Introduction to Predicate Logic

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Plan

• Symbolization of sentences
• Syntax
• Semantics
Propositional logic

The sentence is the smallest syntactic unit

Predicate logic

Sentences are broken further down into

-- constants

-- variables

-- predicates

-- quantifiers
Singular sentences

A *singular* sentence is built from a constant and a predicate

-- the constant refers to a thing

-- the predicate indicates a property

Such a sentence says that the thing referred to has the indicated property

Note: There are no restrictions on what things can be!
An example of a singular sentence

The sentence

"Mars is round"

Is build from the constant

"Mars"

and the predicate

"_ is round"
Symbolization of singular sentences

a, b, c, ... stands for constants

P, Q, R, ... stands for predicates

*Singular sentences are symbolized*

\[ P(a), Q(b), R(c), ... \]

*which respectively stand for*

"a has the property P",
"b has the property Q",
"c has the property R",

etc.
Example of symbolization

If \( a \) is the constant

"Mars"

and \( P \) is the predicate

"_ is round"

then \( P(a) \) symbilizes

"Mars is round"
General sentences

A general sentence is built from a quantifier and a predicate.

Such a sentence either says that

all things have the indicated property

(universal quantification)

or says that

there exists a thing that has the indicated property

(existential quantification)
An example of a general sentence

The sentence

"every thing is round"

is built from the quantifier

"every thing"

And the predicate

"_ is round"
Symbolization of general sentences

\( x, y, z, \ldots \) stands for variables

\( \forall x \) and \( \exists y \) stand for respectively 

"for all \( x \) it is the case that" and "there exists a \( y \) such that"

**General sentences are symbolized**

\[ \forall x \ P(x), \exists y \ Q(y), \ldots \]

**which respectively stand for**

"for all \( x \) it is the case that \( x \) has the property \( P \)”,

"there exists a \( y \) such that \( y \) has the property \( Q \)”,

etc.
Example of symbolization

If $P$ is the predicate

"_ is round"

then $\forall x \; P(x)$ symbolizes the sentence

"for all $x$ it is the case that $x$ is round"

that is

"every thing is round"
There are many more sorts of sentences that singular and general sentences!
A predicate can have an arbitrary number of places, this number is called the **arity** of the predicate.

**Examples**

The sentence "John is taller than Poul" is built from the constants

"John" og "Poul"

and the predicate

"_ is taller than _"

which has the arity 2

Note: A predicate with arity 0 is a complete sentence.
Predicate logic allows

-- predicates of all arities

-- arbitrary composition of formulas using.

  * the connectives of propositional logic

  * quantifiers
Syntax of predicate logic

constants $a, b, c, ...$
variables $x, y, z, ...$
predicates $P, Q, R, ...$

If a predicate $P$ has arity $n$, then $P$ followed by an $n$-tuple of constants and variables is an atomic formula.

Examples of atomic formulas

$P(a), P(x), Q(b, z), Q(y, z)$

where $P$ has arity 1 and $Q$ has arity 2
Compound formulas are built using the connectives

\[ \neg, \land, \lor, \Rightarrow \]

and the quantifiers

\[ \forall x, \exists y \]

together with parantheses

Examples of compound formulas

\[ P(x) \land Q(b, z), \forall x \ P(x), \exists z \ Q(b, z) \]

Sentences can now be symbolized (like in propositional logic)
An occurrence of a variable in a formula is either free or bound

-- all occurrences in atomic formulas are free

-- all free occurrences $\varphi$ are free in $\neg \varphi$

-- all free occurrences in $\varphi$ and $\psi$ are free in $\varphi \land \psi$
  (analogously for the connectives $\lor$ and $\Rightarrow$)

-- all free occurrences in $\varphi$, save occurrences of $x$, are free in $\forall x \varphi$
  (analogously for $\exists x$)

A formula where no variables occur free is closed
Examples

The occurrence of $x$ in the formula $Q(b,x)$ is free whereas the occurrence of $x$ in $\forall x \ Q(b,x)$ is bound.

Thus, $Q(b,x)$ is not closed, but $\forall x \ Q(b,x)$ is closed.
\( \varphi[a/x] \) is the formula \( \varphi \) where all free occurrences of the variable \( x \) are replaced by the constant \( a \)

**Examples**

\( Q(b,x)[a/x] \) is the formula \( Q(b,a) \)

\( Q(x,x)[a/x] \) is the formula \( Q(a,a) \)

\( Q(b,x)[a/y] \) is the formula \( Q(b,x) \)

\( (\forall x \ Q(b,x))[a/x] \) is the formula \( \forall x \ Q(b,x) \)
The semantics of predicate logic

A model is a non-empty set $D$ together with

-- ekstensions $|a|$, $|b|$, $|c|$, ... for the constants $a$, $b$, $c$, ...
  (the extension $|a|$ of a constant is an element in $D$)

-- ekstensions $|P|$, $|Q|$, $|R|$, ... for the predicates $P$, $Q$, $R$, ...
  (the extension $|P|$ of a predicate $P$ with arity $n$ is a set of $n$-tuples of elements in $D$)

*Intuition:*

The expansion of a constant is the thing referred to

The extension of a predicate is the set of things having the indicated property
Given a model

\((D, |a|, |b|, |c|, \ldots, |P|, |Q|, |R|, \ldots)\)

a closed formula can be assigned a truth-value

**Atomic formulas:**

\( P(a_1, \ldots, a_n) \) is true if and only if \((|a_1|, \ldots, |a_n|) \in |P|\)

**Compound formulas:**

The truth-conditions for \(\neg, \land, \lor, \Rightarrow\) are as in propositional logic

\(\forall x \varphi\) is true if and only if \(\varphi[a/x]\) is true for any \(a\)

\(\exists x \varphi\) is true if and only if there exists an \(a\) such that \(\varphi[a/x]\) is true

(Assumption: Any element of \(D\) is referred to by a constant)
Two important definitions

Let $\varphi$ and $\psi_1, \ldots, \psi_m$ be closed formulas

$\varphi$ is **valid** if and only if $\varphi$ is true in any model

$\varphi$ is a **logical consequence** of $\psi_1, \ldots, \psi_m$ if and only if $\varphi$ is true for all models where $\psi_1, \ldots, \psi_m$ are all true

(That is, if and only if the formula $(\psi_1 \land \ldots \land \psi_m) \Rightarrow \varphi$ is valid)