A Hybrid Public Announcement Logic with Distributed Knowledge – Extended Version

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Abstract

In this paper the machinery of Hybrid Logic and the logic of public announcements are merged. In order to bring the two logics together properly the underlying hybrid logic has been changed such that nominals only partially denote states. The hybrid logic contains nominals, satisfaction operators, the downarrow binder as well as the global modality. Following this, an axiom system for the Hybrid Public Announcement Logic is presented and using reduction axioms general completeness (in the usual style of Hybrid Logic) is proved. The general completeness allows for an easy way of adding distributed knowledge. Furthermore it turns out that distributed knowledge is definable using satisfaction operators and the downarrow binder. The standard way of adding distributed knowledge using reduction axioms is also discussed and generalized to other modalities sharing properties with the distributed knowledge modality.

Keywords: Hybrid Logic, Public Announcement Logic, Distributed Knowledge, Completeness, Reduction Axioms, Epistemic Logic.

1 Introduction

When Arthur Prior introduced Hybrid Logic, it was in the context of temporal logics (see [3]), and since then several applications in temporal logics have been found for Hybrid Logic ([5]). However, Hybrid Logic can be viewed as an extension of any kind of modal logic, such as Epistemic Logic. Thus, it is a natural step to extend Epistemic Logic to a hybrid version, but this step has rarely been taken. This paper remedies this insufficiency.

A recent trend in Epistemic Logic is to model the dynamics of knowledge. There are several ways of doing this, and Dynamic Epistemic Logic (DEL) is one type that has received increased attention (see for instance the textbook [16]). The simplest fragment of DEL is Public Announcement Logic (PAL), which adds modalities for the action of public announcement to epistemic logic. The main concern of this paper is to combine PAL with Hybrid Logic.

PAL is obtained by adding modalities of the form $[\varphi]$ (for all formulas φ of the language) to the language of Epistemic Logic. The reading of the formula $[\varphi]\psi$ is "after public announcement of φ, ψ is true" and the semantics specify that $[\varphi]\psi$ is true in a state in a model if, and only if, ψ true at that state in the submodel obtained by restricting the domain to states where φ is true. A central part of Hybrid Logic is the nominals, which are special propositional variables that are interpreted as only being true in one state. In this way we can name and refer to specific states of a model. When combining PAL with Hybrid Logic the immediate problem is that when moving to submodels the states that some nominals name/denote might be removed, and thus conflict with the requirement that nominals must be true in exactly one state. This problem can be overcome by only letting nominals partially denote states. General completeness results from Hybrid Logic can then be transferred to Public Announcement Logic, and this is the first contribution of this paper. A by-product of the general completeness is a straightforward way of adding modal operators such as distributed knowledge to the logic. Indeed adding extra modalities to the language can in many cases be done in a uniform way, this is another contribution of this paper.

Besides this paper, only a handful of other contributions appear to exist on combining Dynamic Epistemic Logic with Hybrid Logic. In the paper [12] all epistemic actions (of full DEL) are internalized. This is done by adding the epistemic actions to the domain of the models, on the same level as epistemic states, and then by using a hybrid language to refer to them. However, in the process of modeling epistemic scenarios, this may result in a blow-up of the models, which must now also contain the epistemic actions. This is not in line with the usual way of using Kripke models, where the states represent different ways the world might be or different states a system might be in. In [11] a public announcement logic with nominals, global modality, modalities for intentions and preferences is introduced. In that paper, to deal with the interplay between nominals and the public announcement operators, the truth condition for nominals is only changed in the updated models. Thus the updated models are not genuine models for the language. We deal with this deficiency in this paper by letting nominals partially denote states in the original model as well; an approach also taken in [7].

In addition to the question of how to combine Hybrid Logic and epistemic modelling, there is the question of the usefulness of Hybrid Logic in epistemic modelling. The usefulness is illustrated by modal logics for games, for instance. [10] introduces a logic with modalities for preferences, knowledge, and intentions as well as the global modality and nominals. It is shown that the notion of Nash equilibrium is definable in this language and that nominals are necessary in this definition (see [10], Fact 5.5.9). In [15] Nash equilibrium is also defined using distributed knowledge, preference modalities and nominals.

Hybrid Logic can also be used to clarify some of the implicit assumptions made when modelling knowledge by Kripke semantics. For instance $@_i\varphi \to K_a @_i\varphi$ is a validity expressing that if φ is true at a state (named by i), then agent a knows this. Furthermore, if the state named by j is accessible from the state named by i all the agents know this, i.e. $@_i \hat{K}_a j \to K_b @_i \hat{K}_a j$ is valid. Thus the hybrid machinery clarifies the implicit assumption that all the agents know what the model looks like. Uncertainty only comes from the fact that they do not necessarily know in which state of the model they are in.

For a hybrid epistemic logic with the downarrow binder $\downarrow x.$ [1] we can express that an agent knows all the (relevant) facts at a given state without specifying what they are. The formula $\downarrow x.K_a x$ thus expresses that agent a is completely informed in the current state. This cannot be expressed in basic Epistemic Logic if there are infinitely many propositional symbols, nor if the intended model is infinite. Imagine a scenario where agent a writes down a natural number (potentially any natural number) and agent b does not see which number. A Kripke model of this scenario will consist of all the natural numbers corresponding to all the possible numbers a could write down. Expressing in classical epistemic logic that agent b knows that a knows what number he writes down would require an infinite disjunction $(K_b(K_a 0 \vee K_a 1 \vee K_a 2 \vee ...))$, where in hybrid logic the formula $K_b \downarrow x.K_a x$ does the trick.

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[1] The intuition behind the operator $\downarrow x.$ is that it names the current state x and by doing so it allows us to return to the state later on.

The main focus of this paper is another advantage of introducing Hybrid Logic machinery into PAL. From a proof theoretical point of view, classical Hybrid Logic fixes a great deal of the problems of classical modal logic. In the case of PAL the proof theory also becomes much nicer when we move to a hybrid version, as already demonstrated by the paper [7].

The structure of this paper is as follows: In section 2 hybrid Logic with Partially Denoting Nominals is introduced and axiomatized. Next, a Hybrid Logic version of PAL is presented, and a sound and complete axiomatization is given (section 3). In section 4 we discuss how distributed knowledge can be added in three different ways. In the process it is shown that distributed knowledge can be defined using satisfaction operators and the downarrow binder. It is also shown how other modalities can be added in a uniform way, generalized from one of the ways distributed knowledge has been added. Finally, concluding remarks and further directions of research are given in section 5.

2 A hybrid logic with partial denoting nominals

The basic idea behind letting nominals partially denote states is that they are true in *at most one* state instead of *exactly one* state. But problems arise with the formula $@_i\varphi$, stating that φ is true at the state denoted by i. If the nominal i does not denote a state, what should the truth value of $@_i\varphi$ be? There seems to be only two obvious answers, either $@_i\varphi$ is true in all states or it is false in all states.[2] We choose the first and thus take the formula $@_i\varphi$ to be true if the nominal i denotes a state and φ is true there. The dual operator of $@_i$, denoted by $\overline{@}_i$ (i.e. $\overline{@}_i\varphi := \neg @_i \neg\varphi$), then corresponds to the second choice. The two choices for $@_i\varphi$ make the logic differ from classical Hybrid Logic, since $@$ is no longer self-dual. Instead the satisfaction operator has been split into an existential modality $@_i$ and a universal modality $\overline{@}_i$.

We will also add the global modality E to the language, where $E\varphi$ is interpreted as "there is some state in the model where φ is true". Since the semantics of this operator do not depend on the nominals, no problem arises by adding this. When adding the modalities E and A (A being the dual of E), the choice of the semantics for $@_i\varphi$ can be seen as the choice between making $@_i\varphi$ equivalent to $E(i \wedge \varphi)$ or $A(i \to \varphi)$. When nominals only partially denote states these two formulas are no longer equivalent. Since we will have that $@_i\varphi$ is equivalent to $E(i \wedge \varphi)$ and $\overline{@}_i\varphi$ is equivalent to $A(i \to \varphi)$, we see that the satisfaction operator has been split into an existential modality $@_i$ and a universal modality $\overline{@}_i$.

Besides the global modality we will also add the downarrow binder. Thus we add formulas of the form $\downarrow x.\varphi$ to the language, having the intuitive reading "naming the current state x makes φ true". In adding $\downarrow x.$, we also allow x and $@_x\varphi$ to occur as formulas and we are thus faced with the same problems of denotation. However now the denotation of a state variable as x is taken care of by assignments and not by the model. Hence we now have to allow partial functions as assignments.

2.1 Syntax and semantics

To define the language, we assume a set of propositional variables PROP, a countable infinite set of nominals NOM, and a countable infinite set of state variables SVAR. Since the enterprise is Epistemic Logic, we will denote the modal box operators by K_a, where a is an agent from a finite set $\mathbb{A}$ of agents. (Thus, we are defining a multi-modal logic.)

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[2] If i does not denote any states in a model it does not point out anything else than the empty set, thus it seems only fair to make $@_i\varphi$ true in the entire model or false in the entire model independent of φ.

Definition 2.1 The syntax of the full language of Hybrid Logic with Partially Denoting Nominals, denoted by $\mathcal{PH}(@, {\downarrow}, E)$, is given by

$$\varphi \ ::= \ p \ \mid \ u \ \mid \ \neg\varphi \ \mid \ (\varphi \wedge \psi) \ \mid \ K_a\varphi \ \mid \ @_u\varphi \ \mid \ {\downarrow}x.\varphi \ \mid \ E\varphi,$$

where $p \in \mathsf{PROP}$, $u \in \mathsf{NOM} \cup \mathsf{SVAR}$, $x \in \mathsf{SVAR}$ and $a \in \mathbb{A}$.[3]

We will also be interested in sub-languages of this full language. The language without the global modality E will be denoted by $\mathcal{PH}(@, {\downarrow})$ and if we also omit the downarrow binder (and thus also omit the cases for the state variable x) we will denote the language by $\mathcal{PH}(@)$. Finally this language added the global modality will be denoted by $\mathcal{PH}(@, E)$.[4] Furthermore we will use the following abbreviations of $\overline{@}_i$ for $\neg @_i \neg$ and $\hat{K}_a$ for $\neg K_a \neg$.

These languages do not differ from classical Hybrid Logic in the syntax, but their semantics differ. The notion of a frame is the usual one; a frame is a pair $\langle W, (R_a)_{a\in\mathbb{A}}\rangle$ such that R_a is a binary relation on the non-empty set W.[5] Given a frame we can build a model upon it and define truth relative to it.

Definition 2.2 Given a frame $\langle W, (R_a)_{a\in\mathbb{A}}\rangle$, a model based upon it is a tuple $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$, such that $V : \mathsf{PROP} \cup \mathsf{NOM} \to \mathcal{P}(W)$ satisfies that $|V(i)| \leq 1$, for all $i \in \mathsf{NOM}$. An assignment in $\mathcal{M}$ is a partial function $g : \mathsf{SVAR} \to W$. (By "$x \in dom(g)$" we will denote that x is in the domain of the partial function g.)

Definition 2.3 Let $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ be a model, $w \in W$ and g an assignment in $\mathcal{M}$. The semantics of φ is inductively defined by:

$$\begin{array}{lll}
\mathcal{M}, w, g \models p & \text{iff} & w \in V(p); \\
\mathcal{M}, w, g \models i & \text{iff} & w \in V(i); \\
\mathcal{M}, w, g \models x & \text{iff} & x \in dom(g) \text{ and } g(x) = w; \\
\mathcal{M}, w, g \models \neg\varphi & \text{iff} & \mathcal{M}, w, g \not\models \varphi; \\
\mathcal{M}, w, g \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, w, g \models \varphi \text{ and } \mathcal{M}, w, g \models \psi; \\
\mathcal{M}, w, g \models K_a\varphi & \text{iff} & \text{for all } v \in W, \text{ if } wR_av \text{ then } \mathcal{M}, v, g \models \varphi; \\
\mathcal{M}, w, g \models @_i\varphi & \text{iff} & \text{there is a } v \in V(i) \text{ s.t. } \mathcal{M}, v, g \models \varphi; \\
\mathcal{M}, w, g \models @_x\varphi & \text{iff} & x \in dom(g) \text{ and } \mathcal{M}, g(x), g \models \varphi; \\
\mathcal{M}, w, g \models {\downarrow}x.\varphi & \text{iff} & \mathcal{M}, w, g' \models \varphi, \text{ where } g' \text{ is as } g \text{ besides that } g'(x) = w; \\
\mathcal{M}, w, g \models E\varphi & \text{iff} & \text{there is a } v \in W \text{ s.t. } \mathcal{M}, v, g \models \varphi.
\end{array}$$

The logic of this semantics will be denoted by $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ (and similar for the sublanguages). The notions of satisfiability and validity are defined as usual. Note, that if we have a language without the downarrow binder, we do not need assignments, and we will simply omit them.

Some classical validities of Hybrid Logic fail in this new semantics. For instance the formula $@_i i$ is no longer valid. Furthermore $@_i @_j \varphi$ is no longer equivalent to $@_j\varphi$, however, $@_i @_j \varphi \to @_j\varphi$ remains valid. As already mentioned, self-duality of @ also fails, and this makes the validity $\neg @_i\varphi \leftrightarrow @_i\neg\varphi$ fail. $@_i\neg\varphi \to \neg @_i\varphi$ is valid though and so is $@_i\varphi \to \neg @_i\neg\varphi$, which can been seen as expressing that the satisfaction operator $@_i$ is functional.

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[3] In the following we will use i, j, k to range over nominals, x, y to range over state variables, and u, s, t to range over both nominals and state variables.

[4] As usual in hybrid logic $@_i\varphi$ can be defined as $E(i \wedge \varphi)$, thus the $@_i$ operators are superfluous when we have E. Still, we prefer to keep the $@_i$ operators in the language to make the forthcoming axiomatization more uniform and easier to read.

[5] Note that we do not require that R_a is an equivalence relation as usually done in epistemic logic. However, this requirement can easily be added and will be discussed later on.

Even though $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ is different from classical Hybrid Logic, we can recover a version of classical Hybrid Logic within $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$. Note that, the formula $@_i i$ (or equivalent Ei) is true exactly when the nominal i denotes a state. Thus putting $@_i i$ as an antecedent to classical hybrid validities will yield validities in $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$, for instance the formulas $@_i i \to (@_j\varphi \leftrightarrow @_i @_j\varphi)$, $@_i i \to (@_i\varphi \leftrightarrow \neg @_i \neg\varphi)$, and $@_i i \to \big(E(i \wedge \varphi) \leftrightarrow A(i \to \varphi)\big)$ becomes valid. Note also that all classical Hybrid Logic models are models for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$, thus all validities of $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ are validities of classical Hybrid Logic.

The validities and equivalences just discussed are used in most proof systems for Hybrid Logic, thus to give a proof system for Hybrid Logic with Partial Denoting Nominals, different axioms and rules are required.

2.2 *Complete proof systems*

We will now give Hilbert-style proof systems for the hybrid logics with partially denoting nominals. We will start by discussing the logic with nominals, satisfaction operators, and downarrow binders $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$ and completeness for this. Completeness of the logic $\mathbf{K}_{\mathcal{PH}(@)}$ can be obtained in a similar manner. Finally we briefly discuss how the global modality can be added as well as how completeness with respect to other classes of frames can be obtained.

The proof systems is shown in figure 1 and follows that of [4] and [1] with some modifications. Existing axioms and rules have been modified to cope with the partially denoting nominals and these have also caused two new axioms to be add (*Denote* and *Collapse*).[6] Note that we are working in a multi-modal language with a modality K_a for each $a \in \mathbb{A}$ and thus for axioms and rules involving K_a we have one axiom/rule for each $a \in \mathbb{A}$.

We use the standard terminology for Hilbert-style proof systems. A proof of φ in $\mathbf{K}_{\mathcal{PH}(-)}$ ("$-$" denotes any combination of @, $\downarrow$, and E) is a finite sequence of formulas ending with φ such that every formula in the sequence is either an axiom of $\mathbf{K}_{\mathcal{PH}(-)}$ or follows from previous formulas in the sequence using one of the proof rules. We denote this by $\vdash_{\mathbf{K}_{\mathcal{PH}(-)}} \varphi$. For a set of formulas Γ, $\Gamma \vdash_{\mathbf{K}_{\mathcal{PH}(-)}} \varphi$ holds if there are $\psi_1, ..., \psi_n \in \Gamma$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(-)}} \psi_1 \wedge ... \wedge \psi_n \to \varphi$. Given a set of formulas Σ, let $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$ denote the logic obtained from $\mathbf{K}_{\mathcal{PH}(-)}$ by adding all the formulas in Σ as axioms. That φ is provable in the logic $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$ will be denoted by $\vdash_{\mathbf{K}_{\mathcal{PH}(-)}+\Sigma} \varphi$. A set of formulas Γ is said to be $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$-inconsistent if $\Gamma \vdash_{\mathbf{K}_{\mathcal{PH}(-)}+\Sigma} \bot$, and $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$-consistent otherwise. A formula φ is pure if it does not contain any propositional variables or state variables. A set of formulas Σ is called substitution-closed, if it is closed under uniform substitution of nominals by nominals.[7]

2.2.1 *The completeness proof for* $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$

We start out by stating a Lindenbaum lemma.

Lemma 2.4 (Lindenbaum lemma) *Let Σ be a set of pure $\mathcal{PH}(@,\downarrow)$-formulas. Every $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+$ Σ-consistent set of formulas Γ can be extended to a maximal $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent set Γ^+ (in a new language obtained by adding countable many new nominals), such that*

(1) Γ^+ contains a nominal.

(2) For all $@_u\hat{K}_a\varphi \in \Gamma^+$ there is a nominal j, such that $@_u\hat{K}_a j \in \Gamma^+$ and $@_j\varphi \in \Gamma^+$.

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[6] Additionally contrary to [4] we have left out a substitution rule. The reason is that the validities of PAL are not closed under substitution ($[p]p$ is a validity for all propositional variables p, but $[\varphi]\varphi$ is not a validity for arbitrary formulas φ.) and thus when we want to add the public announcement machinery we cannot have a substitution rule.

[7] For instance, if Σ is substitution-closed and $@_i(p \to (j \wedge K_a j)) \in \Sigma$ then also $@_k(p \to (l \wedge K_a l)) \in \Sigma$ for all nominals k and l.

Axioms for $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$:

All substitution instances of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	$\mathrm{K}_{\Box}$
$\overline{@}_u(\varphi \to \psi) \to (\overline{@}_u\varphi \to \overline{@}_u\psi)$	$\mathrm{K}_{\overline{@}}$
$@_u\varphi \to \overline{@}_u\varphi$	@-functional
$\overline{@}_u u$	Weak-reflexivity
$@_u @_s\varphi \to @_s\varphi$	Weak-agree
$u \to (\varphi \leftrightarrow @_u\varphi)$	Introduction
$\hat{K}_a @_u\varphi \to @_u\varphi$	Back
$(@_u\hat{K}_a s \wedge @_s\varphi) \to @_u\hat{K}_a\varphi$	Bridge
$@_u\varphi \to @_u u$	Denote
$@_u u \to (\overline{@}_u\varphi \to @_u\varphi)$	Collapse
$\overline{@}_u(\downarrow x.\varphi \leftrightarrow \varphi[x := u])$[1]	DA

Rules for $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$:

From φ and $\varphi \to \psi$, infer ψ	Modus ponens
From φ, infer $K_a\varphi$	Necessitation of $\Box$
From φ, infer $\overline{@}_u\varphi$	Necessitation of $\overline{@}$
From $\overline{@}_u\varphi$, where u does not occur in φ, infer φ	Name
From $(@_u\hat{K}_a s \wedge @_s\varphi) \to \psi$, where $u \neq s$ and s does not occur in φ or ψ, infer $@_u\hat{K}_a\varphi \to \psi$	Paste

Extra axioms for $\mathbf{K}_{\mathcal{PH}(E,-)}$:

$@_i i \to Ei$, (for all $i \in \mathsf{NOM}$)	GM

[1] $\varphi[x := u]$ denotes the formula obtained from φ by substituting all free occurrences of x by u.

Fig. 1. The Hilbert-style proof systems for $\mathbf{K}_{\mathcal{PH}(@)}$ and its extensions.

Proof. Let Σ and Γ be given as in the lemma. Extend the language with a countable infinite set of new nominals (thus we have infinitely many nominals not occurring in Γ). Enumerate the countably many formulas of this extended language as $(\varphi_n)_{n\in\mathbb{N}}$.

Let $\Gamma_0 = \Gamma \cup \{i_0\}$ for a nominal i_0 not occurring in Γ. Now Γ_0 is consistent (in the rest of this subsection consistent means $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent), for assume otherwise: Then there are $\psi_1, ..., \psi_m \in \Gamma$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} i_0 \wedge \psi_1 \wedge ... \wedge \psi_m \to \bot$, hence $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} i_0 \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot)$. Using the $K_{\overline{@}}$-axiom and necessitation of $\overline{@}$ we get $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} \overline{@}_{i_0} i_0 \to \overline{@}_{i_0}((\psi_1 \wedge ... \wedge \psi_m) \to \bot)$. Then using weak-reflexivity and modus ponens it follows that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} \overline{@}_{i_0}((\psi_1 \wedge ... \wedge \psi_m) \to \bot)$. Finally since i_0 did not occur in Γ we get from the Name rule that $\vdash_{\mathbf{K}_{\mathcal{PH}(@)}+\Sigma} (\psi_1 \wedge ... \wedge \psi_m) \to \bot$, which is a contradiction since Γ is assumed to be consistent. Hence Γ_0 must also be consistent.

Now for $n \in \mathbb{N}$ we define Γ_n in the following way:

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if } \varphi_n \text{ is not of the form } @_u\hat{K}_a\psi \text{ and} \\ & \text{the set } \Gamma_n \cup \{\varphi_n\} \text{ is consistent.} \\ \Gamma_n \cup \{\varphi_n, @_u\hat{K}_a j, @_j\psi\}, & \text{if } \varphi_n \text{ is of the form } @_u\hat{K}_a\psi,\ j \text{ is a new} \\ & \text{nominal not occurring in } \Gamma_n \text{ or } \varphi_n, \\ & \text{and the set } \Gamma_n \cup \{\varphi_n\} \text{ is consistent.} \\ \Gamma_n, & \text{otherwise.} \end{cases}$$

Then Γ_n is consistent for all $n \in \mathbb{N}$. The proof of this goes by induction on $n \in \mathbb{N}$ and the start has just been shown for Γ_0. The only non trivial case in the induction step is the case where φ_n is on the form $@_u\hat{K}_a\psi$ and $\Gamma_n \cup \{\varphi_n\}$ is consistent. Assume toward a contradiction that $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n, @_u\hat{K}_a j, @_j\psi\}$ is inconsistent. Then there are $\psi_1, ..., \psi_m \in \Gamma_n$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} (\varphi_n \wedge @_u\hat{K}_a j \wedge @_j\psi \ \wedge \psi_1 \wedge ... \wedge \psi_m) \to \bot$, thus $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} (@_u\hat{K}_a j \wedge @_j\psi) \to (\varphi_n \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot))$. But then since j is new to φ_n and Γ_n it follows from the paste rule that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} @_u\hat{K}_a\psi \to (\varphi_n \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot))$, i.e. $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} \varphi_n \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot)$. This is of course a contradiction to the assumption of $\Gamma_n \cup \{\varphi_n\}$ being consistent. Hence Γ_{n+1} must be consistent and it follows by induction that Γ_n is consistent for all $n \in \mathbb{N}$.

Now it easily follows that $\Gamma^+ := \bigcup_{n\in\mathbb{N}} \Gamma_n$ is also consistent. That Γ^+ contains a nominal follows from the construction of $\Gamma_0 = \Gamma \cup \{i_0\}$. And finally the last property follows from the construction of Γ_{n+1} in the case where φ is on the form $@_i\hat{K}_a\psi$. This completes the proof. □

Before we go on to the completeness proof a small lemma is needed.

Lemma 2.5 *The following are derivable in the logic* $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$*:*

i) $@_u s \to (\overline{@}_u\varphi \leftrightarrow @_u\varphi)$

ii) $@_u s \to @_s u$

iii) $(@_u u \wedge @_s s) \to (@_s\varphi \leftrightarrow @_u @_s\varphi)$

iv) $@_u s \to (@_u\varphi \leftrightarrow @_s\varphi)$

v) $(@_u s \wedge @_s t) \to @_u t$

Proof.

Proof of *i*):

(1)	$@_u\varphi \to \overline{@}_u\varphi$	@-functional
(2)	$@_u s \to (@_u\varphi \to \overline{@}_u\varphi)$	Prop. logic on (1)
(3)	$@_u s \to @_u u$	Denote
(4)	$@_u u \to (\overline{@}_u\varphi \to @_u\varphi)$	Collapse
(5)	$@_u s \to (\overline{@}_u\varphi \leftrightarrow @_u\varphi)$	Prop. logic on (2), (3) and (4)

Proof of *ii*):

(1)	$s \to (u \to @_s u)$	Introduction
(2)	$\overline{@}_u s \to (\overline{@}_u u \to \overline{@}_u @_s u)$	Necessitation of $\overline{@}$ and $K_{\overline{@}_u}$ on (1)
(3)	$\overline{@}_u s \to \overline{@}_u @_s u$	Weak-reflexivity and prop. logic on (2)
(4)	$@_u s \to \overline{@}_u @_s u$	@-functional and prop. logic on (3)
(5)	$@_u s \to @_u @_s u$	*i*) and prop. logic on (4)
(6)	$@_u s \to @_s u$	Weak-agree and prop. logic on (5)

Proof of *iii*):

(1)	$@_u @_s \neg\varphi \to @_s \neg\varphi$	Weak-agree
(2)	$\neg @_s \neg\varphi \to \neg @_u @_s \neg\varphi$	Prop.logic on (1)
(3)	$\overline{@}_s \varphi \to \overline{@}_u \overline{@}_s \varphi$	Definition of $\overline{@}$ on (2)
(4)	$@_s s \to (\overline{@}_s \varphi \leftrightarrow @_s \varphi)$	*i*)
(5)	$(\varphi \leftrightarrow \psi) \to (\overline{@}_u \varphi \leftrightarrow \overline{@}_u \psi)$	Prop. logic and nec. of $\overline{@}$ and $K_{\overline{@}}$
(6)	$@_s s \to (@_s \varphi \to \overline{@}_u @_s \varphi)$	Prop. logic on (3), (4) and (5)
(7)	$@_u u \to (\overline{@}_u \varphi \leftrightarrow @_u \varphi)$	*i*)
(8)	$(@_u u \wedge @_s s) \to (@_s \varphi \to @_u @_s \varphi)$	(6), (7) and prop. logic
(9)	$(@_u u \wedge @_s s) \to (@_u @_s \varphi \to @_s \varphi)$	Weak-agree and prop. logic.
(10)	$(@_u u \wedge @_s s) \to (@_s \varphi \leftrightarrow @_u @_s \varphi)$	Prop. logic on (8) and (9)

Proof of *iv*):

(1)	$s \to (\varphi \leftrightarrow @_s \varphi)$	Introduction
(2)	$\overline{@}_u s \to (\overline{@}_u \varphi \leftrightarrow \overline{@}_u @_s \varphi)$	Necessitation of $\overline{@}$ and $K_{\overline{@}_u}$ on (1)
(3)	$@_u s \to (\overline{@}_u \varphi \leftrightarrow \overline{@}_u @_s \varphi)$	@-functional and prop. logic on (2)
(4)	$@_u s \to (@_u \varphi \leftrightarrow @_u @_s \varphi)$	*i*) and prop. logic on (3)
(5)	$@_u s \to (@_u \varphi \leftrightarrow @_s \varphi)$	*iii*), *ii*), Denote and prop. logic on (4)

Proof of *v*):

(1)	$@_u s \to (@_u t \leftrightarrow @_s t)$	*iv*)
(2)	$(@_u s \wedge @_s t) \to @_u t$	Prop. logic on (1)

□

With this lemma, we can now construct a Henkin style model.

Definition 2.6 Let Γ be a maximal consistent set of $\mathcal{PH}(@,\downarrow)$-formulas. Define $\mathcal{N}_\Gamma = \{u \in \mathsf{NOM} \cup \mathsf{SVAR} \mid @_u u \in \Gamma\}$ and an equivalence relation $\sim$ on $\mathcal{N}_\Gamma$ by $u \sim s$ iff $@_u s \in \Gamma$ (and denote the equivalence class of u by $|u|$). Then the canonical model $\mathfrak{M}_\Gamma = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$ and the canonical assignment g_Γ are defined by

$$\begin{array}{rcl} W & = & \{|u| \mid u \in \mathcal{N}_\Gamma\}; \\ |u|R_a|s| & \text{iff} & @_u\hat{K}_a s \in \Gamma \quad \text{for all } a \in \mathbb{A}; \\ V(p) & = & \{|u| \in W \mid @_u p \in \Gamma\} \quad \text{for all } p \in \mathsf{PROP}; \\ V(j) & = & \{|u| \in W \mid @_u j \in \Gamma\} \quad \text{for all } j \in \mathsf{NOM}; \\ g_\Gamma(x) & = & |x| \quad \text{for all } x \in \mathsf{SVAR} \cap \mathcal{N}_\Gamma. \end{array}$$

A few comments about why this is well-defined are in order. First of all note that by the Denote rule and *ii*) of lemma 2.5, if $@_u s \in \Gamma$ then $u, s \in \mathcal{N}_\Gamma$. That the relation $\sim$ is an equivalence relation (and thus W is well-defined) follows from the construction of $\mathcal{N}_\Gamma$ and *ii*) and *v*) of lemma 2.5. That R_a is well-defined follows from *iv*) of lemma 2.5 and the Bridge axiom. Finally that V is well-defined for $p \in \mathsf{PROP}$ follows from *iv*) of lemma 2.5, and for $i \in \mathsf{NOM}$ by $\sim$ being an equivalence relation. $\sim$ being an equivalence relation also guaranties that g_Γ is a well-defined assignment. Note that if $@_i i \notin \Gamma$ then $V(i) = \emptyset$ and thus i does not denote. Similar for state variables.

An essential part of the completeness proof is the following truth lemma:

Lemma 2.7 (Truth lemma) *Let Γ be a maximal consistent set of $\mathcal{PH}(@,\downarrow)$-formulas that satisfy item* (2) *of the Lindenbaum lemma. Then for all $u \in \mathcal{N}_\Gamma$ and all $\mathcal{PH}(@,\downarrow)$-formulas φ*

$$\mathfrak{M}_\Gamma, |u|, g_\Gamma \models \varphi \quad \textit{iff} \quad @_u\varphi \in \Gamma. \tag{1}$$

Proof. The proof goes by induction on φ. When φ is a p or j for a $p \in \mathsf{PROP}$ or $j \in \mathsf{NOM}$, (1) follows directly from the definition of V. When φ is on the form x for a $x \in \mathsf{SVAR}$, (1) follows from $\sim$ being an equivalence relation. This takes care of the induction basis.

The induction step. In the case φ is on the form $\psi \wedge \chi$, note that $@_u\psi, @_u\chi \in \Gamma$ if and only if $@_u(\psi \wedge \chi) \in \Gamma$, which can be proved using propositional logic and the rules and axioms Denote, Collapse, $K_{\overline{@}}$ and necessitation of $\overline{@}$. In the case φ is on the form $\neg\psi$, the thing to note is that $\neg @_u\psi \in \Gamma \Leftrightarrow @_u\neg\psi \in \Gamma$. "$\Leftarrow$" follows from the axiom @-functional. "$\Rightarrow$" follows using Collapse and the fact that $@_u u \in \Gamma$ by the assumption $u \in \mathcal{N}_\Gamma$.

Assume now that φ has the form $@_s\psi$. First note that if $@_u @_s\psi \in \Gamma$ then $@_s\psi \in \Gamma$ by weak-agree and thus $s \in \mathcal{N}_\Gamma$ by the Denote axiom. Then by induction it follows that $\mathcal{M}_\Gamma, |s|, g_\Gamma \models \psi$, which again implies that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models @_s\psi$. If $\mathcal{M}_\Gamma, |u|, g_\Gamma \models @_s\psi$ then there is a $s' \in \mathcal{N}_\Gamma$ such that $\mathcal{M}_\Gamma, |s'|, g_\Gamma \models \psi$ and $V(s) = |s'|$ if s is a nominal and $g_\Gamma(s) = |s'|$ if s is a state variable. By the definition of V and g_Γ this implies that $@_{s'} s \in \Gamma$ and by the induction hypothesis that $@_{s'}\psi \in \Gamma$. But now it follows from *iv*) of lemma 2.5 that $@_s\psi \in \Gamma$. From the assumption about i and $@_{s'} s \in \Gamma$ and lemma 2.5 *ii*) and Denote it follows that $@_u u, @_s s \in \Gamma$. But then by *iii*) of lemma 2.5, $@_u @_s\psi \in \Gamma$ follows.

The case φ is of the form $\hat{K}_a\psi$. If $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \hat{K}_a\psi$, then there is a $s \in \mathcal{N}_\Gamma$ such that $|u|R_a|s|$ and $\mathcal{M}_\Gamma, |s|, g_\Gamma \models \psi$. By definition of R_a, $@_u\hat{K}_a s \in \Gamma$ and by the induction hypothesis $@_s\psi \in \Gamma$. But then by the bridge axiom it follows that $@_u\hat{K}_a\psi \in \Gamma$. Now assume that $@_u\hat{K}_a\psi \in \Gamma$. Then since Γ satisfies item (2) of the Lindenbaum lemma it follows that there is a nominal j such that $@_u\hat{K}_a j \in \Gamma$ and $@_j\psi \in \Gamma$. Note that by Denote $j \in \mathcal{N}_\Gamma$. Now by the definition of R_a and V and the induction hypothesis it follows that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \hat{K}_a\psi$.

Finally for the case where φ is of the form $\downarrow x.\psi$. First note that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \downarrow x.\psi$ if and only if $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \psi[x := u]$ due to a substitution lemma that can easily be proven.[8] But then

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[8] Let $Int(u)$ stand for $g(u)$ if u is a state variable and $V(u)$ if u is a nominal. Then the substitution lemma can be stated

by the induction hypothesis it follows that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \downarrow x.\psi$ if and only if $@_u\psi[x := u] \in \Gamma$. And finally by the DA axiom it follows that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \downarrow x.\psi$ if and only if $@_u \downarrow x.\psi \in \Gamma$. This concludes the proof. □

A frame $\mathcal{F}$ *validates* a set of formulas Σ, if $\mathcal{M} \models \Sigma$ for all models $\mathcal{M}$ based on $\mathcal{F}$. With this notion we state a Frame lemma:

Lemma 2.8 (Frame lemma) *Let Σ be a substitution-closed set of pure $\mathcal{PH}(@,\downarrow)$-formulas and let Γ be a $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$ maximal consistent set of $\mathcal{PH}(@,\downarrow)$-formulas satisfying item (1) and (2) of the Lindenbaum lemma. Then the underlying frame of $\mathfrak{M}_\Gamma$ validates all the formulas in Σ.*

Proof. See Lemma 7.1 of [2]. □

We are now finally capable of proving the completeness theorem.

Theorem 2.9 (Completeness of $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$) *Let Σ be a substitution-closed set of pure $\mathcal{PH}(@,\downarrow)$-formulas. Every set of $\mathcal{PH}(@,\downarrow)$-formulas that is $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

Proof. Assume that Γ is $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent. Then it can be extended to a maximal $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent set Γ^+ by the Lindenbaum lemma. Since there is a nominal $i \in \Gamma^+$ by item (1) of the Lindenbaum lemma it is easy to see that for all $\varphi \in \Gamma$, $@_i\varphi \in \Gamma^+$ by the Introduction axiom. But then by the truth lemma it follows that $\mathfrak{M}_{\Gamma^+}, |i|, g_{\Gamma^+} \models \Gamma$. By the frame lemma the underlying frame of $\mathfrak{M}_{\Gamma^+}$ validates all the formulas in Σ and the proof is done. □

2.2.2 Completeness for $\mathbf{K}_{\mathcal{PH}(@,E,-)}$ and completeness with respect to other frame classes

In the case of completeness with respect to the global modality E, we once more follow the lines of [4]. We take one of the modalities in our multi-modal logic to be E[9] and add the axiom GM of figure 1. To see why this suffices note that E is just a normal modal operator for which the intended accessibility relation is the universal relation on the domain. The formula $@_i i \to Ei$ is a pure formula, so adding all substitution instances, as in the axiom GM, automatically gives completeness with respect to the class of frames $@_i i \to Ei$ defines. Hence, all that is left to notice is that $@_i i \to Ei$ defines the universal relation on the domain. However, this can easily be proven and we obtain:

Theorem 2.10 (Completeness of $\mathbf{K}_{\mathcal{PH}(@,E,-)}$) *Let Σ be a substitution-closed set of pure $\mathcal{PH}(@,E,-)$-formulas. Every set of $\mathcal{PH}(@,E,-)$-formulas that is $\mathbf{K}_{\mathcal{PH}(@,E,-)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

In Epistemic Logic one usually wants to put extra conditions on the relations R_a, for instance transitivity, reflexivity, and euclideaness. The logic obtained by requiring all these properties will be denoted $\mathbf{S5}_{\mathcal{PH}(-)}$ and if only transitivity and reflexivity are required, the logic will be denoted by $\mathbf{S4}_{\mathcal{PH}(-)}$. When modal logic is used to reason about beliefs, one usually replaces the reflexivity requirement of $\mathbf{S5}_{\mathcal{PH}(-)}$ by requiring seriality of R_a instead, and the logic obtained in this way will be denoted $\mathbf{KD45}_{\mathcal{PH}(-)}$. Now if one wants to work with these logics instead of just $\mathbf{K}_{\mathcal{PH}(-)}$, complete Hilbert-style proof systems can easily be obtained

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as: *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, φ a formula, and $u \in \mathsf{SVAR} \cup \mathsf{NOM}$. Then for all $w \in W$ and all assignments g with $g(x) = Int(u)$: $\mathcal{M}, w, g \models \varphi$ iff $\mathcal{M}, w, g \models \varphi[x := u]$.*

[9] Elaborated, we add an extra agent e to $\mathbb{A}$ and write E instead of $\hat{K}_e$. Thus, in the proof system we also include all the axioms and rules from figure 1 involving K_a, for E.

from theorems 2.9 and 2.10, since all the properties can be defined by pure formulas. $i \to \hat{K}_a i$ defines reflexivity, $\hat{K}_a \hat{K}_a i \to \hat{K}_a i$ defines transitivity, $\hat{K}_a i \to K_a \hat{K}_a i$ defines euclideaness, and $\hat{K}_a \top$ defines seriality, which is all well known in the Hybrid Logic literature.

3 Hybrid Public Announcement Logic

We now combine Hybrid Logic with Partially Denoting Nominals with PAL. As before we assume the sets PROP, NOM and SVAR, and $\mathbb{A}$. The full language $\mathcal{HPAL}(@, \downarrow, E)$ of the Hybrid Public Announcement Logic is given by:

$$\varphi \ ::= \ p \ \mid \ u \ \mid \ \neg\varphi \ \mid \ (\varphi \wedge \psi) \ \mid \ K_a\varphi \ \mid \ @_u\varphi \ \mid \ \downarrow x.\varphi \ \mid \ E\varphi \ \mid \ [\varphi]\psi,$$

where $p \in \mathsf{PROP}$, $u \in \mathsf{NOM} \cup \mathsf{SVAR}$, $x \in \mathsf{SVAR}$, and $a \in \mathbb{A}$. For the sub-languages we will use the same conventions as before.

The notion of a model $\mathcal{M} = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$ is the same as for $\mathcal{PH}(@, \downarrow, E)$. The definition of the semantic entailment $\mathcal{M}, w, g \models \varphi$ is a combination of definition 2.3 for $\mathcal{PH}(@, \downarrow, E)$ and the following clause:

$$\mathcal{M}, w, g \models [\varphi]\psi \quad \Longleftrightarrow \quad \mathcal{M}, w, g \models \varphi \text{ implies that } \mathcal{M}|_\varphi, w, g_\varphi \models \psi,$$

where the definition of the model $\mathcal{M}|_\varphi = \langle W|_\varphi, R|_\varphi, V|_\varphi \rangle$ is:

$$\begin{aligned} W|_\varphi &= \{v \in W \mid \mathcal{M}, v, g \models \varphi\} \\ R_a|_\varphi &= R_a \cap (W|_\varphi \times W|_\varphi) \\ V|_\varphi(p) &= V(p) \cap W|_\varphi \\ V|_\varphi(i) &= V(i) \cap W|_\varphi, \end{aligned}$$

and the assignment g_φ is obtained from g by restricting its domain to the set $\{x \in \mathit{dom}(g) \mid g(x) \in W|_\varphi\}$.

The logic of this semantics will be called the full Hybrid Public Announcement Logic and will be denoted by $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$. Note that $\mathcal{M}|_\varphi$ is just the model $\mathcal{M}$ restricted to the states where φ is true. The problem of adding nominals to PAL now becomes immediately clear: If a nominal i denotes a state where φ is not true, i does not denote any state in the model $\mathcal{M}|_\varphi$. The problem arises for state variables as well. This is the main reason for introducing a Hybrid Logic with partially denoting nominals in this paper. [10]

We will provide the logic with a Hilbert-style proof system and show completeness in the usual way for PAL, i.e we will provide a truth-preserving translation from $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$ into $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$. This is interesting in its own right, since it shows that Hybrid Public Announcement Logic is not more expressive than the underlying hybrid epistemic logic (which is also the case in standard PAL, see [16]). The proof system is given in Figure 2 and is an extension of the one for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ with additional reduction axioms for the public announcement operator. These reduction axioms are the usual ones from PAL plus new ones for the hybrid operators.

Before discussing soundness and completeness of the proof system, we give a few comments on the choice of reduction axioms for the new components. For the *Announcement and satisfaction* axiom, the intuition behind it is: If ψ is true at the state u after an announcement of φ,

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[10] There is another way of defining the semantics for the public announcement operator $[\varphi]$. Instead of removing states where φ is not true, one simply removes access to these states, i.e. restrict the accessibility relations. In standard PAL these approaches are equivalent, but in Hybrid Logic using either satisfaction operators or the global modality, we are capable of reaching states which are not accessible via the accessibility relations and thus the two approaches differ. In the case of Hybrid Logic the approach of only deleting accessibility relations may seem more appealing since the problem of losing denotation of the nominals is not present anymore. However, there are other drawbacks, which to the author's opinion makes the approach with partially denoting nominals much more appealing. For more on these issues see section 6 of the appendix.

Axioms for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$:

All axioms for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$

$[\varphi]\, p \leftrightarrow (\varphi \to p)$	Atomic permanence (propositions)
$[\varphi]\, u \leftrightarrow (\varphi \to u)$[1]	Atomic permanence (states)
$[\varphi]\, \neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	Announcement and negation
$[\varphi]\, (\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	Announcement and conjunction
$[\varphi]\, K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	Announcement and knowledge
$[\varphi]\, [\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$	Announcement composition
$[\varphi]\, @_u\psi \leftrightarrow (\varphi \to @_u(\varphi \wedge [\varphi]\psi))$[1]	Announcement and satisfaction
$[\varphi]{\downarrow}x.\psi \leftrightarrow {\downarrow}x.[\varphi]\psi$[2]	Announcement and downarrow
$[\varphi]\, E\psi \leftrightarrow (\varphi \to E(\varphi \wedge [\varphi]\psi))$	Announcement and global modality

Rules for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$:

All rules for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$

[1] Here $u \in \mathsf{NOM} \cup \mathsf{SVAR}$.
[2] Assuming that x does not occur in φ.

Fig. 2. The Hilbert-style proof system for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$.

this amounts first of all to the state u remaining in the restricted model, i.e. φ is true at u, and secondly to the public announcement of φ at u leading to ψ being true. For the *Announcement and global modality* axiom, almost the same intuition applies. For the downarrow binder a little care has to be taken regarding the reduction axiom. Note that moving a ${\downarrow}x.$-operator from within the scope of a $[\varphi]$-operator outside the scope, can lead to accidental binding of a state variable x in $[\varphi]$, and this might affect the truth value of the formula. Hence the requirement in the *Announcement and downarrow* axiom. However, this is not really a limitation because we can always rename bound variables without changing the truth value of a formula. Thus when encountering a formula $[\varphi]{\downarrow}x.\psi$ where x appears in φ, we can simply replace all occurrences of x in ψ by a new state variable y to get ψ' and obtain an equivalent formula $[\varphi]{\downarrow}y.\psi'$, where y does not occur in φ. With this assumption the reduction axiom for the downarrow binder is sound. The soundness of the reduction axioms for the satisfaction operator, the global modality and the downarrow binder is stated in the following lemma:

Lemma 3.1 *The following holds for all $\mathcal{HPAL}(@,\downarrow,E)$ formulas φ and ψ:*

1) $[\varphi]@_u\psi$ is equivalent to $\varphi \to @_u(\varphi \wedge [\varphi]\psi)$.

2) $[\varphi]E\psi$ is equivalent to $\varphi \to E(\varphi \wedge [\varphi]\psi)$.

3) If the state variable x does not occur in the formula φ, then $[\varphi]{\downarrow}x.\psi$ is equivalent to ${\downarrow}x.[\varphi]\psi$.

Proof. 1) Since $\langle\varphi\rangle\psi$ and $\varphi \wedge [\varphi]\psi$ are equivalent (as in standard PAL), one only needs to show that $[\varphi]@_u\psi$ is equivalent to $\varphi \to @_u\langle\varphi\rangle\psi$. This is shown by the following equivalences:

$$\begin{array}{ll}
 & \mathcal{M}, w, g \models [\varphi]@_u\psi \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow \mathcal{M}|_\varphi, w, g_\varphi \models @_u\psi \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow (\exists v \in W|_\varphi \text{ s.t. } \mathcal{M}|_\varphi, v, g_\varphi \models u \wedge \mathcal{M}|_\varphi, v, g_\varphi \models \psi) \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow (\exists v \in W \text{ s.t. } \mathcal{M}, v, g \models \varphi \wedge \mathcal{M}, v, g \models u \wedge \mathcal{M}|_\varphi, v, g_\varphi \models \psi) \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow (\exists v \in W \text{ s.t. } \mathcal{M}, v, g \models u \wedge \mathcal{M}, v, g \models \langle\varphi\rangle\psi) \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow \mathcal{M}, w, g \models @_u\langle\varphi\rangle\psi \\
\text{iff} & \mathcal{M}, w, g \models \varphi \to @_u\langle\varphi\rangle\psi
\end{array}$$

$$\begin{array}{lcl@{\qquad}lcl}
t(p) & = & p & t([\varphi]p) & = & t(\varphi \to p) \\
t(u) & = & u\ ^1 & t([\varphi]u) & = & t(\varphi \to u)\ ^1 \\
t(\neg\varphi) & = & \neg t(\varphi) & t([\varphi]\neg\psi) & = & t(\varphi \to \neg[\varphi]\psi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) & t([\varphi]\psi \wedge \chi) & = & t([\varphi]\psi \wedge [\varphi]\chi) \\
t(K_a\varphi) & = & K_a t(\varphi) & t([\varphi]K_a\psi) & = & t(\varphi \to K_a[\varphi]\psi) \\
t(@_u\varphi) & = & @_u t(\varphi)\ ^1 & t([\varphi]@_u\psi) & = & t(\varphi \to @_u(\varphi \wedge [\varphi]\psi))\ ^1 \\
t({\downarrow}x.\varphi) & = & {\downarrow}x.t(\varphi) & t([\varphi]{\downarrow}x.\psi) & = & t({\downarrow}x'.[\varphi](\psi[x := x']))\ ^2 \\
t(E\varphi) & = & Et(\varphi) & t([\varphi]E\psi) & = & t(\varphi \to E(\varphi \wedge [\varphi]\psi)) \\
 & & & t([\varphi][\psi]\chi) & = & t([\varphi \wedge [\varphi]\psi]\chi)
\end{array}$$

^{1}Where $u \in \mathsf{NOM} \cup \mathsf{SVAR}$. $^2x'$ is a new state variable not occurring in φ or ψ.

Fig. 3. The translation $t : \mathcal{HPAL}(@, \downarrow, E) \to \mathcal{PH}(@, \downarrow, E)$.

2) This is similar to 1.

3) Let a model $\mathcal{M} = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$, a state $w \in W$ and an assignment g in $\mathcal{M}$ be given. Let also formulas φ and ψ be given such that the state variable x does not occur in φ. Note that since x does not occur in φ, for all assignments h and h' such that they only differs on x, $\mathcal{M}, w, h \models \varphi$ if and only if $\mathcal{M}, w, h' \models \varphi$ (for all models $\mathcal{M}$ and states w). We now have the following equivalences:

$$\begin{array}{rll}
\mathcal{M}, w, g \models [\varphi]{\downarrow}x.\psi & \text{iff} & \mathcal{M}, w, g \models \varphi \ \Rightarrow\ \mathcal{M}|_\varphi, w, g_\varphi \models {\downarrow}x.\psi \\
 & \text{iff} & \mathcal{M}, w, g \models \varphi \ \Rightarrow\ \mathcal{M}|_\varphi, w, g'_\varphi \models \psi \\
 & \text{iff} & \mathcal{M}, w, g' \models \varphi \ \Rightarrow\ \mathcal{M}|_\varphi, w, g'_\varphi \models \psi \\
 & \text{iff} & \mathcal{M}, w, g' \models [\varphi]\psi \\
 & \text{iff} & \mathcal{M}, w, g \models {\downarrow}x.[\varphi]\psi,
\end{array}$$

where g' is just like g except that $g'(x) = w$ and g'_φ is just like g_φ except that $g'_\varphi(x) = w$. ☐

The soundness of the proof system follows from the soundness of $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ together with the soundness of the reduction axioms. For the completeness of the proof system, we first define a translation t from the language of hybrid public announcement logic into the language of hybrid logic with partially denoting nominals, i.e. $t : \mathcal{HPAL}(@, \downarrow, E) \to \mathcal{PH}(@, \downarrow, E)$. The definition of t is given in figure 3. The restriction in the case for $[\varphi]{\downarrow}x.\psi$ is there to avoid accidental binding of x in φ as mentioned earlier.

Note that the translation is not defined inductively on the usual complexity of a formula. Therefore we cannot prove results regarding t by induction on this complexity. However, the complexity of the formula immediately succeeding the public announcement operator decreases trough the translation, and this we can use. A new complexity measure $c : \mathcal{HPAL}(@, \downarrow, E) \to \mathbb{N}$ can be defined such that c decreases for every step of the translation, for instance $c([\varphi]@_i\psi) > c(\varphi \to @_i(\varphi \wedge [\varphi]\psi))$. The details of this are omitted, see [16] or [8]. Using this complexity measure we can easily prove that every formula of hybrid public announcement logic is provably equivalent to its translation:

Lemma 3.2 *For all $\mathcal{HPAL}(@, \downarrow, E)$ formulas φ,*

$$\vdash_{\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}} \varphi \leftrightarrow t(\varphi)$$

From this lemma together with soundness of the proof system, it follows that all formulas is also semantically equivalent to their translation:

Lemma 3.3 *For all $\mathcal{HPAL}(@,\downarrow,E)$ formulas φ, all models $\mathcal{M} = \langle W,R,V\rangle$, all $w \in W$, and all assignments g,*

$$\mathcal{M},w,g \models \varphi \iff \mathcal{M},w,g \models t(\varphi)$$

Note that translating pure formulas from $\mathcal{HPAL}(@,\downarrow,E)$ results in pure formulas in $\mathcal{PH}(@,\downarrow,E)$. A general completeness result now follows:

Theorem 3.4 (Completeness for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$) *Let Σ be a substitution-closed set of pure $\mathcal{HPAL}(@,\downarrow,E)$-formulas. Every set of $\mathcal{HPAL}(@,\downarrow,E)$-formulas that is $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

Proof. Assume that Γ is $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)} + \Sigma$-consistent. For a set of $\mathcal{HPAL}(@,\downarrow,E)$-formulas X, let $t(X) := \{t(\varphi) \mid \varphi \in X\}$. Then $t(\Gamma)$ is $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)} + t(\Sigma)$-consistent, for assume otherwise: Then there are $\varphi_1,...,\varphi_n \in \Gamma$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}+t(\Sigma)} t(\varphi_1 \wedge ... \wedge \varphi_n) \to \bot$. But then also $\vdash_{\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}+\Sigma} t(\varphi_1 \wedge ... \wedge \varphi_n) \to \bot$ (using lemma 3.2 on formulas in Σ) and by lemma 3.2, $\vdash_{\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}+\Sigma} \varphi_1 \wedge ... \wedge \varphi_n \to \bot$, which is a contradiction to Γ being $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)} + \Sigma$-consistent. Now by theorem 2.9 $t(\Gamma)$ is satisfiable in a model $\mathcal{M}$ (which is also a model for $\mathcal{HPAL}(@,\downarrow,E)$), and by lemma 3.3 it follows that Γ is also satisfiable in $\mathcal{M}$.

Finally, for all pure formulas $\varphi \in \Sigma$, $t(\varphi)$ is a pure formula. Thus by theorem 2.9 the underlying frame of $\mathcal{M}$ validates all of the formulas $t(\varphi) \in t(\Sigma)$. But by lemma 3.3 the underlying frame then also validates all $\varphi \in \Sigma$. $\square$

Note that we could have left out any of the operators $\downarrow x.$, E, or both and thus got completeness for any of the weaker logics. Theorem 3.4 also provides completeness with respect to other classes of frames defined by pure formulas and thus we obtain epistemic public announcement logics such as $\mathbf{S4}_{\mathcal{HPAL}(@,\downarrow,E)}$ and $\mathbf{S5}_{\mathcal{HPAL}(@,\downarrow,E)}$.

4 Adding distributed knowledge and other modalities

Often notions of group knowledge are important when modeling knowledge in multi-agent settings. Distributed knowledge is such a notion and we will discuss it in detail. Another is common knowledge which we will also shortly mention. We will add distributed knowledge to $\mathbf{K}_{\mathcal{HPAL}(-)}$ in three different ways. The first way is the standard one for public announcement logic; we add distributed knowledge to the underlying logic $\mathbf{K}_{\mathcal{PH}(-)}$ and then give a sound reduction axiom for distributed knowledge. Due to the generality of theorem 3.4 we also have another way of adding distributed knowledge; using pure formulas we can add distributed knowledge directly to $\mathbf{K}_{\mathcal{HPAL}(-)}$ getting the reduction axiom for free. The third way only works for extensions of logics that contain satisfaction operators and the downarrow binder. In these logics distributed knowledge becomes directly definable. The first way is a little more involved compared to the other two, but we included it here because we want to generalize this method to other modalities, which also give insight into why common knowledge cannot be added to public announcement logic using reduction axioms.

To add distributed knowledge we add to the given language a modal operator D_B for every non-empty subset $B \subseteq \mathbb{A}$. The semantics of the distributed knowledge operator is:

$$\mathcal{M},w,g \models D_B\varphi \quad \text{iff} \quad \text{for all } v \in W;\ \text{if } (w,v) \in \textstyle\bigcap_{b\in B} R_b \text{ then } \mathcal{M},v,g \models \varphi.$$

The dual operator of D_B will be denoted by $\hat{D}_B$. Note that the semantics of D_B is given in terms of intersection of relations, which in PDL is not modally definable though it is axiomatizable. [11]

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[11] There is no axiom that for all frames $\langle W, R_1, R_2, R_3\rangle$ can force $R_1 = R_2 \cap R_3$, see for instance [9]. However the

However, with nominals intersection becomes easy to modally define (see [9] for more on these issues).

4.1 Adding distributed knowledge the standard way

Axioms for $\mathbf{K}_{\mathcal{PH}(-,D)}$:

All the axioms for $\mathbf{K}_{\mathcal{PH}(-)}$

All the axioms of $\mathbf{K}_{\mathcal{PH}(-)}$ involving K_a, with K_a replaced by D_B (for every $\emptyset \neq B \subseteq \mathbb{A}$)

$\hat{D}_B i \leftrightarrow \bigwedge_{b \in B} \hat{K}_b i$, (for all $i \in \mathsf{NOM}$ and all $\emptyset \neq B \subseteq \mathbb{A}$) DK

Rules for $\mathbf{K}_{\mathcal{PH}(-,D)}$:

All the rules for $\mathbf{K}_{\mathcal{PH}(-)}$

All the rules for $\mathbf{K}_{\mathcal{PH}(-)}$ involving K_a, with K_a replaced by D_B (for every $\emptyset \neq B \subseteq \mathbb{A}$)

Fig. 4. The Hilbert-style proof system for $\mathbf{K}_{\mathcal{PH}(-,D)}$.

In the standard way of adding distributed knowledge we first add distributed knowledge to the language $\mathcal{PH}(-)$; we will use the same approach as for the global modality, simply take the usual modal axioms and rules for the modalities D_B and add additionally pure axioms. The proof system of the logic $\mathbf{K}_{\mathcal{PH}(-,D)}$ is given in figure 4. The completeness proof follows from the general completeness in theorem 2.9 or 2.10, since the only new axiom DK is a pure formula. All that remains to be shown is that DK defines the right frame property. However, this is easily shown and stated as:

Lemma 4.1 *$\hat{D}_B i \leftrightarrow \bigwedge_{b \in B} \hat{K}_b i$ is valid on a frame $\langle W, (R_a)_{a \in \mathbb{A}}, (R_B)_{B \neq \emptyset, B \subseteq \mathbb{A}} \rangle$ if and only if $R_B = \bigcap_{b \in B} R_b$.*

Theorem 4.2 (Completeness of $\mathbf{K}_{\mathcal{PH}(-,D)}$) *Let Σ be a substitution-closed set of pure $\mathcal{PH}(-,D)$-formulas. Every set of $\mathcal{PH}(-,D)$-formulas that is $\mathbf{K}_{\mathcal{PH}(-,D)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

After adding distributed knowledge to $\mathbf{K}_{\mathcal{PH}(-)}$ we can now add it to $\mathbf{K}_{\mathcal{HPAL}(-)}$ using a reduction axiom. A reduction axiom for distributive knowledge similar to the one for K_a can be used, as already noted in [14]. The axiomatization of the Hybrid Public Announcement Logic, including distributed knowledge $\mathbf{K}_{\mathcal{HPAL}(-,D)}$ is shown in figure 5. The soundness of the reduction axiom for D_B is guaranteed by the following lemma:

Lemma 4.3 *For all non-empty $B \subseteq \mathbb{A}$ and all $\mathcal{HPAL}(-,D)$-formulas φ and ψ, $[\varphi]D_B\psi$ is equivalent to $\varphi \to D_B[\varphi]\psi$.*

Proof. The proof is given by the following equivalences:

---

logic obtained by adding distributed knowledge, interpreted as intersection, to epistemic logic can be axiomatized, see for instance [6].

Axioms for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:
All axioms for $\mathbf{K}_{\mathcal{PH}(-,D)}$
All the relevant reduction axioms from figure 2.
$[\varphi]\, D_B\psi \leftrightarrow (\varphi \to D_B[\varphi]\psi)$ [1] Announcement and distributed knowledge

Rules for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:
All rules for $\mathbf{K}_{\mathcal{PH}(-,D)}$

[1] Where B is a non-empty subset of $\mathbb{A}$

Fig. 5. The Hilbert-style proof system for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E,D)}$.

$$\begin{array}{ll}
 & \mathcal{M}, w \models [\varphi] D_B\psi \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}|_{\varphi}, w \models D_B\psi \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \Big(\forall v \in W|_{\varphi}\big[(w,v) \in \bigcap_{b\in B}(R_b \cap (W|_{\varphi})^2) \Rightarrow \mathcal{M}|_{\varphi}, v \models \psi\big]\Big) \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \Big(\forall v \in W|_{\varphi}\big[(w,v) \in (\bigcap_{b\in B} R_b) \cap (W|_{\varphi})^2 \Rightarrow \mathcal{M}|_{\varphi}, v \models \psi\big]\Big) \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \Big(\forall v \in W\big[\mathcal{M}, v \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \cap (W|_{\varphi})^2 \Rightarrow \mathcal{M}|_{\varphi}, v \models \psi\big)\big]\Big) \\
\text{iff} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \Big(\mathcal{M}, v \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \cap (W|_{\varphi})^2 \Rightarrow \mathcal{M}|_{\varphi}, v \models \psi\big)\Big)\Big] \\
\text{iff}^{*} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \Big(\mathcal{M}, v \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \mathcal{M}|_{\varphi}, v \models \psi\big)\Big)\Big] \\
\text{iff} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \Big((w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \big(\mathcal{M}, v \models \varphi \Rightarrow \mathcal{M}|_{\varphi}, v \models \psi\big)\Big)\Big] \\
\text{iff} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \mathcal{M}, v \models [\varphi]\psi\big)\Big] \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \forall v \in W\Big[(w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \mathcal{M}, v \models [\varphi]\psi\Big] \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}, w \models D_B[\varphi]\psi \\
\text{iff} & \mathcal{M}, w \models \varphi \to D_B[\varphi]\psi.
\end{array}$$

In the equivalence "iff*" we have used the fact that $(w,v) \in (W|_{\varphi})^2$ is equivalent to $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, v \models \varphi$. □

With this lemma in place we have soundness of the logic $\mathbf{K}_{\mathcal{HPAL}(-,D)}$ and a completeness theorem for the logic, in the style of theorem 3.4, can be proven in the same way as done in section 3:

Theorem 4.4 (Completeness for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$) *Let Σ be a set of pure $\mathcal{HPAL}(-,D)$-formulas. Every set of $\mathcal{HPAL}(-,D)$-formulas that is $\mathbf{K}_{\mathcal{HPAL}(-,D)}+\Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

4.2 Adding distributed knowledge directly

As the reader might have guessed, there is nothing to prevent adding distributed knowledge directly to $\mathbf{K}_{\mathcal{HPAL}(@,-)}$ using the pure formulas from the previous subsection. In other words an alternative proof system (however resulting in the same axioms and rules as the proof system of figure 5) can be described as in figure 6. Completeness of this proof system follows directly from theorem 3.4. Thus we do not need to prove soundness of the reduction axiom for D_B, as it follows from the soundness of the reduction axiom for K_a.

Axioms for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:
All the axioms for $\mathbf{K}_{\mathcal{HPAL}(-)}$
All the axioms of $\mathbf{K}_{\mathcal{HPAL}(-)}$ involving K_a, with K_a replaced by D_B (for every $\emptyset \neq B \subseteq \mathbb{A}$)
$\hat{D}_B i \leftrightarrow \bigwedge_{b\in B} \hat{K}_b i$, (for all $i \in \mathsf{NOM}$ and all $\emptyset \neq B \subseteq \mathbb{A}$) DK

Rules for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:
All the rules for $\mathbf{K}_{\mathcal{HPAL}(-)}$
All the rules of $\mathbf{K}_{\mathcal{HPAL}(-)}$ involving K_a, with K_a replaced by D_B (for every $\emptyset \neq B \subseteq \mathbb{A}$)

Fig. 6. The alternative Hilbert-style proof system for $\mathbf{K}_{\mathcal{PH}(-,D)}$.

The only thing that has to be verified in this way of adding distributed knowledge is that we get completeness with respect to the right class of frames. There is a little more subtleness to this than in the case of $\mathbf{K}_{\mathcal{PH}(-)}$. Theorem 3.4 only ensures that the axiom DK becomes valid in the underlying frame and not necessarily in all subframes of that frame. However if a frame satisfies that $R_B = \bigcap_{b\in B} R_b$, then all subframes also satisfy this property. Thus the meaning of D_B does not change after public announcement.

To see that this is a real problem, look at the modality $[a;b]$ defined by:

$$\mathcal{M}, w, g \models [a;b]\varphi \quad \text{iff} \quad \text{for all } v \in W;\ \text{if } (w,v) \in R_a;R_b \text{ then } \mathcal{M}, v, g \models \varphi,$$

where $R_a;R_b$ denotes the composition of the relations R_a and R_b defined by $R_a;R_b = \{(x,y) \mid \exists z : (x,z) \in R_a \wedge (z,y) \in R_b\}$. In classical Hybrid Logic this is definable by the pure axiom $\langle a;b\rangle i \leftrightarrow \langle a\rangle\langle b\rangle i$. This axiom is easily seen to be valid exactly on the class of frames where $R_{a;b} = R_a;R_b$. However, just because $R_{a;b} = R_a;R_b$ holds on a frame, does not necessarily imply that it also holds on all subframes.[12] Thus in the scope of a public announcement operator $[\varphi]$ the modality $[a;b]$ will change its meaning in the sense that it does not quantify over the composition of the relations R_a and R_b in the submodel, but over the composition of the relations R_a and R_b in the original model. The problem lies in the fact that composition is not an operation that is preserved when going to submodels contrary to intersection. We will return to this issue i section 4.4.

4.3 The definability of distributed knowledge using satisfaction operators and the downarrow binder

In the case of the logics $\mathbf{K}_{\mathcal{PH}(@,\downarrow,-)}$ (or $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,-)}$) it turns out that distributed knowledge is locally definable. The following proposition states this formally:

Proposition 4.5 *Let $B \subseteq \mathbb{A}$ contain at least 2 elements*[13] *, let $a \in B$, let φ be a $\mathcal{PH}(@,\downarrow,-)$-formula and let x and y be different state variables that do not occur in φ. Then for all models*

---

[12] Take for instance the frame $\langle W, R_a, R_b, R_{a;b}\rangle$, where $W = \{x,y,z\}$, $R_a = \{(x,y)\}$, $R_b = \{(y,z)\}$ and $R_{a;b} = \{(x,z)\}$. Clearly this frame satisfy that $R_{a;b} = R_a;R_b$, but in the subframe only containing the states x and z, we still have $(x,z) \in R_{a;b}$ although $(x,z) \notin R_a;R_b$.

[13] If B only contains b then clearly D_B is definable as K_b.

$\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$, *all assingments* g *in* $\mathcal{M}$ *and all* $w \in W$*:*

$$\mathcal{M}, w, g \models D_B\varphi \quad \textit{iff} \quad \mathcal{M}, w, g \models {\downarrow}x.K_a{\downarrow}y.\big(@_x(\wedge_{b\in B\backslash\{a\}}\hat{K}_b y) \to \varphi\big).$$

Proof. The proof is given by the following equivalences, where g' is just like g except that $g'(x) = w$ and g'' is just like g' except that $g''(y) = v$ (thus g'' is just like g except that $g''(x) = w$ and $g''(y) = v$):

$$
\begin{array}{ll}
 & \mathcal{M}, w, g \models{\downarrow}x.K_a {\downarrow}y.\big(@_x(\wedge_{b\in B\backslash\{a\}}\hat{K}_b y) \to \varphi\big) \\
\text{iff} & \mathcal{M}, w, g' \models K_a {\downarrow}y.\big(@_x(\wedge_{b\in B\backslash\{a\}}\hat{K}_b y) \to \varphi\big) \\
\text{iff} & \forall v \in W\Big[wR_a v \quad\Rightarrow\quad \mathcal{M}, v, g' \models{\downarrow}y.\big(@_x(\wedge_{b\in B\backslash\{a\}}\hat{K}_b y) \to \varphi\big)\Big] \\
\text{iff} & \forall v \in W\Big[wR_a v \quad\Rightarrow\quad \mathcal{M}, v, g'' \models @_x(\wedge_{b\in B\backslash\{a\}}\hat{K}_b y) \to \varphi\Big] \\
\text{iff} & \forall v \in W\Big[wR_a v \quad\Rightarrow\quad \Big[\mathcal{M}, w, g'' \models \wedge_{b\in B\backslash\{a\}}\hat{K}_b y \quad\Rightarrow\quad \mathcal{M}, v, g'' \models \varphi\Big]\Big] \\
\text{iff} & \forall v \in W\,[wR_a v \quad\Rightarrow\quad [\forall b \in B\backslash\{a\}\,\exists s \in W(wR_b s \text{ and } \mathcal{M}, s, g'' \models y) \quad\Rightarrow\quad \mathcal{M}, v, g'' \models \varphi]] \\
\text{iff} & \forall v \in W\,[wR_a v \quad\Rightarrow\quad [\forall b \in B\backslash\{a\}\,\exists s \in W(wR_b s \text{ and } s = v) \quad\Rightarrow\quad \mathcal{M}, v, g'' \models \varphi]] \\
\text{iff}^* & \forall v \in W\,[\forall b \in B(wR_b v) \quad\Rightarrow\quad \mathcal{M}, v, g'' \models \varphi] \\
\text{iff} & \forall v \in W\,[\forall b \in B(wR_b v) \quad\Rightarrow\quad \mathcal{M}, v, g \models \varphi] \\
\text{iff} & \mathcal{M}, w, g \models D_B\varphi,
\end{array}
$$

where we in “iff*” have used that x and y do not occur in φ. □

Thus when adding distributed knowledge to the logics $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,-)}$ we can simply take the formula $D_B\varphi$ to be an abbreviation for the formula ${\downarrow}x.K_a{\downarrow}y.\big(@_x(\wedge_{b\in B\backslash\{a\}}\hat{K}_b y) \to \varphi\big)$. Furthermore, as a corollary, adding distributed knowledge does not add to the expressive power of $\mathbf{K}_{\mathcal{PH}(@,\downarrow,-)}$ or $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,-)}$:

Corollary 4.6 *The logics* $K_{\mathcal{PH}(@,\downarrow,-)}$ *(*$K_{\mathcal{HPAL}(@,\downarrow,-)}$*) and* $K_{\mathcal{PH}(@,\downarrow,D,-)}$ *(*$K_{\mathcal{HPAL}(@,\downarrow,D,-)}$*) are equally expressive.*

4.4 A general way of adding modalities to public announcement logic

In the paper [8] Barteld Kooi gives a general framework for showing completeness and expressiveness results for logics using already given reduction axioms. However, what seems as a next natural step, which is not considered in [8], is the actual question of how to find reduction axioms for a given modal operator relative to the public announcement operator. In this section we will take a first step towards answering this question by characterizing a class of modalities that have particularly simple reduction axioms. As mentioned in the example with composition in section 4.2 this has to do with whether or not an operation is preserved when moving to submodels.

Note that the axioms for $@_i$ and E look alike and the axioms for K_a and D_B look alike. The difference between these two cases is alone due to the fact that $@_i$ and E are existential modalities, whereas K_a and D_B are universal modalities. Allowing for dual operators, we could write the four reduction axioms as one, namely

$$[\varphi]\Box\psi \quad\leftrightarrow\quad (\varphi \to \Box[\varphi]\psi), \tag{2}$$

where $\Box$ is one of K_a, $\overline{@}_i$, A or D_B. Equivalently we could use the axiom

$$[\varphi]\Diamond\psi \quad \leftrightarrow \quad (\varphi \to \Diamond(\varphi \wedge [\varphi]\psi)), \tag{3}$$

where $\Diamond$ is one of $\hat{K}_a$, $@_i$, E or $\hat{D}_B$. In the proof of soundness of the reduction axiom for distributed knowledge (lemma 4.3) the only property of the semantics of D_B we used was the fact that $\bigcap_{b\in B}(R_b \cap (W|_\varphi)^2) = (\bigcap_{b\in B} R_b) \cap (W|_\varphi)^2$. The soundness of the reduction axioms for K_a, $@_i$ and E can be viewed as consequences of the same property. To show that this property always guarantees reduction axioms of the above form, we need to specify a general framework. Given a model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ when we speak of "a binary relation on $\mathcal{M}$" we simply mean a binary relation on W.

Definition 4.7 A *n-ary model-relation-operation* is an operation that to any model $\mathcal{M}$ and n binary relations on $\mathcal{M}$ assigns a binary relation on $\mathcal{M}$.

An example of such a model-relation-operation is intersection as used in the semantics for distributed knowledge. A n-ary model-relation-operation $\mathbf{Int}_n$ can be defined by $\mathbf{Int}_n(\mathcal{M}, R_1, ..., R_n) = \cap_{i=1...n} R_i$. Let a non-empty $B \subseteq \mathbb{A}$ be given and assume that B has n elements $b_1, ..., b_n$. Then for any model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$, $\cap_{b\in B} R_b = \mathbf{Int}_n(\mathcal{M}, R_{b_1}, ..., R_{b_n})$. Thus the semantics of D_B can alternatively be specified as:

$$\mathcal{M}, w, g \models D_B\varphi \quad \text{iff} \quad \text{for all } v \in W: \text{ if } (w, v) \in \mathbf{Int}_n(\mathcal{M}, R_{b_1}, ..., R_{b_n}) \text{ then } \mathcal{M}, v, g \models \varphi.$$

Fixing a nominal i, a 0-ary model-relation-operation $\mathbf{Nom}_i$ can be defined by $\mathbf{Nom}_i(\mathcal{M}) = W \times \{V(i)\}$, where $W \times \{V(i)\}$ is the relation consisting of all pairs $(w, V(i))$ for $w \in W$ (if $V(i) = \emptyset$ then also $W \times \{V(i)\} = \emptyset$). The semantics of $\overline{@}_i\varphi$ can then be reformulated as:

$$\mathcal{M}, w, g \models \overline{@}_i\varphi \quad \text{iff} \quad \text{for all } v \in W: \text{ if } (w, v) \in \mathbf{Nom}_i(\mathcal{M}) \text{ then } \mathcal{M}, v, g \models \varphi.$$

Similar things can be done for the semantics of E and K_a by defining a 0-ary model-relation-operation $\mathbf{Glo}$ by $\mathbf{Glo}(\mathcal{M}) = W \times W$ and a unary model-relation-operation $\mathbf{Id}_1$ by $\mathbf{Id}_1(\mathcal{M}, R) = R$. Further examples of model-relation-operations are the PDL constructors union ($\cup$), composition (;) and transitive closure (*).

Definition 4.8 An n-ary model-relation-operation $\mathbf{F}$ *respects intersection* if for all models $\mathcal{M}$, all relations $R_1, ..., R_n$ on $\mathcal{M}$ and any $C \subseteq W \times W$:

$$\mathbf{F}(\mathcal{M}, R_1 \cap C, ..., R_n \cap C) = \mathbf{F}(\mathcal{M}, R_1, ..., R_n) \cap C.$$

Note, that all of the model-relation-operations $\mathbf{Int}_n$, $\mathbf{Nom}_i$, $\mathbf{Glo}$ and $\mathbf{Id}$ respect intersection (this is easy to see). This is also the case for the PDL constructor union, but not for the composition and transitive closure.[14]

The property of respecting intersection is the right property to ensure simple reduction axioms. However, before we can state this we need to add modalities based on model-relation-operations to the syntax of the language. To do this we assume a set of function symbols FSYM (all $F \in \mathsf{FSYM}$ is assumed to have a finite arity) which we use to refer to the model-relation-operations. Since these operations also take relations as argument, we need something to fill in as arguments in the function symbols and for this we use the set of agents $\mathbb{A}$ already given. Now to the syntax of our languages $\mathcal{PH}(-)$ and $\mathcal{HPAL}(-)$, we add a new modal operator $[F(a_1, ..., a_n)]$ for each n-ary function symbol $F \in \mathsf{FSYM}$ and each $a_1, ..., a_n \in \mathbb{A}$.

To give semantics for the new modalities we need to interpret the function symbols. For this we assume a *relation-interpretation* $\mathcal{I}$ that assigns a n-ary model-relation-operation $\mathcal{I}(F)$

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[14] In the case of the composition constructor take for instance $W = \{x, y, z\}$, $R_a = \{(x, y)\}$ and $R_b = \{(y, z)\}$. Then $R_a; R_b = \{(x, z)\}$ and thus $(R_a; R_b) \cap \{x, z\}^2 = \{(x, z)\}$. But $R_a \cap \{x, z\}^2 = \emptyset$ and $R_b \cap \{x, z\}^2 = \emptyset$, so $(R_a \cap \{x, z\}^2); (R_b \cap \{x, z\}^2) = \emptyset$.

to each $F \in \mathsf{FSYM}$ of arity n. With this fixed relation-interpretation $\mathcal{I}$ we can define the semantics of the new modalities by:

$$\begin{aligned} & \mathcal{M}, w, g \models [F(a_1, ..., a_n)]\varphi \\ \text{iff} \quad & \text{for all } v \in W: \text{ if } (w, v) \in \mathcal{I}(F)\big(\mathcal{M}, R_{a_1}, ..., R_{a_n}\big) \text{ then } \mathcal{M}, v, g \models \varphi. \end{aligned} \tag{4}$$

Modalities defined using model-relation-operations that respect intersection have very simple reduction axioms in the form of (2) and (3). However, one cannot just add these reduction axioms to get a sound and complete logic, there has to be a sound and complete axiom system for the underlying logic as well (in some cases, such as distributed knowledge, this is easy since the modality is definable by pure formulas). But with this in mind our considerations can be summarized in the following proposition:

Proposition 4.9 *Let $F \in \mathsf{FSYM}$ be of arity n and such that $\mathcal{I}(F)$ respects intersection and let $a_1, ..., a_n \in \mathbb{A}$. Assume furthermore that there is a sound and complete axiom system for the logic $\mathbf{K}_{\mathcal{PH}(-,[F(a_1,...,a_n)])}$, then a sound and complete axiom system for the logic $\mathbf{K}_{\mathcal{HPAL}(-,[F(a_1,...,a_n)])}$ can be obtained by adding the reduction axiom*

$$[\varphi][F(a_1, ..., a_n)]\psi \;\leftrightarrow\; (\varphi \rightarrow [F(a_1, ..., a_n)][\varphi]\psi)$$

(together with the other relevant reduction axioms) to the axiom system of $\mathbf{K}_{\mathcal{PH}(-,[F(a_1,...,a_n)])}$. Furthermore the logic $\mathbf{K}_{\mathcal{HPAL}(-,[F(a_1,...,a_n)])}$ is no more expressive than $\mathbf{K}_{\mathcal{PH}(-,[F(a_1,...,a_n)])}$.

Proof. The proof is similar to the proof of theorem 4.4 based on a rewriting of the proof of lemma 4.3 using the assumption that $\mathcal{I}(F)$ respects intersection. □

This proposition gives a uniform way of adding $@_i$, E and D_B to the logic $\mathbf{K}_{\mathcal{HPAL}(-)}$. Furthermore since the PDL operator "∪" also respects intersection, the epistemic modality E_B reading "everybody amongst B knows that..." can also be added with a reduction axiom of the form $[\varphi]E_B\psi \leftrightarrow (\varphi \rightarrow E_B[\varphi]\psi)$.[15] Since the operator of composition does not respect intersection, we do not obtain a reduction axiom in the style of the ones for D_B or E_B.

We have presented the proposition as an extension of $\mathbf{K}_{\mathcal{HPAL}(-)}$, however, it is clear that it works for any extension of just classical PAL (without common knowledge). Another important remark is that we have presented the proposition in the setting of the basic logic $\mathbf{K}$, and it cannot just be extended to arbitrary extensions of $\mathbf{K}$. If we require that the relations $(R_a)_{a\in\mathbb{A}}$ of our models satisfy a certain property (like reflexivity or transitivity etc.), we cannot always be sure that the restricted relations $R_a \cap (W|_\varphi)^2$ also satisfy this property. Thus if the property is not preserved when taking intersection the proposition does not apply. Nevertheless, if it is preserved we can extend the result beyond $\mathbf{K}$. One example is assuming that all the relations R_a for $a \in \mathbb{A}$ are equivalence relations, which is normally done in Epistemic Logic. Here there is no problem since restricting an equivalence relation R_a to $R_a \cap (W|_\varphi)^2$ gives rise to an equivalence relation. Finally, note that the proposition only provides sufficient and not necessary conditions for the existence of reduction axioms. Finding necessary conditions is left for further research.

4.5 A note on common knowledge

We now turn to common knowledge. For a non-empty subset $B \subseteq \mathbb{A}$, the common knowledge operator C_B is added to the language, with the reading of $C_B\varphi$ as "it is common knowledge

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[15] Note, however, that since $E_B\varphi$ is directly definable as $\bigwedge_{b\in B} K_b\varphi$, it is not necessary to have the modality explicit in the language as in the case of distributed knowledge in the logic $\mathbf{K}_{\mathcal{PH}(@,\downarrow,-)}$.

among the agents in B that φ". C_B has the following semantics:

$$\mathcal{M}, w, g \models C_B\varphi \quad \text{iff} \quad \text{for all } v \in W;\ \text{if } (w,v) \in \left(\bigcup_{b\in B} R_b\right)^* \text{ then } \mathcal{M}, v, g \models \varphi,$$

where R^* denotes the reflexive transitive closure of the relation R.

Problems arise when one wants to combine public announcement logic with common knowledge in the sense that we cannot prove completeness using reduction axioms anymore. Reduction axioms for common knowledge simply do not exist. One solution is to generalize the notion of common knowledge to what is called relativized common knowledge. Relativized common knowledge is exactly the notion needed to get completeness via reduction axioms for public announcement, see [14]. We will not take on the enterprise of adding common knowledge or relativized common knowledge to $\mathbf{K}_{\mathcal{PH}(-)}$ or $\mathbf{K}_{\mathcal{HPAL}(-)}$. We simply mention common knowledge because the concept of respecting intersection makes it clear why reduction axioms such as the ones for K_a, E and D_B do not work for common knowledge. Common knowledge corresponds to the PDL operator of transitive closure, which does not respect intersection and cannot otherwise be defined.

5 Conclusion and further work

In this paper it has been shown that nominals, satisfaction operators, the downarrow binder, the global modality, and distributed knowledge can be added to the Public Announcement Logic. Furthermore general completeness results for extensions with pure formulas, a well celebrated result in Hybrid Logic, also transfer to the case of Hybrid Public Announcement Logic. Properties of both Hybrid Logic and Public Announcement Logic are thus preserved in the combination. The completeness is shown using reduction axioms as in classical Public Announcement Logic. Hence the public announcement operator does not increase the expressive power when added to Hybrid Logic. Using the terminology of [13], classical Hybrid Llogic is not closed under relativization because nominals might lose their references in submodels, but relaxing Hybrid Logic to a logic with only partially denoting nominals, Hybrid Logic does become closed under relativization. Thus, the fact that the $[\varphi]$-operator does not add expressivity is preserved in hybrid extensions of the basic multi-modal logic.

That the nice properties of Hybrid Logic are preserved in the combination with Public Announcement Logic adds significantly to the proof theory of Public Announcement Logic. We have demonstrated this by adding distributed knowledge via pure formulas. It was also shown that distributed knowledge could actually be defined using satisfaction operators and the downarrow binder. That Hybrid Logic has much to offer the proof theory of Public Announcement Logic is also demonstrated by the tableau system developed in [7], but surely there is still much more that Hybrid Logic can offer to the proof theory of Public Announcement Logic. This is left for future research. Finally a sufficient requirement for the existence of reduction axioms in a general setting has been discussed. This naturally leads to the question of whether there is a semantic requirement to put on the operations of section 4.4 that exactly characterizes the operations that allow reduction axioms. We leave this as further work as well.

Another line of further research is to add common knowledge to the Hybrid Public Announcement Logic. However, as mentioned, this might not allow completeness via reduction axioms. Besides adding common knowledge there is also the question of extending the logic from Public Announcement Logic to full Dynamic Epistemic Logic. The problem here is that in full Dynamic Epistemic Logic there are epistemic actions that can expand a state into several states, and thus it is not clear anymore what nominals should denote.

Acknowledgment

This paper has occurred in many draft versions and several people have commented on various versions among others Olivier Roy, Jiahong Guo, Mai Ajspur, and Sine Zambach. The participants of the ILLC seminar on Logics for Dynamics of Information and Preferences 2009, the participants of the HYLOCORE 2009 workshop, and the participants of the HyLo 2010 workshop have also provided useful comments on presentations of this work. A special thanks to Torben Braüner for many useful comments and suggestions on all the versions of the paper. Finally, I would like to thank a number of anonymous reviews for very useful comments.

The author is partially funded by the Danish Natural Science Research Council through the HYLOCORE project.

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6 Appendix: Alternative semantics for the public announcement operator

In this section we discuss the relationship between the standard semantics and an alternative semantics for the public announcement operator.

Let us first fix the terminology. The semantics already introduced will be referred to as *the standard semantics*. By *the alternative semantics* we refer to the semantics of classical hybrid logic together with the following semantics for the public announcement operator $[\varphi]$:

$$\mathcal{M}, w \models [\varphi]\psi \quad \iff \quad \mathcal{M}, w \models \varphi \text{ implies that } \mathcal{M}|_\varphi, w \models \psi,$$

where $\mathcal{M}|_\varphi$ is the model $\mathcal{M} = \langle W, (R_a|_\varphi)_{a \in \mathbb{A}}, V \rangle$, where

$$R_a|_\varphi = R_a \cap (\{w \in W \mid \mathcal{M}, w \models \varphi\} \times \{w \in W \mid \mathcal{M}, w \models \varphi\}).$$

For the global modality it is easy to see that the logic obtained with the alternative semantics differs from the standard one. In the standard semantics the formula $[p]Ap$ is valid, but with the alternative semantics it is no longer valid. In a similar way we have that $[p]E\neg p$ becomes satisfiable. So after updating with the fact p there is still a state where p is false, which seems contra intuitive. In general publicly announcing a formula involving higher order knowledge might lead to it becoming false, but p is a propositional fact about the worlds which normally are assumed to be unchangeable by just announcements. This is also so in classical PAL. Thus when including the global modality it is more reasonable to use the standard semantics for the public announcement operator.

With the alternative semantics the satisfaction operator also get strange properties. Before an announcement it might be the case that $@_i p \wedge @_i \neg K_a p$, i.e. agent a does not know p at the state i. However, after announcing that the actual state is not i (i.e. a public announcement of $\neg i$) it becomes true that $@_i K_a p$ in the alternative semantics. This is essentially due to the following validity in the alternative semantics: $[\neg i](@_i \varphi \to @_i K_a \varphi)$. Thus information about which state is not the case gives complete information about the world at that state to every agent. Note that in the standard semantics announcing $\neg i$ simply makes every formula of the form $@_i \varphi$ false, which may not be a completely pleasing solution, but still the best one to the authors opinion.

Comparing the two logics related to the two semantics, note that $[p]Ap$ is valid in the standard semantics but not in the alternative one. On the other hand $Ep \to [\varphi]Ep$ is valid in the alternative semantics, but not in the standard one. Thus the two logics are simply different; none of them are contained in the other. Furthermore, and also important, it does not seem entirely clear how to derive reduction axioms for the logic of the alternative semantics.