Introduction to Predicate Logic

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Plan

- Symbolization of sentences
- Syntax
- Semantics

Propositional logic

The sentence is the smallest syntactic unit

Predicate logic

Sentences are broken further down into

-- <u>constants</u>

-- <u>variables</u>

-- <u>predicates</u>

-- <u>quantifiers</u>

Singular sentences

A singular sentence is built from a constant and a predicate

-- the constant refers to a thing

-- the predicate indicates a property

Such a sentence says that the thing referred to has the indicated property

Note: There are no restrictions on what things can be!

An example of a singular sentence

The sentence

”Mars is round”

Is build from the constant

”Mars”

and the predicate

”_ is round”

Symbolization of singular sentences

a, b, c, ... stands for constants

P, Q, R, ... stands for predicates

Singular sentences are symbilized

$P(a)$, $Q(b)$, $R(c)$, ...

which respectively stand for

”a has the property P”,
”b has the property Q”,
”c has the property R”,

etc.

Example of symbolization

If `a` is the constant

”Mars”

and `P` is the predicate

”_ is round”

then `P(a)` symbilizes

”Mars is round”

General sentences

A general sentence is built from a quantifier and a predicate

Such a sentence either says that

all things have the indicated property

(universal quantification)

or says that

there exists a thing that has the indicated property

(existential quantification)

An example of a general sentence

The sentence

”every thing is round”

is built from the quantifier

”every thing”

And the predicate

”_ is round”

Symbolization of general sentences

x, y, z, ... stands for variables

$\forall x$ and $\exists y$ stand for respectively

”for all x it is the case that” and ”there exists a y such that”

General sentences are symbolized

$\forall x\, P(x)$, $\exists y\, Q(y)$, ...

which respectively stand for

”for all x it is the case that x has the property P”,
”there exists a y such that y has the property Q”,

etc.

Example of symbolization

If $\mathtt{P}$ is the predicate

”_ is round”

then $\forall \mathtt{x}\ \mathtt{P}(\mathtt{x})$ symbolizes the sentence

”for all $\mathtt{x}$ it is the case that $\mathtt{x}$ is round”

that is

”every thing is round”

There are many more sorts of sentences that singular and general sentences!

A predicate can have an arbitrary number of places, this number is called the <u>arity</u> of the predicate

Examples

The sentence ”John is taller than Poul” is built from the constants

”John” og ”Poul”

and the predicate

”_ is taller than _”

which has the arity 2

Note: A predicate with arity 0 is a complete sentence

Predicate logic allows

-- predicates of all arities

-- arbitrary composition of formulas using.

* the connectives of propositional logic
* quantifiers

Syntax of predicate logic

constants `a`, `b`, `c`, ...
variables `x`, `y`, `z`, ...
predicates `P`, `Q`, `R`, ...

If a predicate `P` has arity n, then `P` followed by an n-tuple of constants and variables is an atomic formula

Examples of atomic formulas

`P(a)`, `P(x)`, `Q(b,z)`, `Q(y,z)`

where `P` has arity 1 and `Q` has arity 2

Compound formulas are built using the connectives

$\neg, \ \wedge, \ \vee, \ \Rightarrow$

and the quantifiers

$\forall x, \exists y$

together with parantheses

Examples of compound formulas

$P(x) \wedge Q(b,z), \forall x\ P(x), \ \exists z\ Q(b,z)$

Sentences can now be symbolized (like in propositional logic)

An <u>occurrence</u> of a variable in a formula is either <u>free</u> or <u>bound</u>

-- all occurrences in atomic formulas are free

-- all free occurrences φ are free in $\neg\varphi$

-- all free occurrences in φ and ψ are free in $\varphi\wedge\psi$
(analogously for the connectives $\vee$ and $\Rightarrow$)

-- all free occurrences in φ, save occurrences of x,
are free in $\forall x\, \varphi$
(analogously for $\exists x$)

A formula where no variables occur free is <u>closed</u>

Examples

The occurrence of x in the formula

$Q(b,x)$

Is free whereas the occurrence of x in

$\forall x\ Q(b,x)$

Is bound

Thus, $Q(b,x)$ is not closed, but $\forall x\ Q(b,x)$ is closed

$\varphi[a/x]$ is the formula φ where all free occurrences of the variable x are replaced by the constant a

Examples

$Q(b,x)[a/x]$ is the formula $Q(b,a)$

$Q(x,x)[a/x]$ is the formula $Q(a,a)$

$Q(b,x)[a/y]$ is the formula $Q(b,x)$

$(\forall x\, Q(b,x))[a/x]$ is the formula $\forall x\; Q(b,x)$

The semantics of predicate logic

A model is a non-empty set *D* together with

-- ekstensions |`a`|, |`b`|, |`c`|, ... for the constants `a`, `b`, `c`, ...
(the extension |`a`| of a constant is an element in *D*)

-- ekstensions |`P`|, |`Q`|, |`R`|, ... for the predicates `P`, `Q`, `R`, ...
(the extension |`P`| of a predicate `P` with arity n
is a set of n-tuples of elements in *D*)

Intuition:

The expension of a constant is the thing referred to

The extension of a predicate is the set of things having the indicated property

Given a model

(D, |a|, |b|, |c|, ..., |P|, |Q|, |R|, ...)

a closed formula can be assigned a truth-value

Atomic formulas:

$\mathtt{P}(\mathtt{a}_1, \ldots, \mathtt{a}_n)$ is true if and only if $(|\mathtt{a}_1|, \ldots, |\mathtt{a}_n|) \in |\mathtt{P}|$

Compound formulas:

The truth-conditions for $\neg$, $\wedge$, $\vee$, $\Rightarrow$ are as in propositional logic

$\forall \mathtt{x}\ \varphi$ is true if and only if $\varphi\,[\mathtt{a}/\mathtt{x}]$ is true for any $\mathtt{a}$
$\exists \mathtt{x}\ \varphi$ is true if and only if there exists an $\mathtt{a}$ such that $\varphi\,[\mathtt{a}/\mathtt{x}]$ is true

(Assumption: Any element of D is referred to by a constant)

Two important definitions

Let φ and $\psi_1, ..., \psi_m$ be closed formulas

φ is <u>valid</u> if and only if φ is true in any model

φ is a <u>logical consequence</u> of $\psi_1, ..., \psi_m$ if and only if φ is true for all models where $\psi_1, ..., \psi_m$ are all true

(That is, if and only if the formula $(\psi_1 \wedge ... \wedge \psi_m) \Rightarrow \varphi$ is valid)