We may claim that this was brought about by the particular personality of Pacioli – neither Borgi nor Ghaligai went as far; but this personality was allowed to unfold, and was certainly shaped, by the new socio-cultural conditions of the late 15th century, where Pacioli was allowed to interact with Alberti the Humanist and to teach not in abbacus schools but at universities. His endeavour if not his personality was probably also shaped by the possibility to go into print; in any case printing was what allowed the *Summa* to have an immense influence in the 16th century.

The German-speaking area

Already around the mid-fifteenth century – thus before the beginning of mathematical printing – abbacus culture started to spread to German lands. The counterpart of *abbacus* naturally became *Rechnen*, "to calculate", and that of the abbacus master therefore *Rechenmeister*. A document from Nürnberg shows that in 1457 three *Rechenmeister* held school there – and other documents from 1486–87 that they were in unfriendly mutual competition [Schröder 1988: 301; Vogel 1949: 243].

At that time, they represented a new profession, for a while only existing in cities engaged in long-distance trade to or through Italy and therefore able to draw on Italian inspiration. As the corresponding school type took root more broadly, the normal institution in German area would be the *Schreib- und Rechenschule*, which taught reading and writing together with commercial arithmetic – thus combining what had been two different levels in Italy (children being enrolled at age 6 or so). To be observed, however, is that the writers whom shall encounter below characterize themselves as *Rechenmeister* alone (when not as *Stadtschreiber*, "municipal secretaries", or the like).

As is the case concerning the earliest decades of the abbacus school, we have no direct testimony of the contents of the teaching of these *Rechenschulen*. The first trace we have of abbacus culture spreading to Germany is thus at the same time evidence that practical arithmetic inspired by the abbacus tradition was in the process of being accepted as fitting for arts-faculty students.[516] In 1467 and 1468 a certain Gottfried Wolack held lectures at Erfurt University, conserved in several Latin manuscript copies, one of which was published by Hermann Emil Wappler [1900]. It begins

> The rule of proportion begins with its [examples].
>
> The first rule is called *de tri* by the Italians for shortness, that is "of three", because it contains three numbers; golden, because it is very fitting to many questions; useful, because several other rules can be reduced to it as their first principle; it is also called "of proportionals". [...]

An explanation of how to organize the numbers in a square follows, and a single example.[517] The second rule, dealt with similarly, is the inverse rule of three, the third is the rule of five, adequately identified as "composite rule of three". The fourth is the

---

516 In the process, certainly, but in a slow process. In 100 German school regulations from the 14th and 15th centuries, "calculation is mentioned in only two cases: in the municipal school in Wesel (1494) and in the Latin school in Nabburg in Oberpfalz, where according to the school regulation from 1480 exercises in calculation should be held at holidays and at other appropriate occasions'" [Vogel 1949: 239].

517 The Arabic names for the four numbers are indicated, agreeing with what is found in Robert of Chester's translation of al-Khwārizmī [ed. Hughes 1989: 64], in a spelling that coincides with that of the Dresden manuscript C 80 (below, p. 362).

partnership rule, and the fifth the same with differing durations.

The sixth deals with mixing in given proportion, the seventh with simple interest in a rule-of-five setting.

Unnumbered is a variant of the "hundred fowls", without the request for an integer solution and thus treatable by the partnership rule. The eighths explains the kind of proportional sharing where the promised shares amount to more than the totality (the "fallacious sharing" of the Pisa *Ragioni*, cf. p. 166); the ninth is the twin problem, the tenth the unknown heritage, the eleventh a pursuit problem with constant speeds, and the twelfth a corresponding meeting problem.

After these rules, claimed to come from the rule of three (evidently not true in the case of the unknown heritage), three more are added – a meeting problem with one participant going with constant, the other with arithmetically increasing speed; a question of combined works; and a tree problem (formulated about a tower). As we see, all problems are familiar from the abbacus tradition, and apart from the unknown heritage (which is solved by means of the usual unexplained rule) all are quite simple (alloying is missing). One innovation will be taken note of: in two of the additional problems, the persons involved are designated by letters, not by the usual "one/the other" or "the first/second/third/fourth". I know of no earlier Italian parallels, but below (p. 358) we shall encounter the same phenomenon in a slightly earlier German print. The idea was taken by other writers, and came to influence the teaching of elementary practical arithmetic for the next half-millennium, as epitomized in Stephen Leacock's pearl about "A, B and C: the Human Element in Mathematics" [1919: 237–245].[518]

The Regensburg *Practica*, Friedrich Amann – and Regiomontanus

Wolack was not the first German-Latin writer to be interested in the mathematics of Italian trade and traders; already around 1450, an unidentified monk at the St Emmeran Benedictine monastery in Regensburg added to an algorism of the traditional Latin type

---

[518] The beginning deserves to be quoted:

> The student of arithmetic who has mastered the first four rules of his art, and successfully striven with money sums and fractions, finds himself confronted by an unbroken expanse of questions known as problems. These are short stories of adventure and industry with the end omitted, and though betraying a strong family resemblance, are not without a certain element of romance. The characters in the plot of a problem are three people called A, B, and C. The form of the question is generally of this sort: "A, B, and C do a certain piece of work. A can do as much work in one hour as B in two, or C in four. Find how long they work at it".

When I was in high school (and thus beyond the reverberations of abbacus mathematics), our mathematics teacher once read it aloud to us – in the last lesson before Christmas, I believe.

a *Tractatus de practica*. It exists in several manuscripts of different length;[519] they are mostly in Latin, but some problems (concentrated in one of the manuscripts) are written in German.

The *Practica* [ed. Vogel 1954: 27] begins almost like Wolack:

> Thus with the help of God the Lord we happily reach from the aforesaid [the algorism] the *practica*. First thus the rule of proportionals, which by the geometers is called golden, by the Italians truly the rule of three [*regula de tre*].

After that, the two texts do not have much in common. They share the tree-problem, having the same numbers, but with the difference that the *Practica* [ed. Wappler 1900: 54; ed. Vogel 1954: 39] speaks of a pole (*falanga*), which makes it more plausible that $^1/_4$ of it is in the ground, but makes the total length of $181\,^9/_{11}$ feet less plausible; they also share the meeting problem with arithmetically increasing speed [ed. Wappler 1900: 53; ed. Vogel 1954: 43]; that they share the unknown heritage [ed. Wappler 1900: 52; ed. Vogel 1954: 64] with the usual fraction $^1/_{10}$, and also the twin problem [ed. Wappler 1900: 52; ed. Vogel 1954: 209] with ratio 1:2 is hardly significant.

The similarity between the introductions shows beyond doubt that the two text are connected. However, the small overlap of the problems seems to exclude that Wolack has drawn on the *Practica*, the connection is likely to go through a shared source, which suggests that the *Practica* builds on still earlier material that was present in the German area.

When we consider the 354 problems of the *Practica* it is obvious that it draws heavily on the Italian tradition – not because similar problems were not known in earlier and more distant cultures but because the German mercantile environment was not in direct contact with these; two problems [ed. Vogel 1954: 67] are borrowed from monastic recreational culture – first this one:

> One goose speaks to the other geese: I greet you, all 30 geese! One goose speaks, we are not 30, if we were as many more as we are, then once more as many and the half part as many, then we would be 30. [...]

The problem is solved by means of a single false position (not identified by name), and leads to fractional geese. No less than seven problems in the *Propositiones ad acuendos iuvenes* ([ed. Folkerts 1978: 45*f*, 69, 71*f*] – above, n. 36, pp. 40 and 103) ascribed to

---

519 Described and edited in [Vogel 1954: 14–25]. The two principal manuscripts are Munich, Clm 14783, written by Friedrich Amann in 1449–1460, with an addition from 1456; and Clm 14908, in which Amann added more material. In both manuscripts Amann has copied many other works that have nothing to do with the algorism or the *Practica*.

See [Gerl 1999: 1] for the reasons to ascribe these and various other manuscripts to Friedrich Amann and not, as once supposed, to Friedrich Gerhart.

Alcuin are of this type (but all of these have integer solutions).

A large part of the problems in the *Practica* are commercial, but we also find widespread recreational types like give-and-take and purchase of a horse. A number of problems refer to Italian locations and currencies, but many are adapted to local circumstances. Most striking is that lenders are supposed to be Jews, which suggests that the ecclesiastical condemnation of interest-taking (which would only affect Christians) was more efficient in Bavaria than in Italy.[520]

The double false position is dealt with, but with a small exception (below, note 525) not algebra. Outside the part of the manuscript Clm 14908 that is dedicated to the *Practica*, however, several sections deal with the topic.[521]

(a) Fols 133^v–134^v contain an introduction to algebra in German, written by Amann in 1461. It explains what "Machmet" says in the book *Algebra und almalcobula*, namely what *census* (afterwards translated into German as *zins* and as *zensus*), *radix* (*wurcz*) and *numerus* (*zal*) stand for, and then repeats the six basic cases in agreement with al-Khwārizmī's order. There can be no doubt that he draws on either Gerard's or Guglielmo's translation of al-Khwārizmī (Robert uses *substantia* instead of *census*); but the orthography *zensus* shows he must also know about the north-Italian tradition (directly or indirectly from earlier German writings that have been lost). No abbreviations are made use of, nor a fortiori any symbolism.

(b) Fols 136^v–146^v contains another introduction, also in German, *Regule dela cose secundum 6 capitula;* it is written by a different hand, except for the control of the example of the fifth case, which is found already on fol. 136^r in Amann's hand. It is thus due to a collaborative effort, in which Amann took part.[522] Here, the basic entities are *numero*,

---

520 In any case, it was no idiosyncracy of Amann's. In 16th-century *Rechenbücher* there was a strong predilection for pushing in a Jew when composite interest was explained – cf. [Ober 1545: Cviii^v, Hvi^r]; [Ulman & Thierfelder 1564: Hvi^r]; and [Weber 1583a: 71^r]. [Schulz 1600: 109^r] gives the section on composite interest the headline *Juden Rechnung*. Only [Gleydtsmann 1600: 179] refers to the creditor simply as *einer*, "somebody".

521 For the following description I rely,

1. on the digitized manuscript (https://www.digitale-sammlungen.de/en/view/bsb00103422, accessed 30.9.2021);
2. on Maximilian Curtze's edition [1895] of the algebraic material contained in the manuscript;
3. on Vogel's description of the manuscript [1954: 12–19].

522 Since this control is found on the recto page of a sheet where these *Regule dela cosa* begin on the verso, the only reasonable explanation is that Amann has discovered that it had been forgotten and then used a free page to insert it. This proves – if need should be – that the two are copying from the same already existing model.

cosa, and *censo* (the former two also appearing as *zal* and *ding*);[523] the cases are in abbacus order as we know it since Jacopo, the rules start by a normalization, and al-Khwārizmī is not mentioned. Here, the background is Tuscan – a north-Italian background would have given *cossa* and *zenso*. From fol. 140^{v} onward, *ding* is mostly abbreviated (a stylized *d* – henceforth I shall use ∂), often superscript. On fols 143^{r-v}, some formal fractions are used, but in a way that indicates that the writer does not understand too well: $\frac{100}{1\partial}$ becomes $\frac{100}{1}\partial$, $\frac{100}{1\partial\ 5}$ becomes $\frac{100}{1}\partial\ 5$ (Curtze, mercifully, corrects).[524]

(c) Four geometric problems in Latin (fols 90^{r-v}, thus much earlier in the manuscript and in a section dated 1459) – make use of second-degree algebra for the solution of problems involving the Pythagorean rule.[525] Vogel [1954: 73*f*] included them in his edition of the *Practica*. They designate the first power of the unknown *radix* and the second power *census*. For the *census*, the abbreviation is used – most likely an abbreviations for *ce*, it was also the standard abbreviation for the unit *centenarius*.[526] The *radix* (both when standing for the first power of the unknown and when a square root is taken) appears as . This does not look much as the "ρ" used in these Florentine treatises, but we shall soon encounter a (just) possible link.

(d) Fols 134^{v}–135^{r} (in German),[527] some problems and the statement of the fifth and sixth algebraic cases make use of *zensus* and *cosa*, abbreviating the former ʒ, the latter *co* – mostly written superscript, as already Vat. lat. 10488 had done (while using ^{cen} for the *censo*).

(e) Fols 146^{v}–153^{r} contain four problems in Latin – the last is a remainder problem, the former three algebraic problems in commercial dress. The powers of the unknown are *res* and *census*, unabbreviated, but formal fractions are made use of, this time well-shaped. There are references to the rules of *algebra*/*algebra arabis* and to the Euclidean notion of binomials.

---

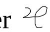
523 A list of rules for products of powers shows that the fourth power is *censo de/di censo*, the fifth *du*^{x} *cubo* (i.e., *duplex cubo*, replacing the *cubo relato* with which we are familiar), and the sixth a multiplicative *cubo di cubo*.

524 Similar traces of failing understanding are also found in certain abbacus books. The *Libro di conti e mercatanzie* from *ca* 1395 (above, p. 30) writes "$\frac{1}{1\ \textit{thing}}$ and 5" instead of "$\frac{1}{1\ \textit{thing}\ \text{and}\ 5}$".

525 All four are of traditional types. Two deal with a tree of known length that either breaks at a known height or whose summit then falls at a given distance when the tree breaks; and two with a ladder first standing against a wall and then sliding down.

526 Amann uses it in the latter function in Clm 14783 (fols 433^{r}, 439^{v}); in Munich, Clm 14111, fol. 306^{v}, it turns up in a problem added (according to [Vogel 1954: 21]) by Hermann Poetzlinger to his selection of problems from the *Practica*.

527 First discussed by C. Immanuel Gerhardt [1870: 142*f*].

(f) Fols 153^{v}–154^{v}, also in Latin with a heading *De regulis per algebram, etc.*, "On the rules by algebra", solves nine first-degree problems (give-and-take, etc.) by means of algebra. The *census* obviously does not occur; the unknown itself is invariably abbreviated, so we cannot know whether *res*, *radix*, *cosa* or (less likely) *ding* was intended. But the abbreviation itself occurs with two shapes,  alternating with the simpler , which *could* have been derived from the shape which the Tuscan "ρ" often took on but rather represents a contracted *re[s]*, as suggested by Kaunzner [1968: 121 n. 17].

Also of interest in this section is the use of letters to designate persons in two of the problems.

(g) Amann continues on fols 155^{r}–156^{v} in a mixture of Latin and German, still with commercially dressed problems: purchase of a horse, give-and-take, etc.; one is of the second degree and correctly described as belonging to the fourth rule. The rules are now spoken of as *dela cosa*, as in section (b), not as "by algebra". This, as well as a new heading, and the change of language, indicates that this is really a new section. The unknown is still not spoken of in a full word, but it is now abbreviated ∂, while the second power (in plural form) is *censy* – both choices also pointing to section (b). A final point of contact with (b) is a badly understood formal fraction on fol. 156^{v}: it should be $\frac{80}{140\ \ 2\partial}$, but Amann first writes "$\underline{80}$ 20∂" and afterwards, discovering that he has forgotten "140", writes it under the line with an $_{\wedge}$ indicating where it is to be inserted (which, if done, would produce "$\underline{80}$ 140 20∂").

All in all it is evident that Amann, with his occasional collaborator, knows that algebra is a branch of mathematics that has to be assimilated, and collects material of disparate origin in the manuscript; but he appears not to have understood everything to perfection – which, given the kind of material we can see to have been at his disposal, would indeed have been a miracle. The vacillating terminology and abbreviations reflect, on one hand, the use of Latin as well as vernacular sources – the latter mainly Tuscan but also North Italian; on the other the lack of agreement even in the Italian vernacular environment about how to abbreviate operations and powers.

Occasionally – for example, in [Kaunzner 1980: 135] – Regiomontanus's use of algebraic abbreviations or symbols is connected to what we have found from Amann's hand, so it may be worthwhile to take a look at what the great astronomer does and appears to know.

As mentioned above (p. 283), Regiomontanus solves a problem about dividing a certain dividend first by one divisor and then by another one exceeding the former divisor by a known amount exactly as does the Ottoboniano *Praticha* – the only difference being that Regiomontanus increases the divisor by 8 instead of 7; the dividend is still 100, and the sum 40. He does so in a private calculation used for a letter to Giovanni Bianchini [ed. Curtze 1902: 235][528] – undated but answering one from Bianchini which he had received 11 February 1464.

Regiomontanus calculates like this:

$$\frac{100}{\rho} \qquad \frac{100}{1\ \rho\ \textit{et}\ 8}$$

$$\begin{array}{l} 100\ \rho\ \text{et}\ 800 \\ \underline{100\ \rho\qquad\qquad} \\ \dfrac{200\ \rho\ \text{et}\ 800}{1\ \sigma\ \text{et}\ 8\ \rho} \text{ —— } 40 \end{array}$$

$$\begin{array}{l} 40\ \sigma\ \text{et}\ 320\ \rho \text{ —— } 200\ \rho\ \text{et}\ 800 \\ 40\ \sigma\ \text{et}\ 120\ \rho \text{ —— } 800 \\ 1\ \sigma\ \text{et}\ \ 3\ \rho \text{ —— } 20 \end{array}$$

ρ here renders Regiomontanus's , *res*/*cosa*, while σ stands for his , *census*; the long stroke still functions as equation sign. When used elsewhere in the calculations (in the four pages rendered in facsimile in [Kaunzner 1971]), "less" is mostly abbreviated , probably meant as *[m]inus*; but we also find , probably meant as *minus*.[529]

There can be no doubt that Regiomontanus has learned from something like the model on which the Ottoboniano *Praticha* was based (in the latter the problem was ascribed to Giovanni di Bartolo, we remember). Bianchini, when stating his question, may have done so too; however, since the *problem type* had circulated in the abbacus environment since Gherardi's time this is far from certain.

Apart from that we observe similarity in the choice of symbols with those used by Amann in his section (c) in 1459.

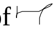
In the *De triangulis* II.xii and II.xxiii, Regiomontanus uses algebra to solve geometric problems.[530] In Johannes Schöner's printed edition [Regiomontanus 1533: 51, 56], the exposition is purely rhetorical, but in the manuscript[531] marginal calculations[532]

528 Facsimile reproduction in [Kaunzner 1980: Abb. 6]. Gerl [1989: 108] states that this sheet is difficult to read, and that use of an ultra-violet lamp allowed him to improve some of Curtze's readings, but only publishes the astronomical calculations. According to the facsimile, Curtze's edition is impeccable in the actual calculation.

529 We also encounter shapes that could be meant as , close to $\widehat{m}$ (regularly found in Italian manuscripts) but could also be a ligatured pen variant of the first shape. The facsimiles do not allow to distinguish, only inspection of the manuscript might perhaps decide.

530 According to [Zinner 1990: 52, 65], book II was probably written in Spring 1463, thus before the letter to Bianchini.

531 Facsimile [Kaunzner 1980: Abb. 2, 3].

make use of  for *res* and  for *census* – both superscript (on one point writing instead ) – both fairly similar to what Regiomontanus had done in the Bianchini calculations, the *census* also to its shape in Amann's section (c), while the *res* can with good will be considered a variant of the secondary shape in his section (f); possibly, they are Regiomontanus's own contractions of *re[s]* and *ce[nsus]*.

The last piece of evidence concerning Regiomontanus's practice of algebra is the manuscript New York, Columbia University, Plimpton 188 (more precisely its first part, which at some later moment was bound together with other documents [Folkerts 1980: 190].

This first part, datable to 1456, was once in Regiomontanus's possession and annotated by him. It is itself composite, the first section being a not quite complete copy of Jean de Murs' *Quadripartitum numerorum*, the second a copy of Gerard of Cremona's translation of al-Khwārizmī's algebra. A third part is clearly related to Amann's section (b).[533]

The annotations to the first two sections show that Regiomontanus, coming out of the university tradition, preferred the classical methods – he often shows how a result reached by algebra can alternatively be grounded in geometry or arithmetic; algebra was obviously not his foremost mathematical tool. That confirms what we can read out of *De triangulis:* he has recourse to algebra when being unable to find a geometric way, *hoc problema geometrico more non licuit hactenus* [Regiomontanus 1533: 51].

Most informative is the third section, which begins with the heading *Regula de cosa et censo sex sunt capitula, per que omnis computatio solet calculari*, "six are the chapters

---

532 These were seemingly added after the main text was written, but we may presume that the calculations were first made on a separate sheet or a slate, as also suggested by the mistake to be mentioned.

533 Paul Lawrence Rose [1975: 93, 112 n. 33] asserts that this part is in Regiomontanus's hand; as Folkerts [1980: 200] points out on the basis of the different ways to write the numerals 4 and 7 in the text and in the annotations, this is far from certain (but according to [Zinner 1990: 31] it *could* mean that the annotations were written after 1459, while the copy itself was made in 1456 or before). The former text was definitely not copied by Regiomontanus. The third section is the one which carries the date 1456 on fol. 85^{r}. Here, the shapes of 4 and 7 are in the style Regiomontanus used by then.

Unfortunately I have only been able to control the few pages (from the present three sections, the first page of each) made available at

https://digital-scriptorium.org/xtf3/search?rmode=digscript;smode=advanced;
field1=shelfmark;term1=Plimpton%20188;join1=token;operator1=and;field2=text;
join2=token;operator2=and;field3=text;join3=token;datetype=range;docsPerPage=1;
startDoc=1;fullview=yes (accessed 7.10.2010).

for the rest I have had to rely on the secondary literature I refer to.

of *cosa* and *census*, by which every calculation is habitually solved". It then gives the basic six rules, in abbacus style (starting with a normalization) and order, each (at least the first two, on fol. 85ʳ) followed by an example with discussion. After the first example rules for the multiplication of powers are inserted, whose most noteworthy feature is that the name for the fifth power is *duplex cubus* (cf. above, note 523).

There can be little doubt that whoever wrote this – probably Regiomontanus, see imminently – used a model descending from the abbacus tradition; an independent compilation produced piece-wise by Regiomontanus would have no reason to restart with basics after having presented them in section 2, and to change the terminology from *res* to Tuscan *cosa*. We may guess that this model was already produced in a German university environment, but we cannot be sure – Bianchini's *Flores almagesti*, written at the Ferrara court around 1440 or after [ed. Heeffer 2015] is also in Latin (so different, however, that a link or just inspiration can be excluded – it was indeed only in 1464 or later that Regiomontanus annotated a copy of that work).

On the first page of this section, "less" is abbreviated four times as and twice as . *cosa* is sometimes written in full, sometimes abbreviated (once superscript, once not).[534] *census* is not abbreviated on this page, but according to [Folkerts 1980: 201] later as superscript , while *radix* (meaning square root) is written ℞, as done by abbacus writers since long. Since Folkerts asserts that all these abbreviations are used generally in all Regiomontanus's algebraic writings, which is clearly not the case, one may suspect Regiomontanus's usage to be unsystematic not only on the first page but throughout the third section.

All in all, there is no reason to see Regiomontanus as the inventor of algebraic symbolism. Mostly we find less than systematic use of abbreviations. In the Bianchini notes and the calculations accompanying the theorems of *De triangulis*, they serve in schemes, one of which (as we have seen) is clearly taken over from an Italian model, while the others have at least adopted a borrowed principle (maybe more even there, maybe not). The material collected in the three sections of part 1 of Plimpton 188 are evidence, first of all, of all-devouring appetite for learning about a new branch of mathematics (with which, as the annotations to section 1 show, he was not yet familiar); in this situation there is no reason to reproach him, nor to hide, that he was an eager eclectic learner, and no systematizer.

Nor is there any reason to believe that Amann should have learned his algebra or algebraic notation from Regiomontanus, as sometimes supposed. The most explicit formulation of the view may be [Folkerts 2006: 8], central to which is the similarity between the heading of section 3 of the Plimpton manuscript and that of section (b) of

---

[534] The lack of system is illustrated by how 2 parts into which 10 is split are written:

1 10 1 *cosa*

Clm 14908, the former

> *Regula de cosa et censo sex sunt capitula, per que omnis computatio solet calculari,*

the latter

> *Regule dela cose secundum 6 capitula, und mit den selben capitel mag man alle rechnung machen.*

Obviously, the two headings are close relatives; but firstly, as we remember, section (b) was *not* written by Amann, even though he must have known the model from which it was copied, since he can provide a proof that was omitted by the copyist. So, if this was copied from Regiomontanus's text, *both* of them must have had access to it. It is not sufficient that Amann *perhaps* visited Vienna in 1456 and there *perhaps* met Regiomontanus.[535] Further, if copying, the anonymous must have known that *de cosa* should be changed into something Tuscan, almost succeeding with his *dela cose*. So, he must in any case have had more information than could be glanced from Regiomontanus's text. At the same time the two texts are too close to each other to allow a hypothesis that what was written in Clm 14908 in 1461 was based on memory of something the writer had seen five years before.

Moreover, as we remember, section (b) used ∂ for the *ding*, nothing like Regiomontanus's . Any idea that Amann and his collaborator learned from Regiomontanus about the *possibility* to use abbreviations for algebraic unknowns can be discarded, since they had been used for well over a century by many abbacus writers.

All in all, the beginning of *Rechenmeister* mathematics in German area (including *Rechenmeister* algebra) was the outcome of multiple exchanges, not of the activity of a single genius – just as the emergence of abbacus mathematics in Tuscany and Lombardy in the late 13th and the early 14th century.

Two manuscripts, a university lecture, and the first prints

Printing came later than the new mathematics in Germany – speaking of it at its beginning as *Rechenmeister* mathematics may be even be a misnomer, since the *Rechenmeister* profession too was still to emerge and hardly yet in possession of a characteristic mathematical style or culture. At first we shall therefore look at another manuscript, Dresden, Sächsische Landesbibliothek, C 80 – consisting of many parts of disparate origin, and once in the possession of Johannes Widmann (*ca* 1462–1498 or later), on whom below.

---

[535] Actually, in 1456 both were primarily interested in astronomy, and the 20-year old Regiomontanus was famous only as a calendar-maker and as an astrologer – cf. [Zinner 1990: 31–44] ; so, *if* they met and discussed, the discussion would hardly have concerned Regiomontanus's attempt to learn about algebra. Moreover, according to Ernst Zinner, there are no traces in Amann's writings that he knew any Vienna astronomy postdating Johann von Gmunden, who died in 1442.

The manuscript is one of those that contain Wolack's lecture (fol. 303$^{\mathrm{v}}$–303$^{\mathrm{r}}$). It is also one of the three manuscript that conserve Robert of Chester's translation of al-Khwārizmī's algebra (fols 340$^{\mathrm{r}}$–348$^{\mathrm{v}}$). Of interest for our present concern is that all three of these were made in south-German area around 1450 [Hughes 1989: 11]. The present copy is incomplete, interrupting before the presentation of the rule of three (which is found separately elsewhere in the manuscript); the others[536] contain an appendix listing the six basic algebraic cases in abbacus order and style (i.e., having a division as first step of the rules). They also use abbreviations for *dragma*, *cossa*/*radix*[537] and *zenso*. Hughes printed rendering does not show their shape unambiguously, but a marginal annotation on fol. 1$^{\mathrm{r}}$ in the Vienna manuscript (in the same hand as the text) which it reproduced in the edition represents the first power by ; the second power is in any case derived from *z* – probably something like  ( as well as  are used in C 80, fol. 350$^{\mathrm{v}}$). Both *cossa* and *zenso* belong to the northern part of Italy, and none of them seem to turn up in abbacus writings before the 15th century.[538] There is thus no reason to ascribe this appendix to Robert, as occasionally suggested.[539] But it is evidence that Robert's translation reached Germany in the company of algebraic material from northern Italy – thus evidence of a further component of the network linking the emergence of *Rechenmeister* mathematics to Italy.

The two constituents of the Dresden manuscript that have most to tell about the import of abbacus algebra into German lands are a *Latin Algebra* and a *German Algebra* (both anachronistic titles under which they are known today, none is found in the manuscript). An edition of the former (fols 350$^{\mathrm{r}}$–364$^{\mathrm{v}}$) was prepared by Wappler in [1887: 11–30], one of the latter (fols 368$^{\mathrm{r}}$–378$^{\mathrm{v}}$) by Vogel in [1981].

The *German Algebra* on its own is evidence of the complex reception: the number term in equations are designated in some five different ways (two of them "symbols", with variations), the first power in equally many, the second to fourth in two to four each – see the scheme in [Vogel 1981: 11]. This is not due to the involvement of two mathematically incompetent copyists [Vogel 1981: 18]; both of them draw on a variety of sources, that in part point to strains in the Italian tradition that we have not encountered

536 Vienna, Österreichische Nationalbibliothek, cod. 4770, fols 1$^{\mathrm{r}}$–12$^{\mathrm{v}}$, and Trier, Stadtbibliothek, cod. 1924/1471, 393$^{\mathrm{r}}$-400$^{\mathrm{v}}$.

537 The Trier manuscripts has both, each in a single plene writing – see [Hughes 1989: 67, apparatus].

538 Dardi, for instance, writes *cosa* and çenso; that at least *çenso* is no invention of his copyists is confirmed by his abbreviation *ç* (above, note 289).

539 One *might* indeed read this out of [Hughes 1989: 26], but I doubt that this is what is intended. Vogel [1981: 11] also lists these abbreviations as belonging to Robert; one has to go to the note on the next page to see that what is meant is an appendix to the Robert text, not in that text itself.

in the preceding discussion of German writings – thus *wurczell von der worczell* ("root of the root") for the fourth power (fol. 368^{r}) and *radix rellata* for the fifth (fol. 373^{r}); further repeatedly from fol. 371^{r} onward the pseudo-fraction notation which we know from Dardi (above, p. 220), where the apparent denominator is meant as a denomination (at times with the misunderstanding that instead the "denominator" is the coefficient and the "numerator" the denomination).

There is no need to go into detail, full documentation is presented by Kurt Vogel in his commentary.

The *Latin algebra* is rather different though still (as we shall see) an eclectic conglomerate. Most remarkable is a systematic notation where the first five powers (starting with power 0, the unit for numbers) are written ∅, ꝛ, ʒ, ꝯ, and ʒʒ. *Radix*, "[square] root", is abbreviated ".", which fortunately does not also serve as punctuation;[540] it is meant to be taken of the ensuing power with coefficient.

The *Latin Algebra* begins (fol. 350^{r}, ed. [Wappler 1887: 11]) by showing how to reduce equations with two or three members to the lowest possible powers;[541] in this connection it is explained that three-member equations are not dealt with in the algebraic cases unless the middle [power] is equally long from the extremes (as we have also seen in Pacioli).

Then follow rules for how the reduced equations are to be solved [ed. Wappler 1887: 11]:

> When ꝯ are made equal[542] to ∅, then the ∅[543] is divided by the ꝯ, the cube root of the outcome is the value of the thing [*res*].
>
> When ʒʒ are made equal to ∅, then the ∅ is divided by the ʒʒ, the square root of the square [root] of the outcome is the value of the thing.
>
> When two neighbouring signs [*signa*, i.e., powers] are made mutually equal, then the minor

540 Neither as a "full stop" separating sentences nor to set off numbers from words, cf. above, note 343.

541 Evidently, this insight had been present in abbacus algebra since the very beginning – but making it explicit was fairly new. It is also done in Modena, Biblioteca Estense, Ital. 578, fols 7^{v}–8^{r} (dated *ca* 1485 by Van Egmond [1980: 171] on the basis of watermarks. See [Høyrup 2019a: 858–860].

542 *assimilatur*, a term not borrowed from any of the Latin translations of al-Khwārizmī's algebra, but shared with the appendix added to the Robert translations [ed. Hughes 1989: 67], which also makes use of the same abbreviations.

543 As we see, the coefficients are understood in the expressions like "the ∅" (where the article, expressed in Italian works, is by necessity absent in this Latin text); on this account, Dardi was more explicit than the present writer.

sign is divided by the major, and the value of the thing will be known.

When two signs are made mutually equal, between which one sign is in between in the series of signs, then the minor is divided by the major, the square root of the outcome is the value of the thing.

[similarly for two or three intervening powers]

When three signs are made mutually equal, that is, when three signs are put into an equation, then they are divided singly by the maximal of the three signs, and the medium sign afterwards is halved; and the half is multiplied in itself. This, however, is done in three ways [...].

[these are then explained[544]].

Paradigmatic examples for 18 cases with indicated solutions (not the usual abstract rules) follow on fols 350^{v}–351^{r} [ed. Wappler 1887: 12*f*] (the six basic cases are omitted) – in order to facilitate comparison I shall use the same notation as above (conserving the explicit *dragma*; the indication of addition by "+" belongs in the manuscript[545]) and start numbering with 7:

$LA_1 7$	$ZZ = 3K$; $t = 3$	$LA_1 16$	$1K+5t = 6C$; $t = 5$
$LA_1 8$	$1CC = 16C$; $t = 4$	$LA_1 17$	$CC+3K = 10t$; $t = 2$
$LA_1 9$	$1CC = 8t$; $t = 2$	$LA_1 18$	$CC = 5K+6C$; $t = 6$
$LA_1 10$	$1CC = 81Ø$; $t = 3$	$LA_1 19$	$CC+3C = 4K$; $t = 3$
$LA_1 11$	$1K = 6C$; $t = 6$	$LA_1 20$	$2C = \sqrt{16C}$; $C = 4$, $t = 2$
$LA_1 12$	$1K = 25t$; $t = 5$	$LA_1 21$	$1C = \sqrt{8t}$; $t = 2$
$LA_1 13$	$1K = 64Ø$; $t = 4$	$LA_1 22$	$1CC+3C = 108Ø$; $t = 3$
$LA_1 14$	$1K+2C = 15t$; $t = 3$	$LA_1 23$	$1CC+16Ø = 17C$; $t = 4$
$LA_1 15$	$1K = 3C+4t$; t = 4	$LA_1 24$	$1CC = 8C+9Ø$; $t = 3$

The copyist (thus informing us that he *is* a copyist mainly following a model) next explains to have found elsewhere as no. 15 (with subsequent numbers thus increased by 1) a case "..12ʒ = 1ʒ+3∅", with solution ꝛ = 3 – later when explained instead appearing as ".12ʒ = 1ʒ+3∅"; the solution should obviously be ʒ = 3, as found correctly but restated wrongly in the explanation that follows.

An explanation is given not only for this extra case but also for cases 14 and 15 –

---

544 In the fifth case the double solution is forgotten.

545 From no on, this symbols entered common use in German writings, not least perhaps because of its use in Widmann's *Behende und hubsche Rechenung auff allen kauffmanschaft* from [1489].

However, we should not fully identify it with the modern arithmetical operator; it also functions as an abbreviation for *et*, "and", as we can see from these passage on fol. 352^{r} of the Dresden manuscript: *resultans in se multiplicare + a* ꝛ *subtrahere*, "multiply the outcome in itself + from ꝛ subtract", and *multiplica in se, + sunt* $^{25}/_{36}$, ",multiply in itself, + they are $^{25}/_{36}$".

those where square roots appear and which therefore cannot be reduced directly to one of the six fundamental case.

This sequence of cases seems not to be known as a whole from Italian sources; the basis idea to let "$\alpha CC = \beta K$" precede "$\alpha CC = \beta C$", etc., is shared with Biagio (above, p. 211); Benedetto follows the principle for the cases involving *CC* but not for those involving *K*, see note 284 – the similarity may hence be accidental. Equations with radicals, on the other hand, might make us think not so much of Dardi's overwhelming exploration of that topic as of the scattered examples of simpler cases involving radicals found in other Italian manuscripts (above, p. 225). However, even on this account there is no strong evidence for any link, independent exploration of such possibilities after the migration of the algebraic technique is quite possible.

After this comes, on fols $351^{\mathrm{r-v}}$ [ed. Wappler 1887: 13–15], a *different* set of 24 cases – even this one different from anything known from the abbacus traditio. This set (now including the basic six cases) is stated as explicit rules and initially explains what would have belonged at the very beginning if this *Latin Algebra* had been coherent treatise and not a conglomerate: namely the meaning of *numerus*, *cossa* and *census* (not, however, identifying the appurtenant abbreviations but going on using them afterwards).

$LA_2 1$	$N = \alpha t$	$LA_2 13$	$\alpha CC = \beta K$
$LA_2 2$	$N = \alpha C$	$LA_2 14$	$\alpha CC = \beta C$
$LA_2 3$	$\beta t = \alpha C$	$LA_2 15$	$\alpha CC = \beta t$
$LA_2 4$	$N = \alpha C+\beta t$	$LA_2 16$	$\gamma C = \beta K+\alpha CC$
$LA_2 5$	$\beta t = \alpha C+N$	$LA_2 17$	$\beta K = \gamma C+\alpha CC$
$LA_2 6$	$\alpha C = \beta t+N$	$LA_2 18$	$\alpha CC = \beta K+\gamma C$
$LA_2 7$	$\alpha K = \beta C$	$LA_2 19$	$\alpha C = \sqrt{}\beta t$
$LA_2 8$	$\alpha K = \beta t$	$LA_2 20$	$\alpha C = \sqrt{}\beta C$
$LA_2 9$	$\alpha K = \beta t$	$LA_2 21$	$\alpha CC = N$
$LA_2 10$	$\gamma t = \beta C+\alpha K$	$LA_2 22$	$N = \beta C+\alpha CC$
$LA_2 11$	$\beta C = \gamma t+\alpha K$	$LA_2 23$	$\beta C = N+\alpha CC$
$LA_2 12$	$\alpha K = \gamma t+\beta C$	$LA_2 24$	$\alpha CC = N+\beta C$

While the preceding examples were all normalized, the present rules presuppose (in abbacus style) that the equations are non-normalized, and therefore have a division as their first step. The first six cases, we observe, are in abbacus order. $LA_2 5$ contains a strange mistake: instead of explaining the double solution, it states that the solution (presupposing $\alpha = 1$) is

$$x = \frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 - \varnothing} \quad ,$$

adding afterwards that "if $\varnothing$ cannot be subtracted it should be added". Similarly in the derived cases $LA_2 11$ and $LA_2 17$.

Then, fols 351^{v}–352^{r} [ed. Wappler 1887: 15*f*], follows yet another set of rules, now

16, under the title *Compendium de* ʒ *et re*, "compendium about ʒ and *thing*" – now with numerical examples:

$LA_3 1$	$\alpha t = N$	$LA_3 9$	$\alpha K+\beta C = \gamma t$
$LA_3 2$	$\alpha C = N$	$LA_3 10$	$\alpha K+\gamma t = \beta C$
$LA_3 3$	$\alpha C = \beta t$	$LA_3 11$	$\alpha K = \beta C+\gamma t$
$LA_3 4$	$\alpha C+\beta t = N$	$LA_3 12$	$\alpha CC = N$
$LA_3 5$	$\beta t = \alpha C+N$	$LA_3 13$	$\alpha CC = \beta t$
$LA_3 6$	$\alpha C = \beta t+N$	$LA_3 14$	$\alpha CC = \beta C$
$LA_3 7$	$\alpha K = N$	$LA_3 15$	$\alpha CC = \beta K$
$LA_3 8$	$\alpha K = \beta t$	$LA_3 16$	$\beta K = \alpha CC+\gamma C$

This time the selection and the order are extremely traditional. The list coincides with Jacopo's cases Ja1–Ja18 (above, pp. 188 and 191), omitting only Ja9 ($\alpha K = \beta t$) and Ja17 ($\alpha CC+\beta K = \gamma C$).

Fols 352^v–364^v [ed. Wappler 1887: 16–30] contain illustrating examples (*aporismata*) for the set LA_2. The first case gets no less than 29 – all traditional recreational or number problems, many of type "divided 10", also some give-and-take. For the remaining basic cases, 4 to 8 are offered. Two of the illustrations of the fourth case deal with composite interest. Many of the solutions make use of formal fractions. After the examples for the fifth case comes (fol. 359^r, ed. [Wappler 1887: 26]) a somewhat better formulation of the principle of double solution: "observe that the fifth rule has this privilege over the others that when the root cannot be subtracted, then it should be added".

The following cases get one illustration each. $LA_2 19$ is merely illustrated by the simple equation, "1 ʒ is worth .8ꝶ"; similarly $LA_2 20$. All the others are formulated around the volume, surfaces and sides of a cube. There is nothing similar to the Italian construction of intricate versions for example of give-and-take problems (for instance, involving square roots or products) that lead to higher-degree problems, nor even examples of the fake-intricate problems around numbers with given ratio. It seems a reasonable assumption that the set of problems illustrating $LA_2 7$–24 have been newly produced in an environment that did not have access to Italian material – that is, in German area. It is thus the first evidence of independent productive work within the new discipline.

After two pages (fols 365^{r-v}) with further algebraic calculations follows (fols 366^{r-v}) an extract from what is claimed to be from "Master Campanus's cautions [*cautele*] from the book about Algebra or about *cossa* and *census*", making use of the same abbreviations for powers and likely to be linked to the rule set LA_2. The ascription to Campanus can be safely discarded – in his times (the mid-13th century), nobody would have referred to neither *cossa* nor *cosa*. After these two pages comes the *German Algebra*, already discussed.

The manuscript Vienna, Österreichische Nationalbibliothek Pal. 5277 contains on fols

2^{r}–33^{r} a section “Regolae Cosae vel Algebrae”, which has been edited by Kaunzner [1972] and was already presented by Gerhardt in [1870: 143–153]; its original predates 1521,[546] we do not know by how much. It makes use of the same abbreviations as the Dresden algebras until ȝȝ (except at times using *N* instead of ∅, at others writing it rather as ϕ), also within a systematic exposition of the arithmetic of polynomials (including that of formal fractions), that is, as genuine symbols on which the operations are performed. It also introduces a numbering of powers (identical with exponents, 𝓇 being the first, etc.[547]) and shows in a scheme how products of powers can be calculated from these. It uses + and – systematically, but not quite as we would do – after ȝȝ the sequence of powers go on *alt* (for *numerus altiore*), ȝ+ꝯ and ꝯ+ȝȝ (which does not make much sense, unless we consider ȝ+ꝯ an unfortunately systematic mistake for ȝ+ȝȝ). In the end of this section (fol. 11^{r}, ed. [Kaunzner 1972: 136]), the rule of three is presented, referring to the quantities involved as the third, the middle, and the first (nothing is said about similarity, which would indeed be out of place when these quantites are polynomials); a counterfactual example is given, “3𝓇+4ϕ are 6 –4𝓇, what are 5𝓇–6ϕ ?”; the outcome is given as an unreduced formal fraction, (30ȝȝ–20ȝ–36ȝ+24𝓇)/(3𝓇+4ϕ). I have never noticed anything similar in Italian sources (the closest being Dardi’s use of the rule of three for computation with *arithmetical*, not algebraic binomials – above, p. 218) and suppose we are confronted here with another innovation produced in the German environment.

Then, like the beginning of the *Latin Algebra*, the present treatise explains (fol. 11^{v}, ed. [Kaunzner 1972: 136]) how to reduce two- and three-member equations to the basic cases. A section with eight rules with examples follow, taking advantage of these reductions; for each rule, various cases that can be reduced to the basic type are included – for the first thus 3𝓇 = 6ϕ, 4ȝ = 8𝓇, 5ꝯ = 10ȝ, and three more. The rules thus reduced lead back to these:

$\mathrm{Wi}_1 1$	$\alpha t = N$	$\mathrm{Wi}_1 5$	$\alpha C+\beta t = N$
$\mathrm{Wi}_1 2$	$\alpha C = N$	$\mathrm{Wi}_1 6$	$\alpha C+N = \beta t$
$\mathrm{Wi}_1 3$	$\alpha K = N$	$\mathrm{Wi}_1 7$	$\beta t+N = \alpha C$
$\mathrm{Wi}_1 4$	$\alpha CC = N$	$\mathrm{Wi}_1 8$	$\beta C+N = \alpha CC$

The rule for Wi6 identifies the double solution correctly. As in the Dresden manuscript, many examples (no less than 27) are provided for the first rule, much fewer for the others

546 Errors that were subsequently corrected leave no doubt that the text is a copy – thus a meaningless *docebunt* which has then been rectified as valebunt [Kaunzner 1972: 130 n. 44].

547 This is thus more “modern” than Pacioli’s *Summa*, but could still be a fresh inspiration from Italy: the same principle is found in the Modena manuscript referred to in note 541. Evidently, independent discovery is just as possible.

(some in Latin, some in German, the latter almost exclusively about commercial problems). In the end of this section (fol. 30r–31v, ed. [Kaunzner 1972: 154–156]) we find examples for two rules that have not been enunciated in the previous scheme, $\alpha C = \surd(\beta t)$ and $\alpha C = \surd(\beta C)$ (which we have already encountered as $LA_2 19$ and $LA_2 20$); a general formulation for three-member equations similar to what is found early on in the *Latin Algebra* (above, p. 364); and a rule for $(n+At)/(\beta t) = m$.

And then in the end (fol. 32r, ed. [Kaunzner 1972: 156]), a list of 24 *Regulae cosse*, *apparently* different from any of the lists found in the *Latin Algebra:*

$Wi_2 1$	$N = \alpha t$	$Wi_2 13$	$\alpha K = \beta C+\gamma t$
$Wi_2 2$	$\beta t = \alpha C$	$Wi_2 14$	$\alpha CC = \beta K+\gamma C$
$Wi_2 3$	$\beta C = \alpha K$	$Wi_2 15$	$\beta t = \alpha C+N$
$Wi_2 4$	$\beta K = \alpha CC$	$Wi_2 16$	$\beta C = \alpha K+\gamma t$
$Wi_2 5$	$N = \alpha C$	$Wi_2 17$	$\beta K = \alpha CC+\gamma C$
$Wi_2 6$	$\beta t = \alpha K$	$Wi_2 18$	$N = \beta C+\alpha CC$
$Wi_2 7$	$\mathrm{B}C = \alpha CC$	$Wi_2 19$	$\alpha CC = \beta C+N$
$Wi_2 8$	$N = \alpha CC$	$Wi_2 20$	$\beta C = \alpha CC+N$
$Wi_2 9$	$N = \alpha t+\beta C$	$Wi_2 21$	$N = \alpha K$
$Wi_2 10$	$\gamma t = \beta C+\alpha K$	$Wi_2 22$	$\beta t = \alpha CC$
$Wi_2 11$	$\gamma C = \beta K+\alpha CC$	$Wi_2 23$	$\alpha C = \beta t$
$Wi_2 12$	$\alpha C = \beta t+N$	$Wi_2 24$	$\alpha CC = \beta CC$

As they stand, $Wi_2 23$ repeats $Wi_2 2$, while $Wi_2 24$ is either trivial or impossible, depending on the values given to the coefficients. Gerhardt [1870: 146 n.1] is doubtlessly right that $Wi_2 23$ is a mistake for $\alpha C = \surd(\beta t)$, while $Wi_2 24$ should have been $\alpha CC = \surd(\beta CC)$.[548] Once more we thus see $LA_2 19$ and $LA_2 20$ repeated, and with this repair the list Wi_2 coincides with LA_2 yet with a new ordering where equation types that coincide after reduction (in agreement with what was taught on fol. 11v) are brought together.

Fols 379r–384r contains treatises about the arithmetic of roots and of binomials – not reaching the depth of what we find in the Florentine encyclopedias but rather similar to Ghaligai's briefer exposition of what he has learned from his master del Sodo and from Benedetto (above, p. 334).

Taken together, however, the algebraic treatises contained in the Vienna manuscript offer evidence of a development from the algebraic eclecticism of Clm 14908 and the Dresden manuscript toward coherence and agreement about notations – not only can the first five "cossic numbers" (as they were to be called) ϕ, ꝛ, ʒ, ꝯ, and ʒʒ now be considered established; so were + and –. Moreover, the use of the algebraic notation within

548 Another piece of evidence that what is contained in the Vienna manuscript is a copy – the small dots standing for square root would easily be overseen.

formal calculations – as a genuine though still incomplete symbolism – was approaching the Italian level.

All of this concerns manuscripts and thus private study. In 1486 Widmann proposed and held a series of lectures over algebraic and related topics at Leipzig university [Gärtner 2000: 6, 34*f*]. According to the announcement, it covered "the 24 rules of algebra, and that which they presuppose" – specified to include algorism for fractions, ratios and surds. Since Widmann possessed and annotated the Dresden manuscript we may safely assume that the lectures were based on the *Latin Algebra* but also drew on other topics covered in the codex.[549] Judging from what he was to publish in print we may also be confident that he presented the matter coherently.

Widmann was indeed to become one of the first to publish within the new *Rechenmeister* area. Not the first, however. The probably earliest "book" to survive[550] is the anonymous "Trento algorism" printed in ca 1475 in Trento in Südtirol (annexed by Italy as war booty in 1919, but definitely of German culture in the 15th century) – that is, some 75 kilometres from Treviso, where the first Italian abbacus book was to be printed three years later, and on the trading route connecting Nürnberg, Regensburg and Bamberg (and further north Leipzig) to Venice.

It consist of six sheets; an edition repairing some disorder in the surviving specimen was made by Vogel in [1963]. Strictly speaking it is no algorism at all, in spite of presenting its topic as *Algorisimus* – all its numbers are expressed in words or in Roman numerals. Instead of the algorithms for calculation by means of Hindu-Arabic numerals it teaches the use of a line abacus and *Rechenpfennige* (counters), the habitual tool of German merchant-calculators, for addition, subtraction, halving, doubling, multiplication, division and (on two lines) arithmetical progression.[551] The inclusion of halving and doubling shows influence from the Latin algorism tradition – they are treated, in the same order, for example in Sacrobosco's *Algorismus vulgaris* [ed. Pedersen 1983: 181–190]. Together with a number of interspersed Latin phrases pointed out by Vogel [1963: 184 n. 8], this awakes the suspicion that the author came from clerical or university culture

549 Widmann is not likely to have known the Vienna manuscript, which however does not exclude familiarity with material that was copied into it, unless the influence goes the other way; suspicious is at least the shared notion of an "algorism of surds".

550 In a single copy, a fact that (together with what se shall se concerning the *Bamberger Blockbuch*, and the first edition of the *Bamberger Rechenbuch*) warns us that others may well have existed but gone lost.

551 On this line abacus, the levels of the lines and of the spaces between them (also used) are 1, 5, 10, 50, 100, etc. That is, they are adapted to Roman numerals.

rather than being a *Rechenmeister*.

This introduction of numerical techniques takes up three of the 12 pages. What follows confirms that Hindu-Arabic numerals were no necessary tool for abbacus-like merchant calculations, as said above (p. 177). Here comes indeed an explanation of the rule of three, called *regula ternari*, followed by four examples. The formulation of the rule is somewhat corrupt, but it is clear that like the probably later Vienna manuscript (above, p. 368) it refers to the given numbers as first, last and middle; but it also points out that the first and last must be similar, *gleich*.

A number of ten more "rules" follow – mostly problems, which however illustrate a principle and can in this sense be considered rules. To these belong partnership without or with different durations of the participation (rule 2 and 3), summation of an arithmetical progression (rule 5), meeting or pursuit with given constant speeds (rules 6 and 8), a "tree problem" about a tower introducing the single false position without giving it a name (rule 9); the rest are applications of the partnership rule, for instance to the twin inheritance problem (rule 4) or to "fallacious" proportional sharing with relative shares that do not add up to 1 (rules 7, 10 and, in general formulation, 11). We may notice a certain similarity with Wolack's lecture.

Slightly longer is an arithmetical block book (a book printed from woodcuts and not by movable type) [Wagner(?) 1475(?); ed. Vogel & Schemmel 1980]. It is undated but almost certainly precedes 1482 – Eberhard Schröder [1988: 303] assumes a date around 1475 and finds it plausible that it was written by Ulrich Wagner, one of three *Rechenmeister* attested in Nürnberg in 1457, remaining active there until his death in 1489/90.

The mathematical substance of this "Bamberger Blockbuch"[552] is still modest, in no way going beyond the immediate basic needs of the students of a *Rechenmeister* – except in one respect: Wagner (assuming he be the author) uses exclusively Hindu-Arabic numerals and does not touch at the use of a line abacus. The first two of its 24 pages (13–15 lines each) contain the multiplication table (going until 10×10). Monetary and metrological conversions follow (3 pages), after which the *regula von dre* is introduced. As the Trento *Algorisimus*, the *Blockbuch* refers to the given numbers as first, last and middle and also points out that the first and last must be similar, *gleich*; since there are no traces of copying one way or the other, this formulation can be supposed to have circulated. The *Blockbuch* adds that the outcome will be as the middle.

The rest of the book is dedicated to problems to be solved by means of this rule. All in all, much less than the basic curriculum of an abbacus school, even if we disregard that the *Blockbuch* does not use space on prescribing how to calculate with the Hindu-Arabic numbers; even the Trento *Algorisimus* has a wider range.

---

552 Thus called after the city where a copy has survived. The book is likely to have been produced somewhere in Franconia, which encompasses Nürnberg as well as Bamberg.

In [1482], Wagner (now identified) published another *Rechenbuch*, of which only a fragment of a single copy survives – just extensive enough to tell the authorship and to show that it is substantially identical with a second edition published anonymously (and unpaginated) in [1483]. This latter *Bamberger Rechenbuch* was published in facsimile with a slightly modernized transcription by Schröder in [1988].

Beyond a number of problems and formulations which it shares with the *Blockbuch*, the exclusion of the line abacus corroborates the hypothesis that they have the same author. (It also has much in common with the Regensburg *Practica* [Vogel 1959: 34], thus demonstrating that shared problems alone is no proof of shared author). Being around six times as long (156 pages of some 16 lines each), it evidently contains much more material than the *Blockbuch*.

After an initial introductory passage about why number is important (referring to Solomo's Wisdom, not to philosophy nor to Boethius), Wagner shows how to calculate in the place-value system and, related to this, with multi-level metrological numbers; difficult matters like casting out sevens, fractions and divisions are included.

About the rule of three – here called "golden rule", though with a reference to Italian usage – it is once more said that the first and third must be similar. The rule is taught with a large number of examples, separated in chapters according to the type of goods dealt with – quite reasonable from a tradesman's point of view, since it determines the metrologies to be used, each with its own peculiarities, and also whether impurities are to be taken into account. For instance (chapter 12), cloves come with a certain fraction of stalk (mathematically of course no different from the loss of weight of wool when washed, or from calculating with tare). Chapter 13, about partnership, transforms the twin problem into a triplet problem, and also has a colourful story about a debtor who runs away from his debts but still leaves behind something to be shared between the creditors. Chapter 14 takes up *tollet* calculation, a method (related to the *welsche Praktik*) allowing to reduce the intricacies arising when a rule-of-three calculation involves different metrologies for the numbers supposedly "the same" – for the first time described in print in the Wagner's book, but certainly a practice current among merchants – also, since *tollet* comes from *tavoletta*, among Italian merchants.[553] The Germanized name indicates, however, that Wagner and his Rechenmeister successors have adopted the technique from the practice of Germanic-speaking merchants, not from Italian traders or books (when speaking of the rule of three, they manage better to render the Italian expression).

Chapter 15 deals with barter, taking into account the different values of goods when

[553] The method is explained in [Tropfke/Vogel et al 1980: 364]. We have already encountered the habit to calculate separately with the different levels of a composite entity in the Italian material, though without the scheme introduced here – for instance when presented by Fibonacci as the "vernacular way" (above, p. 62; cf. also note 34). The complications it eschews can be understood from note 83 above.

sold cash or in barter. Chapter 16 first speaks about gold, explaining how German metrology for gold relates to the Venetian carat system; afterwards it presents four recreational classics: a "tree" problem (as in Wolack's lecture and the Trento *Algorisimus* here about a tower), serving as exposition of the single false position; two pursuit problems, one with constant speeds and one with an arithmetical increase; and a cask emptied through three taps. Chapters 17–21 deal with metrological and monetary conversions.

So, apart from adding *tollet* calculation, Wagner's book stays strictly within what could be taken over from the Italian tradition. But it speaks solely of what is relevant in commercial life (though sometimes in recreational dress), excluding all mathematical extravagances. The Nürnberg *Rechenmeister* might compete, as we have seen, to the point of ending up in court; but they do not appear to have engaged in the more civilized competition through intellectual challenges.

Apart from the appearance of a second edition, there are no traces that Wagner's book was a great success – extremely few copies survive, and there were no further editions. The next *Rechenbuch* was more fortunate. It was published by Widmann in [1489],[554] with new editions in 1500, 1508, 1519 and 1526; [Gärtner 2000] contains a modern edition based on the first edition.

Wagner shows himself so familiar not only with the mathematics of merchants but also with the goods they were trading in that Schröder [1988: 302] suspects him of having been an merchant himself before becoming a *Rechenmeister*. Widmann, instead, as we have seen, was a university scholar.

His *Behende und hubsche Rechenung auff allen kauffmanschaft* ("Skillful and Pretty Calculation for all merchantry") is at least three times as long as the *Bamberger Rechenbuch* – 471 pages, each of which contains somewhat more typographical units than those of Wagner's book.

As we are accustomed to, a part 1 (fols 8^r–38^v) introduces the Hindu-Arabic number system together with its use – in addition, subtraction, doubling, halving, multiplication, division, arithmetical progressions and extraction of square and cube roots. Inspiration from Sacrobosco is not only visible in the substance, he is also referred to twice (fols 5^r, 27^r). The line abacus is not mentioned.

Part 2 is subdivided into three chapters. The first (fols 39^r–56^v) teaches the algorism of fractions – finding no place for progressions but adding *tollet* calculation in the end. The second chapter (fols 56^v–73^v) deals with "proportional number" (pointing also to the connection to the rule of three); after a reference to the treatment of the topic in Campanus's *Elements* (and even an off-hand reference to the binomials of book X) it introduces the Boethian names for ratios, and teaches the reduction to smallest terms and

[554] Still unpaginated; I shall refer to the foliation added in pencil in the specimen in the Bayerische Staatsbibliothek, the title page being fol. 1^r.

shows how to “add” and “subtract” ratios. In the end come *hubsche Fragen*, “pretty questions” – six of which ask for two or three numbers in given ratio that fulfil some other condition, five for the division of a given number into three or five parts; three ask other number questions.

“The third and major chapter of the second part” (fols 74^{r}–203^{v}) speaks of “number directed at merchantry”. At first comes the *Regula Detri*, the rule of three, presented more or less as by Wagner, though omitting the name “golden rule” and explaining matters more broadly, also with reference to proportions and to *Elements* VI and VII. A large number of examples are organized under separate rules (often, once again, determined by the kind merchandise dealt with). These are followed by exchange of monies, barter and partnership. Intermingled with these we find a number of recreational classics – lazy worker, twin and unknown inheritance, purchase of a horse etc. In the very end of the second part the *regula falsi* is explained and exemplified – that is, the double false position.

The third part (fols 203^{v}–234^{v}) mainly deals with geometry, but in the end (fols 234^{r-v}) come a selection of recreational problems meant to refresh the reader who by now, Widmann supposes, must be “bored and exhausted by heart”. As discovered by Kaunzner [1978], the geometry as well as the recreational *coda* are translated uncritically from a Latin precursor conserved (in original or copy) in the manuscript Munich, Clm 26639 – the section in question almost certainly to be dated to the outgoing 15th century [Kaunzner 1978: 7*f*]. It thus illustrates – like the inclusion of *tollet Rechnung* – that the *Rechenmeister* did not restrict themselves to importing abbacus material, they combined these imports with what else they found pertinent for their public (such as *tollet*) of from their own background (the present geometry, Widmann being a university scholar). As pointed out above (note 48), the Latin text which Widmann uses also elucidates some of the puzzling influences in early abbacus geometry.

Since even my present reader may now be “bored and exhausted by heart”, I shall allow myself another frivolous association. My maternal grandfather was a brick mason, and in cold and rainy weather he might ask “wasn’t it a bright idea that the masons built the houses hollow?”. It would appear that this bright idea had not dawned on German masons five hundred years ago. On fol. 226^{r} Widmann at least asks for the comparison of the costs for building two houses in a way that presupposes them to be solid.

***Rechenmeister* culture attains maturity**

The next generation of *Rechenbücher*, produced by *Rechenmeister* proper (some with, some without a university education) went further in the inclusion of local material and adaptation to local needs, by teaching also the use of the line abbacus and describing the *welsche Praktik* and *visieren* (doliometry, the measurement of wine barrels) systematically[555] – thus Heinrich Schreyber, Latinized Grammateus [1521: Biii^{v}–viii^{r}, Eii^{r}–Fv^{v}, Ov^{r}–Qi^{r}].

From then on, we may understanding the German Rechenbücher as expressions of a new mathematical culture, developing on its own and spreading in the course of the 17th century to the whole archipelago of German commercial centres, from the south where it originated to Emden in Nordfriesland and Riga at the Baltic.[556] This culture was not borrowed *as such* from the abbacus tradition but was a local development based in the local socio-economic and commercial situation in southern and Hanseatic Germany; obviously it borrowed material and inspiration from Italy, together with the borrowing of commercial techniques. It would thus be a parallel to, not a descendant of the abbacus culture – even this not borrowed *in toto* from Iberian or North African commercial partners though adopting commercial as well as mathematical techniques.

Obviously, there was much communication and much travel between the homeland of abbacus culture and that of *Rechenmeister culture* – both as the latter emerged and later on. That we still encounter a *new* mathematical culture should merely remind us that *a culture* – even a mathematical culture – is always formed on local conditions, in processes that involve digestion and assimilation of whatever foreign information is at hand and accepted as relevant. But the digestion is performed by the local stomach. Even "Northern Humanism" was something quite different from original, Italian Humanism – and Melanchton's successful Lutheran-orthodox Humanism still something quite different from the Northern Humanism of an Erasmus and a Thomas More.

The independence of *Rechenmeister* culture may be illustrated by the balance between line-abacus and written calculation. Adam Ries's first *Rechenbuch*, written in 1518[557] carries the title *Rechnung auff der Linien*, "calculation on the lines", and teaches the technique extensively; the second, from 1522 [ed. Deschauer 1991], is called *Rechenung auff der linihen und Federn*, "calculation on the lines and [with] pen", and explains the line technique less thoroughly and together with Hindu-Arabic algorithms. The third [Ries 1550] presents itself as *Rechenung nach der Lenge, auff den Linihen und Feder*, "calculation [explained] at the length, on the lines and [with] pen". After four pages' dedication to Moritz, Duke of Saxony, crammed with learned references to the seven Liberal arts, to Plato and Aristotle, to arithmetical and geometric justice, to Vitruvius, etc., and two pages of table of contents, Ries addresses the reader on fols 1$^{r–v}$, explaining that he will first present at length the use of lines in all kinds of mercantile and domestic

---

555 On the latter topic, see [Röttel 1996].

556 The distinctive Riga variant, surviving into the 19th century, is presented in [Deschauer 2010], cf. summary in [Høyrup 2011b].

557 Since no copy seems to survive of the first edition, we do not know when it was printed.

calculation, and in this way prepare and facilitate the teaching of the use of numerals. Indeed, fols 2r–47r are centred on the line abacus – beginning, however, by explaining how to write numbers in the place-value system (Roman numerals are absent). Three-fourths of the section are dedicated to the rule of three, tare, profit and loss, exchange of monies, conversion between metrologies in use in different places, silver-, gold- and copper-metrologies, partnership, inheritance and tutelage (genuine, not recreational as in the abbacus books), and barter. All supposed to be calculated by means of the line abacus.

Jakob Köbel's *Rechenbuch auff Linien und Ziffern*, "on lines and by digits" from [1549],[558] after a thorough description of the algorithms for the line abacus explained with the assistance of Roman numerals, also illustrates (fols 54v–56r) how the *regula de Tri* should be laid out on the lines; after that, the lines only reappear when a technique for finding the day of the new moon is illustrated on lines. Throughout the work, however, no Hindu-Arabic but only Roman numerals appear, showing that the line technique is presupposed everywhere in spite of the *Ziffern* of the title.

Köbel had already used Roman numerals in his much shorter book from [1514]. After him, they become rare. In [1565: Avr], the Leipzig *Rechenmeister* Matthaeus Nefe (who demonstrates Humanist inclinations in the dedication and introduction) presents them (after the Hindu-Arabic numerals and before the line technique) under the names "German numbers" (*deutsche Zalen*) or "imperial numbers" (*Keyser zaln*) since they are sometimes used – also in registers and accounts – because they are difficult to change; but that takes up only two of 596 pages. And then – the last example I have noticed – Roman numerals are presented by Johann Weber in his *New Künstlich und Wolgegründt Rechenbuch, Auff den Linien und Ziffern* [1583a: 4v–5v] after the explanation of the line algorithms (where number words are used) as "letter numbers, or as some call them, the numbers of the ancient Emperors and Romans" (*Buchstaben Zal, oder wie si etliche die altern Keyser oder Römer Zal nennen*) – two pages out of a total of 535.[559]

The line technique, though gradually becoming less conspicuous, does not disappear. Yet of the 30 German *Rechenbücher* from the 16th century I have inspected closely (not counting re-editions), only three present the line technique without offering also the paper algorithms: Johannes Albert's *New Rechenbüchlein auf der Federn, gantz leicht, aus rechtem grund, inn Gantzen und Gebrochen* from [1541][560], which in spite of its title teaches only the line technique "to the benefit of the simple common man as well as the

---

558 Shorter *Rechenbücher* from Köbel's hand had appeared in [1514], [1531], and [1537].

559 They are not mentioned in Weber's much shorter *Gerechnet Rechenbüchlein Auff Erffurdischen* ... [1583b], which starts directly with metrological and commercial matters.

560 Actually a second, enlarged and improved edition according to the title page. According to [Smith 1908: 178] the first edition was printed in 1534; I have not been able to trace it, and what Smith describes is the third edition.

novice lovers of arithmetic" (thus the title page); Johann Ober's *Newgestelt Rechenbüchlin, mit vil schönen exempeln und proben* from [1545]; and Michael Stifel's *Deutsche Arithmetica*, equally from [1545], which however should be thought of together with the counterpart *Rechenbuch von der Welschen und Deutschen Practick*, which he published in 1546, and which initially starts by presenting the paper algorithms and does not teach the line technique.

Even authors with theoretical pretensions continue to present the lines. The Ramist Georg Gleydtsmann [1600: Ai^{r}] explains to do it for the sake of the *Teutschen Rechenschülern*, the "German *Rechenschul* students", not however "for those ostentatious persons who pretend to be masters". For the same reason, the line technique did not disappear at the end of the 16th century. I have not looked systematically at the following generations of *Rechenbuch* authors, but it is taught "at the request of the young" at least as late as [1674] by Christian Trabeth (*auf Begehren der Jugend*, thus the title page).

Tollet as such seems to disappear rather soon, but its cognate *welsche Practica* also survived the 16th century in spite of the objections of certain writers to a technique which was merely a more longwinded way to calculate by the rule of three[561] – maybe also because teachers who had to teach the application of the rule of three to problems dealing with three-level metrologies also found this method more convenient at least when teaching.

The *welsche Praktik* is one example of a field which the *Rechenmeister* shaped as a discipline while in Italy it had been nothing but an informal merchant practice, reflected only occasionally in the abbacus books. But there are more. One is *Faktor Rechnung*, "agent calculation". It is a cognate of partnership calculation, dealing with the situation that a merchant and his agent (*Faktor*) enter into a partnership, the agent investing not only a capital of his own but also his work. Such problems are regularly dealt with in abbacus books, where the role of the partner who invests his person is not specified – but dealt with just as a particular case of the partnership problem (one partner might also have invested florins, the other florins and ducats, at an unknown relative rate). We may presume that the crystallization of a new discipline did not depend so much on internal mathematical dynamics as on a different configuration of commercial arrangements, where precisely the master-*Faktor* combination became important.[562]

Another new incipient "discipline" (which however did not go far) was an evident

---

561 For instance, [Gleydtsmann 1600], 9th page of the unpaginated *Vorrede*. On the other hand Simon Jacob [1565: 89^{r}] (certainly mathematically ambitious and competent), speaks of the advantages of the *welsche Practick* as *unendtlich*, "infinite".

562 Another example is *Bergwerk Rechnung*, "mining calculation", found for instance in [Nefe 1565: Si^{r}], dealing with the specific distribution of profit between the miner and the owner of the ground – in mathematical principle no different from partnership calculation.

outcome of internal mathematical dynamics. As mentioned above (p. 373), already Widmann had included arithmetical progressions among the basic arithmetical operations, and early 16th-century presentations of the line abacus had followed him.[563] The theoretically more ambitious writers of the later 16th century (who only deal with progressions when speaking about paper algorithms) include geometric progressions, which similarly ambitious abbacus writers had dealt with in different contexts. More striking, at least two German books also include the harmonic progression, likely inspired by Boethius's *De institutione arithmetica* II. In [Weber 1583a: Civ^{r}, Fiii^{r}] this is explained as a progression where the first number is to the third as the difference between the first and the second is to the difference between the second and the third. So far this could be taken from Boethius's *De institutione arithmetica* II.47 [ed. Friedlein 1867: 152; trans. Masi 1983: 174], even though Weber's numerical example (3–5–15) differs from what is proposed by Boethius. Then, however, Weber generalizes to longer sequences, and gives the examples 3–4–6–12 and 12–15–20–30–60, and later explains that a harmonic progression can be created by division of the product of all numbers in an arithmetical progression by these numbers singly.[564] A likely source is Stifel's *Arithmetica integra* [1544: 57^{r-v}],[565] which gives the same rule for producing a harmonic progression together with examples similar to those of Weber – though not quite the same, so common inspiration cannot be excluded. A background could be Jordanus's *De elementis arithmetice artis* X.34–51 – well known from Lefèvre d'Étaples' printed version of that work [1496; 1514]; on fol. 79^{v} Stifel refers to Jordanus as well as Lefèvre d'Étaples.

The explanation given by Caspar Thierfelder in the *Arithmetica oder Rechenbuch, auff den Linien und Ziffern* [1587: 337] is independent of Weber. According to this book, a harmonic progression involves only three numbers. He repeats the Boethian definition, and teaches to find the last number in a harmonic progression (say, a–b–c, $a<b$) as $ab/(b-a)$. Thierfelder also claims no originality; if he drew on the same source as Weber, he drew differently.[566]

Retrospectively, after another generalization harmonic progressions (more precisely,

---

563 Thus [Köbel 1531: Fvi^{v}] and [Ober 1545: Ciiii^{r}].

564 The modern explanation would refer to the reciprocals of the members of the arithmetical progression, but dividing the product allows Weber to avoid fractions.

565 A German translation of the whole work can be found in [Knobloch & Schoenberger 2007].

566 There is some evidence that his source was different. In his *Algorithmus* [ed. Tannstetter 1515, unpaginated but p. 85 beginning with the title page], Peurbach also includes the harmonic progression, and states that it has only three members. Since he does not give the formula for finding the third term from the first two, Peurbach cannot be Thierfelder's direct source; but Peurbach's treatise had circulated in Vienna for well over a century when Thierfelder wrote.

their sum, which for the harmonic progressions proper is divergent) became important in modern mathematics – in the shape of Riemann's ζ-function $\zeta(x) = \Sigma(n^{-x})$; but that was far in unforeseeable future. Another rudimentary discipline borrowed from a Latin (but Neolatin) source had absolutely no future, but for a while the mathematically more advanced *Rechenbuch* writers felt obliged to present it – in this case certainly not "to the benefit of the simple common man"; cf. the analysis in [Smeur 1978]. In his *Coss*,[567] on which more below, Christoph Rudolff [1525: Hvi^{v}] had pointed out that the *regula falsi* only solves problems of the first degree – in the formulation that it is restricted to the "first *coß*".[568] In his *Arithmeticae practicae methodus facilis,* Rainer Gemma Frisius [1540: XXIII^{v}] shows that *with this precise formulation* Rudolff is mistaken. The trick is simple – for instance, a problem $\alpha C = \beta$ can be solved as a first-degree problem with unknown C, and once C is known, the square root can be extracted. The first example (fol. XXX^{r}) runs like this:

> There is a certain quadrangular area containing a surface of 200 square cubits, whose length exceeds the width by half; both length and width are asked for. Thus by the rule of false, posit the width to be 4 cubits, the length will be 6, multiply in each other, 24 results, they should be 200, we are thus 176 below the aim. Then posit the width to be 20, the length will be 30, multiply in each other, 600 result, which are 400 above the aim. Until here every rule of false agrees. But now multiply the hypothesis in square, that is, namely 4 and 20, they will be 16 and 400, let these squares be hypotheses for you. [...]

So, in reality, the *square* of the width is posited first to be 16, next to be 400; we may call these the *proper* positions. In this way, the problem is of the first degree.

After this exercise in superfluous complication Gemma then explains that the problem is easily solved by a single false position, which is indeed how Pharaonic and Babylonian calculators would have solved the problem 3000 years before; this is then how Gemma deals with cubic and quartic homogeneous problems. He wisely abstains from problem leading to mixed second-degree equations. In an appendix to book I of his *Arithmetica*

---

567 Thus the writing on the title page. I shall use it consistently when citing Rudolff's book, and speak instead of *coß* when referring to the algebraic discipline of the *Rechenmeister* (in agreement with the prevailing orthography).

568 Rudolff, taking advantage of reducibility, on fol. Gvi^{r}, makes fun of the famous 24 cases and says that he could create 100. Instead, he operates with these eight cases only:

Ru1	$\alpha t^{n+1} = \beta t^{n}$	Ru5	$\alpha t^{n+2}+\beta t^{n+1} = \gamma t^{n}$
Ru2	$\alpha t^{n+2} = \beta t^{n}$	Ru6	$\alpha t^{n+2}+\gamma t^{n} = \beta t^{n+1}$
Ru3	$\alpha t^{n+3} = \beta t^{n}$	Ru7	$\beta t^{n+1}+\gamma t^{n} = \alpha t^{n+2}$
Ru4	$\alpha t^{n+4} = \beta t^{n}$	Ru8	$\alpha t^{2n+p}+\beta t^{n+p} = \gamma t^{p}$

integra, Stifel [1544: 98^r–100^r] reports the method, pointing out that it applies to higher degrees too *in infinitum*.

Gemma's insipid correction of Rudolff was taken up creatively by Jacob [1565: 251^v], who refers to it as *einen lustigen Weg*, "an amusing way", and takes it as a challenge to find a similar way to deal with Rudolff's remaining cases.[569] Before he presents his findings he states to do so mainly to avoid that common calculators waste much time with them, and also that he does not intend to compare them to the much faster rules of the *coß*. Then (after having shown the application of the *regula falsi* to first-degree and homogeneous higher-degree problems) Jacob explains on p. 276^r *why* Gemma's method works.

The examples offered for the mixed cases are quite complex, so I shall just show in modern symbols the mathematical principle of an illustration of the case Ru6 (al-Khwārizmī's fifth, and also the fifth of main the abbacus tradition; fols 277^v–278^r). The age of a son is half of that of his father, and the square root of the age of the father is $^1/_4$ of the age of the son diminished by 100 years – thus

$$x–10 = 4\sqrt{2x} \ ,$$

which reduces to

$$x^2+100 = 52x \ .$$

The usual way to solve this equation transforms it into

$$(x–26)^2 = 476 \ ,$$

which is an linear equation with unknown $(x–26)^2$. That means that the primary positions 30 and 40 for x, the age of the son, must be transformed into the proper positions $(30–26)^2 = 16$ and $(40–26)^2 = 196$. But this implication can only be known when one has *already produced the algebraic equation*, or at least its left-hand side. In the end (fols $279^{r–v}$) Jacob therefore warns against the risk to err, and says that he much prefers to teach the *coß* rather than the *falsi*.

In [1564: b vi^v*ff*], Oswald Ulman and Caspar Thierfelder took up Jacob's idea, without his warning and without pointing out that *coß* would be much better. For the mixed cases they prescribe a way to find the proper positions which depends on the particular structure of the problem that serves as example, without explaining why this should work and thus also without any hints at how to deal with problems of a differens tructure. How much

---

569 [Jacob 1565] is a posthumous re-edition of a book generally supposed to have been first published in 1560 (Jacob died in 1564) – thus [Smith 1908: 295]. Neither Smith nor anybody else I know of appears to have seen that edition, which however can hardly be the first – [Jacob 1557: 110^v] states that the further elaboration of Gemma's idea was presented *im andern theyl meiner Arithmetick*, "in the second part of my arithmetic", which must precede 1557 and seems to be a first version of the *new und wolgegründt Rechenbuch*.

they understood themselves may be questioned; but their text certainly would not help the reader to understand.

They are an exception, however. Weber, also speaking of Gemma's invention as *lustig* [1583a: 161v*f*], chides Ulman and Thierfelder and promises to explain better than Jacob. Whether he succeeds better can be debated, but at least he explains when discussing (fol. 108v) the first mixed case (Ru5) that the solution builds on the algebraic equation.

Bunglers can be found everywhere (even such as are bunglers only occasionally); and we may take Jacob and Weber as representatives of what the *Rechenbuch* tradition at its best could accomplish in this field.

So much about the *mathematics* of the *Rechenmeister* culture. Remains the *why*.

Firstly, primarily, it grew out of the commercial life of the southern German area (Nürnberg, Bamberg, Leipzig, Vienna, etc.) of the late 15th and the 16th centuries and its social base. That explains much of the basic level – the importance of line algorithms, *tollet* and *welsche Practick* – and the creation of new fields like *Faktor Rechnung* and *Bergwerk Rechnung* (there are more) alongside those which were taken over from the Italians because of their links to shared commercial practices (alligation, partnership, barter, etc.). The seminal role of university-educated mathematicians (for instance Widmann, but also Adam Ries and many others) explains the adoption of fields of no commercial importance (doubling, halving, progressions, Boethian names for ratios).

Remain the attempts to develop new theoretical domains (harmonic progressions, more or less successful application of the double false to higher-degree problems). The agonistic culture of abbacus culture, with mathematical challenges directly or indirectly linked to professional competition, explains the proliferation of false algebraic rules as well as the invention of specious "roots" in Italy. The *Rechenmeister* writers apparently did not live in a competitive culture determined by *these* parameters. Theirs was a print culture; an ever-returning element of their titles is that the book is *new* – at best containing matters or methods never seen before (book titles were long and descriptive, leaving ample space for such claims – those I list in the bibliography are shortened). The closest parallel to the Italian situation may be the (modest) unfolding of the inspiration from Gemma's application of the double false to higher-degree problems; what we are presented with by Ulman and Thierfelder can perhaps be seen as a parallel to the proliferation of false algebraic rules in Italy; but the cautious approach of Jacob and Weber, being mathematically sound, is best compared to the Italian introduction of ad-hoc "roots": mathematically sound but rather pointless – as Pacioli as well as Jacob and Weber knew and explained.

Coß

Southern Germany was not the only area where new social and commercial patterns assisted by borrowings from Italy and southern Germany (and by printing) led to interest in commercial arithmetic. Nowhere, however, is the number of surviving books (and, we have reasons to believe, of books produced at the time) large enough to allow us to identify and characterize new local unified mathematical cultures. So, instead of pursuing these transmissions, adoptions, transformations etc., we shall return to the fate of algebra within and in the vicinity of the *Rechenmeister* world.

Above we have traced the algebra of the German area until Widmann's lecture. A long algebraic Latin algebra is contained in the manuscript Munich, Clm 1696. As argued by Menso Folkerts [1996], it was almost certainly written by Andreas Alexander, one of the first specialized mathematics lecturers in Leipzig, which should probably date it no later than 1504 (after which traces of Alexander disappear). The related German text "Initius Algebras" [ed. Curtze 1902: 435–609], known from four manuscripts, may also have been produced by Alexander; what is certain is that the earliest of the known copies was written by Ries. Probably both texts served Ries for his *Coß*, which however also remained in manuscript and did not circulate. Their role in the historical process is therefore restricted to likely influence on Rudolff's *Behend unnd hübsch Rechnung durch die kunstreichen Regeln Algebra, so gemeincklich die Coss genennt werden* ("Skilfull and Pretty Calculations by Means of the Artful Rules of Algebra, Commonly Known as the *Coss*") – *Coss* for short, from [1525]. The two manuscripts deal with the same eight basic cases as Rudolff (above, note 568), and thus had already left behind the tradition of "24 cases". At the same time, references to algebra in *Rechenbücher* later in the century (for instance, those that apply the *regula falsi* to higher-degree problems) all have Rudolff as their fundamental reference.

So, as a discipline, *Rechenmeister* algebra was based on Rudolff's book. Before analyzing it, however, we shall observe that Rudolff was preceded by his teacher Heinrich Schreyber, whose *Ayn new kunstlich Buech, welches gar gewiß und behend lernet nach der gemainen regel Detre, welschen Practic, regel falsi unn etlichen regeln Cosse* was referred to above (p. 374). Schreyber deals with "regula falsi together with several rules of *coß* in [Grammateus 1521: Gvi^{r}–Liv^{v}]. This inclusion of algebra in a general *Rechenbuch* is not Schreyber's only deviation from what was to become the general style. Fols Lv^{r}–Mv^{v} deals with "arithmetic applied to the noble art of music" (starting with the anecdote about Pythagoras and the anvils, here becoming hammers), fols. Mvi^{r}–Oiv^{v} with bookkeeping, and Ov^{r}–Qi^{v} with the preparation of the ruler used in doliometry (which involves the "Delic problem" of finding two intermediate proportionals, which is provided with the familiar geometric proof). Music appears in no other general *Rechenbuch* I have inspected, and book-keeping I have only encountered in [Schulz 1600: Vuu i^{r}–Bbb iii^{v}].[570] Only

570 And of course in later editions of Schreyber's book. Smith [1908: 123] list editions from 1535, 1544 and 1572, of which I have inspected those from [1535] and 1544. It is indeed the bookkeeping chapters that shows the first edition to have been published in 1521 even though the privilege is dated in 1518 and the text thus no later; bookkeeping chapters also show the undated second and

doliometry appears regularly. Schreyber seems to have written at a time when norms had not yet crystallized deciding what belonged in which kind of book.

The *regula falsi* is presented on two pages only in the chapter in question; *coß* is evidently what really interests Schreyber. Before the rules, he offers a thorough introduction, beginning with an explanation of *radix*, *census* and *cubus* in terms of *numerus linealis*, *superficialis* and *corporalis* and correlation of the algebraic powers with geometric progressions and a numbering of the powers corresponding to exponents. This is followed by the arithmetic of algebraic expressions, when adequate making use of computational schemes; remarkably, these make use of the numbering of powers and not of the "cossic symbols" used in the manuscripts we have discussed as well as in the later *coß* tradition – for instance (fol. Gvi^{r}):[571]

$$
\begin{array}{l}
\qquad 6\ 1^{st} + 8\ N \\
\qquad\quad \text{times} \\
\qquad 5\ 1^{st} - 7\ N \\
\hline
\quad 30\ 2^{nd} + 40\ 1^{st} \\
\qquad\qquad - 42\ 1^{st} - 56\ N \\
\hline
30\ 2^{nd} + 40\ 1^{st} - 42\ 1^{st} - 56\ N
\end{array}
$$

Similar schemes we know from the abbacus tradition since the Florentine *Tratato sopra l'arte arismetricha* – see above, p. 242; but the whole preceding context as well as the designation of the powers by their exponents show that Schreyber or those who inspired him, if he/they did not reinvent then at least reformulated; with Schreyber we are clearly beyond the phase of eclectice reception. We also find a systematic explanation of the division by an algebraic expression through formal fractions – for example (fol. Hi^{v}) $\frac{4\ 3^{rd} - 5\ 2^{nd}}{2\ 1^{st} - 4\ N}$. Even this is certainly not new when compared to what Italians had done since long, but once again it is re-expressed. New when compared to Italian algebra is instead the explanation of how to extract square and cube roots – the abbacus tradition, as we remember, dealt with root extraction solely in the context of geometry. It is taken from the Latin algorism tradition.

From fol. Hviii^{v} onward, seven rules for basic cases are presented, coinciding with Rudolff's first seven rules (above, note 568), and thus also with the first seven rules of Alexander's Latin manuscript and the "Initius Algebras" (above, p. 381). These seven

third editions to be from 1535 and 1544.

[571] 1^{st} and 2^{nd} correspond to *pri* and *se* in the text. In a scheme showing the products of powers, these are written 1^{a}, 2^{a}, 3^{a}, ...

rules are initially presented in a format where equations that can be reduced to the same basic equation are put together as a single rule; on fol. Iiiii^r they are recapitulated in lowest powers. In all cases the powers are designated as 1st, 2nd, etc. In the end of the chapter come examples, those corresponding to the first rule are solved by *regula falsi* as well as by *coß*, the others by *coß* alone.

Rudolff's *Coss* was the book that so to speak defined the field, All those who generalized the rule of false refer to that book, not to Schreyber. Moreover, after Stifel had published the German *Deutsche Arithmetica* in [1545] and *Rechenbuch von der Welschen und Deutschen Practick* in [1546] (above, p. 376), based respectively on line and paper algorithms he did not complete the German *triptych* by a book wholly of his own drawn from his own *Arithmetica integra* from [1544] but instead by producing an "improved much augmented" version of Rudolff's work [Stifel 1553].[572]

Rudolff's *Coss* is divided into two parts, further subdivided into chapters. Part I can be described as providing general and specific foundations:

Ch. 1 is a Hindu-Arabic algorism for integers, leaving out doubling and halving and going on until arithmetical and geometric progressions.

Ch. 2 is "the common algorism for fractions", starting with a distinction between "simple fractions" (*schlechte prüch*) like $^2\!/_5$ and "part of parts" (*teil von teilen*) like $^3\!/_4$ of $^5\!/_7$. It further teaches how to reduce the latter to the former and how to perform all arithmetical operations on them.

Ch. 3 presents the rule of three in integer and broken numbers, illustrated by an abundance of mercantile examples.

Ch. 4 teaches how to extract square and cube roots.

Ch. 5 presents the "algorism of the *coß*", organized after the model of the common algorism: numeration explaining the names and symbols for the algebraic powers (quite justified, since these were spoken of as "cossic numbers"); addition, subtraction, multiplication and division of algebraic monomials and binomials (and trinomials under

---

572 It is probably only in our perspective that the re-edition of Rudolff's book was part of a triptych. According to Stifel's *Vorrede* he had been asked to undertake the work because Rudolff's book was no longer to be bought "even if one wanted to pay three or four times" the price.

Actually, already one of the three parts of Stifel's *Deutsche Arithmetik* is "*coß* or artful calculation". Part of it, however, is an algorism for fractions; even the rest is a rather rudimentary presentation, and has the double purpose (thus explained on fols 17^{r-v}), on one hand, to show that what Adam Ries solves by means of the *regula falsi* is solved more conveniently by *coß*; on the other, to replace the Latin loanwords used by Rudolff by terms used currently in other *Rechenbücher* – thus *sum* instead of *radix*, *sum.sum* instead of *zensus*, *sum.sum.sum* instead of *cubus*. The first purpose leads to first-degree problems only; the second confirms the role of Rudolff's book as defining the field. No symbols are used beyond + and –.

addition and subtraction). All calculations with bi- and trinomials are made in schemes similar to those of Schreyber (and Italian predecessors), and everywhere the symbolism we know from the Vienna manuscript Pal. 5277 is used: Firstly, + and –. For the powers (with a new symbols ß for the fifth power) the sequence is extended

ϕ	*dragma* or *numerus*
ꝛ	*radix* (henceforth I shall use *r*)
ʒ	*zensus* (henceforth *z*)
ꝯ	*cubus* (henceforth *c*)
ʒʒ	*zensdezens*
ß	*sursolidum*
ʒꝯ	*zensicubus*
bß	*bissursolidum*
ʒʒʒ	*zensdezensdezens*
ꝯꝯ	*cubus de cubo*

As we see, names are now (as in Pacioli's *Summa*, above, p. 338) made by embedding – *zensus* and *cubus* are functions, *cc* means *c*(*c*). These symbols were used by Stifel, taken over by Clavius, and encountered by René Descartes in the Jesuit school (he did not like them and preferred to make his own, indeed much more flexible and still with us – cf. below, p. 413). Last in the algorism comes "the rule of three in integer numbers" – that is, involving binomials with integer coefficients.

Ch. 6 is the "algorism for added and subtracted in [formal] fractions" – once more going until the rule of three.

Ch. 7–11 are dedicated to radicals: the extraction of square, cube and fourth roots, "algorism" for binomials and apotomes, and finally the extraction of the square root of "binomial numbers", that is, arithmetical binomials and apotomes (nothing approaching *Elements* X, to be sure).

Ch. 12 presents the Boethian classification of ratios – going beyond the tradition only by considering the ratios between broken numbers.

Part II is the algebra proper:

Ch. 1 presents "the 8 rules of the *coß* " together with a few examples. They were already listed in note 568, and there is no need to repeat.

Ch. 2 teaches the reduction of equations – also equations involving roots and formal fractions – for example

$$\frac{8}{5}\phi \qquad \frac{896\phi}{48r-1z}$$

(as in many late-15th-century Italian works there is no symbol for equality).

Ch. 3 (70% of the whole work) is a collection of examples for the rules, subdivided after which powers are actually involved. For the fifth rule, for instance, 33 examples are of type $z+r$ = ϕ, 5 of type $c+z$ = r, and two of type $zz+c$ = z. For the first rule, the first 39 examples are pure-number problems, afterwards 7 deal with "binomial numbers". Then come 141 problems of commercial character, divided as in other *Rechenbücher:* purchase; mixing; exchange; testaments; money; profit and customs; barter; partnership and similar partition; and alloying.

Then the *regel quantitatis* is taught, that is, how to introduce a second unknown designated *quantitet* – sometimes abbreviated *quant*, *quantit* or *q* in equations). This rule is presented on fol. I.vi[v] as "a completion of the *coss*, indeed in truth a completion without which it would not be worth much more than a trifle [*pfifferling*]". Although this name of the rule is not found in Pacioli's *Summa* it must have been in more widel spread circulation. It is spoken of (as *regle de la quantite*) by Étienne de la Roche in his *Larismethique nouvellement composée* [1520: 42[r], 61[r]], printed in Lyon in [1520], and also referred to as a name currently used in a marginal note in Chuquet's *Triparty*, apparently in the hand of de la Roche [Heeffer 2012: 134].[573]

The first rule is illustrated by 30 examples, mostly with commercial (but recreational) themes. 23 final examples cover the equation types $z = r$, $c = z$ and $zz = c$. Many make use of the *regula quantitatis* – many even deal with three unknowns, but as Pacioli (above, p. 351) in a way that allows Rudolff to operate with only two at a time.[574]

If we leave aside the problems illustrating the first rule, the problems purportedly dealing with commercial life are often quite complex and highly artificial – how else to transform situations where everything can be solved by first-degree methods (occasionally

[573] Paris, BNP, français 1346, fol. 169[v]. Because of a number of similarities, Heeffer supposes Rudolff to have borrowed from de la Roche. In view of the way *both* refer to the phrase as already existing and of the demonstrable vast lacunae in the surviving record I have my doubts.

Since de la Roche uses the "Florentine ρ" as basic unknown, he can be assumed to known Italian material (beyond Pacioli's *Summa*, where it does not occur).

[574] The first instance is a horse purchase with three participants, which may be summarized

$$A+\tfrac{1}{2}(B+C) = 34\ ,\quad B+\tfrac{1}{3}(A+C) = 34\ ,\quad C = \tfrac{1}{4}(A+B) = 34\ .$$

Identifying A with r allows Rudolff to conclude that

$$B+C = 68–2r\ .$$

Identification of B with the *quantity* (q) then leads in several steps to

$$q = 17+\tfrac{1}{5}\ r\ .$$

Now C is identified with the *quantity* (for clarity I shall use Q). Then tacit use of the first equation changed into

$$2A+B+Q = 68$$

leads to

$$A+B = 68–r–Q\ ,$$

which is inserted in the third original equation, whence

$$C = Q = \tfrac{1}{3}r+22\tfrac{2}{3}\ ,$$

etc.

a root extraction when composite interest is the theme) in such a way that they give rise to higher-degree problems? The number problems are also often constructed with great fantasy.

If this is considered together with the very orderly organization of the book and its resonance with prevailing mathematical interests it is not strange that is came to define the field.

Nine years before publishing his "improved much augmented" edition of Rudolff's work, Stifel produced the much more extensive Latin *Arithmetica integra* – that is, in [1544]. Stifel knows Rudolff, acknowledges the importance of the work, and takes over Rudolff's symbolism; but he knows much more, for instance *Elements* X, which he deals with in depth in Book II (fols 103^r–223^v) – reordering the material under the conditions of arithmetization; Euclid's order, indeed, "now nobody can present in a satisfactory manner" – thus the beginning of chapter 13, fol. 143^v). Stifel also knows what had been made recently in Italy: Cardano is referred to repeatedly (fols 101^r, 251^v, 252^r, 256^r and 301^v) – evidently not the *Ars magna*, which was only printed by the Stifel's Nürnberg printer (Petreius) the following year, but his *Practica arithmetice, et mensurandi singularis* from 1539. Stifel combines Rudolff's more systematic approach with matters far beyond what Rudolff had dealt with, and almost everything advanced that had been developed in Italian algebra.

Almost only, however. Scipione del Ferro's solution of certain irreducible cubic equations was as yet a secret, only to be divulged by Cardano in the *Ars magna* – cf. [Masotti 1971]. Nor does Stifel present anything similar to the equation transformations hidden behind Dardi's four rules of limited validity" (above, p. 223) or those (equally hidden) in the *Tratato sopra l'arte della arismetricha* (above, p. 245).

On the other hand, Stifel presents something like an innovation. He solves a number of problems making use of several algebraic unknowns under the heading "on second roots". He knows Rudolff's (and Cardano's) use of *quantity* abbreviated *q* for the second unknown but describes (fols 251^v–252^r) a way to give names to as many unknowns as wanted. The unknowns beyond *r*, the *res*, will be "1*A* (that is, 1*Ar*), 1*B* (that is, 1*Br*), 1*C* (that is, 1*Cr*), 1*D* etc", their second powers 1*Az* etc. For the product of *r* and *A* he suggests *rA*, while that of *A* and *B* will be written *AB*. His first example is borrowed from Rudolff and thus only uses *r* and *A*, the next is similar. The third (fol. 253^v, however, is superficially similar to what we have encountered in Benedetto's *Praticha:*

> Seven men owe me money in this way. The first and second, third, fourth, fifth and sixth owe 142 florins. (Here observe, that only the debt of the seventh debtor is excluded from this amount of florins. I posit therefore that the amount of the seventh is 1r, and thus that the amount of all the debts will be 142+r.) The second, third, fourth, fifth, sixth and seventh owe 126 florins. (Here the debt of the first is excluded. I posit therefore for the amount of the first is1A florins. And thus again the amount of all results, making 126+1A. [...].

As we have seen numerous times, the problem formulation is cyclical. Therefore, the following numbers are equal:

$$142+1r$$
$$126+1A$$
$$136+1B$$
$$128+1C$$
$$130+1D$$
$$120+1E$$
$$148+1F$$

At first, from the equality of the first two amounts follows $A = 16+1r$. Similarly, $B = 6+1r$, $C = 14+1r$, $D = 12+1r$, $E = 22+1r$, $F = 1r–6$. Therefore $A+B+C+D+E+F+r = 7r+64$, which must therefore equal $142+1r$. The calculations are much simpler than those of Benedetto; the writing of the equations also more rudimentary, and the reasoning closer to rhetorical algebra; but what we see is indubitable algebraic reasoning with seven unknowns. However, only one unknown beyond the *res* is made use of at a time – Pacioli's and Rudolff's method would have sufficed.

The problem is indeed so simple that it is easily solved without the use of algebra, as done time and again in abbacus books, with the only difference that their problems speak about possessions and not about debts. We encountered the type already in the *Liber abbaci* (above, p. 96), and again in the Ottoboniano and Palatino *Pratiche* (above, p. 261). Fibonacci as well as the abbacus writers would have observed that $142+126+136+128+130+120+148 = 930$ corresponds to all debts taken six times, whence the sum of the debts is 155. Therefore the seventh debtor owes $155–142 = 13$ florins, etc. Stifel evidently knows. Fibonacci and the abbacus writers had introduced extra algebraic variables to solve intricate problems sometimes presented as challenges, sometimes perhaps only imagined as possible challenges. Stifel instead uses a simple situation to illustrate the method; *his* aim is pedagogical, and the agonistic character of abbacus mathematics had not survived the transfer to the *Rechenmeister* environment and its expression in printed books.

The last problem presented in the section (fol. 254^{v}) uses three unknowns in a (reducible) quartic problem. Stifel asks for two numbers P and Q fulfilling (in our symbolism) the conditions that

$$P^2+Q^2–(P+Q) = 78\ ,\quad PQ+(P+Q) = 39\ .$$

He identifies the first number with the unknown r and the second with A, and for convenience their sum with a third unknown B. His argument, however, is based on a geometric diagram and not algebraic; the introduction of the seemingly superfluous "variable" B thus illustrates that in a geometric diagram there is no need to determine a minimal set of unknowns, all entities that are present serve on an equal footing. Antonio's use of two *algebraic* unknowns in second-degree equations has no parallel in the

Arithmetica integra.

Fols 292^r–301^r bring another batch of problems using several unknowns. Most are of the first degree (for instance, give-and-take). One (fol. 300^v deals with a rectangle and looks at first as if it were of higher degree and is solved by algebraic methods; during the procedure, however, it turns out to allow the elimination of the second unknown via a linear equation. Only one (fol. 292^r) is properly of higher degree, and even this one is solved by geometric arguments, not by algebraic method.[575]

Also innovative is Stifel's discussion of negative ("absurd") numbers (fols 248^v–250^v); not because the acceptance of these was unprecedented (cf. above, p. 348); but Stifel explores these numbers systematically in a new way.

So, without going through the book in detail we may summarize that the *Arithmetica integra* integrates everything that was publicly known in algebra and related topics *somewhere* and brings it to order in a way that was found *nowhere*. This work may thus be regarded as the starting point for the next phase, and at the same time be taken as an adequate point to stop the story about the transforming afterlife of abbacus algebra. The year after the appearance of the *Arithmetica integra* Cardano published his *Ars magna* which, as can be argued, set the stage for the development of equation theory for at least two centuries, In the words of the late Jacqueline Stedall [2010: 207*f*]:

> When Cardano in 1545 turned his attention to the problem of transforming equations without actually solving them, he too had been engaging in a process of generalization, from particular techniques of solution to a more all-embracing vision of equations as mathematical objects in their own right. In the centuries between Cardano and Lagrange, algebra took on a variety of names, forms, and applications, but always one of its characteristic features was the process of increasing abstraction from one level of thinking to another. Cardano, in embarking on that path, transformed not just equations but algebra itself. Lagrange two centuries later looked back to Cardano and his successors, and in doing so he too produced ideas that were again to change the nature and scope of algebra, this time from the study of equations to the investigation of the abstract structures that later became known as groups. Lagrange rightly recognized Cardano's work as the beginning of a key period in algebra; his own work in turn initiated another.

This (together with what was done by Tartaglia and Bombelli) is clearly a *new* story, which should not be pursued within the present framework (though the final section will look at some aspects in generalizing perspective). Nor is there any reason to analyze Scheubel's integration of *coß* in his edition of *Elements* I–VI [1550] or such French algebraic writers

---

[575] In his re-edition of Rudolff's *Coss*, Stifel makes use of his new notation. In the problem reported in note 574, he still [Stifel 1553: 309^v–310^v] uses *r* for the possession of the first participant, but 1*B* and 1*C* for the possessions ob *B* and *C*, thus avoinding the recycling of the same name.

as Jacques Peletier [1554] and Buteo [1559], both building on Stifel and other German writers though only the former recognizes it; even Pedro Nuñez can be left in peace in spite of his *Libro de algebra in arithmetica y geometria* from [1567]; all were direct or indirect descendants of the abbacus algebra and the *coß* traditions, but none of them contributed to creating a new synthesis like the *coß* tradition, not did they have much influence on what happened from Viète onwards.

Nor will we learn much of *general* interest from looking at the influence of the abbacus and *Rechenmeister* traditions in other regions (the Iberian peninsula, England, etc. – my apologies to patriots of bygone world empires). We shall therefore now return to abbacus mathematics but in new perspectives.

VI. A double conclusion

Chapter IV told the story of abbacus mathematics until the late 15th century in chronological order and in details – not all details, for sure, but enough perhaps to cloud the general view. Though in less detail but still with many details, chapter V dealt with its transformation when going into print and when re-emerging as *Rechenmeister* mathematics. This final chapter, drawing on these, presents a kind of summary and a perspective. The summary, synchronous, deals with the global "anthropology" of the abbacus endeavour – how can it be understood in its own world? The perspective, diachronic, looks at the role, not of abbacus mathematics as a whole but of abbacus algebra – its mathematically though not economically most influential component – in the longer historical process.

A mathematical practice *sui generis*

Mathematics, as anything produced by human beings, is made not in a void but within a *particular* practice, within a particular socio-cultural space and with particular aims and norms. What was the mathematical practice of abbacus mathematics?[576]

Different from ours, though perhaps not as different as we might at first believe (provided that ours be a single practice, which is hardly the case – at least it is not the practice which post-Weierstraß ideology prescribes). Yet on the other hand not so different that we cannot recognize the undertaking as mathematics.

At a first glance, this similarity might be doubted. How does it fit the production and lasting transmission of false algebraic rules from Gherardi onward (above, p. 200)? How does it fit Giovanni di Davizzo's use of roots as negative powers (above, p. 208)? Our present mathematical practice does not accept the principle *anything goes* and ostracizes those who insist that it does.

For the moment we shall observe that false solutions as well as the conflation of roots and negative powers belong within algebra, a prestige but marginal topic not taught in the abbacus school. There is more to say about this, but at first we shall concentrate on the "practice of the practical", the level connected to commercial life and to the teaching of the school.

Even here we seem to encounter phenomena that modern mathematics would find unacceptable. Certain rules are mostly presented just as rules, without any explanation.

This is first of all the case of the rule of three. Abbacus writers with scholarly ambitions (the authors of the Florentine encyclopedias and Pacioli) might connect it to proportion theory, but Jacopo's presentation (above, p. 16) corresponds to the way of the overwhelming majority of abbacus books. The presentation of the partnership rule is similar, and so is that of the determination of circular circumference and area from the diameter. Here, however, the passage from the Vatican manuscript of Jacopo's *Tractatus algorismi* that was quoted on p. 34 is illuminating:

> And if you should want to know for which cause you divide and multiply by 3 and $^1/_7$, then I say to you that the reason is that every round of whatever measure it might be is around 3 times and $^1/_7$ as much as is its diameter, that is, the straight in middle. And for this cause you have to multiply and divide as I have said to you above.

That almost amounts to the statement of an axiom, and this axiom indeed serves in other geometric calculations. In the same vein, the rule of three is also used axiomatically as the foundation for other explanations – for instance, as we have seen (above, p. 218), by Dardi when he explains how to divide 8 by $3+\sqrt{4}$. Similarly, the *Istratti di ragioni* (above, note 371, ed. [Arrighi 1964: 26]) uses the rule of three to perform the division of $^4/_5$ by $^1/_3$.

---

576 This section is largely a summary of [Høyrup 2009].

The partnership rule too, itself mostly unexplained, often serves as a tool for other proportional sharings; on p. 22 we have encountered its use in the twin problem.

Evidently this does not amount to an axiomatic *system*. But there is no doubt that abbacus mathematics was generally reasoned, taking certain truths as secure foundations on which others arguments could be based. Such truths tended to be those which oft-repeated use had made extremely familiar.

Abbacus mathematics was also expected to give true answers that would not have been different if a different but still valid way for the calculations had been chosen. The truth of abbacus mathematics thus included coherence. Often proofs (in the sense of "verifications") are offered showing that a result that has been obtained fulfils the conditions of a problem; but not rarely, proofs are given that show the same outcome to result from a different procedure (or a reverse calculation is made that does not reverse the steps performed but goes a different way). From p. 53 we remember this charming commentary to such a proof in the Milan-Florence version of Jacopo's *Tractatus:*

> We have thus alloyed well, since we precisely found again the said 700 δ. It would have been a pity if we had found more or less.

Since abbacus mathematics used true, not rounded decimal fractions when leaving the integer domain, proofs could mostly verify absolute agreement of results with what was requested. At times they could not, and approximations had to be introduced. That happens, for example, when an iterative procedure is used to solve a discounting problem (above, p. 29) that stops when the last contribution is either negligible or invisible. In such cases I do not remember to have ever seen a proof. It appears that the request of absolute truth was recognized to belong *within* mathematics, and that a similar imperative could not be imposed once the conditions of real commercial life had inserted themselves in the considerations.[577]

Consistency was not only a norm imposed on those who practised abbacus mathematics but also expected to characterize its *object* (the latter an obvious precondition for the former). That is why formal fractions could be introduced and expected to behave just like ordinary fractions (cf. the explanation of the *Trattato dell'alcibra amuchabile* on p. 228), and that is what Dardi presupposed in his demonstration that less times less makes plus (above, p. 218). Somehow, even Giovanni di Davizzo' mistaken intuition (p. 208) probably depends on that expectation.

We may summarize the norms or expectations which characterized abbacus mathematics:

– it should, *in so far as authors and users could do it respectively follow it*, be argued;

[577] Another personal aside inserted as an interruption of the all-too-serious: When teaching physics to future building engineers half a century ago I learned the principle that "the difference between theory and practice is condensed water".

– it should be consistent, in agreement with the consistency of its object;
– and it should be exact, unless some real-world application asked for approximation.

This does not fit the false rules too well, so we need to return to them. One can doubt (but not be sure) that the one who invented Gherardi's false rules believed that they were true, and also that those who added similarly false rules did so. The inventor of the rules for Dardi's irregular cases certainly knew why they were true for the specific situations from which they were derived; less certain is that he was aware of their limited validity (Dardi at least knew, though he probably was not able to describe precisely *when* they the rules would be valid). But we should remember that abbacus teaching was a liberal profession, where competition for jobs and students decided who could earn a living and who not. In that situation, anything goes *as long as it goes*, and the commands derived from the conditions of survival might easily overwhelm those derived from professional norms (a familiar situation known from other epochs and norm systems – cf. [Høyrup 2000: 351]). This is as true in the present century as in the 14th-15th centuries – and as true then as now, *vide* the conflict between the condemnation of usury and the eager teaching of the topic. In itself it tells nothing specifically about mathematics.

Yet the specifics of the use of false rules and of Giovanni di Davizzo's wonderful inventions does tell something.

Firstly, norm systems are double-edged. Most participants in a social system obey the norms, if not in all respects then largely; if not, norms would have no effect. If everybody was always lying, everybody would soon learn that words are not to be trusted, and lying would be nothing but futile sounds. Thus, a competitor presenting a false solution involving radicals (never approximated, as we remember) at a competition or in an exchange of challenges would not be exposed unless he had the rare bad luck to encounter an opponent who really understood algebra. So, just as physicians are supposed to obey professional ethos and to care for their patients (and fortunately do so generally), abbacus masters were *supposed* by those who were unable to control to pursue mathematical truth and consistency. Norms regulate the behaviour of most of us but protect the cheats.

Even this is not specific for mathematics, even though it explains some of the specifics of the invention and survival of the false roots.[578] But there is more to the matter, better illustrated by Giovanni di Davizzo.

---

578 It goes by itself that the introduction of printing was to change the situation. There were always *some* who understood the false rules to be false, and once communication became more dense they might get a public; in Italy, the false rules were killed if not before then by Pacioli's *Summa*, and they never invaded Germany. Only his being located in the extreme periphery allowed Bento Fernandes to repeat the false rules and the conflation of roots and negative powers in a printed book in 1555 (above, p. 200) – probably believing in their veracity and not possessing the competence to discover otherwise.

Solving problems which opponents could not solve might already carry a personal premium. Moreover, after the first introduction of reducible higher-degree problems, for instance in Jacopo's algebra, there seems to have been an urge to do better and to expand the scope of algebra, and that may have carried an extra premium (irrespective of whether the expansion was fake or not as long as it was believed). Accordingly we see more and more false solutions propping up over the next 150 years; but we also see writers who produced true expansions – Dardi, Antonio, the anonymous author of the Florentine *Tratato sopra l'arte arismetricha* – and the introduction of special roots (pronic, etc.). So, at its theoretical level at least (algebra, Pacioli's *Pratica speculativa*, see above, p. 348), abbacus mathematics was seen as an *expanding* body of knowledge (and a body of knowledge *to be expanded*), at a time where the prevailing general attitude was that the ancients had known better. But there is evidence that this idea of expansion was not restricted to those who were engaged in "theoretical practice". From Jacopo's introduction (above, p. 9) we may think of this passage:

> As in this treatise the mind and good intelligence grants us to know the great subtlety of the prophecies and the philosophies and the celestial and temporal writings, it will grant us to know even more henceforth, since by mind and good and subtle intelligence men make many investigations and compose many treatises which were not made by other people, and know to make many artifices and written arguments which for us bring to greater perfection things that were made by the first men.

This is certainly Jacopo's own formulation. But no other introduction was copied as often in other abbacus books, so we may conclude that in was in harmony with widespread attitudes in the abbacus ambience. Accordingly, somebody seriously engaged in abbacus mathematics should not only produce true and coherent knowledge, he should also produce *expand* knowledge.

That is also what Giovanni di Davizzo did, or at least tried to do. Retrospectively we may say that his innovation was futile; but since it went into a direction with no application, not even within abbacus algebra, he was not forced to discover, nor were those who copied from him.

All in all, the norms governing abbacus practice were not fundamentally different from ours – or, more important when it comes to historical impact, from those governing 17th-century theoretical mathematics.

Different they were, of course. "Precision" in commercial arithmetic is not the same thing as precision in a world debating which kind of constructions are precise – only those made by ruler and compass, or also those that can be made pointwise. But the similarity was sufficient to allow communication – not least of course when mediated by writers like Schreyber, Rudolff and Stifel, who did not care for virtuosity but rather for agreement with the norms of university mathematics. Without that communicability, abbacus mathematics (specifically abbacus algebra) would hardly have made the impact that shall be the topic of the following section.

The Zilsel thesis and the transformation of mathematics[579]

In [1942], Edgar Zilsel, an Austrian sociologist and member of the Vienna circle and at the time a refugee in the U.S., published a paper on "The Sociological Roots of Science". The abstract runs as follows:

> In the period from 1300 to 1600 three strata of intellectual activity must be distinguished: university scholars, humanists, and artisans. Both university scholars and humanists were rationally trained. Their methods, however, were determined by their professional conditions and differed substantially from the methods of science. Both professors and humanistic literati distinguished liberal from mechanical arts and despised manual labor, experimentation, and dissection. Craftsmen were the pioneers of causal thinking in this period. Certain groups of superior manual laborers (artist-engineers, surgeons, the makers of nautical and musical instruments, surveyors, navigators, gunners) experimented, dissected, and used quantitative methods. The measuring instruments of the navigators, surveyors, and gunners were the forerunners of the later physical instruments. The craftsmen, however, lacked methodical intellectual training. Thus the two components of the scientific method were separated by a social barrier: logical training was reserved for upper-class scholars; experimentation, causal interest, and quantitative method were left to more or less plebeian artisans. Science was born when, with the progress of technology, the experimental method eventually overcame the social prejudice against manual labor and was adopted by rationally trained scholars. This was accomplished about 1600 (Gilbert, Galileo, Bacon). At the same time the scholastic method of disputation and the humanistic ideal of individual glory were superseded by the ideals of control of nature and advancement of learning through scientific co-operation. In a somewhat different way, sociologically, modern astronomy developed. The whole process was imbedded in the advance of early capitalistic society, which weakened collective-mindedness, magical thinking, and belief in authority and which furthered worldly, causal, rational, and quantitative thinking.

So, according the Zilsel, the university tradition did not give rise to the scientific revolution – that had, in gross abridgement, been Pierre Duhem's idea. Nor did Renaissance Humanism or technicians do so. What was decisive was the *interaction* between the three groups, their mutual fecundation.

Zilsel died by suicide in 1944. Inspection of his *Nachlass* half a century later revealed that this essay and a handful of others (some published, others not) belonged within a larger project on "The social roots of modern science" [Raven & Krohn 2000: xxx–xxxiv]. In Zilsel's outline for this project, mathematics only enters in section IV, "The rise of the quantitative spirit", subsection 2, "mathematics and its relation to commerce, military

---

[579] This concluding section could be believed to be a contracted version of an article under way in a *Festschrift* – or that article to be an expanded version of what appears here. Actually, both descend from an article written for a volume that after ten years remains stuck in editorial process.

engineering, technology, and painting 1300–1600". It might none the less be worthwhile to ask whether Zilsel's thesis can be applied to the 17th–18th-century metamorphosis of mathematics, in which "analysis" (algebra, and soon infinitesimal analysis) replaced Greek-style geometry as the central discipline.

Applied *mutatis mutandis*, of course. Of Zilsel's groups, Renaissance Humanists can be taken over directly. Something like his university scholars also recur – not natural philosophers like Bradwardine, Swineshead and Albert of Saxony, however, but the readers of Euclid and of other ancient mathematicians; therefore, the group should be extended so as to include also the pre-university translators of the twelfth century. The role of Zilsel's higher artisans (artist-engineers, surgeons, instrument makers, surveyors, navigators and gunners), finally, will have to be taken over by the abbacus masters and the *Rechenmeister*.

The discussion is most conveniently ordered according to gross chronology.

12th-13th-century reception of algebra

Arabic algebra reached the Latin world through a number of channels during the 12th century:

– The second part of the *Liber alchorismi de pratica arismetice* (the "Toledan *Regule*"), probably compiled shortly after 1150. It contains a short fragment on the topic [ed. Burnett, Zhao & Lampe 2007: 163–165]; it had no influence whatsoever on later algebra, as can be seen from the lack of emulation of its particular terminology (Arabic *māl* is translated *res*, "thing", not *census*).
– the *Liber mahameleth* [ed. Vlasschaert 2010; ed. Sesiano 2014], probably a free translation of an Arabic original made in the Toledo environment around 1160 – see [Høyrup 2021a: 42–44]. There are cross-references to a systematic exposition of algebra, which may have been omitted already from the translation; in any case it is absent from all known manuscripts. A number of problems in the text we possess are solved by means of algebra; they are so different from anything else we find in later Latin writings that we may safely conclude that even they had no impact.
– Two translations of al-Khwārizmī's algebra have survived. One, from *ca* 1145, is due to Robert of Chester. It is known from three manuscripts, all prepared in south-German area around 1450 (above, p. 362). It uses *substantia* for *māl*, which is found nowhere else; it may have been read by Widmann, but the main influences on Widmann's algebraic thinking point to Italy. It appears to have had no influence on anybody else, even though copies must have circulated during the three centuries between 1145 and 1450.
– The other was made by Gerard of Cremona around 1170. An appreciable number of manuscripts survive – one was in the possession of Regiomontanus; it was almost certainly also consulted by Fibonacci [Miura 1981].
– Guglielmo de Lunis also translated al-Khwārizmī's algebra during the first half of

the 13th century (cf. above, p. 313) – whether into Latin or an Italian vernacular we cannot know. As we have seen, this translation was used by the three Florentine encyclopedists and seems to have been known to Giovanni del Sodo, but apart from that we know nothing about it. We may assume its impact to have been rather limited. In any case, Guglielmo's translation belongs to the 13th century.

– In the beginning of chapter 14 of the *Liber abbaci* Fibonacci makes use of a work that uses *solidatio* for *al-jabr*. It *could* be the lost algebra chapter of the *Liber mahameleth*, but the only argument in favour of that hypothesis is the absence of contrary evidence – cf. above, note 143.
– Finally, Fibonacci must have drawn on a Romance-vernacular collection of algebra problems using *avere* for Arabic *māl* when the word does not serve in its algebraic function.

Neither of the final two had any influence beyond the *Liber abbaci*. We may conclude that algebra reached Latin Europe through Gerard's translation, and that all other channels were unimportant.

Even Gherardo's translation, however, made no strong impression – for good reasons on the receiving side, we may say. There were two main motives for translating philosophical and scientific works from the Arabic. One was the aspiration to get hold of those central works that were known by name and fame from Martianus Capella's *Marriage of Mercury and Philology* and other encyclopedias but were otherwise unknown. This would not call for translation of al-Khwārizmī, nor indeed for appropriation of algebra. The other is "medico-astrological naturalism", which had astronomy subservient to astrology as an essential ingredient (other ingredients being medicine, magic and astrology *stricto sensu*). Those who were familiar with the Arabic tradition would know that al-Khwārizmī's algebra was reckoned among the "middle books" (together with Euclid's *Data* and various works on spherics): those books that were to be read between the *Elements* and the *Almagest* [Steinschneider 1865]. The translators probably knew, and for them it would be an obvious choice to translate it; al-Khwārizmī's introduction to the Hindu-Arabic numerals was similarly translated as an essential tool for astronomical table-making and calculation. There is a difference, however. Work with astronomical tables was only possible if one was familiar with Hindu-Arabic numerals and algorism. Algebra, on the other hand, was of no use in astronomical or astrological practice – the closest we come is Regiomontanus's 15th-century use of algebra in two demonstrations in *De triangulis* (above, p. 359). In consequence university scholars had no reason to be interested in algebra.

The algebra contained in the *Liber abbaci* as chapter 15 was discussed at length above. As we saw, Fibonacci draws to some extent on al-Khwārizmī but also on other works circulating in the Arabic world. There are no traces, however, of the algebraic symbolism that was created in the (presumably outgoing) twelfth century in the Maghreb, nor of al-

Karajī's elaboration of a theory of polynomials or his approaches to a purely algebraic proof technique.

Fibonacci, like al-Khwārizmī, gives geometric proofs of the rules for solving the mixed second-degree equations, of a similar kind though not the same. Further on in his algebra section, geometric proofs abound which have no counterpart in al-Khwārizmī. For Fibonacci, *proof* was *geometric proof*, in agreement with his orientation toward Greek theory (probably already that of mathematicians from al-Andalus from whom he borrowed).

With one possible exception, we have no evidence that the *Liber abbaci* was read outside Italy before Jean de Murs did so in the mid-14th century.

The possible exception is Jordanus de Nemore. Perhaps in the later 1220s,[580] he wrote the treatise *De numeris datis* [ed. Hughes 1981]. It emulates the format of Euclid's *Data* and applies it to the arithmetical domain. It is deductively organized and contains propositions of the form "if certain arithmetical combinations of certain numbers are given, then the numbers themselves are also given",[581] and formulates the proofs in an abstract letter symbolism. Jordanus does not mention algebra at all, but he gives numerical examples that often coincide with what can be found in corresponding problems in properly algebraic works. He thus leaves no doubt that he had undertaken to reformulate algebra as a demonstrative arithmetical discipline, leaving so many traces that those who knew algebra would recognize the undertaking.

In many cases, the numerical examples coincide with those of al-Khwārizmī. In others, they point to either Abū Kāmil or Fibonacci [Høyrup 1988: 310 n.10] – and since the known Latin translation of Abū Kāmil's algebra may have been made in the 14th century – thus is the disputed claim of Sesiano [1993: 315–317] – Jordanus may have known the *Liber abbaci* (*may* – but this is not a case of the excluded third). A further suggestion (still nothing beyond a mere suggestion) in the same direction comes from what Jordanus presents in II.27 as "the Arabic method" to solve a pure-number version of a problem of type "purchase of a horse", which has some similarity (namely non-trivial parameters) to what we find in the *Liber abbaci* [B245–248;G400–403].

De numeris datis goes beyond mere reformulation. The quest for deductivity as well

---

[580] The treatise is written after his *De elementis arithmetice artis*, to which it refers, and the latter after the second version of the algorism treatises. Here, indeed, the letter symbolism is first developed in primitive form which was then used to the full in the *De elementis*. One of the algorism treatises is copied (apparently by Robert Grosseteste) in 1215/16 [Hunt 1955: 133*f*].

[581] For instance, I.17, "When a given number is divided into two parts, if the product of one by the other is divided by their difference, and the outcome is given, then each part will also be given". IV.9 indicates the existence of a double solution to what we would express $x^2+b = ax$ as follows: "a square which with the addition of a given number makes a number that is produced by its root multiplied by a given number, can be obtained in two ways".

as Jordanus's general inclinations cause the outcome of his undertaking to be at least as much of a piece of coherent theory as the Euclidean model. It also goes beyond what was done in Arabic algebra – book II, starting with what corresponds to the rule of three (thus the equivalent of the final chapter of Gherardo's translation of al-Khwārizmī's algebra) develops into a wide-ranging exploration of proportion theory. Book III contains further elaboration of the same topic.[582]

A small circle seems to have existed around Jordanus, comprising Campanus and Richard de Fournival while being at least known to Roger Bacon [Høyrup 1988: 343–351]. It is regularly claimed that *De numeris datis* became the standard algebra textbook in the scholastic university. Unfortunately, there is no documentary basis whatsoever for assuming that there *was* any algebra teaching there before Widmann, and *a fortiori* not for assuming that Jordanus's treatise was used. What we know from the 14th century is that Oresme cites the *De elementis* and the *De numeris datis* in three of his works[583]. Oresme being without competition the foremost Latin mathematician of his century,[584] *his* use of another eminent mathematician proves little concerning his contemporaries – except perhaps that, if even he drew nothing beyond elementary non-algebraic matters from Jordanus, nobody probably did.

In the 15th century, Peurbach and Regiomontanus demonstrate that they not only knew the work but also understood in what way it was related to the Arabic art and in which way it differed. In a poem, Peurbach [ed. Größing 1983: 210] refers to "the extraordinary ways of the Arabs, the force of the entirety of numbers so beautiful to know, what algebra computes, what Jordanus demonstrates". The other is Regiomontanus, in whose Padua lecture on the mathematical sciences from 1464 [ed. Schmeidler 1972: 46] we read about the "three most beautiful books about given numbers" which Jordanus

> had published on the basis of his *Elements of arithmetic* in ten books. Until now, however, nobody has translated from the Greek into Latin the thirteen most subtle books of Diophantos,

---

[582] A few of the propositions from book III coincide with what can be found in chapter 15 part 1 of the *Liber abbaci* (e.g., III.14 and III.15). The two contexts are so different, however – both texts are systematic, but they are organized according to different principles – that this coincidence is likely to be accidental.

[583] The former in *Algorismus proportionum* [ed. Curtze 1868: 14], in *De proportionibus proportionum* [ed. Grant 1966: 140, 148, 180] and in *Tractatus de commensurabilitate vel incommensurabilitate motuum celi* [ed. Grant 1971: 294] (merely a complaint that Jordanus's subtle work is inapplicable to the presumably irrational ratios of celestial speeds); the latter in *De proportionibus proportionum* [ed. Grant 1966: 164, 266] – both passages refer to propositions about elementary proportion theory and not to Jordanus's crypto-algebra.

[584] Antonio may have been his mathematical peer in as far as such things can be measured across disciplines, but he wrote in Tuscan and not in Latin.

> in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name.

The reference to Diophantos anticipates Regiomontanus's interaction with the Humanist current; for the moment we shall take it as another way to specify the relation between Jordanus's treatise and the algebraic discipline.

In the list of books left by the later less famous Vienna astronomer Andreas Stiborius in *ca* 1500 we find as neighbouring items Euclid's *Data*, Jordanus's *De numeris datis* and *Demonstrationes cosse* (an unidentifiable work on algebra but evidently in Italo-German tradition) [Clagett 1978: 347]. Either Stiborius or Georg Tannstetter (who made the list) thus understood *De numeris datis* as belonging midway between Euclid's *Data* and algebra.

Jordanus was certainly an eminent representative of the universitarian mathematical environment, even if his work had little impact on the further development of university mathematics. Fibonacci is less easy to categorize. He wrote in Latin; as we have seen, much of the material he presents is similar to what we find later in the abbacus tradition; he often applies methods belonging to the "magisterial" mathematical tradition, in particular geometric reasoning in Euclidean style; and he refers to abbacus-type methods as "vernacular".[585] His reception within the university tradition was negligeable; and so was his impact on the abbacus tradition at least until the mid-15th century. It is thus impossible to locate him within any of the three traditions; what we can say is that he is a witness of the existence of something close to the later abbacus tradition already around 1200 (though hardly in Italy), and that his book is a first attempt at synthesis between the practical and the scholarly tradition – an impressive and heroic but premature attempt.

The 14th century – early abbacus algebra, and first interaction

Abbacus teachers and schools are mentioned in the sources from 1265 onward, teaching young boys a curriculum encompassing the following (cf. above, p. 4) – in summay:

– First the practice of numbers: writing numbers with Hindu-Arabic numerals; the multiplication tables and their application; division by divisors known from the multiplication tables, and next by multi-digit divisors; calculation with fractions.
– Then topics from commercial mathematics: the rule of three; monetary and metrological conversions; simple and composite interest, and reduction to interest per day; partnership; simple and composite discounting; alloying; the technique of a "single false position"; and area measurement.

Everything, from the multiplication tables onward, was accompanied by problems to be solved as homework. Manuscript books being expensive, the teaching was evidently oral.

---

[585] [B63,111,114,115,127,170,204,364;G107,190,198,219,290,342,563,564].

The “abbacus books” written by many teachers were thus not meant as textbooks for the school. Some were written explicitly as gifts to patrons or friends, some perhaps as teachers’ handbooks (that is at best an educated guess), some claim to be suited for self-education. They often include topics that go beyond the curriculum, such as the double false position and algebra. These may have served in the training of assistant-apprentices, but this is a speculation with no support in the sources; in any case we know that proficiency in such difficult matters was important in the competition for employment or for paying pupils.

The probably earliest extant abbacus book (though extant as a 14th-century copy only) is the “Columbia algorism” (above, p. 170). It reveals some puzzling affinities with Iberian 14th-century material. The *Livero de l’abbecho* (above, p. 159), probably slightly but not very much later, claims in its introductory lines to be written “according to the opinion” of Fibonacci. As close analysis shows, the treatise moves on two levels. One, elementary and corresponding to the curriculum of the school, borrows nothing at all from Fibonacci; the other consists almost exclusively of sophisticated problems borrowed from Fibonacci – but demonstrably often borrowed without understanding, and without the compiler having followed the calculations. Fibonacci cannot have inspired the actual teaching of the compiler; his role is that of prestigious decoration. The incomplete Pisan *Libro di ragioni* probably belongs to the earliest 14th century. What could at first suggest inspiration from Fibonacci turns out at closer scrutiny to point instead to the commercial connections of Pisa in the Maghreb.

Later ordinary abbacus treatises owe no more to Fibonacci; actually, they do not even refer to him. The Florentine encyclopedias do refer to and borrow from him; but they are not run of the mill but exceptions, and then in their own mathematics they not really in debt to Fibonacci). The abbacus tradition did not (as often claimed without the slightest support in the sources) derive from Fibonacci’s *Liber abbaci* and *Pratica geometrie*. It had its roots in the larger Mediterranean tradition for commercial calculation – in Arabic *muʿamalāt* mathematics, in particular perhaps in Iberian practices. Fibonacci had been acquainted with the same practices a small century earlier, but by presenting what he had learned from them according to scholarly norms he had effectively barred diffusion to the mathematically humble abbacus teachers.

Further details about the origin of the abbacus tradition are of no concern for the present question. All that has to be taken note of is that it existed as an independent tradition.

Algebra was no part of the early abbacus tradition – the compiler of the *Livero* demonstrates by occasional misunderstandings of Fibonacci’s words that he has never heard about it. The earliest abbacus algebra is likely to be the one contained in Jacopo da Firenze’s *Tractatus algorismi*, written in Montpellier in 1307 (above, p. 186).

This algebra is very different, both from that of the *Liber abbaci* and from anything we know (in the original language or in translation) from the hands of al-Khwārizmī, Abū

Kāmil and al-Karajī (although it has more in common with al-Karajī's elementary *Kafī* than with his advanced *Fakhrī* and *Badī'* and with the other two authors). Its descent from Arabic algebra is indubitable, and the use of the term *census* (Tuscanized as *censo*) for *māl* is shared by various Iberian twelfth-century translations.

Jacopo first presents rules for the six basic cases (those of the first and second degree), already dealt with by al-Khwārizmī. These are provided with examples. Then follow fourteen that can either be solved by simple root extraction or reduced to one of the initial six examples.They are not followed by examples.

The "root" has disappeared from the rules, being everywhere replaced by the "thing" (Tuscan *cosa*), and all rules are formulated so as to cover non-normalized equations. More significant, all references to geometric proofs have disappeared.

Finally, Jacopo's examples not only differ in actual content from those encountered in al-Khwārizmī (etc.) and the *Liber abbaci*, many of them also differ in character. Those of al-Khwārizmī and Fibonacci (and of Abū Kāmil too) are either pure-number problems or, at most, deal with an unspecified "capital" or with an amount of money divided between a number of men. Half of Jacopo's ten examples pretend to deal with real commercial problems – and one with a square root of *real* money, not merely a formal *māl*. Commercial problems, we may observe, abound in Ibn Badr's *Ikhtisār al-jabr wa'l-muqābalah* [ed. Sánches Pérez 1916], *possibly* of Iberian origin and in any case known in the Iberian world, and square roots of real money are copious in the *Liber mahamaleth*.

The further development of algebra was to build on this foundation – not all of it directly on Jacopo, but on the same source tradition. Within a couple of decades, however, new elements were added, presumably borrowed directly or indirectly from what had been developed in the Maghreb and/or al-Andalus in the twelfth century: calculation with "formal fractions" (e.g., $\frac{100}{1\ cosa}+\frac{100}{1\ cosa\ -\ 5}$); use (mostly unsystematic use) of abbreviations for *root*, *cosa* and *censo*; and schemes for the calculation with arithmetical and algebraic binomials. More problematic, and probably no borrowing but a local development, is that the field becomes infested with false solutions to irreducible third- and fourth-degree cases, surviving into the mid-16th century.

Some authors understood that the false solutions *were* false. In 1344, Dardi wrote the earliest extant treatise dedicated exclusively to algebra (above, p. 216). He solves no less than 194 cases correctly – a huge number he attains by including complicated radicals (e.g., $\alpha c+\beta\sqrt{K} = \gamma Ç$), whose correct treatment shows that he understood the nature of the sequence of algebraic powers well. He also includes 4 rules for irreducible cases which only hold under special circumstances (as he says without specifying these); they are almost certainly not his own brew, but the one who derived them from obviously reducible cases by changes of variable must have had a very good understanding of polynomial algebra.

The Florentine *Tratato sopra l'arte della arismetricha* from the outgoing 14th century contains a very long chapter on algebra (above, p. 240). Here, the nature of the sequence of algebraic powers as a geometric progression is set out explicitly, and it is shown how equations of the types $K+\beta Ç = m$, $K = \beta Ç+m$ and $\beta Ç = K+m$ can be reduced to the form $K = n+\alpha c$. The transformed non-reduced coefficients show beyond doubt that the author makes the change of variable and the consecutive operations exactly as we would perform them. We also find schemes for the multiplication of trinomials, modelled after the algorithm *a scacchiera* ("on chessboard") for multiplying multi-digit numbers (see above, p. 65).

15th-century copies of Antonio de Mazzinghi's late 14th-century writings show that *his* insights went even deeper (above, p. 230). But they were also exceptional and had no impact of relevance for our theme, and we may leave them aside.

14th-century Humanism, as represented by such outstanding figures as Petrarca and Boccaccio (not to speak of the mere teachers of *studia humanitatis*, good Latin style – the Humanist in the proper sense), was purely literary. It did not make any attempt to get in touch with mathematics, neither universitarian nor of abbacus type; nor did university mathematicians or abbacus masters take any interest in what these Humanists were doing.

One well-known university mathematician, however, took up algebra, in part from Gherardo's translation of al-Khwārizmī, in part from the *Liber abbaci*, in part from familiarity with unidentified abbacus writings: Jean de Murs, in his *Quadripartitum numerorum* from c. 1343 [ed. l'Huillier 1990], which in Regiomontanus's prospectus of books he intends to get into print [ed. Schmeidler 1972: 531] stands alongside Jordanus's *De numeris datis*. Regiomontanus does not characterize it as an algebraic work, nor is it indeed one when taken as a whole. It consists of four books and a "half-book" (*semiliber*). Book I is in a mixed Boethian-Euclidean tradition, whereas book II deals with calculation with the Hindu-Arabic numerals and with fractions. These two books are thus firmly rooted in the scholarly mathematical tradition as it had been shaped from the twelfth century onward – the fraction-part of book II, however, rooted in twelfth-century works which we know from annotations to have been consulted by Jean[586] rather than in the university tradition, which (because Hindu-Arabic numerals served astronomy) was primarily interested in "physical" or "philosophical", that is, sexagesimal fractions.

Book III, the first to deal with algebra, is also in the scholarly tradition. At first it takes up proportion theory (chapters 1–8); next follows an exposition of algebra, not copied from Gerard yet in its beginning close to him – but omitting his geometric proofs. However, while writing this chapter Jean must have come across the *Liber abbaci*: the

---

586 Namely the "Toledan regule" and a truncated copy of the *Liber mahameleth*, both contained in the manuscript Paris, latin 15461 [l'Huillier 1990: 35, 37]. Both deal not only with ordinary fractions but also with "fractions of fractions", the *Liber mahameleth* further with ascending continued fractions – both types customarily used in Arabic mathematics (and in the *Liber abbaci*). Jean takes up both types [ed. l'Huillier 1990: 204, 250].

first three problems following after the general presentation are from al-Khwārizmī, but the rest are borrowed from Fibonacci, as shown by Ghislaine l'Huillier.

Between book III and book IV, Jean now inserts a *semiliber* or "half-book", stated to be an "explanation of what preceded and presentation of what comes".[587] Here, and also in book IV, the inspiration from the *Liber abbaci* is conspicuous – not only from its algebra section but also from chapter 12, the collection of mixed predominantly recreational problems. Even the *regula recta* turns up under the name *ars rei*, "the art of the thing" [ed. l'Huillier 1990: 418, 420*f*], mostly but not always in borrowed problems where Fibonacci already uses it. Jean also promises to propose many questions in book IV illustrating the method, but actually does not do so.[588]

But not everything in the *Quadripartitum* that is new to the school tradition comes from Fibonacci. Quite striking it the appearance and discussion of formal fractions [ed. l'Huillier 1990: 468*f*], for instance $\frac{res}{10\ re\ diminuta}$, that is, $\frac{thing}{10\ diminished\ by\ a\ thing}$. Jean even operates on them, adding (using our above abbreviations) $\frac{10-c}{c}$ and $\frac{c}{10-c}$ and finding the correct result $\frac{100-20c-2Ç}{10c-Ç}$ – precisely as done in contemporary advanced abbacus algebra. Besides that, we find systematic (but dubious) work on the products of algebraic powers and roots [ed. l'Huillier 1990: 463–469], going beyond what had been done by al-Khwārizmī (but related to what Dardi must have known – and known better – a few years before). In addition, Jean uses the powers of 2 as an explanatory parallel to the algebraic powers – a device that was to be used commonly in the late 15th century and which may have had 14th-century abbacus antecedents unknown to us.[589]

So, Jean adopts into a scholarly treatise material both from Fibonacci and from what was produced in his own times in the abbacus environment, and attempts to subject it to the methodological norms of scholarly mathematics (not always with great success, he is no outstanding mathematician and tends to err when working on his own on difficult matters). But he does more. The methods by which "recreational" problems about pursuit are treated in book IV are applied afterwards to the astronomical problem of conjunctions (Jean was an eager practising astrologer no less than a mathematician, particularly interested in conjunctions – cf. [Poulle 1973: 131]). So, his aim is multi-faceted synthesis,

---

[587] L'Huillier [1990: 13] argues that from the problem section of book III onward it is far from certain that things were written in the order they appear in the treatise. For our present purpose, this is immaterial. In any case, her main argument does not hold water, see imminently.

[588] Non noticing that a particular method is spoken of, l'Huillier [1990: 13] believes that the questions referred to are those actually located at the end of book III, which is one of her arguments for doubting the order of the material. However, none of these problems make use of the "art" in question, for which reason this hypothesis must be rejected.

[589] Another instance of use by Jean of abbacus material unknown to us is found in his *De arte mensurandi* [ed. Busard 1998: 187*f*], see [Høyrup 1999].

not just incorporation.

According to Regiomontanus's prospectus, the work was "gushing with subtleties". Unfortunately (if we allow ourselves a disputable moral judgement of history – better, unfortunately for Jean and his project), not many tended to see his work in that way, neither in his own nor in Regiomontanus's century. "Time was not yet ripe" – that is, those who had such interests were too rare to get into direct or indirect contact and to develop a common undertaking.

The 15th century – the beginnings of a *ménage à trois*

In the 15th century, some abbacus teachers took over norms both from the Humanist movement and from scholarly mathematics – a few Humanists showed interest in mathematics (including abbacus mathematics) – and some mathematicians with university education and career took interest in "Humanist" (to be explained) as well as abbacus mathematics.

Two of the three "abbacus encyclopedias" (above, p. 249) written around 1460 – Benedetto's *Praticha d'arismetricha* and the homonymous Palatino manuscript – show evidence of Humanist orientation. Both authors, when writing on their own, are fully immersed in the abbacus tradition – specifically in a particular Florentine school tradition reaching back over Antonio de' Mazzinghi and Paolo dell'Abbacho to Biagio "il vecchio". However, both also demonstrate Humanist interest in the foundations of their discipline. In the initial presentation of algebra, they choose not to base themselves on Fibonacci (whose problems they give later in separate chapters) nor on more recent authors from their school tradition (equally quoted at length with due reference) but on al-Khwārizmī (in Guglielmo's version) – according to Benedetto because his proofs are *piu antiche*. Their way to render Fibonacci's algebraic problems is also evidence of Humanistic deference to a venerated text – no changes are made, no new marginal commentaries are added, the margins only contain Fibonacci's own diagrams. Their respectful copying from predecessors in their school tradition points in the same direction.

But both also have ambitions to wrap their mathematics in scholarly garments. Book 2 of Benedetto's treatise, dealing with "the nature and properties of numbers" is a presentation of speculative arithmetic in the Boethian tradition. It also offers an exposition and explanation of the complicated way ratios are named in this tradition. The first part of Benedetto's book 5 (on "the nature of numbers and proportional quantities") builds on the Campanus version of *Elements* V–IX and on Campanus's *De proportione et proportionalitate* about the composition of ratios (the second part treats of metrological conversions). The first part of his Book XI presents *Elements* II. The Palatino writer is less ambitious, but his chapter II.8 still deals with "the way to express as part, and, first, the definition", initially quoting Boethius's, Euclid's and Jordanus's definition of a ratio (*proportione*) as a relation between two numbers or quantities, going on afterwards with the Boethian names. This is not unproblematic: according to the definition a ratio is *not* a (possibly broken) number, as is the "part" the author wishes to express. He sees the difficulty but chooses to regard it as a mere question of language: "we in the schools do not use such terms [*vocaboli*] but say instead [...] that 8 is $\frac{2}{3}$ of 12 and 12 is $\frac{3}{2}$ of 8" (fol. 17^v). He also points to the necessity that the two magnitudes in a ratio be of the same kind, but overlooks that this should create difficulties when, later, the concept is used to explain the rule of three.

This illustrates well the limited ambition (and actual reach) of this integration of abbacus and scholarly mathematics (Benedetto's as well as that of the Palatino anonyme): when it seems fitting, abbacus procedures or concepts are explained within the framework of scholarly mathematics; but the authors reinterpret concepts as needed, and the contradictions that arise are disregarded.

Pacioli (above, p. 334) had similar aims, and in his case we also see them reflected in his biography: he rose socially from being a teacher of abbacus mathematics to having the rank of a court mathematician (until Ludovico Sforza was driven out from Milan by the French) and to being a lecturer on and translator and editor of Euclid.

His *Divina proportione* – written while he was in Milan but printed in [1509a] – is obviously inspired by Humanism (and by the wish to flatter the princely protector) in its long introduction (Pacioli is always longwinded) and elsewhere, and attempts to make mathematics a legitimate courtly-Humanist subject.

Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalita* from [1494] is different in orientation. The contents is primarily an encyclopedic presentation of abbacus mathematics. However, the authorities from whom Pacioli pretends initially to have borrowed most of the material are Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco and Prosdocimo de' Beldomandi[590] – all Latin writers (Pacioli's Euclid is the Campanus edition), and all except Fibonacci bright stars on the heaven of university mathematics (but, excepting instead Euclid and Boethius, not exactly luminaries on that of contemporary Humanists). The work is thus (as also confirmed by the contents) in its general orientation a parallel to the *Liber abbaci*, submitting abbacus material to the norms of scholarly mathematics. The algebra to which Pacioli had access and which he presents is certainly much more sophisticated than what we find in Fibonacci, as illustrated by their different lists of "keys" (above, pp. 117 and 342, respectively). Among unidentifiable others, Antonio de' Mazzinghi plays a role [Høyrup 2009d: 99]. But while Antonio may have understood at least in practice that purely algebraic demonstration was feasible, Pacioli stoops (like Fibonacci) to the idea that proof has to be geometric proof – apparently a regression if we look at matters in the perspective of the development of algebra as an autonomous branch of scientific mathematics, but perhaps less so if we think of

[590] Borrowed he certainly has, but beyond Euclid and Fibonacci his main sources are earlier abbacus masters.

Cardano's proof of the solutions for the cubic equations (below, note 595).

Among university mathematicians taking up algebra, the central figure is Regiomontanus. At least after coming in close contact with Bessarion in 1460–61, he clearly worked intensely to connect mathematics with Humanist ideals. In his Padua lecture from 1464, as we remember, he observed that until then "nobody [had] translated from the Greek into Latin the thirteen most subtle books of Diophantos, in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name". He thus wants to understand algebra as a legitimately ancient and Greek art, or to make the audience see it thus.[591]

We also know the kind of algebra he practised when calculating in private, namely from his notes to the correspondence with Bianchini (above, p. 359). This is in the style of Florentine abbacus algebra of his own times. He also uses algebra twice in *De triangulis* (above, p. 359). Even this algebra is in abbacus style. The Humanist connection certainly had no impact on his algebraic *practice*, neither inspiring transformation nor preventing its use – *nor could the problems to which Regiomontanus applied the technique ask for more than what its traditional shape had to offer*.

Another university-trained mathematician is Wolack, who held university lectures in Erfurt in 1467 and 1468 – in 1486 followed by Widmann teaching in Leipzig (above, pp. 353 and 370). Wolack's lecture may have been the first *public* exposition of abbacus mathematics in German area, and shows that this kind of mathematics was gaining acceptance in the universities of southern German. However, "cossic" algebra (that is, Germanized algebra in abbacus style) already circulated in German manuscripts around a decade before, whereas Wolack taught no algebra (Widmann did).

Some Humanists, most notably Bessarion, already thought around 1460 that mathematics (but in particular mathematical astronomy) was an important part of the ancient legacy. The earliest Humanist of reputation to *take up* mathematics was probably Leone Battista Alberti. When looking at his treatises on perspective, in particular at his *Elementi di pittura* [ed. Grayson 1973: 109–129] we may find a merger of a broadly Humanist (but more precisely, artistic) and a mathematical outlook. Since this has nothing to do with algebra, I shall not pursue the question. His *Ludi rerum mathematicarum* [ed. Grayson 1973: 131–173] turns out not to offer more. Most of the work concerns elementary sighting geometry and area measurement. There is no trace of it being taken from

[591] In order to know that the work should have 13 books, Regiomontanus must have read at least (in) Diophantos's preface to book I. Since he believes all 13 books were actually present in the manuscript, he can not have inspected the whole of it. If after the preface he has read no more than book I, he will have had no occasion to discover that Diophantos mainly investigates indeterminate problems, and thus presupposes but hardly presents techniques similar to those of Arabic algebra.

contemporary abbacus geometry, even though that would have been possible. The authorities that are cited [ed. Grayson 1973: 153] are Columella and Savazorda [*sic*] among the ancients[592] and "Leonardo Pisano among the moderns". Through its conception of mathematics as noble leisure, the work may have served to provide mathematics with Humanist legitimacy; but it did nothing for mathematics beyond that.

The first Humanist edition of ancient mathematical texts is Giorgio Valla's posthumous *De expetendis et fugiendis rebus* from [1501], where quadrivial matters, including Euclidean excerpts but also music and astronomy cum astrology are dealt with in books I-XIX; the second is Bartolomeo Zamberti's translation of the *Elements* from an inferior Greek manuscript, printed in [1505]. The former is a *florilège* and the second a mere text edition (made moreover, as pointed out by Maurolico, by a translator who knew Greek but was so far from being up to the topic that he did not discover the blunders of his inferior manuscript[593]).

Around 1500, at the very beginning of French Humanism, we also have Lefèvre d'Étaples mathematical editions. Their character is well illustrated by the purely medieval-quadrivial-universitarian contents of the volume he brought out in [1496]:

– Jordanus's *Arithmetica decem libris demonstrata*;
– Lefévre d'Étaples' own *Elementa musicalia* in Boethian tradition;
– his *Epitome in duos libros arithmeticos divi Severini Boecii;*
– his description of *rithmomachia*, a board game invented around 1000 and serving to train the concepts of Boethian arithmetic.

No wonder, perhaps, that Humanism had had nothing to offer to mathematics in the preceding century – a fortiori to algebra.

1500–1575: a changing scenery

After Pacioli's time, the abbacus environment *per se* was no longer theoretically productive in algebra – Cardano and Stifel, certainly advancing algebra in continuation of the abbacus tradition, were scholars; Tartaglia, like Pacioli, worked hard and successfully to become one; Bombelli was an engineer-architect. Printed books linked directly to abbacus-teaching (like [Borgi 1484], serving as "introduction for any youth dedicated to trade" – above, p. 327) tend to include no algebra (thus agreeing with the school curriculum). At most they would repeat what had been made before 1500 – like Ghaligai's

---

592 Since he can only have known Savasorda through Plato of Tivoli's translation, his sensibility to the "barbaric" Latin of the twelfth century cannot have been as acute as Humanists would like it to be.

593 For instance in two letters *Illustrissimo Domino D. Ioanni Vegae* and *Illustrissimo Ac Reverendissimo Domino D. Marco Antonio Amulio* [Maurolico 1556; 1568]. See also [Rose 1975: 165].

Summa de arithmetica from [1521] (above, p. 332), where the last chapters (those that deal with algebra) are drawn from what Ghaligai had been taught about that topic by his master Giovanni del Sodo in the late 15th century.

The first to find the solution to certain irreducible third-degree equations – Scipione del Ferro, around 1505 – was a university professor, but his way to communicate it confidently to friends and students who could then use it in competitions shows vicinity to the abbacus norm system. However, since we have no knowledge about the deliberations that led him to the goal,[594] he is uninteresting for our purpose.

Let us therefore first look at a physician and intellectually omnivorous scholar who had turned his interest to abbacus-type mathematics, namely Cardano (already referred to on page 389). Most of his mathematical writings have problems or methods from abbacus mathematics as their starting point. But they are written in Latin, and their shared aim is to produce scholarly mathematics, mostly in agreement with (some sort of) Euclidean norms. Further, Cardano was versed in Humanist culture; this is already obvious from the language and the rhetoric of his *Encomium geometriae*, "Praise of Geometry", read at the Academia Platina in Milan in 1535 [Cardano 1663: 440–445] – not to speak of his non-mathematical writings.

That Cardano implored Tartaglia to give him the solution to the cubic equations and then found the proofs (publishing them with due reference once he discovered that Tartaglia had no priority)[595] does not set him apart from what had been done in abbacus algebra at its best since centuries; nor does the fact that he went on and showed in the *Ars magna* [1545] how other mixed cubic equations could be reduced to these types (the necessary techniques were already used in the algebra contained in the Florentine *Tratato*, as we have seen on p. 245). But Cardano went on from here to questions that had not been raised by any abbacus writer as far as we know, investigating the relation between coefficients and roots, and using the theory of irrationals of *Elements* X in order to find conditions which solutions would have to fulfil. He was not the first to work with negative numbers – Pacioli had done so, as well as the Florentine *Tratato*; Stifel had also done so a year

---

594 Masotti [1971: 596], following [Vacca 1930], shows how mere experimentation with cubic binomials might suffice. Since this had been in focus since the earlier 14th century, and since it had been known by the more insightful abbacus algebraists for almost as long that the solutions that circulated were false, an attentive del Ferro may well have taken note if such play suddenly gave an interesting result.

595 This is Cardano's version of the story; given his work on the problems to find two numbers from their product and either their sum or their difference in [Cardano 1539] it sounds plausible – once you see the solution formulae for the equations $K+\alpha c = n$ and $K = \alpha c+n$, it is almost immediately clear that they have the corresponding structure, and from there the road to the geometric proof is also easy – in particular because of the importance such proofs had reacquired in algebra since Pacioli.

before, as we have observed on p. 389. But Cardano did so more effortless that any precursor, and in the end of the book he even introduces their roots and operates with them – possibly because he had run into them when working on cubic problems, but actually on the basis of the second-degree problem $r+t = 10$, $rt = 40$, which everybody before him would just have dismissed as "impossible". A similar experimental spirit had been present in the university environment in Oresme's time, but certainly not after 1400. Nor was it common in 15th-century *Humanism* – Lorenzo Valla is the only exception that comes to (my) mind. But it was not foreign to the spirit that developed in the Humanism of the mid-16th century – we may think of two works written around the same time in Humanist style and famous in the history of science, Vesalius's *De fabrica humanis corporis* from [1543] and Agricola's *De re metallica* from [1556]. Both are respectful toward Antiquity – Agricola even shapes his title after Columella's *De re rustica* – but both are also proud to follow tracks which (as they point out) had never been explored by the ancients.

In 1545, it was possible for Cardano to pursue revolutionary novelties in algebra. Other famous writers in the field were less revolutionary. Stifel's *Arithmetica integra* from [1544] (above, p. 386) introduced some innovations, but on the whole Stifel set forth and systematized "all that was then known about arithmetic and algebra" [Vogel 1976: 59] and generalized in a way his predecessors had not done (in his development of polynomial algebra as well as in his use of symbolism). He presented everything (or almost) developed or used by *some* Italian abbacus algebraist and deployed it systematically in a way *none* of them (including Pacioli) had done. We may say that Stifel brought the project of abbacus algebra to completion, as also recognized by those who borrowed from him – Tartaglia in Italy, Peletier in France.[596]

In 1535, Cardano had referred to Grynaeus's edition of the Greek Euclid with Proclos's commentary, published two years earlier. The *editio princeps* of Pappos's *Collection* appeared in Basel in 1538 (Commandino's Latin translation in 1588, after having circulated in manuscript), that of Archimedes in Basel in 1543; Memmo's Latin edition of books I–IV of Apollonios's *Conics* appeared in 1537 (Commandino's in 1566); Xylander's Latin translation of Diophantos was published in 1575 (the Greek *editio princeps* only in 1621). Only from the 1530s or 1540s onward is it thus possible to distinguish a genuine Humanist interest in mathematics. Maurolico's and Commandino's work in mathematics also began around this time.

---

596 Petrus Ramus is of course an exception; in [1569: 66] he goes as far as to mask Stifel's very existence, which for somebody with Ramus's psychological constitution amounts to a confession that his algebra from [1560] depends (in all its poverty, and maybe indirectly) on the *Arithmetica integra*.

However, being a mathematically interested Humanist was not sufficient to be able to contribute actively to the development of algebra. A good example is Peletier's *L'algebre* from [1554], which is decent but brings nothing new with respect to Stifel (in spite of Peletier's engagement in *linguistic* symbolization [1555]). Even being actively interested in Greek mathematics was not enough – here we may think of Buteo's *Logistica* from [1559]. A perfect precursor of Molière's *précieuses ridicules* (who also existed outside comedy), he finds the term *Arithmetica* too vulgar, and introduces *Logistica*. He writes ρ for the first power of the unknown (so had Benedetto done in 1463, and the Florentine *Tratato* around 1400); for the second power he uses ♦, for the third , and P respectively M where Stifel, following Widmann, had used + and –; even these two signs he may have considered vulgar since mercantile. His geometric proofs for the solution of the mixed cases refer explicitly to *Elements* II, and he adds and subtracts polynomials in schemes (as done in Italy since the 14th century). But he has no further theory, only problems, and none of his problems go beyond what could be found among 14th–15th century abbacists. In all probability he had no intention to go beyond his predecessors, his aim may well have been to submit the elementary textbook genre to linguistic and notational purification (we may perhaps think of the father of Humanists, Petrarca, who would rather be ill than cured by Arabic-inspired university medicine).[597]

Even Maurolico, a far better mathematician than Buteo and not burdened by linguistic prudishness, did little more in his short manuscript *Demonstratio algebrae* [ed. Fenaroli & Garibaldi, n.d.], and probably intended to do no more. The treatise is an orderly presentation of the sequence of algebraic powers as a geometric progression, with rules for multiplication and division. As had been formulated in many more words by Pacioli (and by other abbacists before him), Maurolico states that the traditional rules for the mixed second-degree cases can be used for all three-term equations where the middle power is "equidistant" (Maurolico uses Pacioli's word and makes no attempt to show off by speaking of geometric means) from the other two; and his geometric proofs refer to *Elements* II.

From the mid-16th century onward, Boethius's *Arithmetica* and *De musica* gradually lost ground in university curricula, being replaced not by anything Humanist but rather by such works as Gemma Frisius's *Arithmeticae practicae methodus facilis* [1540]; but Humanism, with its emphasis on civic utility and polite leisure, may have contributed to preparing the ground for this (slow) change.

The take-off of Modern algebra

One effect of the reception of the full Greek mathematical heritage and the first creative work based on it was that *problems* moved into the focus of scholarly

[597] *Lettere senili* XII, 2 [ed. Fracassetti 1869: II, 260*f*].

mathematics,[598] in contrast to the emphasis on *theory* of the high and late medieval Euclidean tradition.[599] This change of focus was due in part to what was found in the ancient texts themselves (not least Pappos), in part also to the kind of activity that came out of attempts to work creatively within the new theoretical framework provided by these texts; but it was certainly furthered by the type of social interaction in which players like Viète, Fermat and Mersenne participated, competitive and communicative at the same time.

But the change was not solely toward the solution of problems taken in isolation; it also implied interest in the general conditions for solvability and the general character of solutions – that is, in a new kind of theory. Inspiration for this theory and some answers (the classification of plane, solid and linear problems) could come from Pappos; but that did not suffice. Diophantos provided challenges rather than answers. Algebra, on the other hand, had always been primarily a technique for solving problems, and it already had some successes to exhibit within the kind of mathematics that now had the foreground. It will therefore have seemed obvious to re-investigate it in order to draw from it *not* just isolated problem solutions but also higher-level information.

This is indeed what we read in Descartes' introduction to his *Discours de la Methode* [ed. Adam/Tannery 1902: 17f] (to which *La geometrie* is one of three appendices purportedly meant to test the new method): he had hoped to get assistance for his project from logic, and, among the branches of mathematics, from "the analysis of the geometers and from algebra". But he immediately discards logic as an art which only serves to explain to others what one already knows, or even to speak of what one does not know. Analysis is too intimately bound up with the consideration of geometrical figures; algebra, finally has been so much "subjected to certain rules and certain signs that one has made out of it a confused and obscure art that puts the mind in difficulty instead of a science that cultivates it" (Stifel's rules and cossic symbols, still used in Clavius's *Algebra* from [1608], which Descartes had been taught after in the Jesuit school). Algebra thus seems to offer some hope, if only it could be liberated from these rules and signs – which is indeed one of the things done in *La geometrie*.[600]

---

598 See [Bos 1996: 186–188] and, in general, [Bos 2001].

599 Beyond the theologically tainted preference for the "speculative" over the "active", the importance of which should perhaps not be overstated, the teaching style of universities certainly played a major role in the creation of this emphasis. Lectures would allow the exposition of theory, and disputation invited metatheoretical reflections on the status and ontology of the discipline. Even written *quaestiones*, emulating the style of the disputation, invited philosophy of mathematics and not eristic work on problems.

600 In contrast, we may think of the more modest ambitions expressed by Nuñez in his *Libro de algebra en arithmetica y geometria*, published in [1567] but possibly written well before that year.

The algebra that was taken over was not directly that of the abbacus masters but abbacus algebra as ordered and developed by Rudolff, Stifel and Cardano (and by Niccolò Tartaglia, Pedro Nuñez, Simon Stevin and Christoph Clavius), and as it was also known through French writers like Jacques Peletier and Guillaume Gosselin) – a creative synthesis the abbacus tradition with the meta-theoretical norms of high and late medieval university mathematics.

Evidently, Viète's famous reference to "a new art, or rather so old and so defiled and polluted by barbarians that I have found it necessary to bring it into, and invent, a completely new form" [Viète 1591: 2^{v}] is in itself a Humanist confession. As demonstrated by Regiomontanus, however, such confessions might be mere lip service.[601] The mere wish to distinguish himself from the Arabs was certainly not what inspired Viète's reformulation of the whole discipline, at most what induced him to use the terms *logistica*, *analysis*, *zetetics* and *poristics* – and it did not keep him from using also the term *algebra*

Nuñez's aim is to show the wonders algebra can perform. However, in the geometry section we find statements like these (in total, Nuñez offers 77):

– If the side of a square is known, the area will also be known;
– if the sum of the diameter and the side is known, each of them will also be known;
– if the side and the diameter and the area joined together, each of them will be known;
– if the product of the diameter and the area of the square is known, each of them will be known;
– if one of the sides [of a rectangle] and the diameter are known, the area will be known;
– if the area [of a rectangle] is known, and the two sides containing a right angle joined in one sum is known, each of the sides will be known;
– if the ratio of the two sides is known, and the magnitude of the diameter is known, or the ratio of the diameter to one of the sides as well as the other side are known, each of the others will be known;
– if the sides of a right triangle joined in one sum is known, and they are in [continued] proportion [...], each of the three will be known;
– if the two sides of a triangle are known, and the ratio between the parts of the base where the perpendicular falls is known, the base and the perpendicular and the area will be known;
– if the three sides of the triangle are known, and a circle is described which touches its three sides, the semi-diameter of the circle will be known, and the parts of the sides divided by the touching points, and the distances from the centre to the angles of the triangle will also be known.

Only those about non-right triangles go beyond what we could find, for instance, in Abu Bakr's *Liber mensurationum*, translated by Gerard of Cremona 400 years earlier [ed. Busard 1968]. These triangle problems, however, illlustrate why algebra might look as a promising tool already for somebody like Viète with his aim to leave no problem unsolved (*nullum non problema solvere*, [1591: 9^{r}]), and which indeed was to allow him to go far beyond what Nuñez had done.

601 Actually, Clavius [1608: 4] quotes Regiomontanus' ascription to Diophantos as more verisimilar than the belief that the art be Arabic. But what is found in his book is quite in Stifel's style.

albeit *nova*, instead of leaving it (like Jordanus) to readers to discover. Such rhetoric characterizes him as a scholar of Humanist constitution. But what caused his *mathematics* (and that of Descartes and Fermat, and of others who did not contribute to the reshaping of algebra) to be Humanist, or rather post-Humanist, was their participation in an endeavour made possible (and next to compulsory for active theoretical mathematicians) by that relatively full access to the best ancient mathematicians that had been provided by 16th-century Humanism.

I shall not undertake a detailed analysis of the aims and the novelties of Viète's and Descartes' algebra – I would not be able to add anything of importance to Richard Witmer's "Translator's Introduction" [1983] nor to [Bos 2001] (to name but these two). As an argument that the reformulation of the discipline was really needed for the post-Humanist mathematical project, and instead of drowning myself in a study of Fermat, I shall point to an episode that took place a small decade after the publication of Descartes' *Geometrie*. In 1645–46, the adolescent Huygens studied mathematics under the guidance of Franz van Schooten – during the same years editor of Viète's *Opera mathematica* [van Schooten 1646] and soon-to-be Latin translator of Descartes' *Geometrie* [Descartes 1649]. Vol. 11 of Huygens' *Oeuvres* contains a number of problems he investigated in this period by means of Cartesian algebra, many of which deal with matters inspired by Archimedes and Apollonios [Huygens 1908: 27–60]. Another sequence of problems (pp. 217–275), dated around 1650, is derived in part from Pappos. It is difficult to imagine that they could have been efficiently dealt with by algebraic notations in Cardano's or Stifel's style.

The total recomposition of mathematics brought about by the new (double) *analysis* may not have been totally comprehended at the time – cf. [Malet 2005]. How it was soon to be seen, however, is well illustrated by this excerpt from Jean-Pierre de Crouzas' preface to his *Traite de l'algebre* [1726: á iii^{v}]:

> Algebra is of so great utility in the rest of the mathematics; by its help one solves geometric problems with so much facility; the calculations one needs in physics, and their demonstration, are made by its help in such a brief and such a short way, that one has to know it and grasp how to use it.
>
> Before the discovery of differential calculus, [algebra] was considered almost the last effort of the human mind. There has even been a time where this study was held to be dangerous, and where it was counted as secure proof of a good head to have succeeded in it without being troubled, and without the reason being damaged by it.

Abbacus algebra, filtered through Cardano, Stifel and others, provided the material for the emergence of the "new algebra" of the 17th century, the first branch of the new analysis – and soon, after further interaction with natural philosophy, for the creation of the second branch, infinitesimal analysis. But it could only do so when applied in a context where problems were taken from or inspired by what was found in Archimedes, Apollonios and Pappos. Works like Marten Wilken's *Flores algebraici, das ist Algebra oder Coss* *mit schönen ausserlesenen künstlichen theils resolvirten new erfundenen Quaestionen und Exemplen* [1622] exemplify what mathematically ambitious authors would bring forth in the earlier 17th century in situations where this context was lacking; on his title page, Wilken presents himself as "*Rechenmeister* in Emden" and his high point is a large collection of higher-degree algebraic problems that can be solved by *cossam quadratam*. As scores of *Rechenbuch* authors before him, Wilken competes *on the book market* by claiming that "such things have never been seen before".

The context of intellectual competition based on problems inspired by Archimedes, Apollonios and Pappos, finally, had materialized only when (a small current within) Humanism had taken care to make the ancient mathematical texts accessible and interesting.

All in all, Zilsel's general thesis appears to receive unexpected and uninvited support from the creation of the new algebra and analysis, the central constituents of the transformed mathematical undertaking – first in mathematical practice but at least with d'Alembert's generation also in ideology. In this *ménage à trois*, all were fathers, mothers and midwives simultaneously, nobody was a mere bystander.

Sigla

D_1: Dardi, *Aliabraa argibra*. Vatican, Chigi M.VIII.170 .
D_2: Dardi, *Aliabraa argibra*. Siena, Biblioteca Comunale I.VII.17, used via [Franci 2001].
D_3: Dardi, *Aliabraa argibra*. Tempe, Arizona State University. used via Van Egmond's personal transcription.
F: Jacopo da Firenze, *Tractatus algorismi*. Florence, Riccardiana MS 2236.
L: Fibonacci, *Liber abbaci*, chapter 12. Florence, Biblioteca Medicea Laurenziana, Ms. Gaddi 36.
M: Jacopo Firenze, *Tractatus algorismi*. Milan, Trivulziana MS 90.
T_f: *Trattato di tutta l'arte dell'abacho*. Florence, BNC, fond. prin. II,IX.57.
T_r: *Trattato di tutta l'arte dell'abacho*. Rome, Accademia Nazionale dei Lincei, Cors. 1875.
V: Jacopo da Firenze, *Tractatus algorismi*. Vatican, Vat. Lat. 4826.
V_F: Fibonacci, *Liber abbaci*. Vatican, Pal. Lat. 1343.
ω: Fibonacci's working copy for the *Liber abbaci*.

Bibliography

Ordered according to first author. Multi-volume works are keyed to the year where the first volume was published.

Abdeljaouad, Mahdi, 2005. “Le manuscrit mathématique de Jerba: Une pratique des symboles algébriques maghrébins en pleine maturité”, pp. 9–98 *in Actes du 7ème Colloque Maghrébin sur l’Histoire des Mathématiques Arabes*, Marrakech, 30 mai – 1 juin 2002, vol. II. Marrakech: École Normale Supérieure Marrakech.

Adam, Charles, & Paul Tannery (eds), 1902. *Oeuvres* de Descartes. VI: *Discours de la méthode* & *Essais*. Paris: Léopold Cerf.

Ageron, Pierre, & Hmida Hefdi, 2020. “Ibrahīm al-Balīshṭār’s Book of Arithmetic (*ca.* 1575): Hybridizing Spanish Mathematical Treatises with the Arabic Scientific Tradition”. *Historia Mathematica* **52**, 26–50.

Agricola, Georgius, 1556. *De re metallica* libri xii. Eiusdem *De animantibvs subterraneis liber* Basel: Froben.

al-Farghānī, Abū al-ʽAbbās, 1546. *Alfragani astronomorum peritissimi compendium*. Paris: Andreas Wechelum.

Albert, Johan, 1541. *New Rechenbüchlein auf der Federn, gantz leicht,aus rechtem grund, inn Gantzen und Gebrochen*. Wittenberg.

Allard, André (ed., trans.), 1981. Maxime Planude, *Le Grand calcul selon les Indiens*. Louvain-la-Neuve: [Faculté de Philosophie et Lettres].

Allard, André (ed.), 1992. Muḥammad ibn Mūsā al-Khwārizmī, *“Le Calcul indien” (Algorismus).* Histoire des textes, édition critique, traduction et commentaire des plus anciennes versions latines remaniées du XIIe siècle. Paris: Blanchard / Namur: Société des Études Classiques.

Arrighi, Gino (ed.), 1964. Paolo dell’Abbaco, *Trattato d’aritmetica*. Pisa: Domus Galilæana.

Arrighi, Gino, 1966a. “La matematica del Rinascimento in Firenze: L’ereditá di Leonardo Pisano e le «botteghe d’abaco»”. *Cultura e scuola* **5**:18, 287–294.

Arrighi, Gino (ed.), 1966b. Paolo dell’Abbaco, *Regoluzze*. Secondo la lezione del codice 2511 della Biblioteca Riccardiana di Firenze. Prato: Azienda autonoma di Turismo di Prato.

Arrighi, Gino (ed.), 1967a. Antonio de’ Mazzinghi, *Trattato di Fioretti* nella trascelta a cura di Mº Benedetto secondo la lezione del Codice L.IV.21 (sec. XV) della Biblioteca degl’Intronati di Siena. Siena: Domus Galilaeana.

Arrighi, Gino, 1967b. “L’aritmetica speculativa nel ‘Trattato' di M° Benedetto”. *Physis* **9**, 311–336.

Arrighi, Gino, 1967c. “Un «programma» di didattica di matematica nella prima metà del Quattrocento (dal Codice 2186 della Biblioteca Riccardiana di Firenze)”. *Atti e memorie dell’Accademia Petrarca di Lettere, Arti e Scienze di Arezzo*, Nuova Serie **38** (1965–67), 117–128.

Arrighi, Gino (ed.), 1970a. Francesco di Giorgio Martini, *La praticha di gieometria*. Dal codice Ashburnam 361 della Biblioteca Laurenziana di Firenze. Firenze: Giunti.

Arrighi, Gino (ed.), 1970b. Piero della Francesca, *Trattato d’abaco*. Dal codice ashburnhamiano

280 (359*-291*) della Biblioteca Medicea Laurenziana di Firenze. A cura e con introduzione di Gino Arrighi. Pisa: Domus Galilæana.

Arrighi, Gino (ed.), 1973. Scuola lucchese, *Libro d'abaco.* Dal Codice 1754 (sec. XIV) della Biblioteca Statale di Lucca. Lucca: Cassa di Risparmio di Lucca.

Arrighi, Gino (ed.), 1974. Pier Maria Calandri, *Tractato d'abbacho*. Dal codice Acq. e doni 154 (sec. XV) della Biblioteca Medicea Laurenziana di Firenze [the ascription is mistaken, the author is Benedetto da Firenze]. Pisa: Domus Galilaeana.

Arrighi, Gino, 1980. "Una importante lezione dell'opera di m. Paolo dell'abaco (Il Cod. 2511 della Biblioteca Riccardiana di Firenze)". *Atti della Fondazione Giorgio Ronchi* **35**:6, 3–19.

Arrighi, Gino (ed.), 1982. Tommaso della Gazzaia, fl. 1387–1415, *Praticha di geometria e tutte misure di terre* dal ms. C. III. 23 della Biblioteca comunale di Siena. Trascrizione di Cinzia Nanni. Introduzione di Gino Arrighi. (Quaderni del Centro Studi della Matematica Medioevale, 1). Siena: Servizio editoriale dell'Università di Siena.

Arrighi, Gino (ed.), 1985. Giovanni de' Danti Aretino, *Tractato de l'algorisimo*. Dal Cod. Plut. 30. 26 (sec. XIV) della Biblioteca Medicea Laurenziana di Firenze. *Atti e Memorie della Accademia Petrarca di Lettere, Arti e Scienze*, nuova serie **47**, 3–91.

Arrighi, Gino (ed.), 1987. Paolo Gherardi, *Opera mathematica: Libro di ragioni – Liber habaci*. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze. Lucca: Pacini-Fazzi.

Arrighi, Gino (ed.), 1989. "Maestro Umbro (sec. XIII), *Livero de l'abbecho*. (Cod. 2404 della Biblioteca Riccardiana di Firenze)". *Bollettino della Deputazione di Storia Patria per l'Umbria* **86**, 5–140.

Arrighi, Gino (ed.), 1991. "Maestro Umbro (sec. XIII), *Amastramento de l'arte de la geometria*. (Cod. 2404 dalla Biblioteca Riccardiana di Firenze)". *Bollettino della Deputazione di Storia Patria per l'Umbria* **88**, 5–31.

Arrighi, Gino, 2004/1965. "Il codice L.IV.21 della Biblioteca degl'Intronati di Siena e la «bottega dell'abaco a Santa Trinita di Firenze»", pp. 129–159 *in* Gino Arrighi, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci & L. Toti Rigatelli. Pisa: Edizioni ETS. First published in *Physis* **7** (1965), 369–400.

Arrighi, Gino, 2004/1967. "Nuovi contributi per la storia della matematica in Firenze nell'età di mezzo: Il codice Palatino 573 della Biblioteca Nazionale di Firenze", pp. 159–194 *in* Gino Arrighi, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci & L. Toti Rigatelli Pisa: Edizioni ETS. First published in *Istituto Lombardo. Accademia di scienze e lettere. Rendiconti, Classe di scienze (A)* **101**, 395–437.

Arrighi, Gino, 2004/1968. "Il trattato d'abaco di M° Benedetto da Firenze del codice 5570 (sec. XV) della Biblioteca Marciana di Venezia", pp. 125–128 *in* Gino Arrighi, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci & L. Toti Rigatelli Pisa: Edizioni ETS. First published in *Bollettino della Unione Matematica Italiana*, s. IV *1* (1968), 146–150.

Barnes, Jonathan (ed.), 1984. The Complete *Works* of Aristotle. The Revised Oxford Translation.

2 vols. (Bollingen Series, 71:2). Princeton: Princeton University Press.

Bartolozzi, Margherita, & Raffaella Franci, 1990. “La teoria delle proporzioni nella matematica dell’abaco da Leonardo Pisano a Luca Pacioli”. *Bollettino di Storia delle Scienze Matematiche* **10**, 3–28.

Black, Robert, 2007. *Education and Society in Florentine Tuscany: Teachers, Pupils and Schools, c. 1250–1500*. Leiden & Boston: Brill.

Bocchi, Andrea, 2006. “Un libro d’abaco pisano del primo trecento”. *Studi linguistici italiani* **32**, 15–77, 177–209.

Bocchi, Andrea (ed.), 2017. *Lo livero de l’abbacho*. I. *Introduzione e testo critico*. Pisa: ETS.

Bonaini, F., 1857. “Memoria unica sincrona di Leonardo Fibonacci, novamente scoperta”. *Giornale Storico degli Archivi toscani* **1**, 239–246.

Boncompagni, Baldassare, 1851. “Della vita e delle opere di Leonardo Pisano matematico del secolo decimoterzo”. *Atti dell’Accademia pontificia de’ Nuovi Lincei* **5** (1851–52), 5–91, 208–246.

Boncompagni, Baldassare, 1852. “Della vita e delle opere di Leonardo Pisano matematico del secolo decimoterzo”. Roma: Tipografia delle Belle Arti.

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Boncompagni, Baldassare (ed.), 1862. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. II. *Practica geometriae* ed *Opusculi*. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Borgi, Piero, 1484. *Opera de arithmethica*. Venezia: Ratdolt.

Borghi, Piero, 1540. *Libro de abacho*. Venezia: Francesco Bindoni & Maffeo Pasini.

Bos, Henk J. M., 1996. “Tradition and Modernity in Early Modern Mathematics: Viète, Descartes and Fermat”, pp. 183–204 *in* Catherine Goldstein, Jeremy Gray & Jim Ritter (eds), *L’Europe mathématique. Mathematical Europe*. Paris: Éditions de la Maison des sciences de l’homme.

Bos, Henk J. M., 2001. *Redefining Geometrical Exactness: Descartes’ transformation of the Early Modern Concept of Construction*. New York etc.: Springer.

Brucker, Gene, 1969. *Renaissance Florence*. New York etc.: Wiley.

Bubnov, Nicolaus (ed.), 1899. Gerberti postea Silvestri II papae *Opera mathematica* (972 – 1003). Berlin: Friedländer.

Burnett, Charles, 2002. “Indian Numerals in the Mediterranean Basin in the Twelfth Century, with Special Reference to the ‘Eastern Forms' ”, pp. 237–288 *in* Yvonne Dold-Samplonius et al (eds), *From China to Paris: 2000 Years Transmission of Mathematical Ideas*. Stuttgart: Steiner.

Burnett, Charles, Ji-Wei Zhao & Kurt Lampe, 2007. “The Toledan *Regule:* a Twelfth-Century Arithmetical Miscellany”. *SCIAMUS* **8**, 141–231.

Busard, Hubert L. L. (ed.), 1967. “The Translation of the *Elements* of Euclid [books I–VI] from the Arabic into Latin by Hermann of Carinthia (?)”. *Janus* **54**, 1–142.

Busard, H. L. L., 1968. “L’algèbre au moyen âge: Le «Liber mensurationum» d’Abû Bekr”. *Journal des Savants*, 1968 no. 2, 65–125.

Busard, H. L. L. (ed.), 1983. *The First Latin Translation of Euclid’s ‘Elements' Commonly Ascribed to Adelard of Bath*. (Studies and Texts, 64). Toronto: Pontifical Institute of Mediaeval Studies.

Busard, Hubert L. L. (ed.), 1984. *The Latin Translation of the Arabic Version of Euclid's* Elements *Commonly Ascribed to Gerard of Cremona*. (Asfār, 2). Leiden: Brill.

Busard, H. L. L. (ed.), 1987. *The Mediaeval Latin Translation of Euclid's Elements made directly from the Greek.* Wiesbaden: Franz Steiner.

Busard, H. L. L., & M. Folkerts (eds), 1992. *Robert of Chester's (?) Redaction of Euclid's Elements, the so-called Adelard II Version*. 2 vols. Boston etc.: Birkhäuser.

Busard, H. L. L., 1997. "A Latin Translation of an Arabic Commentary on Book X of Euclid's *Elements*". *Mediaeval Studies* **59**, 19–110.

Busard, H. L. L. (ed.), 1998. Johannes de Muris, *De arte mensurandi*. A Geometrical Handbook of the Fourteenth Century. Stuttgart: Franz Steiner.

Busard, Hubert L. L. (ed.), 2001. *Johannes de Tinemue's Redaction of Euclid's Elements, the So-Called Adelard III Version*. 2 vols. Stuttgart: Franz Steiner.

Busard, H. L. L., 2005. *Campanus of Novara and Euclid's Elements*. 2 vols. Stuttgart: Franz Steiner.

Buteo, Joannes, 1559. *Logistica, quae et arithmética vulgò dicitur in libros quinque digesta*. Lyon: Guillaume Rouillé.

Calandri, Filippo, 1491. *De arimethrica opusculum*. Firenze: Lorenzo de' Morgiani & Giovanni Thedesco da Maganza, 1491,

Calandri, Filippo, 1518. *Ad nobilem & studiosum Iulianum Laurentii Medicem de Arimethrica opusculum*. Firenze: Bernardo Zucchecta, 1518.

Calzoni, Giuseppe, & Gianfranco Cavazzoni (eds), 1996. Luca Pacioli, *"tractatus mathematicus ad discipulos perusinos"*. Città di Castello: Delta Grafica.

Cantor, Moritz, 1875. *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst. Eine historisch-mathematische Untersuchung*. Leipzig: Teubner.

Cantor, Moritz, 1892. *Vorlesungen über Geschichte der Mathematik*. Zweiter Band. *Von 1200–1668*. Leipzig: Teubner.

Cantor, Moritz, 1900. *Vorlesungen über Geschichte der Mathematik.* Zweiter Band, *von 1200–1668*. Zweite Auflage. Leipzig: Teubner.

Cardano, Girolamo, 1539. *Practica arithmetice, et mensurandi singularis*. Milano: Bernardini Calusco.

Cardano, Girolamo, 1545. *Artis magnae sive de regulis algebraicis, liber unus*. Nürnberg: Johan Petreius.

Cardano, Girolamo, 1663. Hieronymo Cardani Mediolanensis Philosophi ac Medici celeberrimi *Operum* tomus quartus; quo continentur *Arithmetica, Geometrica, Musica*. Lyon: Jean Antoine Huguetan & Marc Antoine Ragaud.

Cassinet, Jean, 2001. "Une arithmétique toscane en 1334 en Avignon dans la citè des papes et de leurs banquiers florentins", pp. 105–128 *in Commerce et mathématiques du moyen âge à la renaissance, autour de la Méditerranée*. Actes du Colloque International du Centre International d'Histoire des Sciences Occitanes (Beaumont de Lomagne, 13–16 mai 1999). Toulouse: Éditions du C.I.H.S.O.

Caunedo del Potro, Betsabé, & Ricardo Córdoba de la Llave (eds), 2000. *El arte del alguarismo.*

Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV. (Ms. 46 de la Real Colegiato de San Isidoro de León). Salamanca: Junta de Castilla y León, Consejeria de Educación y Cultura.

Caunedo del Potro, Betsabé, 2004. "*De Arismetica*. Un manual de aritmética para mercaderes". *Cuadernos de Historia de España* **78** (2003–04), 35–46.

Chasles, Michel, 1841. "Histoire de l'algèbre. – I. Sur l'époque où l'algèbre a été introduite en Europe" – II. Sur les expressions *res* et *census*. Et sur le nom de la science, *Algebra et almuchabala*. *Comptes Rendus hebdomadaires des séances de l'Académie des Sciences* **13**, 497–524.

Chemla, Karine, & Guo Shuchun (eds), 2004. *Les neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

Chiarini, Giorgio, et al (eds), 1972. [Pierpaolo Muscharello], *Algorismus. Trattato di aritmetica pratica e mercantile del secolo XV*. 2 vols. Verona: Banca Commerciale Italiana.

Clagett, Marshall, 1978. *Archimedes in the Middle Ages*. Volume III, *The Fate of the Medieval Archimedes 1300–1565*. Philadelphia: The American Philosophical Society.

Clavius, Cristopher, 1608. *Algebra.* Roma: Bartolomeo Zanetti.

Corominas, Joan, & José A. Pascual, 1980. *Diccionario Crítico etimológico castellano a hispánico*. 6 vols. Madrid: Gredos, 1980–1983.

Corpus thomisticum. http://www.corpusthomisticum.org (Accessed 29.1.2006).

Cossali, Pietro, 1797. *Origine, trasporto in Italia, primi progressi in essa dell'algebra. Storia critica*. 2 vols. Parma: La Reale Tipografia.

Cowley, Elisabeth Buchanan, 1923. "An Italian Mathematical Manuscript", pp. 379–405 *in* Christabel F. Fiske (ed.), *Vassar Mediaeval Studies*. New Haven: Yale University Press.

Crusca: Vocabolario degli accademici della Crusca. Venezia: Giovanni Alberti.

Cullen, Christopher, 2007. "The *Suàn shù shū*, 'Writings on Reckoning': Rewriting the History of Early Chinese Mathematics in the Light of an Excavated manuscript". *Historia Mathematica* **34**, 10–44.

Curtze, Maximilian (ed.), 1868. Der *Algorismus Proportionum* des Nicolaus Oresme zum ersten Male nach der Lesart der Handschrift R. 4º. 2. der königlichen Gymnasial-Bibliothek zu Thorn herausgegeben. Berlin: Calvary & Co.

Curtze, Maximilian, 1895. "Ein Beitrag zur Geschichte der Algebra in Deutschland im 15. Jahrhundert". *Abhandlungen zur Geschichte der Mathematik* **7** (=*Supplement zur Zeitschrift für Mathematik und Physik* **40**), 31–74.

Curtze, Maximilian (ed.), 1902. *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance*. (Abhandlungen zur Geschichte der mathematischen Wissenschaften, vol. 12–13). Leipzig: Teubner.

d'Ooge, Martin Luther (trans.), 1926. Nicomachus of Gerasa, *Introduction to Arithmetic*. With Studies of Greek Arithmetic by Frank Egleston Robbins and Louis Charles Karpinski. New York: The MacMillan Company / London: MacMillan.

da Silva, Maria do Céu Pereira, 2008. "The Algebraic Contents of Bento Fernandes' *Tratado da* *arte de arismetica* (1555)". *Historia Mathematica* **35**, 190–219.

de Crouzas, Jean-Pierre, *Traite de L'Algebra*. Paris: François Montalant, 1726.

de la Roche, Etienne, 1520. *Larismethique novellement composee*. Lyon: Constantin Fradin.

Delle Donne, Fulvio, 2007. "Un'inedita epistola sulla morte du Guglielmo de Luna, Maestro presso lo studium di Napoli, e le traduzione prodotte alla corte du Manfredi di Svevia". *Recherches de Théologie et Philosophie médiévales* **74**, 225–245.

Descartes, René, 1649. *Geometria*. Leiden: Ian Maire.

Deschauer, Stefan (ed.), 1991. *Das 2. Rechenbuch von Adam Ries*. Nachdruck der Erstausgabe Erfurt 1522 mit einer Kurzbiographie, bibliographischen Angaben und einer Übersicht über die Fachsprache. München: Institut für Geschichte der Naturwissenschaften.

Devlin, Keith, 2017. *Finding Fibonacci: The Quest to Rediscover the Forgotten Mathematical Genius Who Changed the World*. Princeton and Oxford: Princeton University Press.

Di Teodoro, Francesco Paolo, 2014. "Pacioli, Luca". *Dizionario Biografico degli Italiani* **80**, online. https://www.treccani.it/enciclopedia/luca-pacioli_%28Dizionario-Biografico%29/ (accessed 18.9.2021).

Diels, Hermann, 1951. *Die Fragmente der Vorsokratiker, Griechisch und Deutsch*. Herausgegeben von Walther Kranz. 3 vols. 6. Auflage. Berlin: Weidmann, 1951–52.

Du Cange, Charles du Fresne, G. A. Louis Henschel, Pierre Carpenter, Léopold Favre, Johann Christoph Adelung, 1883. *Glossarium mediae et infimae latinitatis*. 10 vols. Niort: L. Favre, 1883–1887.

Edler, Florence, 1934. *Glossary of Mediaeval Terms of Business: Italian Series 1200-1600*. Cambridge, Mass.: The Mediaeval Academy of America.

Eneström, Georg, 1906. "Über zwei angebliche mathematische Schulen im christlichen Mittelalter". *Bibliotheca Mathematica*, 3. Folge **7**, 252–262.

Euclidis Megarensis mathematici clarissimi Elementorum Geometricorum libri XV. Cum expositione Theonis in Priores XIII à Bartholomeo Veneto Latinitate donata, Campani in omnes & Hypsiclis Alexandrini in duos postremos. His adiecta sunt Phaenomena, Catoptrica & Optica, deine Protheoria Marini et Data. Postremum verò, Opusculum de Levi & Ponderoso, hactenus non visum, eiusdem autoris. Basel: Johannes Herwagen, 1537.

Euler, Leonhard, 1774. *Élémens d'algebre*. I. *De l'analyse déterminée*. II. *De l'analyse indéterminée*. Traduit de l'allemand, avec des notes et des annotations. Lyon: Jean-Marie Bruyset.

Evans, Allan (ed.), 1936. Francesco Balducci Pegolotti, *La pratica della mercatura*. Cambridge, Mass.: The Medieval Academy of America.

Fanfani, Amintore, 1951. "La préparation intellectuelle à l'activité économique, en Italie, du XIV^{e} au XVI^{e} siècle". *Le Moyen Age* **57**, 327–346.

Fenaroli, Giuseppina, & Antonio Carlo Garibaldi (eds.), n.d.. Francesco Maurolico, *Demonstratio algebrae*. http://maurolico.free.fr/edizioni/arithmet/algebra/alg-1.htm (accessed 21.8.2011).

Figliucci, Felice (ed., trans.), 1563. *Divine lettere del gran Marsilio Ficino*. Nuovamente ristampate. 2 vols. Venezia: Gabriel Gioliti de' Ferrari.

Folkerts, Menso (ed.), 1978. "Die älteste mathematische Aufgabensammlung in lateinischer Sprache:

Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*". *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band, 6. Abhandlung (Wien).

Folkerts, Menso, 1980. "Die mathematischen Studien Regiomontans in seiner Wiener Zeit", pp. 175–209 *in* Günther Hamann (ed.), *Regiomontanus-Studien*. (Österreichische Akademie der Wissenschaften, Phil.-Hist. Klasse, Sitzungsberichte, Bd. 364). Wien: Österreichische Akademie der Wissenschaften.

Folkerts, Menso, 1996. "Andreas Alexander – Leipziger Universitätslehrer und Cossist?" pp. 53–61 *in* Rainer Gebhardt, *Rechenmeister und Cossisten der frühen Neuzeit.* Beiträge zum wissenschaftlichen Kolloquium, 21. September 1996, Annaberg-Buchholz, Deutschland. Freiberg: Technische Universität Bergakademie Freiberg.

Folkerts, Menso (ed.), 1997. *Die älteste lateinische Schrift über das indische Rechnen nach al-Ḫwārizmī.* Edition, Übersetzung und Kommentar. Unter Mitarbeit von Paul Kunitzsch. (Bayerische Akademie der Wissenschaften. Philosophisch-historische Klasse. Abhandlungen, neue Folge, 113). München: Verlag der Bayerischen Akademie der Wissenschaften.

Folkerts, Menso, 2004. "Leonardo Fibonacci's Knowledge of Euclid's *Elements* and of Other Mathematical Texts". *Bollettino di Storia delle Scienze Matematiche* **24**, 93–113.

Folkerts, Menso, 2006. *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*. Aldershot: Ashgate.

Fracassetti, Giuseppe (ed., trans.), 1869. Francesco Petrarca, *Lettere senili*, volgarizzate e dichiarate con note. 2 vols. Firenze: Le Monnier, 1869–70.

Franci, Raffaella, & Laura Toti Rigatelli, 1983. "Maestro Benedetto da Firenze e la storia dell'algebra". *Historia Mathematica* **10**, 297–317.

Franci, Raffaella, 1984. "Numeri congruo-congruenti in codici dei secoli XIV e XV". *Bollettino di Storia delle Scienze Matematiche* **4**, 3–23.

Franci, R., & L. Toti Rigatelli, 1985. "Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli". *Janus* **72**, 17–82.

Franci, Raffaella, & Marisa Pancanti (eds), 1988. Anonimo (sec. XIV), *Il trattato d'algibra* dal manoscritto Fond. Prin. II. V. 152 della Biblioteca Nazionale di Firenze. (Quaderni del Centro Studi della Matematica Medioevale, 18). Siena: Servizio Editoriale dell'Università di Siena.

Franci, Raffaella (ed.), 2001. Maestro Dardi, *Aliabraa argibra*, dal manoscritto I.VII.17 della Biblioteca Comunale di Siena. (Quaderni del Centro Studi della Matematica Medioevale, 26). Siena: Università degli Studi di Siena.

Franci, Raffaella, 2002. "Trends in Fourteenth-Century Italian Algebra". *Oriens–Occidens* 2002 n° 4, 81–105.

Franci, Raffaella, 2015. "Un trattato d'abaco pisano della fine del XIII secolo (Ms. L.VI.47 della Biblioteca degl'Intronati di Siena)". *Bollettino di storia delle scienze matematiche* **25**, 9-96.

Friedlein, Gottfried (ed.), 1867. Anicii Manlii Torquati Severini Boetii *De institutione arithmetica* libri duo. *De institutione musica* libri quinque. Accedit *Geometria* quae fertur Boetii. Leipzig: Teubner.

Gärtner, Barbara (ed.), 2000. *Johannes Widmanns »Behende vnd hubsche Rechenung«. Die Textsorte »Rechenbuch« in der frühen Neuzeit.* Tübingen: Max Niemeyer.

Gemma Frisius, Rainer, 1540. *Arithmeticae practicae methodus facilis*. Antwerpen: Gregorius Bontius, 1540

Gerhardt, Carl Immanuel, 1870. "Zur Geschichte der Algebra in Deutschland. Zweiter Theil". *Monatsberichte der Königlich Preussischen Akademi der Wissenschaften zu Berlin* (1870), 141–153.

Gerl, Armin, 1989. *Trigonometrisch-astronomisches Rechnen kurz vor Copernicus. Der Briefwechsel Regiomontanus-Bianchini*. Stuttgart: Franz Steiner.

Gerl, Armin, 1999. "Fridericus Amann", pp. 1–12 in Rainer Gebhardt (ed.), *Rechenbücher und mathematische Texte der frühen Neuzeit*. Tagungsband zum wissenschaftlichen Kolloquium anläßlich des 440. Todestages des Rechenmeisters Adam Ries, Annaberg-Buchholz, Deutschland, 16.–18. April 1999 (Schriften des Adam-Ries-Bundes Annaberg-Buchholz, 11). Annaberg-Buchholz: Adam-Ries-Bund.

Germano, Giuseppe, & Nicoletta Rozza (ed., trans.), 2019. Leonardo Pisano detto il Fibonacci, *Liber abaci. Il libro del calcolo*. Epistola a Michele Scoto – Prologo – Indice – Capitoli I–IV. Napoli: Paoli Loffredo.

Ghaligai, Francesco, 1521. *Summa de arithmetica*. Firenze: impr. Bernardo Zucchetta.

Giusti, Enrico, 1991. "L'algebra nel trattato d'abaco di Piero della Francesca: osservazioni e congetture". *Bollettino di Storia delle Scienze mathematiche* **11**, 55–83.

Giusti, Enrico, 1993. "Fonti medievali dell'*Algebra* di Piero della Francesca". *Bollettino di Storia delle Scienze matematiche* **13**, 199–250.

Giusti, Enrico, 2017. "The Twelfth Chapter of Fibonacci's *Liber abbaci* in Its 1202 Version". *Bollettino di Storia delle Scienze Matematiche* **37**, 9–216.

Giusti, Enrico (ed.), 2020. Leonardi Bigolli Pisano *vulgo* Fibonacci *Liber Abbaci*. Firenze: Leo S. Olschki.

Gleydtsmann, Georg, 1600. *Künstliches Rechenbuch So wol auff den Linien durch alle Species, als mit Ziffern*. Frankfurt a. M.: Zacharias Palthenius.

Goldthwaite, Richard A., 1972. "Schools and Teachers of Commercial Arithmetic in Renaissance Florence". *Journal of European Economic History* **1**, 418–433.

Goldthwaite, Richard A., 2009. *The Economy of Renaissance Florence*. Baltimore: Johns Hopkins University Press.

Grammateus, Heinrich, 1521. *Ayn new kunstlich Buech, welches gar gewiß und behend lernet nach der gemainen regel Detre, welschen Practic, regel falsi unn etlichen regeln Cosse*. Wien: Lucas Alantsee, [1521].

Grammateus, Heinrich, 1535. *Ayn new kunstlich Buech, welches gar gewiß und behend lernet nach der gemainen regel Detre, welschen Practic, regel falsi unn etlichen regeln Cosse*. Frankfurt a. M.: Christian Egenolff, [1535].

Grant, Edward (ed., trans.), 1966. Nicole Oresme, *De proportionibus proportionum* and *Ad pauca respicientes*. Madison etc.: University of Wisconsin Press.

Grant, Edward (ed., trans.), 1971. *Nicole Oresme and the Kinematics of Circular Motion. Tractatus de commensurabilitate vel incommensurabilitate motuum celi*. Madison etc.: University of Wisconsin Press.

Grayson, Cecil (ed.), 1973. Leon Battista Alberti, *Opere volgari*. Volume terzo. *Trattati d'arte, Ludi rerum mathematicarum, grammatica della lingua toscana, Opuscoli amatori, Lettere.* Bari: Laterza.

Gregori, Silvano, & Lucia Grugnetti (eds), 1998. Anonimo (sec. XV), *Libro di conti e mercatanzie*. Parma: Università degli Studi di Parma, Facoltà di Scienze Matematiche, Fisiche e Naturali, Dipartimento di Matematica.

Grendler, Paul F., 1989. *Schooling in Renaissance Italy: Literacy and Learning, 1300–1600*. Baltimore & London: Johns Hopkins University Press.

Grimm, Richard E., 1976. "The Autobiography of Leonardo Pisano". *Fibonacci Quarterly* **21**, 99–104.

Größing, Helmuth, 1983. *Humanistische Naturwissenschaft. Zur Geschichte der Wiener mathematischen Schulen des 15. und 16. Jahrhunderts*. Baden-Baden: Valentin Koerner.

Guy, Richard K., 2004. *Unsolved Problems in Number Theory*. Third edition. New York etc.: Springer.

Haskins, Charles Homer, 1924. *Studies in the History of Mediaeval Science*. Cambridge, Mass: Harvard University Press.

Heath, Thomas L. (ed., trans.), 1926. *The Thirteen Books of Euclid's Elements*. 2nd revised edition. 3 vols. Cambridge: Cambridge University Press / New York: Macmillan.

Hecht, Wolfgang (ed.), 1987. Jean Pauls *Werke*. 2 vols. Fünfte Auflage. Berlin & Weimar: Aufbau Verlag, 1987.

Heeffer, Albrecht, 2010. "From the Second Unknown to the Symbolic Equation", pp. 57–101 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*. London: College Publications.

Heeffer, Albrecht, 2012. "The Rule of Quantity by Chuquet and de la Roche and its Influence on German Cossic Algebra", pp. 127–147 *in* Sabine Rommevaux, Maryvonne Spiesser & Maria Rosa Massa Esteve (eds), 2012. *Pluralité de l'algèbre à la Renaissance*. Paris: Honoré Champion.

Heeffer, Albrecht, 2015. Draft transcription of Bianchini, *Flores Almagesti*, algebra section. 2015.

Heiberg, Johan Ludvig (ed., trans.), 1883. Euclidis *Elementa*. 5 vols. Leipzig: Teubner, 1883, 1884, 1886, 1885, 1888.

Herz-Fischler, Roger, 1987. *A Mathematical History of Division in Extreme and Mean ratio*. Waterloo, Ontario: Wilfrid Laurier University Press.

Hissette, Roland, 1997. "Guillaume de Luna ou de Lunis ou Lunense: un même traducteur d'Averroès et de traités d'*al-Jabr*?" *Bulletin de philosophie médiévale* **29**, 121–129.

Hoche, Richard (ed.), 1866. Nicomachi Geraseni Pythagorei *Introductionis arithmeticae* libri II. Leipzig: Teubner.

Hochheim, Adolf (ed., trans.), 1878. *Kâfî fîl Hisâb (Genügendes über Arithmetik)* des Abu Bekr

Muhammed ben Alhusein Alkarkhi. I-III. Halle: Louis Nebert, 1878–1880.

Hockey, Thomas, Virginia trimble & Thomas R. Williams (eds), 2007. *The Biographical Encyclopedia of Astronomers*. New York: Springer.

Høyrup, Jens, 1988. “Jordanus de Nemore, 13th Century Mathematical Innovator: an Essay on Intellectual Context, Achievement, and Failure”. *Archive for History of Exact Sciences* **38**, 307–363.

Høyrup, Jens, 1990a. “On Parts of Parts and Ascending Continued Fractions”. *Centaurus* **33**, 293–324.

Høyrup, Jens, 1990{c}. “*Dýnamis*, the Babylonians, and Theaetetus 147c7 – 148d7”. *Historia Mathematica* **17** (1990), 201–222.

Høyrup, Jens, 1998. “‘Oxford' and ‘Gherardo da Cremona': on the Relation between Two Versions of al-Khwārizmī’s Algebra”, pp. 159–178 *in Actes du 3me Colloque Maghrébin sur l’Histoire des Mathématiques Arabes, Tipaza (Alger, Algérie), 1–3 Décembre 1990*, vol. II. Alger: Association Algérienne d’Histoire des Mathématiques.

Høyrup, Jens, 1999. “The Finer Structure of the Old Babylonian Mathematical Corpus. Elements of Classification, with some Results”, pp. 117–177 *in* Joachim Marzahn & Hans Neumann (eds), *Assyriologica et Semitica*. Festschrift für Joachim Oelsner anläßlich seines 65. Geburtstages am 18. Februar 1997, Münster 1999. Münster: Ugarit Verlag.

Høyrup, Jens, 2000. *Human Sciences: Reappraising the Humanities through History and Philosophy*. Albany, New York: State University of New York Press, 2000.

Høyrup, Jens, 2001. “On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six ‘Algebras' ”. *Science in Context* **14**, 85–131.

Høyrup, Jens, 2002a. *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*. New York: Springer.

Høyrup, Jens, 2002b. “Seleucid Innovations in the Babylonian ‘Algebraic' Tradition and Their Kin Abroad”, pp. 9–29 *in* Yvonne Dold-Samplonius et al (eds), *From China to Paris: 2000 Years Transmission of Mathematical Ideas*. Stuttgart: Steiner.

Høyrup, Jens, 2004. “Conceptual Divergence – Canons and Taboos – and Critique: Reflections on Explanatory Categories”. *Historia Mathematica* **31**, 129–147.

Høyrup, Jens, 2005. “Leonardo Fibonacci and *Abbaco* Culture: a Proposal to Invert the Roles”. *Revue d’Histoire des Mathématiques* **11**, 23–56.

Høyrup, Jens, 2007. *Jacopo da Firenze’s Tractatus Algorismi and Early Italian Abbacus Culture*. Basel etc.: Birkhäuser.

Høyrup, Jens, 2008. “The ‘Unknown Heritage': Trace of a Forgotten Locus of Mathematical Sophistication”. *Archive for History of Exact Sciences* **62**, 613–654.

Høyrup, Jens, 2009a. “What Did the Abbacus Teachers Aim At When They (Sometimes) Ended Up Doing Mathematics? An Investigation of the Incentives and Norms of a Distinct Mathematical Practice”, pp. 47–75 *in* Bart van Kerkhove (ed.), *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics*. Singapore: World Scientific.

Høyrup, Jens, 2009b. “A Response to Van Egmond on Høyrup, *Jacopo da Firenze's* Tractatus Algorismi”. *Aestimatio* **6** (2009; actually 2010), 116–126.

Høyrup, Jens, 2009c. “The Rare Traces of Constructional Procedures in ‘Practical Geometries'”, pp. 367–377 *in* Horst Nowacki & Wolfgang Lefèvre (eds), *Creating Shapes in Civil and Naval Architecture*. Leiden & Boston: Brill.

Høyrup, Jens, 2009d. “‘Proportions’ in and around the Italian Abbacus Tradition”. *Physis* **46** (2009; published 2011), 55–110.

Høyrup, Jens, 2010. “Hesitating progress – the slow development toward algebraic symbolization in abbacus- and related manuscripts, c. 1300 to c. 1550”, pp. 3–56 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*. London: College Publications.

Høyrup, Jens, 2011a. “«Proportions» in the *Liber abbaci*”, pp. 83–100 *in* Sabine Rommevaux, Philippe Vendrix & Vasco Zara (eds), *Proportions: Science – Musique – Peinture & Architecture*. Actes du LI[e] Colloque International d’Études Humanistes, 30 juin – 4 juillet 2008. Turnhout: Brepols.

Høyrup, Jens, 2011. [Review of Stefan Deschauer, *Die Rigischen Rechenbücher. Spiegel einer lokalen mathematischen Tradition im Ostseeraum.* (Algorismus 73). Augsburg: ERV Dr. Erwin Rauner Verlag, 2010]. *Zentralblatt MATH* Zbl 1205.01008.

Høyrup, Jens, 2012. “Sanskrit-Prakrit Interaction in Elementary Mathematics As Reflected in Arabic and Italian Formulations of the Rule of Three – and Something More on the Rule Elsewhere”. *Gaṇita Bhārati* **34** (2012; published 2014), 144–172.

Høyrup, Jens, 2016. “Seleucid, Demotic and Mediterranean Mathematics versus Chapters VIII and IX of the *Nine Chapters:* Accidental or significant Similarities?”. *Studies in the History of Natural Sciences* **35**, 463–476.

Høyrup, Jens, 2017. “Practitioners – School Teachers – ‘Mathematicians': The Divisions of Pre-Modern Mathematics and Its Actors”, pp. 207–224 *in* John M. Steele & Mathieu Ossendrijver (eds), *Studies on the Ancient Exact Sciences in Honor of Lis Brack-Bernsen*. Berlin: Edition Topoi.

Høyrup, Jens, 2018. “When is the Algorithm Concept Pertinent – and When Not? Thoughts about Algorithms and Paradigmatic Examples, and about Algorithmic and Non-Algorithmic Mathematical Cultures”. *AIMS Mathematics* **3**, 211–232.

Høyrup, Jens, 2019a. *Selected Essays on Pre- and Early Modern Mathematical Practice*. Cham etc.: Springer.

Høyrup, Jens, 2019b. “Reinventing or Borrowing Hot Water? Early Latin and Tuscan Algebraic Operations with Two Unknowns”. *Gaṇita Bhārati* **41** (2019, published 2021), 115-159.

Høyrup, Jens, 2019c. “Hippocrates of Chios – His Elements and His Lunes: A Critique of Circular Reasoning”. *AIMS Mathematics* **5**, 158–184

Høyrup, Jens, 2021a. “Advanced Arithmetic from Twelfth-Century Al-Andalus, Surviving Only (and Anonymously) in Latin Translation? A Narrative That Was Never Told”, pp. 33–61 *in* Sonja Brentjes and Alexander Fidora (eds), *Premodern Translation: Comparative Approaches* *to Cross-Cultural Transformations*. Turnhout: Brepols.

Høyrup, Jens, "Observations and Reflections Concerning the First Volume of a New Edition of the *Liber abbaci*". *Athenaeum. Studi di Letteratura e Storia dell'Antichita* **109** (2021), 618–627.

Hughes, Barnabas B., O.F.M. (ed., trans.), 1981. Jordanus de Nemore, *De numeris datis*. A Critical Edition and Translation. Berkeley etc.: University of California Press.

Hughes, Barnabas, O.F.M. (ed.), 1986. "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*: A Critical Edition". *Mediaeval Studies* **48**, 211–263.

Hughes, Barnabas B. (ed.), 1989. Robert of Chester's Latin translation of al-Khwārizmī's *Al-jabr*. A New Critical Edition. Wiesbaden: Franz Steiner.

Hughes, Barnabas B., 2004. "Fibonacci, Teacher of Algebra. An Analysis of Chapter 15.3 of *Liber Abbaci*". *Mediaeval Studies* **66**, 313–361.

Hughes, Barnabas B., 2010. "An Early Abridgement of Fibonacci's *De practica geometrie*". *Historia Mathematica* **37**, 615–640.

Hultsch, Friedrich (ed., trans.), 1876. Pappi Alexandrini *Collectionis* quae supersunt. I, II, III.1-2. Berlin: Weidmann, 1876, 1877, 1878.

Hunger, Herbert, & Kurt Vogel, 1963. *Ein byzantinisches Rechenbuch des 15. Jahrhunderts. 100 Aufgaben aus dem Codex Vindobonensis Phil. Gr. 65*. (Österr. Akad. d. Wiss., phil.-hist. Kl., Denkschriften, Bd. 78, Abh. 2). Wien.

Hunt, Richard William, 1955. "The Library of Robert Grosseteste", pp. 121–145 *in* Daniel A. Callus, O.P. (ed.), *Robert Grosseteste, Scholar and Bishop*. Essays in Commemoration of the Seventh Centenary of his Death. Oxford: Clarendon Press.

Huygens, Christiaan, 1908. *Oeuvres complètes*. XI. *Travaux mathématiques 1645–1651*. La Haye: Martinus Nijhoff.

Jacob, Simon, 1565. *Ein neu und Wolgegründt Rechenbuch, auff den Linien und Ziffern, sampt der Welschen Practic und allerley Vortheilen, neben der Extraction Radicum, und von den Proportionen, mit vilen lustigen Fragen und Auffgaben, etc. Deßgleichen ein vollkommner Bericht der Regel Falsi* [...] *Und dann von der Geometria* [...]. Frankfurt am Main: Sigmund Fewyerabend & Simon Hüters.

Johns, Jeremy, 2002. *Arabic Administration in Norman Sicily: The Royal Diwān*. Cambridge: Cambridge University Press.

Karpinski, Louis C., 1910. "An Italian Algebra of the Fifteenth Century". *Bibliotheca Mathematica*, 3. Folge **11** (1910–11), 209–219.

Karpinski, Louis Charles, 1911. "The Algebra of Abu Kamil Shojaʿ ben Aslam". *Bibliotheca Mathematica*, 3. Folge **12** (1911–12), 40–55.

Karpinski, Louis C., 1929. "The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307". *Archeion* **11**, 170–177.

Kaunzner, Wolfgang, 1968. "Über das Eindringen algebraischer Kenntnisse nach Deutschland", pp. 91–122 *in* Kurt Elfering et al, *Rechenpfennige. Aufsätze zur Wissenschaftsgeschichte*. Kurt Vogel zum 80. Geburtstag am 30. September 1968 gewidmet von Mitarbeitern und Schülern. München: Forschungsinstitut des Deutschen Museums für die Geschichte der

Naturwissenschaften und der Technik.

Kaunzner, Wolfgang, 1971. “Deutsche Mathematiker des 15. und 16. Jahrhunderts und ihre Symbolik: Ein Brückenschlag in der Mathematik vom Altertum zur Neuzeit”. *Veröffentlichungen des Forschungsinstituts des Deutschen Museums für die Geschichte der naturwissenschaften und der Technik*. Reihe A. *Kleine Mitteilungen*, 90 (München).

Kaunzner, Wolfgang, 1972. “Über einige algebraische Abschnitte aus der Wiener Handschrift Nr. 5277”. *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band, 4. Abhandlung (Wien, 1972).

Kaunzner, Wolfgang, 1978. *Über die Handschrift Clm 26639 der Bayerischen Staatsbibliothek München: Eine mögliche Quelle zu Widmanns deutschem Rechenbuch von 1489. Ein Beitrag zur Geschichte der Mathematic im ausgehenden Mittelalter*. (Arbor Scientiarum. Beiträge zur Wissenschaftsgeschichte. Reihe B: texte, 1). Hildesheim: Gerstenberg.

Kaunzner, Wolfgang, 1980. “Regiomontanus als Mathematiker”, pp. 125–145 *in* Günther Hamann (ed.), *Regiomontanus-Studien*. (Österreichische Akademie der Wissenschaften, Phil.-Hist. Klasse, Sitzungsberichte, Bd. 364). Wien: Österreichische Akademie der Wissenschaften.

Kaunzner, Wolfgang, 1985. “Über eine frühe lateinische Bearbeitung der Algebra al-Khwārizmīs in MS Lyell 52 der Bodleian Library Oxford”. *Archive for History of Exact Sciences* **32**, 1–16.

Kaunzner, Wolfgang, 1986. “Die lateinische Algebra in MS Lyell 52 der Bodleian Library, Oxford, früher MS Admont 612”, pp. 47–89 *in* G. Hamann (ed.), *Aufsätze zur Geschichte der Naturwissenschaften und Geographie*. (Österreichische Akademie der Wissenschaften, Phil.-Hist. Klasse, Sitzungsberichte, Bd. 475). Wien: Österreichische Akademie der Wissenschaften.

Kichenassamy, Satyanad, 2015. “Continued Proportions and Tartaglia’s Solution of Cubic Equations”. *Historia Mathematica* **42**, 407–435.

Klein, Felix, 1908. *Elementarmathematik vom höheren Standpunkt aus*. Teil I: *Arithmetik, Algebra, Analysis*. Vorlesung gehalten im Wintersemester 1907–08. Ausgearbeitet von E. Helinger. Leipzig: Teubner.

Knobloch, Eberhard, 1973. “Zur Überlieferungsgeschichte des Bachetschen Gewichtsproblems”. *Sudhoffs Archiv* **57**, 142-151.

Knobloch, Eberhard, & Otto Schönberger (ed., trans.), 2007. Michael Stifel, *Vollständiger Lehrgang der Arithmetik*. Würzburg: Königshausen & Neumann.

Köbel, Jacob, 1514. *Ain new geordnet Rechenbiechlin auf den Linien mit Rechenpfennigen*. Augsburg: Erhart Öglin.

Köbel, Jacob, 1531. *Ein new geordenet Künstlich Rechenbüchlin*. Frankfurt a. M.: Christian Egenolff.

Köbel, Jacob, 1537. *Zwey rechenbüchlin: uff der Linien und Zipfer, Mit eym angehenckten Visirbuch*. Leipzig: Christian Egenolph, 1537.

Köbel, Jacob, 1549. *Rechenbuch Auff Linien und Ziffern. Mit einem Visir Büchlin Klar unnd verstendtlich fürgeben*. Frankfurt a. M.: Christian Egenolff.

Kokian, P. Sahak (ed., trans.), 1919. “Des Anania von Schirak arithmetische Aufgaben”. *Zeitschrift für die deutsch-österreichischen Gymnasien* **69** (1919–20), 112–117.

l’Huillier, Ghislaine (ed.), 1990. Jean de Murs, le *Quadripartitum numerorum*. (Mémoires et documents publiés par la Société de l'École des Chartes, 32). Genève & Paris: Droz.

Lafont, Robert, & Guy Tournerie (eds), 1967. Francès Pellos, *Compendion de l'abaco*. Montpellier: Édition de la Revue des Langues Romanes.

Lamassé, Stéphane, 2007. "Les problèmes dans les arithmétiques commerciales en langue française et occitane de la fin du Moyen Âge". *Thèse*, Université de Paris 1 Panthéon-Sorbonne.

Larte de labbacho. Treviso, 1478.

Lefèvre d'Étaples, Jacques (ed.), 1496. *In hoc opere contenta. Arithmetica decem libris demonstrata. Musica libris demonstrata quatuor. Epitome in libros Arithmeticos divi Severini Boetii. Rithmimachie ludus qui et pugna numerorum appellatur*. Paris: Johannes Higman & Wolfgang Hopyl.

Lefèvre d'Étaples, Jacques (ed.), 1514. *In hoc opere contenta. Arithmetica decem libris demonstrata. Musica libris demonstrata quatuor. Epitome in libros Arithmeticos divi Severini Boetii. Rithmimachie ludus qui et pugna numerorum appellatur*. Secundaria aeditio. Paris: Henricus Stephanus.

Lefèvre-d'Étaples, Jacques, 1516. *Euclidis megarensis Geometricorum elementorum libri XV*. Paris: Henricus Stephanus.

Libbrecht, Ulrich, 1973. *Chinese Mathematics in the Thirteenth Century: The Shu-shu chiu-chang of Ch'in Chiu-Shao*. Cambridge, Mass., & London: MIT Press.

Libri, Guillaume, 1838. *Histoire des mathématiques en Italie*. 4 vols. Paris: Jules Renouard, 1838–1841.

Lübke, Wilhelm, & Max Semrau, 1905. *Die Kunst der Renaissance in Italien und im Norden*. Stuttgart: Paul Neff.

Lüneburg, Heinz, 1993. *Leonardi Pisani liber abbaci oder Lesevergnügen eines Mathematikers*. 2. überarb. und erw. Aufl. Mannheim: B. I. Wissenschaftsverlag.

Lüneburg, Heinz, 2003. [Review of Laurence Sigler (ed., trans.), 2002. Fibonacci's *Liber Abaci*. A Translation into Modern English of Leonardo Pisano's Book on Calculation. New York: Springer, 2002]. Web publication, http://www.mathematik.uni-kl.de/~luene/miszellen/abbaci.html (accessed 3.5.07).

Malet, Antoni (ed.), 1998. Francesc Santcliment, *Summa de l'art d'Aritmètica*. [Facsimile and very slightly modernized transcription] Vic: Eumo Editorial.

Malet, A., Antoni, 2005. "Algebra as Language: Wallis and Condillac on the Nature of Algebra", pp. 311–332 *in* Patricia Radelet-de-Graves, *Symétries*. Contributions au séminaire de Han-sur-Lesse, septembre 2002. Turnhout: Brepols, 2005.

Manin, Yuri Ivanovic, & Alexei A. Panchishkin, 2005. *Introduction to Modern Number Theory: Fundamental Problems, Ideas and Theories*. Second Edition. Berlin etc.: Springer.

Masi, M., 1983. *Boethian Number Theory. A Translation of the* '*De Institutione Arithmetica'*. Amsterdam: Rodopi.

Masotti, Arnaldo, 1971. "Ferro (or Ferreo, dal Ferro, del Ferro), Scipione". pp. 595–597 *in Dictionary of Scientific Biography*, vol. IV. New York: Scribner.

Maurolico, Francesco, 1556. "Illustrissimo Domino D. Ioanni Vegae". http://people.dm.unipi.it/

maurolic/edizioni/epistola/propriae/prop-12.htm (accessed 23.10.2021).

Maurolico, Francesco, 1568. “Illustrissimo Ac Reverendissimo Domino D. Marco Antonio Amulio”. http://people.dm.unipi.it/maurolic/edizioni/epistola/propriae/prop-22.htm (accessed 23.10.2021).

Maurolico, Francesco, 1568. “Illustrissimo Ac Reverendissimo Domino D. Marco Antonio Amulio”. http://people.dm.unipi.it/maurolic/edizioni/epistola/propriae/prop-22.htm (accessed 23.10.2021).

McCuaig, William, 1990. “On the Fondo Magliabechiano”. *Aevum* **64**, 431-434.

McGarry, Daniel D. (ed., trans.), 1971. The *Metalogicon* of John of Salisbury. *A Twelfth-Century Defence of the Verbal and Logical Arts of the Trivium*. Gloucester, Massachusetts: Peter Smith.

Miura, Nobuo, 1981. “The Algebra in the *Liber abaci* of Leonardo Pisano”. *Historia Scientiarum* **21**, 57–65.

Moyon, Marc, 2017. *La géométrie de la mesure dans les traductions arabo-latines médiévales*. Turnhout: Brepols.

Moyon, Marc, 2019. “The *Liber Restauracionis:* A Newly Discovered Copy of a Mediaeval Algebra in Florence”. *Historia Mathematica* **46**, 1–37.

Narducci, Enrico, 1858. *Saggio di voci italiane derivate dall’arabo*. 2 parts. Roma: Tipografia delle scienze matematiche e fisiche.

Needham, Joseph, 1959. *Science and Civilisation in China*. III. *Mathematics and the Sciences of the Heaven and the Earth*. Cambridge: Cambridge University Press.

Nefe, Mattheus, 1565. *Arithmetica: Zwey newe Rechenbuecher, das erste auff der Linien und Federn, darinn die gebreuchlichsten Regeln mit gruendlichem Unterricht erklert... das ander ist die kuenstliche Rechnung, wie man rechnet, wie weit von einer Stadt zur andern ist*. Breslau.

Niermeyer, J. F., 1976. *Mediae latinitatis lexicon minus. Lexique latin médiéval - Français-anglais. A Medieval Latin - French/English dictionary*. Leiden: Brill.

Nissen, Hans J., Peter Damerow & Robert Englund, 1993. *Archaic Bookkeeping: Writing and Techniques of Economic Administration in the Ancient Near East*. Chicago: Chicago University Press.

Nuñez, Pedro, 1567. *Libro de Algebra en Arithmetica y Geometria*. Anvers: En casa de los herederos d’Arnaldo Birckman.

Oaks, Jeff A., 2017. “Irrational ‘Coefficients' in Renaissance Algebra”. *Science in Context* **30**, 141–172.

Ober, Johann, 1545. *Newgestelt Rechenbüchlin, mit vil schönen exempeln und proben, seer behender gryff unn vorteyl, auf den lynien und zyffern zuvor der weys nie aufgangen*. Augsburg: Heynrich Steyner.

Pacioli, Luca, 1494. *Summa de Arithmetica Geometria Proportioni et Proportionalita.* Venezia: Paganino de Paganini.

Pacioli, Luca, 1509a. *Divina proportione*. Venezia: Paganius Paganinus.

Pacioli, Luca (ed.), 1509b. *Euclidis megarensis philosophi acutissimi mathematicorumque omnium sine controversia principis opera a Campano interprete fidissimo tralata*. Venezia: Paganinus de Paganino.

Pancanti, Marisa (ed.), 1982. Giovanni di Bartolo, *Certi chasi nella trascelta a cura di Maestro*

Benedetto secondo la lezione del Codice L.IV.21 (sec. XV) della Biblioteca degli Intronati di Siena. (Quaderni del Centro Studi della Matematica Medioevale, 3). Siena: Servizio Editoriale dell'Università di Siena.

Paton, W. R. (ed., trans.), 1918. *The Greek Anthology*. Vol. V. New York: G. P. Putnam / London: Heinemann.

Pedersen, Fritz Saaby (ed), 1983. Petri Philomena de Dacia et Petri de S. Audomaro *Opera quadrivialia*. Pars I. *Opera* Petri Philomenae. København: Gad.

Peet, T. Eric, 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058*. Introduction, Transcription, Translation and Commentary. London: University Press of Liverpool.

Peletier, Jacques, 1554. *L'algebre*. Lyon: Ian de Tournes.

Peletier, Jacques, 1555. *Dialogue de l'Ortografe e Prononciacion Françoese, departi an deus livres*. Lyon: Ian de Tournes.

Picutti, Ettore (ed.), 1979. "Il libro dei quadrati di Leonardo Fibonacci e i problemi di analisi indeterminata". *Physis* **21**, 195–339.

Picutti, Ettore (ed., trans.), 1983. "Il «Flos» di Leonardo Pisano: dal codice E.75 P. sup. della Biblioteca Ambrosiana di Milano". *Physis* **25**, 293–387.

Picutti, Ettore, 1989. "Sui plagi matematici di frate Luca Pacioli". *Le Scienze* **40**:246 (febbraio 1989), 72–79.

Pieraccini, Lucia (ed.), 1983. M^{o} Biagio, *Chasi exenplari alla regola dell'algibra nella trascelta a cura di M^{o} Benedetto* dal Codice L. VII. 2Q della Biblioteca Comunale di Siena. (Quaderni del Centro Studi della Matematica Medioevale, 5). Siena: Servizio Editoriale dell'Università di Siena.

Pirenne, Henri, 1929. "L'instruction des marchands au Moyen Âge". *Annales d'Histoire économique et sociale* **1**, 13–28.

PL: *Patrologiae cursus completus, series latina*, acccurante J. P. Migne. 221 vols. Paris, 1844–1864.

Plofker, Kim, 2009. *Mathematics in India: 500 BCE–1800 CE*. Princeton & Oxford: Princeton University Press.

Poulle, Emmanuel, 1973. "John of Murs", pp. 128–133 *in Dictionary of Scientific Biography*, vol. VII. New York: Scribner.

Preclarissimus liber elementorum Euclidis perspicacissimi. Venezia: Ratdolt, 1482.

Procissi, Angiolo (ed.), 1954. "I Ragionamenti d'Algebra di R. Canacci". *Bollettino Unione Matematica Italiana*, serie III, **9**, 300–326, 420–451.

Raṅgācārya, M. (ed., trans.), 1912. The *Gaṇita-sāra-saṅgraha* of Mahāvīrācārya with English Translation and Notes. Madras: Government Press.

[Ramus, Petrus], 1560. *Algebra*. Paris: Andreas Wechelum.

Ramus, Petrus, 1569. *Scholarum mathematicarum libri unus et triginta*. Basel: Eusebius Episcopius.

Rankin, Fenella K. C., 1992. "The Arithmetic and Algebra of Luca Pacioli". *Thesis*, Warburg Institute, University of London.

Rashed, Roshdi, 1984. *Entre arithmétique et algèbre. Recherches sur l'histoire des mathématiques arabes*. Paris: Les Belles Lettres.

Rashed, Roshdi, 2003. "Fibonacci et le prolongement latin des mathématiques arabes". *Bollettino di storia delle scienze matematiche* **24**, 55–73.

Rashed, Roshdi (ed., trans.), 2007. Al-Khwārizmī, *Le Commencement de l'algèbre*. (Collections Sciences dans l'histoire). Paris: Blanchard.

Rashed, Roshdi (ed., trans.), 2012. Abū Kāmil, *Algèbre et analyse Diophantienne*. Berlin & Boston: de Gruyter.

Raven, Diederick, & Wolfgang Krohn, 2000. "Edgar Zilsel: His Life and Work (1891–1944)", pp. xxi–lxii *in* Edgar Zilsel, *The Social Origins of Modern Science*, ed. Diederick Raven, Wolfgang Krohn & Robert S. Cohen. Dordrecht etc.: Kluwer.

Rebstock, Ulrich, 1992. *Rechnen im islamischen Orient. Die literarischen Spuren der praktischen Rechenkunst*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Regiomontanus, Johannes, 1533. *De triangulis omnimodis libri quinque*. Nürnberg: Johan Petreius.

Ries, Adam, 1550. *Rechenung nach der lenge, auff den linihen und feder. Darzu forteil und behendigkeit durch die Proportiones, Practica genannt, Mit grüntlichem unterrichts des visierens*. Leipzig: Jacob Berwalt.

Ristori, Renzo, 1979. "Cegia, Francesco", in *Dizionario biografico degli italiani*, vol. 23. Roma: Instituto della Enciclopedia Italiana, 1979. https://www.treccani.it/enciclopedia/francesco-cegia_(Dizionario-Biografico)/ (accessed 5.4.2021).

Rivali, Luca, 2019. "Tagliente, Giovanni Antonio". *Dizionario Biografico degli Italiani* **94**. https://www.treccani.it/enciclopedia/giovanni-antonio-tagliente_(Dizionario-Biografico).

Robinson, Chase F., 2010. "The Rise of Islam, 600–750", pp. 173–225 *in New Cambridge History of Islam*, vol. I. Cambridge: Cambridge University Press.

Rose, Paul Lawrence, 1975. *The Italian Renaissance of Mathematics. Studies on Humanists and Mathematicians from Petrarch to Galileo*. Genève: Librairie Droz.

Röttel, Karl, 1996. "'Arithmetica applicirt': Visierkunst, Buchhaltung, Kartographie und Astronomie bei Henricus Grammateus", pp. 67–89 *in* Manfred Weidauer (ed.), *Heinrich Schreyber aus Erfurt, genannt Grammateus. Festschrift zum 500. Geburtstag*. München: Institut für Geschichte der Naturwissenschaften.

Rudolff, Christoff, 1525. *Behend unnd hübsch Rechnung durch die kunstreichen Regeln Algebra, so gemeincklich die Coss genennt werden*. Straßburg.

Saidan, Ahmad S., 1974. "The Arithmetic of Abū'l-Wafāʾ". *Isis* **65** , 367–375.

Saidan, Ahmad S. (ed., trans.), 1978. The *Arithmetic* of al-Uqlīdisī. The Story of Hindu-Arabic Arithmetic as Told in *Kitāb al-Fuṣūl fī al-Ḥisāb al-Hindī* by Abū al-Ḥasan Aḥmad ibn Ibrāhīm al-Uqlīdisī written in Damascus in the Year 341 (A.D. 952/53). Translated and Annotated. Dordrecht: Reidel.

Salomone, Lucia (ed.), 1982. M° Benedetto da Firenze, *La reghola de algebra amuchabale* dal Codice L.IV.21 della Biblioteca Comunale de Siena. (Quaderni del Centro Studi della Matematica Medioevale, 2). Siena: Servizio Editoriale dell'Università di Siena.

Salomone, Lucia (ed.), 1984. Leonardo Pisano, *E' chasi della terza parte del XV Capitolo del Liber Abaci della trascelta a cura di Maestro Benedetto* secondo la lezione del Codice L.IV.21 (sec.

XV) della Biblioteca Comunale di Siena. (Quaderni del Centro Studi della Matematica Medioevale, 10). Siena: Servizio Editoriale dell'Università di Siena.

Sánchez Pérez, José A. (ed., trans.), 1916. *Compendio de algebra de Abenbéder*. Texto arabo, traduccion y estudio. Madrid: Centro de Estudios Históricos.

Sapori, Armando, 1955. *Studi di storia economica: (secoli XIII-XIV-XV)*. 3 vols. Firenze: Sansoni, 1955-67.

Scheubel, Johann (ed., trans.), 1550. Euclidis Megarensis, Philosophi et mathematici excellentissimi, *Sex libri priores, De Geometricis principiis, Graeci et latini, una cum demonstrationibus propositionum .... Algebrae porro regulae, propter numerorum exempla, passim propositionibus adiecta, his libris præmissae sunt, eademque demonstratae*. Basel: Herwagen.

Schmeidler, Felix (ed.), 1972. Joannis Regiomontani *Opera collectanea*. Faksimiledrucke von neun Schriften Regiomontans und einer von ihm gedruckten Schrift seines Lehrers Purbach. Zusammengestellt und mit einer Einleitung herausgegeben. Osnabrück: Otto Zeller.

Schneider, Ivo, 1989. *Die Entwicklung der Wahrscheinlichkeitstheorie von den Anfängen bis 1933. Einführungen und Texte*. Berlin: Akademie-Verlag.

Schöne, Hermann (ed., trans.), 1903. Herons von Alexandria *Vermessungslehre* und *Dioptra*. Griechisch und deutsch. (Heronis Alexandrini Opera quae supersunt omnia, vol. III). Leipzig: Teubner.

Schröder, Eberhard (ed.), 1988. Ulrich Wagner, *Das Bamberger Rechenbuch*. Facsimile-Druck der Ausgabe von 1483. Weinheim: VCH Verlagsgesellschaft.

Schulz, Anthonius, 1600. *Arithmetica Oder Rechenbuch, Dorinn mit begründeter ausführung die Species, schöne nützliche Reguln [...] so wol die Kunst und sinnreiche Regul Coß, gelehret, mit außerlesenen Exempeln erkleret*. Liegnitz: Nikolaus Schneider.

Schwartz, Randy K., 2006. "Issues in the Origin and Development of *Hisab al-Khata'ayn* (Calculation by Double False Position)" pp. 275–296 *in Actes du Huitième Colloque Maghrébin sur l'Histoire des Mathématiques Arabes*, Tunis, le 18–19–20 Décembre 2004. Tunis: Association Tunisienne des Sciences Mathématiques.

Sesiano, Jacques (ed.), 1993. "La version latine médiévale de l'Algèbre d'Abū Kāmil", pp. 315–452 *in* M. Folkerts & J. P. Hogendijk (eds), *Vestigia Mathematica*. Studies in Medieval and Early Modern Mathematics in Honour of H. L. L. Busard. Amsterdam & Atlanta: Rodopi.

Sesiano, Jacques, 2000. "Un recueil du XIII[e] siècle de problèmes mathématiques". *SCIAMUS* **1** , 71-132.

Sesiano, Jacques, 2014. *The* Liber mahameleth: *A Twelfth-Century Mathematical Treatise*. Heidelberg etc.: Springer.

Sesiano, Jacques (ed., trans.), 2018. *L'arithmétique de Pamiers: Traité mathématique en langue d'oc du XV[e] siècle*. Lausanne: Presses Polytechniques et Universitaires Romandes.

Sigler, Laurence E. (ed., trans.), 1987. *Leonardo Pisano Fibonacci, The Book of Squares*. An Annotated Translation into Modern English. Boston etc.: Academic Press.

Sigler, Laurence (ed., trans.), 2002. Fibonacci's *Liber Abaci*. A Translation into Modern English of Leonardo Pisano's *Book on Calculation*. New York: Springer.

Simi, Annalisa (ed.), 1992. Anonimo fiorentino, *Regole di geometria e della cosa* dal Codice Palatino 575 (sec. XV) della Biblioteca Nazionale di Firenze. (Quaderni del Centro Studi della Matematica Medioevale, 20). Siena: Servizio Editoriale dell' Università di Siena.

Simi, Annalisa, & Laura Toti Rigatelli, 1993. "Some 14th and 15th Century Texts on Practical Geometry", pp. 453–470 *in* Menso Folkerts & Jan P. Hogendijk (eds), *Vestigia mathematica: Studies in Medieval and Early Modern Mathematics in Honor of H. L. L. Busard*. Amsterdam & Atlanta: Rodopi.

Simi, Annalisa (ed.), 1994. Anonimo (sec. XIV), *Trattato dell'alcibra amuchabile* dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze. (Quaderni del Centro Studi della Matematica Medioevale, 22). Siena: Servizio Editoriale dell' Università di Siena.

Simi, Annalisa, 1995. "Trascrizione ed analisi del manoscritto Ricc. 2236 della Biblioteca Riccardiana di Firenze". *Università degli Studi di Siena, Dipartimento di Matematica. Rapporto Matematico* Nº 287.

Simi, Annalisa, 1999. "La geometria nel Rinascimento: il codice Ottoboniano Latino 3307 della Biblioteca Apostolica Vaticana". *Contributi de Filologia dell'Italia Mediana* **13**, 41–109.

Smeur, Alphons Johannes Emile Marie, 1978. "The *Rule of False* Applied to the Quadratic Equation, in Three Sixteenth Century Arithmetics". *Archives Internationales d'Histoire des Sciences* **28**, 66–101.

Smith, David Eugene, 1908. *Rara arithmetica. A Catalogue of the Arithmetics Written Before the Year MDCI with a Description of Those in the Library of George Arthur Plimpton of New York*. Boston & London: Ginn.

Smith, David Eugene, 1923. *History of Mathematics*. I. *General Survey of the History of Elementary Mathematics*. II. *Special Topics of Elementary Mathematics*. Boston: Ginn, 1923–1925.

Smith, David Eugene, 1926. "The First Great Commercial Arithmetic". *Isis* **8**, 41–49.

Sombart, Werner, 1919. *Der moderne Kapitalismus*. 2 vols in 4 tomes. Dritte unveränderte Auflage. München & Leipzig: Duncker & Humblot.

Soubeyran, Denis, 1984. "Textes mathématiques de Mari". *Revue d'Assyriologie* **78**, 19–48.

Souissi, Mohamed, 1968. *La langue des mathématiques en arabe*. Thèse principale pour le Grade de Docteur d'État, Université de Paris, Faculté des lettres et des sciences humaines. Tunis: Université de Tunis.

Souissi, Mohamed (ed., trans.), 1969. Ibn al-Bannāʾ, *Talkhīṣ a ʿmāl al-ḥisāb*. Texte établi, annoté et traduit. Tunis: L'Université de Tunis.

Souissi, Mohamed (ed., trans.), 1988. Qalaṣādī, *Kašf al-asrār ʿan ʿilm ḥurūf al-ġubār*. Carthage: Maison Arabe du Livre.

Spiesser, Maryvonne (ed.), 2003. *Une arithmétique commerciale du XV^{e} siècle*. Le *Compendy de la praticque des nombres* de Barthélemy de Romans. Turnhout: Brepols.

Stabile, Giorgio, 1971. "Bragadin, Domenico". *Dizionario Biografico degli Italiani* **13** (1971), online. https://www.treccani.it/enciclopedia/domenico-bragadin_%28Dizionario-Biografico%29/ (accessed 18.9.2021).

Stedall, Jacqueline, 2010. *From Cardano's Great Art to Lagrange's Reflections: Filling a Gap in* *the History of Algebra*. Zürich: European Mathematical Society.

Steele, Robert (ed.), 1922. *The Earliest Algorisms in English*. London: Humphrey Milford, Oxford University Press.

Steinschneider, Moritz, 1865. “Die ‘mittleren' Bücher der Araber und ihre Bearbeiter”. *Zeitschrift für Mathematik und Physik* **10** (1865), 456–498.

Stifel, Michael, 1544. *Arithmetica integra*. Nürnberg: Johan Petreius.

Stifel, Michael, 1545. *Deutsche Arithmetica, inhaltend Die Hausrechnung, Die Deutsche Coß, Die Kirchrechnung.* Nürnberg: Johan Petreius.

Stifel, Michael, 1546. *Rechenbuch von der Welschen und Deutschen Practick, auff allerley vorteyl [...] zu machen*. Nürnberg: Johan Petreius.

Stifel, Michael (ed.), 1553. *Die Coss* Christoffs Rudolffs. Die schönen Exemplen der Coss gebessert und gemehrt. Königsberg in Preussen: Alexander Lutomyslensis.

Suter, Heinrich, 1908. “Die Abhandlung Qosṭā ben Lūqās und zwei andere anonyme über die Rechnung mit zwei fehlern und mit der angenommenen Zahl”. *Bibliotheca Mathematica*, 3. Folge **9** (1908–1909), 111–122.

Swetz, Frank J., 1987. *Capitalism and Arithmetic: The New Math of the 15th Century*, Including the Full Text of the *Treviso Arithmetic* of 1478, Translated by David Eugene Smith. La Salle, Illinois: Open Court.

Szabó, Árpád, 1969. *Anfänge der griechischen Mathematik*. München & Wien: R. Oldenbourg/Budapest: Akadémiai Kiadó.

Tagliente, Girolamo, 1520x. *Libro dabaco che insegna a fare ogni ragione mercadantile, et pertegare le terre con l’arte della Geometria* .... Venezia: Luca Antonio de Uberti, 1520(?).

Tagliente, Girolamo, 1530. *Libro dabaco che insegna a fare ogni ragione mercadantile, et pertegare le terre con l’arte della Geometria* .... Venezia: Luca Antonio de Uberti, 1530(?).

Tagliente, Girolamo, 1532. *Libro dabaco che insegna a fare ogni ragione mercadantile, et pertegare le terre con l’arte della Geometria* .... Venezia: Luca Antonio de Uberti, 1532(?).

Tagliente, Girolamo, 1579. *Libro de abbaco che insegna a fare ogni ragione mercadantile, et pertegare le terre con l’arte della Geometria* .... Milano: Valerio Meda.

Tagliente, Giovanni Antonio, 1524. *Lo presente libro insegna la vera arte delo excellente scrivere de diverse varie sorti de litere le quali se fano per geometrica ragione*. [Venezia], 1524.

Tagliente, Giovanni Antonio, 1527. *Opera nova che insegna a fare ogni ragione di mercantia*. Venezia: Bernardin Venetian de Vidali.

Taisbak, Christian Marinus, 1996. “Det gyldne snit – eller den større og den mindre”, pp. 229–236 *in* Thomas Heine Nielsen et al (eds), *Hvad tales her om? 46 artikler om græsk-romersk kultur*. Festskrift til Johnny Christensen. København: Museum Tusculanum.

Tangheroni, Marco, 2002. “Pisa e il Mediterraneo all’epoca di Fibonacci”, pp. 45–57 *in* Enrico Giusti (ed.), *Un ponte sul mediterraneo: Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente*. Firenze: Edizioni Polistampa.

Tannery, Paul (ed., trans.), 1893. Diophanti Alexandrini *Opera omnia* cum graecis commentariis. 2 vols. Leipzig: Teubner, 1893–1895.

Tannstetter, Georg (ed.), 1515. [Jean de Murs] *Arithmetica communis*. [Thomas Bradwardine] *Tractatus brevis proportionum*. [Nicole Oresme] *De latitudine formarum*. Georg Peurbach, *Algorithmus in integris*. Johannes von Gmunden, *Algorithmus de minuciis phisicis*. Wien: Johannes Singrentus.

Thomson, William, & Gustav Junge (eds, trans.), 1930. The *Commentary* of Pappus on Book X of Euclid's *Elements*. Arabic Text and Translation, with Introductory Remarks, Notes, and a Glossary of Technical Terms. Cambridge, Mass.: Harvard University Press.

Trabeth, Christian, 1674. *Compendium arithmedicae. Das ist: Ein kurz, jedoch nutzbares Rechen-Büchlein, Darinnen die 4 Haupt-Species, Als Addiren, Subtrahiren, Multiplicierenm und Dividieren, aufden Linien, mit der Feder, und durch die Brüche, ganz vernehmlich ausgeführet, auch mit exempeln zur gnüge angwiesen, wie die Regel Detri recht zu gebrauchen ist*. Osterode: Christian Trabeth.

Travaini, Lucia, 2003. *Monete, mercanti e matematica. Le monete medievali nei trattati di aritmetica e nei libri di mercatura*. Roma: Jouvence.

Travaini, Lucia, 2020. *Monete, mercanti e matematica. Le monete medievali nei trattati di aritmetica e nei libri di mercatura*. Seconda edizione ampliata con nuove liste inedite. Roma: Jouvence.

Tropfke, J./Vogel, Kurt, et al, 1980. *Geschichte der Elementarmathematik.* 4. Auflage. Band 1: *Arithmetik und Algebra*. Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.

Ulivi, Elisabetta, 1994. "Luca Pacioli: una biografia scientifica", pp. 21–78 *in* Enrico Giusti & Carlo Maccagni (eds), *Luca Pacioli e la matematica del Rinascimento*. Firenze: Giunti.

Ulivi, Elisabetta, 1996. "Per una biografia di Antonio Mazzinghi, maestro d'abaco del XIV secolo". *Bollettino di Storia delle Scienze Matematiche* **16**, 101-150.

Ulivi, Elisabetta, 2002a. "Benedetto da Firenze (1429–1479), un maestro d'abaco del XV secolo. Con documenti inediti e con un'Appendice su abacisti e scuole d'abaco a Firenze nei secoli XIII–XVI". *Bollettino di Storia delle Scienze Matematiche* **22**, 3–243.

Ulivi, Elisabetta, 2002b. "Scuole e maestri d'abaco in Italia tra Medioevo e Rinascimento", pp. 121–159 *in* Enrico Giusti (ed.), *Un ponte sul mediterraneo: Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente*. Firenze: Edizioni Polistampa.

Ulivi, Elisabetta, 2004. "Maestri e scuole d'abaco a Firenze: La Bottega di Santa trinità". *Bollettino di storia delle scienze matematiche* **24**, 43–91.

Ulivi, Elisabetta, 2011. "Su Leonardo Fibonacci e sui maestri d'abaco pisani dei secoli XIII–XV". *Bollettino di Storia delle Scienze Matematiche* **31**, 247–286.

Ulivi, Elisabetta, 2015. "Masters, Questions and Challenges in the Abacus School". *Archive for History of Exact Sciences* **659**, 651–670.

Ulivi, Elisabetta, 2017. *Giovanni del Sodo, un maestro d'abaco fiorentino nel* Tractatus mathematicus ad discipulos perusinos *di Luca Pacioli*. Firenze: Università di Firenze, Dipartimento di Matematica e Informatica.

Ulman, Oswald, & Caspar Thierfelder, 1564. *Ein New Künstlich Rechenbuch, auff den Linien und Ziffern, von mancherley schönen Regeln*. Leipzig: Jacob Berwaldt.

Valla, Giorgio, 1501. *De expetendis et fugiendis rebus opus*. 2 vols. Venezia: Manutius.

van Schooten, Frans (ed.), 1646. François Viète, *Opera mathematica*. Leiden: Elzevier.

Van Egmond, Warren, 1976. *The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence*. Dissertation, Indiana University, 1976.

Van Egmond, Warren, 1977. "New Light on Paolo dell'Abbaco". *Annali dell'Istituto e Museo di Storia della Scienza di Firenze* **2**:2, 3–21.

Van Egmond, Warren, 1978. "The Earliest Vernacular Treatment of Algebra: The *Libro di ragioni* of Paolo Gerardi (1328)". *Physis* **20**, 155–189.

Van Egmond, Warren, 1980. *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600*. Firenze: Istituto e Museo di Storia della Scienza.

Van Egmond, Warren, 1983. "The Algebra of Master Dardi of Pisa". *Historia Mathematica* **10**, 399–421.

Van Egmond, Warren, 2008. "The Study of Higher-Order Equations in Italy before Pacioli", pp. 303–320 *in* Joseph W. Dauben et al (eds), *Mathematics Celestial and Terrestrial*: Festschrift für Menso Folkerts zum 65. Geburtstag. Halle: Wissenschaftliche Verlagsgesellschaft.

Van Egmond, Warren, 2009. [Review of Jens Høyrup, *Jacopo da Firenze's* Tractatus Algorismi *and Early Italian Abbacus Culture*. Basel etc.; Birkhäuser, 2007]. *Aestimatio* **6** (2009; published 2010), 37–47.

Verlinden, Charles, 1970. *Wo, wann und warum gab es einen Großhandel mit Sklaven während des Mittelalters?* (Kölner Vorträge zur Sozial- und Wirtschaftsgeschichte, 11). Köln: Forschungsinstitut für Sozial- und Wirstschaftsgeschichte an der Universität Köln.

Vesalius, Andreas, 1543. *De humani corporis fabrica libri septem*. Basel: [Johann Oporinus].

Victor, Stephen K., 1979. *Practical Geometry in the Middle Ages. Artis cuiuslibet consummatio* and the *Pratike de geometrie.* Edited with Translations and Commentary. Philadelphia; The American Philosophical Society.

Viète, François, 1591. *In artem analyticem isagoge*. Tours: Jamet Mettayer.

Villani, Giovanni, 1823. *Cronica*. 12 vols. Firenze, 1823–1832.

Vlasschaert, Anne-Marie (ed.), 2010. *Liber Mahameleth*. Wiesbaden: Franz Steiner.

Vogel, Kurt, 1930. "Die algebraischen Probleme des P. Mich. 620". *Classical Philology* **25**, 373–375.

Vogel, Kurt, 1949. "Das älteste deutsche Gedruckte Rechenbuch Bamberg 1982", pp. 231–277 *in Gymnasium und Wissenschaft*. Festschrift des Maximiliansgymnasiums in München. München.

Vogel, Kurt (ed.), 1954. Die *Practica* des *Algorismus Ratisbonensis*. Ein Rechenbuch des Benediktinerklosters St. Emmeram aus der Mitte des 15. Jahrhunderts. München: C. H. Bech.

Vogel, Kurt, 1959. "Adam Riese, der deutsche Rechenmeister". *Deutsches Museum. Abhandlungen und Berichte* **27**:3, 4–37.

Vogel, Kurt, 1963. "Der Trienter Algorismus von 1475". *Nova Acta Leopoldina. Abhandlungen der Kaiserlich Leopoldinisch-Carolinisch deutschen Akademie der Naturforscher*, Neue Folge **27**:167, 183-200.

Vogel, Kurt (ed., trans.), 1968. *Ein byzantinisches Rechenbuch des frühen 14. Jahrhunderts*. Wien:

Institut für Byzantinistik der Universität Wien/Hermann Bohlau.
Vogel, Kurt, 1976. “Stifel”, pp. 58–62 *in Dictionary of Scientific Biography*, vol. XIII. New York: Scribner.
Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13).* (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München.
Vogel, Kurt (ed.), 1981. *Die erste deutsche Algebra aus dem Jahre 1481*, nach einer Handschrift aus C 80 Dresdensis herausgegeben und erläutert. (Bayerische Akademie der Wissenschaften. mathematisch-naturwissenschaftliche Klasse. Abhandlungen. Neue Folge, Heft 160). München: Verlag der Bayerischen Akademie der Wissenschaften.
Vogel, Kurt, 1982. “Zur Geschichte der Stammbrüche und der aufsteigenden Kettenbrüche”. *Sudhoffs Archiv* **66**, 1–19.
Wagner, Ulrich, 1482. [*Bamberger Rechenbuch*]. Bamberg: Heinrich Petzensteiner.
[Wagner, Ulrich], 1483. [*Bamberger Rechenbuch*]. Bamberg: Heinrich Petzensteiner.
Wagner(?), Ulrich(?), 1475(?). [*Bamberger Blockbuch*]. Franken(?), *ca* 1475.
Wappler, Hermann Emil, 1887. “Zur Geschichte der deutschen Algebra im 15. Jahrhundert”, pp. 1–32 *in Gymnasium zu Zwickau. Jahresbericht über das Schuljahr von Ostern 1886 bis Ostern 1887*. Zwickau: R. Zückler.
Wappler, E., 1900. “Zur Geschichte der Mathematik im 15. Jahrhundert”. *Zeitschrift für Mathematik und Physik. Historisch-literarische Abteilung* **45**, 47–56.
Weber, Johann, 1583a. *Ein New Künstlich vnd Wolgegründt Rechenbuch, Auff den Linien vnd Ziffern, von vielen nützlichen Regeln zu allerley Handtierunge, Gewerben und Kaufmanschlag dientlichen*. Leipzig: Jacob Berwaldt.
Weber, Johann, 1583b. *Gerechnet Rechenbüchlein Auff Erffurdischen Weit, Tranck, Centner, Stein und Pfund kauff*. Erfurt: Esaias Mechler.
Widmann, Johannes, 1489. *Behende und hubsche Rechenung auff allen kauffmannschafft*. Leipzig: Konrad Kacheloffen.
Wilken, Marten, 1622. *Flores algebraici, das ist Algebra oder Coss mit shcönen auserlesenen künstlichen theils resolvierten new erfundenen Quaestionen und Exempeln*. Emden: Helwich Kallenbach, 1622.
Witmer, T. Richard (ed., trans.), 1983. François Viète, *The Analytic Art: Nine Studies in Algebra, Geometry, and Trigonometry from the Opus restitutae mathematicae analyseos, seu algebrâ novâ*. Translated with Introductions and Notes. Kent, Ohio: Kent State University Press.
Woepcke, Franz, 1852. “Notice sur une théorie ajoutée par Thâbit ben Korrah à l’arithmétique spéculative des Grecs”. *Journal Asiatique*, 4e série **20**, 420–429.
Woepcke, Franz, 1853. *Extrait du Fakhrî, traité d’algèbre* par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; *précédé d’un mémoire sur l’algèbre indéterminé chez les Arabes.* Paris: L’Imprimerie Impériale.
Woepcke, Franz, 1854. “Sur un essai de déterminer la nature de la racine d’une équation du troisème degré, contenu dans un ouvrage de Léonard de Pise découvert par M. la prince Balthasar

Boncompagni”. *Journal des Mathématiques Pures et Appliquées*, 1ère série **19** (1854), 401–406.

Zamberti, Bartolomeo (ed.), 1505. *Euclidis megarensis philosophi platonici, Mathematicarum disciplinarum janitoris, habent in hoc volumine quicunque ad mathematicam substantiam aspirant: Elementorum libros xiii cum expositione insignis mathematici* ... Venezia: Giovanni Tacuini.

Zilsel, Edgar, 1942. “The Sociological Roots of Science”. *American Journal of Sociology* **47**, 544–562.

Zinner, Ernst, 1990. *Regiomontanus: His Life and Work*. Amsterdam etc.: North-Holland.

Zupko, Ronald Edward, 1981. *Italian Weights and Measures from the Middle Ages to the Nineteenth Century*. Philadelphia: American Philosophical Society.

Source index

NB: Sources are only listed where there is at least a minimal information about then. For instance, if it is said about source *A* that it does not draw on source *B*, only *A* will be listed in this index.

Page numbers of sections where a specific work is described systematically – for instance, Fibonacci's *Liber abbaci* – appear as boldface.

Manuscripts referred to as such

Books, Treatises and other documents

Name and subject index