

MESOPOTAMIAN MATHEMATICS

Contribution to *The Cambridge History of Science*

The term “Mesopotamian mathematics” refers to the mathematical knowledge and the mathematically based practices of the cuneiform tradition from the mid-fourth millennium BC until its disappearance around the beginning of the Christian era.¹ All dates in the following should thus be understood to be BC when AD is not indicated explicitly.

The reference to the writing system is not peripheral. Throughout its history, the development and orientation of Mesopotamian mathematics was intimately bound up with written administration and the scribal craft, and all documentation we possess derives from documents written on clay tablets (here as mostly, the mathematical regularities of buildings and other artefacts tell us little conclusive about the kind of mathematical *knowledge* which was involved in their production).

Attentive reading of the written sources reveals, however, that the written tradition must have received important inspiration from traditions carried by non-scribal (and, at least until the advent of Aramaic alphabetic literacy in the first millennium, non-literate) specialists: surveyors, master-builders, traders, and/or similar groups. In all likelihood, these “lay” practitioners have also borrowed from the literate tradition, but this is more difficult to document.

Long-term developments

So-called protoliterate writing was created in Southern Mesopotamia (the later “Sumerian” area) after the mid-fourth millennium, in connection with the earliest formation of a bureaucratic state (understood as a social system characterized by an at least three-tiered system of control and by extensive specialization of social roles) headed by a temple institution. The root of the invention was an accounting system based on clay tokens (probably standing for various measures of grain, for livestock, etc.) that had been used in the Near East since the eighth millennium, and various transformations and extensions of this system introduced in response to the needs

¹ The changing approach to the field and the increasing awareness that Mesopotamian mathematics has a *history* is described in Jens Høyrup, “Changing Trends in the Historiography of Mesopotamian Mathematics: An Insider’s View,” *History of Science*, 34 (1996), 1–32. An exhaustive annotated bibliography until 1982 is Jöran Friberg, “A Survey of Publications on Sumero-Akkadian Mathematics, Metrology and Related Matters (1854–1982),” *Department of Mathematics, Chalmers University of Technology and the University of Göteborg* No. 1982–17. A very detailed account of mathematical knowledge and procedures is Friberg, “Mathematik,” in *Reallexikon der Assyriologie und Vorderasiatischen Archäologie*, vol. 7 (Berlin & New York: de Gruyter, 1990), pp. 531–585.

created by increasing social complexity.²

In the protoliterate period, metrological notations were created that depicted the traditional tokens. A notation for “almost abstract” number may have been created by adaption of the system of grain or hollow measures to existing spoken numbers, with basic signs for 1, 10, 60 and 3600, and composite signs for 60·10 and 3600·10. Sub-unit extensions of all metrologies, an administrative calendar and a combined metrology for length and area measurement replacing older “natural” (ploughing or irrigation) measures may also be new creations.³

Mathematics was fully integrated with its bureaucratic applications – school texts are “model documents”, distinguishable from real administrative documents only by lacking the name of a responsible official and by the prominence of nice numbers. But the integration was mutual: bureaucratic procedures, centered on accounting, were mathematically planned, for instance around the new area metrology and the calendar. The cognitive integration corresponds to social integration – the literate and numerate class seems to coincide with the stratum of temple managers.

The third millennium continued the mutual fecundation of administrative procedures and the development of mathematics (in a process whose details we are unable to follow). The reach of accounting systems increased gradually, and metrologies were modified intentionally so as to facilitate managerial planning and accounting. At the same time, there is a trend toward “sexagesimalization”, expanding use of the factor 60 – see below.

Around 2600, however, when a distinct scribal profession emerged, numeracy and literacy gave up the full cognitive subservience to accounting and management. For the first time, writing served to record literary texts (proverbs, hymns and epics); and we find the first instances of “pure” or (better) “supra-utilitarian” mathematics – mathematics starting from applicable mathematics but going beyond its usual limits. It seems as if the new class of professional intellectuals set out to test the potentialities of the professional tools – the absolute favorite problem was the division of very large

² On the token system and its development, see for instance Denise Schmandt-Besserat, *Before Writing. I. From Counting to Cuneiform* (Austin: University of Texas Press, 1992).

³ A broad summary of fourth and third millennium mathematical techniques (including the details of metrologies) is Hans J. Nissen, Peter Damerow & Robert Englund, *Archaic Bookkeeping: Writing and Techniques of Economic Administration in the Ancient Near East* (Chicago: Chicago University Press, 1994). The interplay between state formation and the shaping of mathematical techniques and thought is analyzed in Høyrup, *In Measure, Number, and Weight. Studies in Mathematics and Culture* (New York: State University of New York Press, 1994), pp. 52–57, 68–74. Pp. 45–87 of the same volume may serve as a general reference (with extensive bibliography) for the links between statal bureaucracy, scribal craft and culture, and the transformations of mathematics until the mid-second millennium.

round numbers by divisors that were more difficult than those handled in normal practice.⁴

The language of the protoliterate texts is unidentified, whereas the language of the third-millennium southern city states was Sumerian. Toward 2300, however, an Akkadian-speaking dynasty conquered the whole Sumerian region, and soon the entire Syro-Iraqian area (Akkadian is a Semitic language, later split into the Babylonian and Assyrian dialects). Sumerian remained the administrative language (and hence the language of scribal education), but new problem types suggest inspiration from a lay, possibly non-Sumerian surveyors' tradition – area computations that are very tedious unless one knows that $\square(R-r) = \square(R) + \square(r) - 2\square(R,r)$ (\square and \square stand for rectangle and square, respectively), and the bisection of a trapezium by means of a parallel transversal.

The 21st century is of particular importance. After a breakdown of the “Old Akkadian” empire, a new territorial state (“neo-Sumerian” or “Third Dynasty of Ur”) established itself in 2112 (“middle chronology”). A military reform under king Šulgi in 2074 was followed immediately by an administrative reform, in which scribal overseers were made accountable for the outcome of every $\frac{1}{60}$ of a working day of the labor force allotted to them according to fixed norms; at least in the South, the majority of the working population was subjected to this regime, probably the most meticulous large-scale bureaucracy that ever existed.

Several mathematical tools were apparently developed in connection with the implementation of the reform (all evidence is indirect): A new book-keeping system – not double-entry book-keeping, but provided with similar built-in controls; a place-value system with base 60 used in intermediate calculations; and the various mathematical and technical tables needed in order to make the place-value system useful (described below).

No space seems to have been left to autonomous interest in mathematics; once again, the only mathematical school texts we know are “model documents”.

For several reasons (among which probably the exorbitant costs of the administration) even the Ur-III state collapsed around 2000. A number of smaller states arose in the beginning of the succeeding “Old Babylonian” period (2000 to 1600), all to be conquered by Hammurapi around 1760. Without being a genuine market economy, the new social system left much space to individualism, both on the socio-economic and the ideological level. In the domain of scribal culture, this individualism expressed itself in the ideal of “humanism” (*šic – n a m - l ú - u l ù*, Sumerian for “being human”): scribal *virtuosity* beyond what was needed in practice. This involved the ability to read and speak Sumerian, now a dead language known only by scribes, as well as supra-

⁴ Two specimens with divisor 7 are analyzed in Høyrup, “Investigations of an Early Sumerian Division Problem, c. 2500 B.C.,” *Historia Mathematica*, 9 (1982), 19–36. A similar problem with divisor 33 from Ebla in Syria (whose mathematics was borrowed from Sumer) is analyzed in Friberg, “The Early Roots of Babylonian Mathematics. III: Three Remarkable Texts from Ancient Ebla,” *Vicino Oriente*, 6 (1986), 3–25.

utilitarian mathematical competence.

The vast majority of Mesopotamian mathematical texts come from the Old Babylonian school (teacher's texts or copies from these, not student exercises as the third millennium specimens). They are invariably in Akkadian (notwithstanding sometimes heavy use of Sumerian word signs), another indication that the whole genre of "humanist" mathematics had no Ur-III antecedents. Its central discipline was a geometrically based second-degree algebra, probably inspired from a collection of geometrical riddles circulating among lay, Akkadian-speaking surveyors (to find the side of a square from [the sum of] "the side and the area" or from "all four sides and the area", etc.), but transformed into a genuine mathematical discipline and a general analytical technique.

A first classification divides the text corpus into table texts and problem texts. The second category can be subdivided in different ways: into (1) theme texts whose problems have a common theme, (2) anthology texts which have no common theme, and (3) single-problem texts; or into (I) procedure texts that tell how to obtain a solution, and (II) catalogue texts listing mere problem statements (most catalogues are theme texts). It is noteworthy that anthology texts, even if mixing different kinds of mathematics, do not mix mathematics with other topics (not even sacred numerology); Old Babylonian mathematics was clearly a cognitively autonomous field.

Some of the texts come from excavations, but most from illegal diggings.⁵ For these, provenience and dating must be derived from paleography, orthography and characteristic differences in terminology. In a region encompassing the former Sumerian South, the Center (Babylon and surroundings) and the Center-to-North-East (Ešnunna), and even the eastern periphery (Iranian Susa), the global character of Old Babylonian mathematics is grossly the same (from the Assyrian North, never dominated by Ur III or Babylonia, no mathematical texts but only accounts are known). Close attention to language and procedures reveals, however, that the adoption of lay material has taken place simultaneously in Ešnunna and in the South; that pre- and post-Šulgi-reform Sumerian mathematics coexisted in 18th-century Ešnunna without being fully merged; that a number of schools tried to develop a strict terminological canon but did not agree in their choices; that all texts that try to explain procedures abstractly and not only through paradigmatic numerical examples are close to the lay oral tradition – the school

⁵ The basic text editions are O. Neugebauer, *Mathematische Keilschrift-Texte*, 3 vols (Berlin: Julius Springer, 1935–1937) = MKT; O. Neugebauer & A. Sachs, *Mathematical Cuneiform Texts* (New Haven, Connecticut: American Oriental Society, 1945) = MCT; and E. M. Bruins & M. Rutten, *Textes mathématiques de Suse* (Paris: Paul Geuthner, 1961) = TMS. MKT and MCT are very careful editions, TMS alas not. Only TMS contains archeologically excavated texts. Single texts with known provenience have been published by Taha Baqir, Jöran Friberg, and others, many in the journal *Sumer*.

seems to have given up abstract formulation as pedagogically inefficient.⁶

Inner weakening followed by a Hittite raid put an end to the Old Babylonian state in 1600. A warrior tribe (the Kassites) subdued the Babylonian area, for the first time rejecting that managerial-functional legitimization of the state which, irrespective of suppressive realities, had survived since the protoliterate phase and made mathematical-administrative activity an important ingredient of scribal professional pride. The school institution disappeared, and scribes were trained henceforth as apprentices. Together, these events had the effect that mathematics disappears almost completely from the archeological horizon for a millennium or more (one Kassite problem text and one table text have been found; the problem text seems to derive from the style of the Old Babylonian northern periphery); metrologies were modified in a way that would fit practical computation in a mathematically less sophisticated environment (e.g., making use of normalized seed measures in area mensuration; though no longer an object of pride, mathematical administration did not disappear).

Around the “Neo-Babylonian” mid-first millennium, mathematical texts turn up again, for instance concerned with area mensuration, the conversion between various seed measures, and some supra-utilitarian problems of the kind that had once inspired Old Babylonian algebra. This and other features may reflect renewed interaction between the scribal and the lay traditions, which so far cannot be traced more precisely.

One of the Neo-Babylonian texts starts by telling the sacred numbers of the gods before going on with genuinely mathematical topics. This break-down of cognitive autonomy corresponds to what the texts tell us about their owners and producers (such information is absent from the Old Babylonian tablets); they identify themselves as “exorcists” or “omen priests” (another reason to believe that their practical geometry was borrowed from lay surveyors).

A final development took place in the Seleucid era (311 onwards). Even this phase is only documented by utterly few texts: some multi-place tables of reciprocals probably connected to astronomical computation, one theme text, an anthology text focusing on practical geometry and an unfocused anthology text. The unfocused anthology text shows some continuity with the Old Babylonian tradition (including its second-degree algebra) but also fresh developments (e.g., formulas for $\Sigma 2^n$ and Σn^2). The theme text contains “algebraic” problems about rectangles and their diagonals of which only one

⁶ The analysis is presented in Høyrup, “The Finer Structure of the Old Babylonian Mathematical Corpus. Elements of Classification, with some Results”, in Joachim Marzahn & Hans Neumann (eds), *Assyriologica et Semitica. Festschrift für Joachim Oelsner anlässlich seines 65. Geburtstages am 18. Februar 1997*. (Altes Orient und Altes Testament, 252). Münster: Ugarit Verlag, 2000, pp. 117–177. The idea that traces of pre-Šulgi-mathematics might be present in texts from the northern periphery was first proposed by Eleanor Robson in her dissertation from 1995, now published as *Mesopotamian Mathematics 2100–1600 BC. Technical Constants in Bureaucracy and Education*. (Oxford Editions of Cuneiform Texts, 14).

type is known from the earlier record, but where even this is solved in a different way. It seems to be a list of *new* problem types, either borrowed from elsewhere or fresh inventions in the area. The Seleucid texts make heavy use of Sumerian word signs, but in a way that sometimes directly contradicts earlier uses. To some extent at least they represent a new translation into Sumerian of a tradition that must have been transmitted outside an erudite scribal environment.

In connection with the creation of a planetary astronomy based on arithmetical schemes, the Neo-Babylonian period (in particular the Seleucid phase) developed a set of highly sophisticated numerical techniques; these are dealt with in the chapter ???.

Numbers, number systems, tables, and their computational use

The original number system was based on specific signs for 1, 10, 60, 600, 3600 and 36000, multiples of which were produced by repetition in fixed patterns (, , , , :: , :: , etc.). In the third millennium, adjunction of the sign ga l, “great”, allowed upwards extension of the system by a factor 60, whereas the sign gín, borrowed from weight metrology, was used in the sense of $\frac{1}{60}$ (the same tricks were used to expand the reach of metrological sequences). From the Old Akkadian epoch onward, calculators can be seen to experiment with the system, thus approaching the place-value principle – but all the relevant texts commit errors, thus showing that no place-value *system* was yet available.⁷

The *system* seems to have been created in the wake of the Ur-III administrative reform. It employed the traditional sign for 1 for any integer power 60^n , and the sign for 10 for any $10 \cdot 60^n$ – still with repetitions in fixed patterns to express 2, 3, ..9, and 20, 30, ..., 50. It was a floating-point system, with no indication of absolute order of magnitude, as the slide rule engineers would use until recently. Nor were “intermediate zeroes” indicated. For both reasons, the notation could only be used for intermediate calculations, final results had to be inserted in the documents in the traditional, unambiguous notation (and for the same reason, it is extremely difficult to pinpoint the inception of the system, documents that contain only numbers are not fit for paleographic analysis).

The place-value notation did not facilitate additions and subtractions, the reason to introduce it was the importance of multiplications in Ur-III planning and accounting. If, e.g., the labor needed to produce a wall of given dimensions of bricks of a given type was to be found, one (“metrological”) table would translate a thickness measured in cubits and fingers into the standard length unit (a “rod” \approx 6 m), after which the volume of the wall could be found in standard units. A “technical” table of “constant

⁷ See Marvin A. Powell, “The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics,” *Historia Mathematica*, 3 (1976), 417–439.

factors” would tell the number of the bricks in question to a unit volume, another the number of bricks produced by a worker per day, a third the number carried a given distance per man-day, etc. The total consumption of labor could then be found by means of multiplications and additions.⁸

Beyond metrological conversion tables and tables of technical constants, the system depended on the availability of multiplication tables and of tables of reciprocals – the latter because division by n was performed as a multiplication by $\frac{1}{n}$. The important step in the invention of the place value system was thus not the inception of the idea, which had been in the air for centuries; it will have been the government decision to have it spread in teaching and to produce (mass-produce!) the tables needed for its implementation.

Once introduced, the place-value system could survive in less bureaucratic settings. It became the standard system of Old Babylonian mathematical texts (only occasionally will the units of “real” life turn up in statements or final results), and of late Babylonian mathematical astronomy. It is quite uncertain whether the Indian decimal place-value system for integers depends on it, but undisputed that it was taken over in the minute-second fractions of Greek and later astronomy, whence it inspired the introduction of decimal fractions.

It may seem a drawback of the Babylonian division method that it only works for “regular” divisors of the form $2^p \cdot 3^q \cdot 5^r$ (p , q and r positive, negative or 0). In practice this was no trouble, firstly because all metrological step factors were regular, secondly because the margin on technical factors was always large enough to allow representation by a simple regular number. Factors that might turn up as divisors were always chosen thus.

Beyond the tables already mentioned, other arithmetical tables occur: n^2 (with inversions as \sqrt{N} , where N itself is square), n^3 , $n^2 \cdot (n+1)$, and a^n . Tables of squares (viz square areas expressed in metrological units) go back to before the mid-third millennium and thus antedate the place-value system by c. 500 years.

Geometry

Very few third-millennium texts reveal the actual mathematical knowledge and procedures that went into their results. The post-Old-Babylonian texts at our disposal are also too few to suggest any global picture. For these reasons, this and the following two sections deal primarily with the mathematics and the mathematical thought of the Old Babylonian period.

In geometry, no concept of the quantifiable angle existed. In order to find the area of a rectangle the Babylonians would multiply the length with the width – as mentioned, the area metrology had been adapted to this already in the fourth millennium. In

⁸ The most thorough treatment to date of the technical factors and their use as reflected in mathematical texts is Robson, *op. cit.*

practice, they would choose as length and width the legs of an approximately right angle (as opposed, we may say, to a “wrong” angle). If opposite sides were slightly different, average length would be multiplied by average width (the “surveyors” formula; also since the fourth millennium). This would always yield too large results, but with one known exception it was only used when the error was negligible (when used as a mere pretext for supra-utilitarian problems in the Old Babylonian school, the formula might be employed in cases where it is blatantly absurd).

The area of approximately right triangles was found as the product of the bisected width with “the length” – as opposed to “the long length”, i.e., the hypotenuse. More complex shapes would be split up in quasi-rectangles and quasi-right triangles (this is seen in Ur-III field plans). A Seleucid text computes the height of an equilateral trapezium; a text from Old Babylonian Susa suggests that the same *could* be done in earlier times when regular polygons were investigated.

The absence of the notion of the quantifiable angle did not prevent the understanding of similarity relations. It was also routinely used that the areas of similar figures relate as the squares on the linear dimensions.

It was known that the square on the diagonal of a rectangle augmented by the doubled area equals the square on the sum of the sides, whereas the squared diagonal minus the doubled area equals the square on their difference – and, probably as a sequel, that the squared diagonal itself equals the sum of the squared sides. The latter, of course, is what we know as the “Pythagorean theorem”.

The fundamental circle parameter was the perimeter p – the area was found as $\frac{1}{12}p^2$, and the diameter as $\frac{1}{3}p$. In one text group from the northern region (in general close to the lay tradition, where both the separate treatment of the semicircle and the very same formula turn up in later ages) the area of the semicircle is found as $\frac{1}{4}$ of the product of diameter and arc.

In volume metrology, the area units were thought of as provided with a “standard thickness” of 1 cubit. In order to determine a prismatic or cylindrical volume, the calculator would first find the base (with this implicit thickness) and then “raise it to”, i.e., multiply it with, the height. This operation was so important that “raising” became the standard term for any multiplication which was based on similar considerations of proportionality (only concrete repetition and the laying-out of rectangular areas employ other terms); all multiplications with factors taken from metrological tables or tables of technical constants were thus “raisings”.

The volume of a truncated cone was calculated as the height times the mid-cross-section, that is, as that of a cylinder with the average diameter. In one case, the volume of a truncated pyramid is determined as the average base raised to the height – in another, the correct value is found, whether from a correct formula or not is unclear (a correct formula *can* be derived from relatively simple intuitive arguments).

The simple area and volume formulas are without doubt based on such intuitive insights. The restricted use of the “surveyors’ formula” indicates that it was known to be only an approximation, but nothing suggests any precise idea as to the importance

of the error; probably the Babylonians would see no difference between this kind of approximation and the treatment of an inevitably uneven terrain as if it were a perfect plane.

Formal demonstration seems to be absent from Babylonian geometry. There was a certain interest in striking geometrical configurations – e.g., systems of concentric squares; reflections on a concentric two-square system and on the appurtenant “average square” may have led to the discovery of how to bisect a trapezium by a parallel transversal. Apart from this, the only important kind of supra-utilitarian geometry was the area technique which has become known as “Babylonian algebra”.

“Algebra” and other pure pursuits

When it was discovered in the late 1920s that the sequence of numbers in certain texts corresponded to the solution of second-degree equations, the appurtenant technical terminology was as yet uninterpreted. It was assumed – and generally accepted for 60 years – that the underlying conceptualizations were arithmetical; that the operations involved were therefore numerical additions, subtractions, multiplications, and extractions of roots; and that the persistent references to lengths, widths and areas were nothing but metaphors for numerical unknowns and their products.⁹

Close attention to the vocabulary and the organization of the texts demonstrates, however, that two presumed additions are kept strictly apart; that there are two different subtractive operations; that two different “halves” are distinguished; and that “multiplications” are four in number. All of this concerns *concepts* – several of the concepts are covered by two or more synonymous *terms*. This makes no sense in the arithmetical interpretation, but everything becomes obvious if we take the words of the texts (lengths, widths, squares, areas) seriously.

Babylonian “algebra” turns out to be a cut-and-paste technique which manipulates measurable line segments and areas in analytical processes which, in their numerical steps, correspond to the procedures of our equation algebra. As an example we may look at the simplest of all mixed second-degree problems: The sum of a square area and the side is 45´ (i.e., $45 \cdot \frac{1}{60} = \frac{3}{4}$). The sequence of numerical steps in the solution (with added indications of absolute magnitude, ´ indicating minutes or sixtieths, ¨ seconds) is as follows: 45´ -- 1 -- 1 -- 30´ (= $\frac{1}{2}$) -- 30´ -- 15´ (= $\frac{1}{4}$) -- 45´ -- 1 -- 1 --

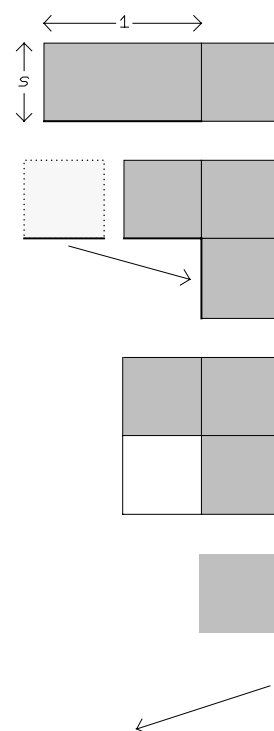


Figure 1. The procedure of BM 13901 n° 1

⁹ The discovery and the story of interpretations is analyzed in Høyrup, “Changing trends” (*cit.* n.1), pp. 1–10.

30' -- 1 -- 30'.¹⁰ What goes on can be followed in the diagram: At first the side is represented by a rectangular area $\square\square(1,s)$, which is glued to the square $\square(s)$. Its length 1 is bisected, and the outer $\frac{1}{2}$ is moved so as to span with the $\frac{1}{2}$ that remains in place a quadratic complement with the area $\frac{1}{4}$. This is joined to the gnomonic area $\frac{3}{4}$ consisting of the square and the bisected rectangle. The resulting square has the area 1, and thus also the side 1. The $\frac{1}{2}$ that was moved is detached from this 1, and $\frac{1}{2}$ remains as the side of the original square. The method, as we see, is analytical in the same sense as equation algebra: the unknown side s is treated as if it were known, and the complex relation subjected to manipulations until s stands back in isolation.

This is one of the original surveyors' riddles that were apparently borrowed by the early Old Babylonian scribe school. In the scribe school it was only one of many problems dealing with areas and segments. In non-normalized cases, a proportional scaling of figures along one dimension was used along with the cut-and-paste procedures.

Line segments and areas constituted the basis of the technique. They could then be used to *represent* entities of other kinds: numbers from the table of reciprocals, prices – or segments might represent areas or volumes. The technique thus served as a general tool for finding unknown entities involved in complex relationships. Even in this sense, it was similar to equation algebra; its “basic representation” was not numerical, it is true, but the segments and areas of this representation were as *functionally abstract* as the numbers of equation algebra. The closest kin of Babylonian algebra, however, is pre-Viète algebra: it was and remained a *technique*, and was never associated with any algebraic theory about solvability conditions or the classification of problems (classification was based on geometrical object and not on algebraic type, as revealed by the organization of theme texts). Nor was it used to solve “real-life” problems – no single practical problem presenting itself to Babylonian calculators was of the second degree. The only “practical” purpose of treating second-degree problems in school was as a pretext for training calculation with sexagesimal numbers (much as second-degree equation algebra has served in the schools of recent centuries to train the manipulation of algebraic letter symbols).

Practical first-degree problems were solved without recourse to algebraic techniques, often by variants of the “single false position” (also used in homogeneous problems of the second and third degree). However, second-degree algebraic systems might include a genuine first-degree equation, of the type “the sum of the length and the width, from which $\frac{1}{4}$ of the width is detached, is 45’). A couple of texts discuss such equations and their transformation, identifying most pedagogically the coefficients of length and width and the contribution of each to the sum.

Some higher-degree problems (of biquadratic and similar types) were solved by means of the algebraic technique. Mixed third-degree problems were treated occasionally – e.g., to find the side of a cubic excavation if the sum of the volume and

¹⁰ BM 13901 n° 1, in MKT III, 1.

the base is known. Here the algebraic technique would forsake. Instead the calculator resorted to a combination of false-position considerations and factorization or (in the case just mentioned) the table of $n^2 \cdot (n+1)$. The trick is elegant but only works because a simple solution is known in advance to exist (*all* school problems were constructed backwards from known solutions).

The cubic problems are found in theme texts together with other “excavation” problems of the first or the second degree, solved on their part with algebraic methods. As regards their method, however, they are rather linked with another kind of supra-utilitarian mathematics: Investigations of the properties of the regular numbers of the sexagesimal place value system. In simple cases, it involved factorizations, continued products of simple factors, etc. The high point is a tabulation, not directly of Pythagorean triplets $a-b-c$ but of $???-\bar{c}^2-b-c$, where $???$ stands for one or more missing columns, and $\bar{c} = c/a$. All sets $(\bar{b}, \bar{c}) = (\frac{t'-t}{2}, \frac{t'+t}{2})$ are listed for which $\sqrt{2}-1 < t < \frac{5}{9}$, t being the quotient between two regular integer numbers no greater than 125, $t' = 1/t$.¹¹

The headings of the b - and c -columns speak about width and diagonal, and it is thus certain that a geometric rectangle and its diagonal is involved. Apart from that, the purpose of the text is obscure. As shown by Friberg, it is not the result of a pure number-theoretical investigation. Instead, he proposes, it might serve as a tool for finding an array of data that would permit some mathematical problem (e.g., concerning right triangles) to be solvable. Unfortunately, Old Babylonian texts always have very simple solutions and often stick to the same solution in many consecutive problems; all available evidence therefore speaks against this proposal, but no more convincing alternative is at hand. The text adds an important shade to our knowledge about *what* the Babylonians could do but so far nothing to our understanding of *why* they would do it.

“Mathematics” or “computation”? A global characterization

In the Old Babylonian period, mathematics was a cognitively autonomous subject, and it may therefore be considered legitimate to speak of it precisely as *mathematics*, as done above. In contrast, the term “mathematician” appears nowhere. All we know with some certainty about the authors of the mathematical texts is that they will have been teachers of future scribes. Much of what we find in the texts is supra-utilitarian – but its ultimate legitimacy always rests on its link to scribal activity. The scribe, however, when using mathematics, would always be interested in *finding a number*,

¹¹ The tablet has been much discussed in the literature. Analysis and summary of earlier work is found in Jöran Friberg, “Methods and Traditions of Babylonian Mathematics. Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations,” *Historia Mathematica*, 8 (1981), 277–318. A new profound analysis is Eleanor Robson, “Neither Sherlock Holmes nor Babylon: a reassessment of Plimpton 322”, forthcoming in *Historia Mathematica*, 28 (2001).

not, e.g., in geometrical regularities; artists might have this interest, but with the exception of the above-mentioned concentric squares nothing permits us to link patterns with mathematically interesting symmetries to the mathematical texts.

Strictly speaking, Old Babylonian (and, in general, Mesopotamian) mathematics might therefore better be characterized as *computation*; instead of “mathematicians” we should speak of “calculators” and “teachers of calculation”; supra-utilitarian activities represent “pure calculation” rather than “pure mathematics”. The ultimate interest in finding a number is of course also a characteristic of most contemporary applications of mathematics; but it remains a feature which distinguishes both the Mesopotamian and the contemporary calculating orientation from the *investigation of the properties of mathematical objects* which (since the Greeks) constitutes our ideal type of mathematics proper.

The italicized passage contains a veiled reference to another difference between our ideal type and the Mesopotamian type: “investigation”. In principle, theoretical mathematics has *the problem* as its core, and then sets out to construct methods and a conceptual apparatus that permit its solution. The same characteristic holds for applications of mathematics (Mesopotamian as well as contemporary), with the difference that the defining problem is no mathematical problem. The core of Mesopotamian supra-utilitarian mathematics, on the contrary, was always *the method*. When the mid-third millennium calculators were testing the potentialities of the professional tools, these tools were the starting point, and the aim was to find out how far they would reach. Similarly, the “scribal humanism” of the Old Babylonian period, aiming at handling with *virtuosity* the tools and techniques of the scribe, would be centred on these.

This does not preclude the practical existence of mathematical research, in the form of search for problems that could be treated by available techniques and tricks. The difference between the surveyors’ riddles and the algebraic discipline created in the school is indeed the outcome of this kind of search. Nor did it preclude the invention of new techniques of scarce practical utility; such inventions might be needed if new problem types were to be transformed so as to be solvable – the “quadratic completion” used to solve mixed quadratic problems is an example, already conceived in the lay surveyors’ environment and then adopted into the early Old Babylonian scribe school. Once devised, such techniques would themselves become part of the stock of professional tools, and serve in the search for problem types that might now be solved – as the quadratic completion became the basis for the whole fabulous development of second-degree algebra in the school.

Reverberations

After the discovery of Babylonian second-degree algebra in the late 1920s, Neugebauer proposed that the geometry of *Elements* II (characterized by Zeuthen as “geometric algebra” already in the 1880s) should be understood as a geometrical translation of the supposedly numerical algebra of the Babylonians, prompted by the

discovery of irrationality and the ensuing “foundation crisis” of Greek mathematics.

The foundation crisis turned out to be a projection of the 1920s on Greek Antiquity, and even the translation theory proved problematic as it was formulated. *Elements* II solve no problems, at most they can be said to prove algebraic identities of a kind Babylonian algebra seemed to be based on. Worse was the disappearance of the main stock of Babylonian algebra a millennium before the creation of Greek geometry and the failing evidence that any Greek mathematician knew about Babylonian mathematics.

The geometric reinterpretation of the Babylonian technique transforms the question: Euclid’s diagrams coincide with those of which the Babylonians had made use (II.6 thus with the procedure shown above), and his proofs may be said to provide a “critique” of the Babylonian procedures – verification of their legitimacy and investigation of the conditions under which they are valid. But it does not invalidate the second objection to Neugebauer’s thesis.

Comparative analysis of the Babylonian material and a number of later sources – mostly treatises on practical geometry containing supra-utilitarian material, many from the Islamic Middle Ages but others belonging to classical Antiquity or to the stock of Italian borrowings from lost Arabic sources – now allows us to delineate a new scenario.¹²

The original stock of quasi-algebraic surveyors’ riddles can be said with fair certainty to have encompassed at least the following problems on a single square (area A , side s , “all four sides” ${}_4s$; Greek letters indicate given numbers):

$$A \pm s = \alpha \quad // \quad A + {}_4s = \beta \quad // \quad A = {}_4s .$$

On rectangles (length l , width w , all sides ${}_4s$, diagonal d) the following can be identified:

$$A = \alpha, \quad l \pm w = \beta \quad // \quad A + (l \mp w) = \alpha, \quad l \pm w = \beta \quad // \quad A = \alpha, \quad d = \beta ,$$

and seemingly also

$$A = l + w \quad // \quad A = {}_4s .$$

On two squares, finally,

$$A_1 + A_2 = \alpha, \quad s_1 \pm s_2 = \beta \quad // \quad A_1 - A_2 = \alpha, \quad s_1 \pm s_2 = \beta .$$

The lay tradition – whose geographical extension may have outranged Mesopotamia – survived the collapse of the Old Babylonian scribe school, and conserved its stock of riddles. It may have borrowed from the scribe school, but only marginally, and never anything “algebraic” of a more advanced character than the original riddles. Some of its characteristic riddles turn up in Diophantos’s *Arithmetica* I, some are found in pseudo-Heronian or agrimensur treatises, and some are referred to in the *Theologoumena*

¹² The details of the scenario and fairly full arguments from the sources will be found in Høyrup, “On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six ‘Algebras’”, forthcoming in *Science in Context*. Supplementary material is in “Hero, Ps.-Hero, and Near Eastern Practical Geometry. An Investigation of *Metrica*, *Geometrica*, and other Treatises”, pp. 67–93 in *Antike Naturwissenschaft und ihre Rezeption*, Band 7, ed. Klaus Döring, Bernhard Herzhoff & Georg Wöhrle (Trier: Wissenschaftlicher Verlag Trier, 1997), pp. 67–93.

arithmeticae – enough, indeed, to demonstrate that Greek theoretical geometry would have had no difficulty in running into the tradition (whether during contacts with Syro-Phoenician practitioners or in Egypt, to where it may have arrived in the wake of the Assyrian or the Persian conquest). It seems that some geometers did so before Theodoros’s time (thus probably in the fifth century) and submitted the old procedures to a “critique” whose results turn up in *Elements* II, propositions 1–10; all of these, indeed, are related to the basic riddles or to the formulas $\square(R \pm r) = \square(R) + \square(r) \pm 2\square(R, r)$, apparently known already in the Old-Akkadian school. In contrast, nothing in Euclid relates to the particular creations of the Old Babylonian scribe school: the treatment of biquadratics and other higher-order problems, the scaling of non-normalized problems (*Elements* VI.28–29 is likely to represent an independent though similar generalization).

In the Islamic world, the tradition and at least some of the riddles are still encountered around 1200 AD. In the ninth century AD, the cut-and-paste technique was borrowed by al-Khwārizmī for his demonstrations of the algorithms of *al-jabr*. When this discipline was borrowed into Latin Europe as *algebra*, these geometric proofs came to be regarded as the very heart of the discipline.

References to the old tradition are also found in Mahāvīrā’s ninth-century *Gaṇita-Sāra-Sangraha*. Since they do not correspond to what occurs in Islamic sources, Mahāvīrā is likely to draw on the Jaina tradition. He is thus a witness of a possible link between the Near Eastern tradition and Indian medieval algebra – a link which is invisible in the numerical algebra of Āryabhata and Brahmagupta. If not directly then at least through this lay tradition, the Babylonian algebra discovered by Neugebauer and his collaborators thus had even even wider repercussions than he ever dared imagine in print.

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