Further questions to the historiography of Arabic (but not only Arabic) mathematics from the perspective of Romance abbacus mathematics

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Abstract

For some years I have been engaged in a close reading of early Italian abbacus books and related material from the Ibero-Provençal orbit and in comparison of this material with Arabic mathematical writings. At the 7th North African Meeting on the History of Arab Mathematics in Marrakesh in 2002 I presented the first outcome of this investigation: namely that early Italian abbacus algebra was not influenced by the Latin algebraic writings of the 12th-13th centuries (neither by the translations of al-Khwārizmī nor by the works of Fibonacci); instead, it received indirect inspiration from a so far unknown link to the Arabic world, viz to a level of Arabic algebra (probably integrated with *mu*ʻ*āmalāt* mathematics) of which very little is known. At the 8th Meeting in Tunis in 2004 I presented a list of linguistic clues which, if applied to Arabic material, might enable us to say more about the links between the Romance abbacus tradition and Arabic *mu*ʻ*āmalāt* teaching.

Here I investigate a number of problem types and techniques which turn up in some but not necessarily in all of the following source types:

– Romance abbacus writings,
– Byzantine writings of abbacus type,
– Arabic mathematical writings of various kinds,
– Sanskrit mathematical writings,

in order to display the intricacies of the links between these – intricacies which force us to become aware of the shortcomings of our current knowledge, and hence formulate questions that go beyond the answers I shall be able to present.

Over the last ten years, I have worked (along with other things) on early Italian abbacus books and related material from the Ibero-Provençal orbit. Time and again, this has led me to comparison with Arabic mathematical writings – often, however, with the outcome that good arguments derived from the Romance texts spoke in favour of a connection to the Arabic world while no Arabic sources I knew of allowed a more precise anchorage of this connection.

The first area where I was struck by this puzzle was algebra. At the 7th North African Meeting on the History of Arab Mathematics in Marrakesh in 2002 I presented the result that early Italian abbacus algebra (1300–1350) was not influenced by the Latin algebraic writings of the twelfth and thirteenth centuries (neither the translations of al-Khwārizmī nor the works of Fibonacci); instead it received indirect inspiration from a so far unknown link to the Arabic world, viz to a level of Arabic algebra (probably integrated with the teaching of *mu*ʿ*āmalāt* mathematics) of which very little is known. At the 8th Meeting in Tunis in 2004 I presented a list of linguistic clues which, if applied to Arabic material, might hopefully enable us to say more about the links between the Romance abbacus tradition(s) and Arabic *mu*ʿ*āmalāt* teaching.

At the present occasion I intend to investigate some problem types and techniques – the rule of three, certain aspects of algebra, and a strange problem about an unknown heritage – which turn up in some but (as far as can be judged from known extant writings) not necessarily in all of the following source groups:

– Romance (i.e., Italian, Provençal and Iberian) abbacus writings,
– Byzantine writings of abbacus type,
– Arabic mathematical writings of various kinds,
– Sanskrit mathematical writings,
–Byzantine writings of abbacus type

in order to display the intricacies of the links between these – intricacies which force us to become aware of the shortcomings of our current knowledge, and hence to formulate questions that go beyond the answers I shall be able to present.

The rule of three

In many of our sources, this rule is characterized as *the* rule, the rule *par excellence* for merchant mathematics. Before we approach the sources, however, a note on terminology.

It is not uncommon that writings on the history of mathematics conflate the *problem type* and the *rule*. However, this problem about coins minted in Tours and Paris,[1]

> 7 *tornesi* are worth 9 *parigini*. Say me, how much will 20 *tornesi* be worth

is *not* an instance of the rule, simply because it is no rule and can indeed be solved by means of several different rules. Instead, the rule of three is exemplified by the calculation which follows:

> Do thus, the thing that you want to know is that which 20 *tornesi* will be worth. And the not similar is that which 7 *tornesi* are worth, that is, they are worth 9 *parigini*. And therefore we should multiply 9 *parigini* times 20, they make 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $\frac{5}{7}$. And 25 *parigini* and $\frac{5}{7}$ will 20 *tornesi* be worth.

[1] Like all translations in the following where no translator is identified, the responsibility is mine. Here I translate from Jacopo da Firenze's *Tractatus algorismi* from 1307 (ms. Vat. Lat. 4826, fol. 17r), ed., trans. [Høyrup 2007a: 237].

This refers to a preceding formulation of the rule in abstract terms,

> If some computation should be given to us in which three things were proposed, then we should always multiply the thing that we want to know against that which is not similar, and divide in the other thing, that is, in the other that remains.[2]

Other Italian treatises refer instead to the "other thing" as "the similar thing".

Alternatively, one might have found the solution by saying that 20 *tornesi* must be worth $^{20}/_{7}$ times as much as 7 *tornesi*, and hence $(^{20}/_{7})\cdot 9$ *parigini*. This is the way which Ibn Thabāt [ed., trans. Rebstock 1993: 43–45] calls "by ratio" (*nisbah*), and which al-Karajī found preferable to the one by "multiplication and division" (i.e., the rule of three) in the *Kāfī* [ed., trans. Hochheim 1878: II, 17].

The latter procedure is rarely described in Italian abbacus writings, but it is not totally absent. It is found in a *Libro di conti e mercatanzie* from c. 1395 [ed. Gregori & Grugnetti 1998: 3*f*] and in very similar though corrupt words in a *Libro di molti ragioni* from c. 1330 [ed. Arrighi 1973: 17*f*], which shows a shared source of the two to antedate 1330. This early date could suggest a borrowing from a non-Italian (and ultimately Arabic) source, but independent "discovery" is too near at hand to allow any certain conclusion – not least because these two Italian works also suggest a third method, namely (if we apply it to Jacopo's example) to calculate that 1 *tornese* is worth $^{9}/_{7}$ *parigino*, and 20 *tornesi* therefore $20\cdot(^{9}/_{7})$ *parigini*. I have not seen any name given to this approach in Arabic sources, which could mean that it was not considered a standard method.[3]

We may learn more from the name given to the method and the terms in which it was discussed. Most Italian abbacus books refer to it as *regola delle tre cose*, "rule of the three things" – including what is likely to be the second-earliest extant text, a *Livero de l'abbecho* from c. 1300 [ed. Arrighi 1989: 9].[4] The same name is used by Sanskrit writers like Āryabhaṭāa,

2 Understood thus, as a rule prescribing to make the multiplication first and the division afterwards, the rule is likely to be of Indian origin. In the *Vedāṅgajyotiṣa* from around 500 BCE if not before we find the rule "The known result is to be multiplied by the quantity for which the result is wanted, and divided by the quantity for which the known result is given" (trans. Kuppanna Sastry, quoted from [Sarma 2002: 135]). Another possible origin is in China, where the rule is formulated in the *Nine Chapters on Arithmetic* half a millennium later or so [trans. Vogel 1968a: 17*f*; trans. Chemla 2004: 225] but already used in the *Suàn shù shū* from no later than c. 186 BCE [trans. Cullen 2004: 62*f* and *passim*] – the earliest extant source for Chinese mathematics. The earliest known source west of India is al-Khwārizmī's algebra.

3 As we shall discuss in a moment, most Arabic authors link the rule of three and the problems to which it was applied to Euclidean proportion theory. In this perspective the *nisbah* method is perfectly legitimate, since it considers the ratio between entities of the same kind. A ratio between (say) *tornesi* and *parigini*, on the other hand, is as troublesome as the rule of three itself with its product of (e.g.) *tornesi* and *parigini*. Though pedagogical reasons might speak in favour of the *parigini*-per-*tornese* approach, reasons of principle may have spoken against discussion of it. Actually, Ibn Thabāt mentions this method, but only when treating *numbers* in proportion, before getting to the *mu'āmalāt*-questions.

4 Conventionally, this compilation is dated as early as 1290 on the basis of supposed internal evidence. At closer analysis this internal evidence – (probably sham) loan contracts – turns out to be copied from an earlier treatise [Høyrup 2005]. From the compiler's total ignorance of even the most basic level of algebra we may still suppose the date to be not much if at all later than 1300.

Brahmagupta and Mahāvīra.[5]]

In Arabic writings I have not found this name for the rule. Al-Karajī, when distinguishing it from his favourite *nisbah*-method in the *Kāfī* [ed., trans. Hochheim 1878: II, 18] refers to it simply as "multiplication and division", but that is probably just a reference to his previous description and not meant to be a name of general validity. Many others, from al-Khwārizmī onward, state that all, or many, *muʿāmalāt* transactions present us with four magnitudes in proportion,[6] often identified as the given and the requested price and the corresponding magnitudes (*thaman*, *muthaman*, etc.), and then go on with a reference to the proportion theory of *Elements* VII, taking from there that the product of the first and the fourth magnitude equals that of the second and the third – from which the rule of three is easily derived irrespective of which magnitude is unknown.[7]

The Italian abbacus treatises introduce the rule, either as does Jacopo (coin against coin) or as do almost all Arabic writings, coin against commodity (many of them treat coin against coin separately). In this respect, all Ibero-Provençal writings I know of are different, introducing the rule by means of counterfactual pure-number problems of the types "If 3 were 4, what would 5 be?" and "if $4\frac{1}{2}$ are worth $7\frac{2}{3}$, what are $13\frac{3}{4}$ worth?".[8] It is therefore of some interest that the Castilian *Libro de arismética que es dicho alguarismo*[9] (from 1393, but a copy of an earlier treatise) presents the rule (by way of the problem type) in a different way, "If so much is worth so much, how much is so much worth?". Two later abbacus-type writings from the Ibero-Provençal area, Francesc Santcliment's *Summa de l'art d'aritmética* from 1482 [ed. Malet 1998: 163] and Francés Pellos's *Compendion de l'abaco* from 1492 [eds Lafont & Tournerie 1967: 101–103], have the heading "rule of three" ("regla de tres" respectively "regula de tres causas"). Both, however, refer afterwards to the question type "If so much is worth so much, how much is so much worth?" – Santcliment explaining that this is how it is spoken of *en nostre vulgar*, "in our vernacular". The Castilian phrasing is thus representative of the whole Ibero-Provençal region. Exactly the same phrase is used by al-Qurašī [ed., trans. Rebstock 2001: 64] to describe the kind of problems to be dealt with by the rule of three before he refers to the rule (on Euclidean base). It is likely to be significant that al-Qurašī's first example (after the Euclidean reference) is in pure

[5] For Āryabhaṭa, see [Elfering (ed., trans.) 1975: 140], for Brahmagupta [Colebrooke (ed., trans) 1817: 283]. Regarding Mahāvīra, see [Raṅgācārya (ed., trans.) 1912: 86].

[6] Al-Khwārizmī also speaks of two "modes", *wajh*, depending on whether the price of a given quantity or the quantity corresponding to a given price is asked for.

[7] Al-Khwārizmī, [ed., trans. Rosen 1831: 68; ed. Hughes 1986, trans. Gherardo da Cremona: 255]; Ibn al-Khiḍr al-Qurašī [ed., trans. Rebstock 2001: 64]; ps.-Ibn al-Samh (actually a Persian writer – Ahmed Djebbar, personal communication) [ed., trans. Moreno Castillo 2006: 101–103]; Ibn Thabāt [ed., trans. Rebstock 1993: 44]; Ibn al-Bannā' [ed., trans. Souissi 1969: 88]; al-Qalaṣādī [ed., trans. Souissi 1988: 67]; and, as far as can be seen from the description in [Rebstock 1992: 107], Abū'l-Wafā'.

[8] Similar counterfactual problems also appear in many Italian abbacus books, but either as secondary illustrations of the rule of three or wholly outside that heading though dealt with in the same manner. This holds even for those treatises that were written by Italians working in Provence and otherwise apparently learning from the local environment .

[9] Ed. Caunedo del Potro, in [Caunedo del Potro & Córdoba de la Llave 2000: 147].

numbers (though only as numbers in proportion).[10] Possibly, there is also a link to Persian (pre-Islamic?) ways: according to A. S. Saidan (Mahdi Abdeljaouad, personal communication), al-Baghdādī refers to the way profit and loss are calculated by the Persian expressions *dah yazidah*, "ten (is) eleven", and *dah diyazidah*, "ten (is) twelve".[11]

So much about the name and other identifications of the rule. Next the terms in which it is discussed. The reference to the similar and the not-similar is standard in Italian abbacus writings. It is not to be found in the *Libro de arismética que es dicho alguarismo*, but Santcliment as well as Pellos refer to "the contrary" and "the similar"; since these terms are not well adapted to problems formulated in terms of pure numbers, one might be tempted to assume them to represent an Italian influence, which then *could* also be responsible for the reference to a "rule of three things". However, the fact that both refer to "the contrary" instead of the "not similar" (which I do not remember to have seen in any Italian treatise) might rather suggest that the Italian and the Iberian traditions adopted both from elsewhere independently of each other.

In agreement with their primary reference to problems confronting commodity and price, Arabic writings mostly refer to these entities in their exposition of the rule. However, many of them also contain hint of what we have encountered as the Italian standard formulation.

More than a hint, indeed the same rule, is presented by Ibn Thabāt [ed., trans. Rebstock 1993: 45; my English]:

> The fundament for all *mu'āmalāt* -computation is that you multiply a given magnitude by one which is not of the same kind, and divide the outcome by the one which is of the same kind.

Fairly close to this, though more elaborate, is al-Karajī's *Kāfī* [ed., trans. Hochheim 1878: II, 16*f*, my English]; his four magnitudes can be rendered price, measure, (purchase) sum and (corresponding) quantity:

> You find the unknown by multiplying one of the known magnitudes, for instance the sum or the quantity, by that which is not similar to it, namely the measure or the price, and dividing the outcome by the magnitude which is of the same kind.

Ibn al-Bannā' [ed., trans. Souissi 1969: 88, my English] also expands though differently, but leaves out that the third magnitude is "of the same kind":

[10] So is al-Khwārizmī's example of the first mode in the published Arabic text (and thus in Rosen's translation [1831: 69]) and in Robert of Chester's translation [ed. Hughes 1989: 65]. However, in Gherardo's translation [ed. Hughes 1986: 256] this example deals with *dragmae* and *cafficii* (*qafīz*, a hollow measure, mostly between 4 and 60 litres [Hinz 1970: 48–50]). As I have argued elsewhere [Høyrup 1998: 172–174], Gherardo's text is, as far as it goes, closer to the original than Robert's, which translates a (twice) revised version. The published Arabic text has been revised yet another time, in agreement with the stylistic habits of a later time.

The example for the second mode is in pure numbers in all texts and thus also, we must presume, in al-Khwārizmī's original. Al-Qurašī thus seems to promote to primary level a formulation which was also around in the early ninth century (al-Khwārizmī's words strongly suggest that he is quoting a current formula – "Suppose that someone asks .... This is also sometimes expressed thus, `What must be the price of four of them´" [ed., trans. Rosen 1831: 69]).

[11] Unfortunately, neither Abdeljaouad nor I have so far been able to get hold of Saidan's edition of al-Baghdādī.

> You multiply the isolated given number, (that is, the one which is) dissimilar from the two others, by the one whose counterpart one does not know, and divide by the third known number.

Al-Qalaṣādī [ed., trans. Souissi 1988: 67, my English] demonstrates that the resemblance is not enforced by the subject-matter (he refers to the previous presentation of the proportion, but the proportion is also discussed first by the three authors just quoted):

> If one of the extremes is unknown one finds the product of the middle terms and divides the outcome by the known extreme. And similarly, if one of the middles is unknown, one multiplies the extremes and divides the outcome which was reached by the known middle.

Well before anybody wrote about the rule of three in the West (under which term I include the Islamic core region), Brahmagupta [ed., trans. Colebrooke 1817: 283] referred to similar magnitudes:

> In the rule of three, argument, fruit and requisition: the first and last terms must be similar. Requisition, multiplied by the fruit, and divided by the argument, is the produce.

Mahāvīra [ed., trans. Raṅgācārya 1912: 86], contemporary with Al-Khwārizmī or roughly so, explains the rule in a way which includes inverse proportionality,

> in the rule-of-three, *Phala* [Colebrooke's "fruit"] multiplied by *Icchā* [Colebrooke's "requisition"] and divided by *Pramāṇa* [Colebrooke's "argument", by others translated "measure" or "norm"], becomes the answer, when the *Icchā* and the *Pramāṇa* are similar; and in the case of this being inverse, this operation is reversed.

It is difficult to imagine that the Italian presentation of the rule should descend from the Arabic standard presentation, beginning with a discussion of proportions, then followed by names like measure and price, and finally having more or less explicit references to similar and dissimilar magnitudes. It is much more likely that the rule which we know from Italy and Ibn Thabāt was the basis, which theoretically schooled Arabic authors from al-Khwārizmī onward then inserted into and explained by means of the framework of proportion theory. Since the Italian abbacus writers did not borrow the framework but only offer the basic rule, it is obvious that they did not take it from the Arabic theoretical writers. The presentation in the *Libro de arismética que es dicho alguarismo*, on the other hand, is so close to what we find in al-Qurašī (but not in any of the other Arabic works discussed) that a connection to *this family* of literate Arabic mathematics is plausible.

It is also highly implausible that rudimentary references to the similar like those of Brahmagupta or Mahāvīra should be the origin. These are, indeed, *rudiments* glued onto a formulation like that of Āryabhaṭa [ed., trans. Elfering 1975: 140, my English],

> in the three magnitudes, after one has multiplied the magnitude *phala* with the magnitude *icchā*, the intermediate outcome is divided by the *pramāṇa*.

However, *qua* rudiments, they have to come from an environment which differs from that of Sanskrit astronomer-mathematicians, and in which the rule of three was used routinely. This can only be an environment where commercial arithmetic was taught and practised – which should not astonish us, only the existence of such an environment can explain that a large part of the mathematical preliminaries taught by the astronomer-mathematicians (and the Jaina-sage Mahāvīra) consists of commercial arithmetic. In spite of the postulate of the *Laws of Manu* I, 101,[12]

> The Brahmana eats but his own food, wears but his own apparel, bestows but his own in alms; other mortals subsist through the benevolence of the Brahmana,

Brahmin mathematics (that of the *Vedāṅgajyotiṣa* as well as that of later times) certainly subsisted (though not only) through the generosity of the mathematics of the subordinate castes, not least those engaged in trade, in the handling of money, and in the measurement of land.

The rudimentary Sanskrit reference to the similar in the writings of the scholars and sages point to a full form in which the similar and the not-similar were core distinctions. This is also the form which, probably through trading contacts, reached the Arabic world, its merchants as well as its scholars. By the scholars, as we have seen, it was integrated in the framework of proportion theory, and the original form reduced in various ways; al-Khwārizmī does not mention the categories of similar and not-similar; others do so these more or less fully. The fact that they do so even though al-Khwārizmī does not and that they do it in different ways (most fully Ibn Thabāt around 1200) shows that they could still draw directly on the rule in "merchant shape" – a primarily legal scholar like Ibn Thabāt more directly and more fully than those whom we would characterize primarily as "mathematicians" or "astronomer-mathematicians". The Italian abbacus environment also received the rule in this shape and rendered it faithfully; the corresponding Ibero-Provençal environment may have been more influenced by *madrasah* learning similar to what we find in al-Qurašī's book (as perhaps also reflected in the change of al-Khwārizmī's first example – see note 10).

This is as much as analysis of only a handful of sources from each cultural area allows us to conclude. Inclusion of more sources (in areas where they exist, that is, first of all in the various parts of the Arabic and Persian region) would probably allow us to distinguish the pattern more clearly.

Algebra – or almost

The *Liber abbaci* [ed. Boncompagni 1857: 191] presents us with a riddle which nobody appears to have noticed to be one. Here, Fibonacci tells us that

> there is a certain rule, which is called *recta*, which the Arabs make use of. And this rule is very praiseworthy, since it allows to solve a boundless number of questions.

It is then explained that the rule consists in positing that a certain magnitude is a *res*, a "thing". The same rule turns up time and again in Chapter 12 of the work [ed. Boncompagni 1857: 198, 203*f*, 207, 213, 258, 260, 264, 280]. All these examples are of the first degree, and the *regula recta* thus seems to be nothing but first-degree algebra of the kind which (together with the second and sometimes higher degrees) is explained in Chapter 15, Part 3.[13]

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[12] I quote from George Bühler's translation, as found on http://www.sacred-texts.com/hin/manu.htm.

[13] One reference will suffice to show that the riddle has not been discovered to be one. Heinz Lüneburg [1993: 143] quotes the passage where the rule is introduced together with the whole problem solution which makes use of it. He uses 13 lines on pointing out that *summa* may be interpreted either as sum of money or as number, and observes that *res* is declinated through all cases, singular and plural (which is not strictly true); he claims, in gratuitous polemics with [Tropfke/Vogel et al 1980: 377], that Fibonacci uses names for the unknown which are suggested by his phantasy and follows no norm (which is blatantly false, since he *always*

So it seems, indeed – *for us*. Apparently Fibonacci did not agree, *he* never makes the connection to *algebra et almuchabala*. Why?

If nobody else had referred to the rule, it might be a quirk of his and not worthwhile pursuing. But he is not alone.

Firstly, the rule also occurs systematically (as *regula* simpliciter) in the twelfth-century translation *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304–371][14] ascribed to an unidentified Abraham, probably *not* to be identified as Abraham Ibn Ezra.[15] This cannot be derived from Fibonacci, nor can Fibonacci have used this source.

Secondly, the rule turns up as *modo retto*/*repto*/*recto* in Benedetto da Firenze's *Tractato d'abbaco*, ed. [Arrighi 1974: 153, 168, 181].[16] Benedetto speaks of the unknown as *una quantità*. None of Benedetto's problems are taken over from Fibonacci, and both for this reason and because of the particular name given to the unknown Benedetto can be presumed to have other sources for the separateness of the method and for its name.

In all three authors, *regula recta* is first-degree one-variable algebra, with the unknown designated *res* or (in Benedetto's examples) *quantità*. The name – "forward rule" – opposes the method to reverse or backward calculation; in an informal way, it refers to the method the ancient Greeks called *analysis*.

There is no reason to doubt Fibonacci's assertion that he took the method from the Arabs, nor (given how it is also mentioned by "Abraham" and Benedetto) that the methhod was seen even in the Arabic world as being separate from *al-jabr*. However, I do not know of any Arabic source which speaks of it in this way. Although it would be most adequate in the case of algebraic inheritance calculation, this is, if I am not mistaken (which I may easily be, since my knowledge is second-hand except when al-Khwārizmī is concerned), always referred to as *al-jabr wa'l muqābalah*.

The reason that it would be interesting to find Arabic traces of the method is its possible origin independently from *al-jabr* and, if so, its almost certain role in the formation of *al-jabr*. Fibonacci introduces the method as a secondary approach to this problem [ed. Boncompagni

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uses that *res* in the *regula recta,* which was the established translation of Arabic *šayʾ*); but he does not even mention the term *regula recta* outside the quotation. Then, without backward reference, on p. 180 it is claimed that *regula recta* is a technical term for "the procedure to provide unknown magnitudes with names" – that is, at most covering the whole process of formulating of problems (of any degree) as algebraic equations, but not their solution.

14 Cf. Barnabas Hughes' supplement [2001], which transforms Libri's edition into a critical one. The Latin text must be a translation, since it refers to an initial praise of God but does not bring it ("hic post laudem Dei inquit"). The same happens, e.g., in Gherardo da Cremona's translation of al-Khwārizmī's algebra [ed. Hughes 1986: 233].

15 This identification was proposed hypothetically by Libri [1838: I, 304 n. 1]; many later writers have dropped Libri's doubts – without ever advancing good reasons for the identification, which a stylistic comparison with Ibn Ezra's *Sefer ha-mispar* [ed., trans. Silberberg 1895] makes highly implausible. Actually, Shlomo Sela [2001] does not even mention the *Liber augmenti et diminutionis* in his discussion of Ibn Ezra's complete scientific corpus, which otherwise includes *dubia* which are then rejected.

16 For the identification of the author (whom Arrighi had believed to be Pier Maria Calandri), see [Van Egmond 1980: 96] and [Ulivi 2002: 54–56].

1857: 190]:

> One [of two men] asks from the other 7 *denarii*; and then he will have the quintuple of what the other has. And the second asks from the first 5 *denarii*, and then he will have seven times as much as the other.

He posits that the second man has (originally) a *thing* plus the seven *denarii* which the first man asks for. This is a sagacious choice, but also noteworthy from another perspective. Diophantos, in *Aritmetica* I,xv [ed. Tannery 1893: I, 36] asks for "two numbers such that each, after having received a given number from the other, will be in a given ratio to the remainder". This looks very much as a transposition of Fibonacci's problem type into the realm of abstract numbers and generality – as, in its entirety, *Arithmetica* I consists of traditional "recreational" mathematical riddles stripped of their concrete reference. The two problems are obviously identical in mathematical structure. Moreover, the ensuing numerical example states that if the first number receives from the second 30 units it becomes its double, and if the second receives 50 from the first, it becomes its triple. In order to solve this, Diophantos makes the position that the ἀριθμός is the second number diminished by that number which is to be transferred to the first – that is, that the second number is the ἀριθμός plus the transfer, exactly Fibonacci's choice.

Papyrological evidences shows that the ἀριθμός-technique was in general use in late Antiquity, at least for first-degree problems.[17] There is no reason to believe that it should suddenly have disappeared among for instance Egyptian reckoners at the Islamic conquest (in many other respects they carried their methods into the new political context). In view of the strong similarity between Diophantos's and Fibonacci's solutions, it is therefore reasonable to assume that the *regula recta* is indeed a continuation of the ancient ἀριθμός-technique. Within the Caliphate, it then got the occasion to merge with the *al-jabr* riddles about a possession and its roots, thereby giving rise to the technique and discipline which al-Khwārizmī presents as *al-jabr wa'l-muqābalah*.

A captivating and plausible scenario, I believe. Identification of the *regula recta* in Arabic sources might perhaps corroborate or destroy it.

Missing a good point – and making others

The earliest Italian treatise dedicated to algebra alone is Dardi of Pisa's *Aliabraa argibra* from 1344.[18] The first part of this impressive work is a "Treatise on the rules which belong to the multiplications, the divisions, the summations and the subtractions of roots".[19] In this opening treatise, a number of procedures are illustrated by polynomials containing rational roots (e.g., $36/(\sqrt{4}+\sqrt{9}+\sqrt{16})$, treating them *as if* they were surds (formulated thus in $\mathbf{D_1}$ fol. 3^v, similarly $\mathbf{D_2}$ p. 62), the obvious point being that this allows control of the correctness of the result; however, such control is never made, nor is any other advantage taken of the choice of rational roots, except an unproven statement that the result coming from a calculation (in the example $\sqrt{40\frac{24}{25}}+\sqrt{92\frac{4}{25}}+\sqrt{5\frac{19}{25}}-\sqrt{163\frac{21}{25}}-\sqrt{10\frac{6}{25}}$) *can* be reduced.

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17 See [Robbins 1929], [Karpinski & Robbins 1929] and [Vogel 1930].

18 I have used the Vatican manuscript Chigi M.VIII.170 from c. 1395 ($\mathbf{D_1}$); Raffaella Franci's edition [2001] of the Siena manuscript I.VII.17 from c. 1470 ($\mathbf{D_2}$); and Warren Van Egmond's personal transcription of the Arizona manuscript, written in Mantova in 1429 ($\mathbf{D_3}$), for which I want to express my gratitude; the datings of $\mathbf{D_1}$ and $\mathbf{D_2}$ are based on watermarks and according to [Van Egmond 1980]; that of $\mathbf{D_3}$ is stated in the

Already from Dardi's own text it is not plausible that the idea was his own. Nobody would prepare time and again on his own initiative for a proof and then never perform it. It is quite possible, however, that an author takes over a style which prepares for such proofs and then misses the opportunity to perform them.

But Dardi's own text is not our only evidence. The manuscript Vat. Lat 10488 of the Vatican Library, itself written in 1424, contains six pages with the heading "Algebra", told to be copied from a book written by Giovanni di Davizzo de l'Abacho da Firenze on September 15th, 1339; such information will have been taken from the *incipit* of Giovanni's manuscript and can therefore be considered trustworthy.[20] Giovanni's algebra is so much below Dardi's level (and so different from Dardi's in its orientation) that Dardi cannot have learned from him; already for chronological reasons, influence the other way can be excluded. However, Giovanni also deals with the arithmetic of roots and binomials, and in five out of nine examples[21] he operates here with the roots of square numbers, *without taking advantage of this particular choice*, exactly as Dardi.

In a third treatise, however, the trick is used *and explained*, namely in an anonymous *Trattato d'algibra* from the 1390s or so [ed. Franci & Pancanti 1988: 6, 12]. That this is interesting and relevant follows from the way this treatise treats certain other problems.

In Jacopo da Firenze's *Tractatus algorismi*, four problems deal with the successive salaries of the manager of a *fondaco*, supposed to increase geometrically.[22] If we use the successive letters of the alphabet to designate the yearly salaries, the problems are the following:

$$a+c = 20\,,\ \ b = 8 \qquad (1)$$

$$a = 15\,,\ \ d = 60 \qquad (2)$$

$$a+d = 90\,,\ \ b+c = 60 \qquad (3)$$

$$a+c = 20\,,\ \ b+d = 30 \qquad (4)$$

None of the solutions make use of *al-jabr*/*cosa* algebra. Without using the concept explicitly, (2) finds the rate of yearly increase as the cube root of $\frac{60}{15}$, while (4) finds this rate r as $\frac{30}{20}$, a then as $20/(1+r^2)$, etc. (1) uses that $b^2 = ac$, and then applies the method used to solve analogous rectangle problems in Abū Bakr's *Liber mensurationum* (and in similar treatises, and indeed since Old Babylonian "algebra"). (3) makes use of the formulae

$$a{\cdot}d = b{\cdot}c = \frac{(b+c)^3}{3(b+c)+(a+e)} \qquad (5)$$

and then proceeds as (1).

Problems of the same kind – some still dealing with the salaries of a *fondaco* manager or a servant, others formulated as pure-number problems – are found in various later abbacus writings until Cardano. Only two occurrences need to be mentioned.

Benedetto da Firenze's selection from 1463 from maestro Biagio's collection of algebra

manuscript.

19 **D_1** fol. 3^v; **D2** p. 38. **D_3** does not have this general caption but separate captions for the single sections.

20 An edition and translation of the relevant part of Giovanni's text can be found in [Høyrup 2006].

21 Namely $\sqrt{9}\cdot\sqrt{9}$; $\sqrt{25}/\sqrt{9}$; $(5+\sqrt{4})\cdot(5-\sqrt{9})$; $(7+\sqrt{9})\cdot(7+\sqrt{9})$; and $35/(\sqrt{4}+\sqrt{9})$.

22 An edition of these problems with English translation and mathematical commentary can be found in [Høyrup 2007a: 324–331, 115–121].

problems, the latter dated before c. 1340, repeats (4) but uses *cosa*-algebra from the point in the solution where *r* has been found as $^{60}/_{40}$ [ed. Pieraccini 1983: 89–91].

In our anonymous *Trattato d'algibra* [ed. Franci & Pancanti 1988: 80] we also find an example of type (4); it shares Biagio's numbers, uses almost the same words in the statement and follows the same path in the solution, apart from positing the salaries of the first two years to be, respectively, 2 *censi* and 3 *censi*, not 2 *cose* and 3 *cose*. In a context where the *censo* (originally introduced as the introduction of Arabic *māl*) was no longer remembered to represent an amount of money, a spontaneous replacement of the *cosa* by a *censo* is so unexpected that it is unbelievable (not least because the author of the *Trattato d'algibra* feels obliged to find the *cosa* from the *censo*, only to square it again). There seems to be no doubt that the *Trattato d'algibra* had its words not from those of *Biagio* (which would ask for such a spontaneous replacement to have taken place) but directly from an environment closer to an Arabic source – from which follows, *firstly*, that the problem type might be of Arabic origin, in spite of its apparent absence from extant Arabic sources;[23] and *secondly*, that the *Trattato d'algibra* might also in other respects reflect Arabic ways. This supposition is further corroborated by the use in the *Trattato* of schemes for the multiplication of polynomials; these schemes emulate the algorithm for the multiplication of multi-digit numbers in a way which is very close to what can be found in the Jerba manuscript [Abdeljaouad 2002: 35].

In consequence, it might be rewarding to look into Arabic sources, both for expositions of the arithmetic of polynomials which make use of "rational roots as if they were surds" and for problems about wages increasing in geometric proportion – not least for such advanced insights as are revealed by formula (5).[24]

The unknown heritage – a heritage from unknown parents

In the end I shall consider a case where influences appear to have moved in a different direction – but that picture might still be changed by the appearance of unexpected sources.[25]

In the *Liber abbaci* [ed. Boncompagni 1857: 279] we find the following:

> Somebody coming to his end told his oldest son: Divide my movable property among yourselves in this way, you take one *bezant* and a seventh of those that remain. And to the second son he said, you take 2 bezants, and a seventh of those that remain. And the third he ordered that he should take 3 bezants, and $^1/_7$ of those that remained. And in this way he called all his sons in order, giving each of them one more than the other; and afterwards, always $^1/_7$ of those that remained. The last, however, had those that remained. It happened, however, that each of them had the same part of the possession of the father, under the said condition. It is asked, how many were the sons; and how much was their money. You do thus: for the seventh which he gave to each, you retain 7; from which you detract 1, 6 remain; and so many were the sons: which 6 multiplied by itself make 36; and so many were their bezants.

Jacopo da Firenze has a similar problem:

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23 Since Biagio's words are so close to those of the anonymous *Trattato*, he must have drawn, either on the same environment or on an Italian precursor of the problem in the *Trattato d'algibra*.

24 Certainly no more advanced than what could be grasped by for instance al-Karajī – but to my knowledge not found in any of al-Karajī's writings.

25 A full presentation of the topic can be found in [Høyrup 2007b].

I go to a garden, and come to the foot of an orange. And I pick one of them. And then I pick the tenth of the remainder. Then comes another after me, and picks two of them, and again the tenth of the remainder. Then comes another and picks 3 of them, and again the tenth of the remainder. Then comes another and picks 4 of them and the tenth of the remainder. And thus come many. Then the one who comes last, that is, behind, picks all that which he finds left. And finds by this neither more nor less than we others got. And one picked as much as the other. And as many men as there were, so many oranges each one got. I want to know how many men there were, and how many oranges they picked (each) one, and how many they picked all together. Do thus, detract one from 10, 9 is left, and there were 9 men, and 9 oranges (each) one picked. And they picked in all 81 oranges. And if you want to verify it, do thus,

the first picked 1 of them, left
80. The tenth is eight, and you have that this one got9, left
72. The second2, left 70, the tenth is 7, and he got 9, left
63. The third 3, left 60, the tenth is 6, and he got 9, left
54. The fourth 4, left 50, the tenth is 5, and he got 9, left
45. The fifth 5, left 40, the tenth is 4, and he got 9, left
36. The sixth 6, left 30, the tenth is 3, and he got 9, left
27. The seventh7, left 20, the tenth is 2, and he got 9, left
18. The eighth8, left 10, the tenth is 1, and he got 9, left
9. The ninth, that is, the last one, picked these 9, neither more nor less, as there were no more. So that you see that it is well done. And it goes well. And thus are done the similar computations.

The problem is overdetermined and, as Euler said in his didactical *Élémens d'algebre* [1774: 489], "of a quite particular nature, and therefore deserves attention".

Actually, a simple argument can lead to the solution, but only *because* a solution exists: Each son gets an absolute contribution equal to his number in the row, and then (in Fibonacci's case) $\frac{1}{7}$ of the remainder, leaving $\frac{6}{7}$ of that same remainder. Since the last – say, number N – takes everything, he leaves nothing, that is, he gets his absolute contribution N and nothing more. Therefore, what each son gets (Δ) is equal to the number of sons, that is, N. The second-last gets an absolute contribution of N–1, and then $\frac{1}{7}$ of the remainder, which must be 1 (in total, he should have N). But if $\frac{1}{7}$ of the remainder is 1, then $\frac{6}{7}$ must be 6, and that is what the last son gets. Thus $N = \Delta = 7–1$.

It is quite obvious from the sources, however, that no medieval calculator (at least, none of those whose version of the problem we possess) has seen that (nor any more recent historian of mathematics).

Such problems turns up fairly regularly in abbacus treatises until Tartaglia, almost always in inheritance dress (Jacopo's orange-picking is his alone). In the later fifteenth and the sixteenth century, they are found in German cossist works, and from the sixteenth onward in Portuguese and French treatises. Sometimes the absolutely defined contributions are α, 2α, ..., in which case $N = d–1$ (the fraction being $\frac{1}{d}$), $\Delta = \alpha N$; sometimes the fraction is taken before the absolute contribution, in which case $N = d$, $\Delta = \alpha(d–1)$. Sometimes the absolute contributions are $p\alpha$, $(p+1)\alpha$, $(p+2)\alpha$, ..., which corresponds to leaving out the first p–1 sons. In a few cases, a reasoned solution is given, built on the equality of the first two shares (which of course does not guarantee that the following shares will have the same value, for which reason Jacopo's proof is still adequate); this can be done by *cosa*-algebra or by a double false position.

Fibonacci also has examples where d is not integer, and where the absolute contributions are α, α+ε, α+2ε, ... (ε not dividing α). Then the number of shares is no longer integer (that is, the last share is not equal to the others). For the first of these Fibonacci gives an algebraic solution based on the equality of the first two shares, and then a rule *which pretends to be but is not derived from his algebra* (the fraction being $^{p}/_{q}$, *T* being the total):

$$T=\frac{[(\varepsilon-\alpha)q+(q-p)\alpha]\cdot(q-p)}{p^2}\ ,$$

$$N=\frac{(\varepsilon-\alpha)q+(q-p)\alpha}{\varepsilon p}\ ,$$

$$\Delta=\frac{\varepsilon(q-p)}{p}\ .$$

For the following cases he gives only rules, no algebra (there are four rules in total, depending on whether ε is larger or smaller than α, and on whether the absolute or the relative part of the shares is taken first). It is evident that Fibonacci does not know where the formulae come from, and does not understand why they work.[26]

Similar sophisticated cases are treated in Barthélemy de Romans' *Compendy de la praticque des nombres*[27]. Barthélemy's rules are sufficiently different from Fibonacci's to exclude direct use of the *Liber abbaci*, although they are obviously algebraically equivalent. Even Barthélemy does not understand why the rules work.

As far as I know, the problem type is *not* present in any Arabic treatise, but two treatises (which I know about thanks to a personal communication from Mahdi Abdeljaouad) contain a distorted echo:

One of these is Ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l ʿamali bi rušūm al-ghubār* ("Fecundation of thoughts through use of *ghubār* numerals") – written in Marrakesh in c. 1190. It runs as follows:[28]

> An inheritance of an unknown amount. A man has died and has left at his death to his six children an unknown amount. He has left to one of the children one dinar and the seventh of what remains, to the second child two dinars and the seventh of what remains, to the third three dinars and the seventh of what remains, to the fourth child 4 dinars and the seventh of what remains, to the fifth child 5 dinars and the seventh of what remains, and to the sixth child what remains. He has required the shares be identical. What is the sum?
>
> The solution is to multiply the number of children by itself, you find 36, it is the unknown sum. This is a rule that recurs in all problems of the same type.

The other is *Al-Maʿūna fī ʿilm al-ḥisāb al-hawāʾī* ("Assistance in the science of mental

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26 The rules *can* be derived and shown to be valid by means of a technique with which Fibonacci was familiar, and which is also used amply in the twelfth-century *Liber mahamaleth*: namely the representation of numerical magnitudes by line segments and use of proportion theory.

27 Barthélemy probably wrote this treatise around 1467, but what we possess is a revised redaction from 1476 due to Mathieu Préhoude – see [Spiesser 2003: 26, 30]. The sophisticated cases are also treated in the problem collection which Nicolas Chuquet attached to his *Triparty en la science des nombres* [ed. Marre 1881: 448–451], but since Chuquet may have drawn all his sophisticated cases from Barthélemy (whom he cites) he can be left out of the picture.

28 My translation from Mahdi Abdeljaouad's French translation.

calculation") written by Ibn al-Hā'im (1352–1412, Cairo and Jerusalem, and familiar with Ibn al-Yāsamīn's work):[29]

> An amount of money has been diminished by one dirham and the seventh [of what remains]; by two dirhams, and then the seventh of what remains; then three dirhams and the seventh of what remains; then four dirhams and the seventh of what remains; then five dirhams and the seventh of what remains. In the end, six remain.
> Take the square of the six that remain, it is the amount which was asked for.

The number of shares is thus given in both versions; none the less, both still use the same rule as the "Christian" version of the simple problem. Ibn al-Yāsamīn leaves out that the last share is determined according to the same rule as the preceding ones, whereas Ibn al-Hā'im does not require the shares to be equal. Both informations are indeed superfluous.

As we see, Ibn al-Hā'im's version is not overdetermined; it can be solved backwards step by step, in this way:

The fifth share is $5+\frac{1}{7}A$, where $A+5$ is what is left after the taking of the fourth share; but this remainder is also the sum of the fifth and sixth shares. Hence,

$$A+5 = 6+5+\frac{1}{7}A,$$

from which follows $A = 7$. The fourth share is $4+\frac{1}{7}B$, where $B+4$ is what is left after the taking of the third share; but this is also the sum of the fourth, fifth and sixth shares; etc.

Obviously, a similar backward calculation could be made for changing fractions and for absolutely defined contributions that are not in arithmetical progression. However, *the rule* is only valid for a constant fraction $\frac{1}{N+1}$, where N is the given number of shares, and if the absolutely defined contributions are $1+(i-1)$. There is hence no doubt that Ibn al-Hā'im's problem descends from the "Christian" problem and results from an attempt to assimilate it to the more familiar structure of "nested boxes".

Ibn al-Yāsamīn's problem *is* overdetermined, but the evident way to solve it would still be a backward calculation: if S is what is left when the fifth share is to be taken, the fifth share is $5+\frac{1}{7}(S-5)$, and the sixth share is what is left, i.e., $S-5-\frac{1}{7}(S-5)$. From their equality follows that S is 12, each share thus 6, and the total $6\cdot6$. The rule, once again, is valid but not naturally adapted to the actual problem.

The conclusion is that mathematicians from the Maghreb or al-Andalus[30] had come to know about the problem type already before the *Liber abbaci* was written; but their use of a rule which is better adapted to the "Christian" version of the problem shows that this latter version with its unknown value of N was not derived from the "Islamic" nested-box versions but was indeed the original form. Whether Ibn al-Hā'im knew the problem from the Maghreb mathematicians or through other channels cannot be decided at present. In any case, the aberrant character of the two Arabic problems is strong evidence that Fibonacci did not get the problem from the Arabic world – if it was known and accepted there, why should our two authors need to make it more familiar by transforming N into a given magnitude? Ibn al-Yāsamīn confirms on the

29 Still my translation from the French. Biographical information from [Djebbar 2005: 131].

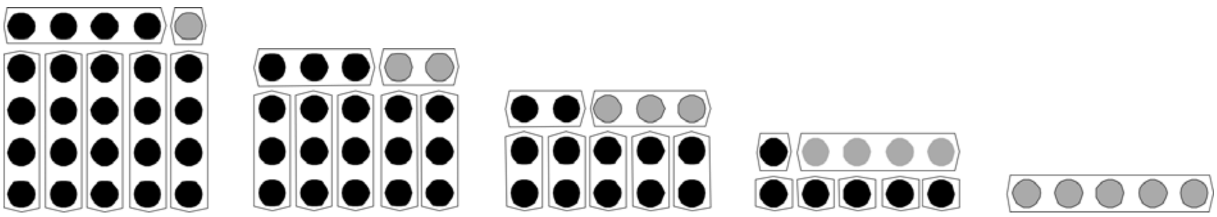
30 Ibn al-Yāsamīn's "all problems of the same type" seems to prove that he was not the only mathematician in his area to know about them. He had been active in Morocco and in al-Andalus; he may have encountered the derived problem type in either place.

other hand that the problem type which inspired him was indeed familiar (in *a place* that might inspire him and where he expected to find readers) before the *Liber abbaci* was thought of.

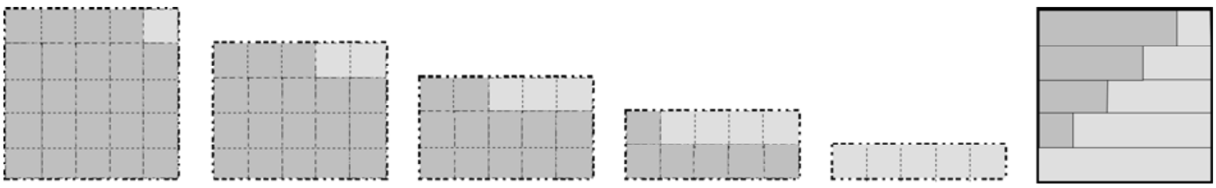
The problem also turns up in several Byzantine sources: Maximos Planudes's late thirteenth-century *Calculus According to the Indians, Called the Great* [ed., trans. Allard 1981: 191–194]; a problem collection from the early fourteenth century [ed., trans. Vogel 1968: 102–105]; and Elia Misrachi's *Sefer ha-mispar* from c. 1500 [ed., trans. Wertheim 1896: 59*f*]. The latter two might in principle be derived from Italian works (even though the fourteenth-century collection is otherwise not influenced by Italian material, at least not in its choice of metrology and currency). But Planudes is interesting. After having presented the problem and stated the rule (coinciding with Fibonacci's first example) he gives this explanation of the procedure:[31]

> When a unit is taken away from any square number, the left-over is measured by two numbers multiplied by each other, one smaller than the side of the square by a unit, the other larger than the same side by a unit. As for instance, if from 36 a unit is taken away, 35 is left. This is measured by 5 and 7, since the quintuple of 7 is 35. If again from 35 I take away the part of the larger number, that is the seventh, which is then 5 units, and yet 2 units, the left-over, which is then 28, is measured again by two numbers, one smaller than the said side by two units, the other larger by a unit, since the quadruple of 7 is 28. If again from the 28 I take away 3 units and its seventh, which is then 4, the left-over, which is then 21, is measured by the number which is three units less than the side and by the one which is larger by a unit, since the triple of 7 is 21. And always in this way.

This is true, but not obviously so if one does not make use of a representation in symbols (which Planudes could not do) or by means of counters or something similar in this way:



We may also use this version, in which the summary to the right shows that the square is divided into the sum of two triangular numbers, one of which – namely 1+2+...+n – consists of the absolutely and the other – namely (n–1)+(n–2)+...+1 – of the relatively defined contributions.



Pebble arithmetic, and play with rectangular and triangular numbers, suggest a root in ancient or late ancient mathematics, and Planudes's problem stands indeed at the point of his treatise where he has finished his presentation of Hindu-Arabic reckoning and before a problem of probably late ancient origin, which is found in almost exactly the same words in the pseudo-Heronic

[31] I try to make a very literal translation, conserving all quasi-logical particles even when they offend the modern ear; a somewhat less literal French translation accompanies Allard's edition of the Greek text.

Geometrica Ch. 24 [ed., trans. Heiberg 1912: 414–417], cf. [Sesiano 1998: 284–286]. Since diagrams of the kind just shown could provide *the idea* behind the problem (which the above solution could never do), there are fair reasons to believe that Planudes reveals in this way where the problem comes from. Whether Fibonacci's reference to Byzantine coin implies that *he* encountered the problem in Byzantium is not to be decided with certainty. We know, however that he discussed mathematics with mathematics teachers in Byzantium – the problem for which the *regula recta* is presented as a secondary solution was posed to him by *apud Constantinopoli a quodam magistro,* "in Byzantium by some master" [ed. Boncompagni 1857: 190]; moreover, *every* time Constantinople is mentioned in the work together with coin, this coin is the bezant [Boncompagni (ed.) 1857: 94, 161, 203, 249, 274, 276].[32]

In any case, Ibn al-Yāsamīn cannot have encountered the unknown heritage in Byzantium, so the problem type must have been more widely diffused in the late twelfth century. Moreover, no Byzantine source which I know of contains the slightest hint that the sophisticated versions were known in Byzantium. Where then?

Beyond Byzantium, Bejaïa, Egypt and Syria (according to the above to be left out), Fibonacci says in the introduction to the *Liber abbaci* [ed. Boncompagni 1857: 1] that he learned about the Indian numerals (and what went with them) in Provence and Sicily. Among these places, Barthélemy's familiarity with the sophisticated cases speak in favour of Provence or some region close by. But the evidence is hardly compelling; perhaps it is after all easier to imagine that one of the mathematicians from al-Andalus whom we know by name only could have derived these intricate rules than to believe in the existence before 1200 of a mathematical environment in the non-Arabic Mediterranean world which was capable of feats that went beyond even Fibonacci's wits.

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[32] This observation suggests that the *Liber abbaci* might be a general source for economic history, giving evidence for which coin (and which metrology) was used in the various locations it speaks of.

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